Pareto envelopes in simple polygons*

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Abstract

For a set T of n points in a metric space (X, d), a point $y \in X$ is *dominated* by a point $x \in X$ if $d(x, t) \leq d(y, t)$ for all $t \in T$ and there exists $t' \in T$ such that $d(x, t') < d(y, t')$. The set of non-dominated points of X is called the *Pareto envelope* of T. H. Kuhn (1973) established that in Euclidean spaces, the Pareto envelopes and the convex hulls coincide. Chalmet et al. (1981) characterized the Pareto envelopes in the rectilinear plane $(\mathbb{R}^2, d_1)$ and constructed them in $O(n \log n)$ time. In this note, we investigate the Pareto envelopes of point-sets in simple polygons P endowed with geodesic d_2- or d_1-metrics (i.e., Euclidean and Manhattan metrics). We show that Kuhn's characterization extends to Pareto envelopes in simple polygons with d_2-metric, while that of Chalmet et al. extends to simple rectilinear polygons with d_1-metric. These characterizations provide efficient algorithms for construction of these Pareto envelopes.

1 Introduction

Convex hulls, in particular convex hulls in 2- and 3-dimensional spaces, are used in various applications and represent a basic object of investigations in computational geometry. They host such remarkable points as center, barycenter, and median as well as the optimal solutions of some NP-hard problems like the Steiner tree, the p-median, and the p-center problems. H. Kuhn [13] noticed that conv(T) can be described in truly distance terms: a point $p \in \mathbb{R}^m$ belongs to conv(T) if and only if the vector of Euclidean distances of p to the points of T is not dominated by the distance vector of any other point of $\mathbb{R}^m$. Inspired by this characterization of conv(T), one can define analogous geometric objects by replacing the Euclidean distance d_2 by any other distance d on $\mathbb{R}^m$, or by replacing $\mathbb{R}^m$ by a polygonal or a polyhedral domain endowed with an intrinsic distance. This leads to the following general concept of Pareto envelope. Given a set T of n points in a metric space (X, d), a point $y \in X$ is *dominated* by a point $x \in X$ if $d(x, t) \leq d(y, t)$ for all $t \in T$ and there exists $t' \in T$ such that $d(x, t') < d(y, t')$. The set of non-dominated points of X is called the *Pareto envelope* of T and is denoted by $\mathcal{P}_d(T)$.

Pareto envelopes have been investigated in several papers under the name of "sets of efficient points". Thisse, Ward, and Wendell [17] proved that $\mathcal{P}_{d_2}(T) =$ conv(T) holds for all distances induced by round norms. The investigation of Pareto envelopes for particular polyhedral norms has been initiated by Wendell, Hurter, Lowe [21] and continued by Chalmet, Francis, Kolen [2] and Durier, Michelot [6, 7]. The main result of [2] is the following nice characterization of Pareto envelopes in the Manhattan plane:

$$\mathcal{P}_{d_1}(T) = \cap_{i=1}^n (\cup_{j=1}^n I_{d_1}(t_i, t_j)), \tag{1}$$

where $I_{d_1}(t_i, t_j)$ is the smallest axis-parallel rectangle with diagonal $[t_i, t_j]$. This result was used in [2] to establish the correctness of an optimal $O(n \log n)$ sweeping-line algorithm for constructing $\mathcal{P}_{d_1}(T)$ in $\mathbb{R}^2$. Consequently, Pelegrin and Fernandez [14] described an algorithm for constructing Pareto envelopes in the plane endowed with a polygonal norm. Recently, Chepoi and Nouioua [5] characterized $\mathcal{P}_{d_1}(T)$ in $(\mathbb{R}^3, d_1)$ and showed that the characterization of Chalmet et al. [2] holds for $\mathcal{P}_{d_\infty}(T)$ in $(\mathbb{R}^m, d_\infty)$. They also presented efficient algorithms for constructing $\mathcal{P}_{d_1}(T)$ and $\mathcal{P}_{d_\infty}(T)$ in $\mathbb{R}^3$. We refer to [5] for other references on Pareto envelopes in normed spaces and their applications.

In this note, we characterize and efficiently construct the Pareto envelopes of sets in simple polygons endowed with the geodesic d_2 and d_1-distances. Distance problems for simple polygons constitute a classical subject in computational geometry; [9, 11, 15, 16, 18] is a small sample of papers devoted to this subject. We show that, like in Euclidean spaces, Pareto envelopes of finite sets in simple polygons with d_2-distance coincide with their geodesic convex hulls and therefore can be constructed using an algorithm of Toussaint [18]. On the other hand, we show that Pareto envelopes in simple rectilinear polygons can be characterized using equality (1). This characterization is used to design an efficient algorithm for constructing these envelopes. Due to space constraints, the proofs of several results in last section are postponed to the full version.

We conclude this section with some definitions. Let (X, d) be a metric space. The *interval* $I(x, y)$ between two points $x, y \in X$ consists of all points *between* x

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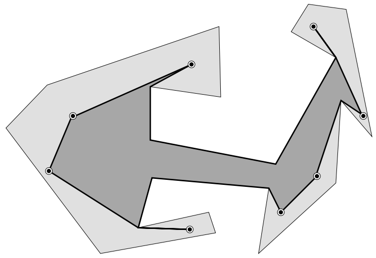

Figure 1: Example of $\mathcal{P}_{d_2}(T)$

and y: $I(x,y) := \{u \in X : d(x,u)+d(u,y) = d(x,y)\}$. A set M of X is *convex* if $I(x,y) \subseteq M$ for all $x,y \in M$. The *convex hull* $\text{conv}(S)$ of a set $S \subset X$ is the smallest convex set containing S.

2 Simple polygons

In this section, P is a simple polygon with m sides endowed with the geodesic d_2-metric. For two points $x,y \in P$, $\gamma(x,y)$ is the unique geodesic path inside P between x and y, and $d_2(x,y)$ is the length of this path. For a set of n points $T \subset P$, we denote by $\text{conv}(T)$ and $\mathcal{P}_{d_2}(T)$ the *geodesic convex hull* and the Pareto envelope of T. Since two points of a simple polygon P are connected by a unique geodesic, (P,d_2) is a metric space of global non-positive curvature, i.e. a *CAT(0)-space* [1]. CAT(0) spaces are characterized in several ways (in particular, by uniqueness of geodesic paths, convexity of the distance function, etc.) and have many important properties, placing them in the center of modern geometry; for results and definitions the reader can consult the book [1]. Below we will show that $\mathcal{P}_d(T) \subseteq \text{conv}(T)$ holds for any finite subset of a CAT(0)-space (X,d) and we conjecture that in fact $\mathcal{P}_d(T) = \text{conv}(T)$ holds.

2.1 $\mathcal{P}_{d_2}(T) = \text{conv}(T)$

We aim to establish the following result:

Proposition 1 $\mathcal{P}_{d_2}(T) = conv(T)$. *Consequently, $\mathcal{P}_{d_2}(T)$ can be constructed in $O(m + n\log m)$-time.*

The inclusion $\mathcal{P}_{d_2}(T) \subseteq \text{conv}(T)$ follows from the following more general result:

Lemma 1 $\mathcal{P}_d(T) \subseteq conv(T)$ *for any finite set of a CAT(0) metric space (X,d).*

Proof. Let $x \notin \text{conv}(T)$. By Proposition 2.4(1) of [1] there exists a unique point $\pi(x)$ (the metric projection of x) such that $d(x,\pi(x)) = \inf_{y\in \text{conv}(T)} d(x,y)$. As in the case of Euclidean spaces, $\pi(x)$ can be viewed as the orthogonal projection of x on $\text{conv}(T)$, because by Proposition 2.4(3) the Alexandrov angle α at $\pi(x)$ between the geodesics $\gamma(x,\pi(x))$ and $\gamma(y,\pi(x))$ is at least $\pi/2$ for any point $y \in \text{conv}(T), y \neq \pi(x)$. By law of cosines which holds in CAT(0) spaces (page 163 of [1]), if $a = d(x,\pi(x)), b = d(y,\pi(x))$, and $c = d(x,y)$, then $c^2 \geq a^2 + b^2 - 2ab\cos\alpha \geq a^2 + b^2 > b^2$ for any $y \in \text{conv}(T), y \neq \pi(x)$. Hence $d(x,y) > d(\pi(x),y)$, i.e., x is dominated by $\pi(x)$. Since x is an arbitrary point outside $\text{conv}(T)$, this implies that $\mathcal{P}_d(T) \subseteq \text{conv}(T)$. □

Now we show the converse inclusion $\text{conv}(T) \subseteq \mathcal{P}_{d_2}(T)$. Pick $q \in \text{conv}(T)$. If q belongs to the boundary of $\text{conv}(T)$, then q belongs to the geodesic path $\gamma(t,t')$ between two vertices t,t' of $\text{conv}(T)$. Since $t,t' \in T$, if q is dominated by some point p, then $d_2(p,t) \leq d_2(q,t)$ and $d_2(p,t') \leq d(q,t')$. Since $q \in \gamma(t,t')$, this is possible only if these inequalities hold as equalities, thus $p \in \gamma(t,t')$, yielding $p = q$. Thus $q \in \mathcal{P}_{d_2}(T)$ in this case. Now, suppose that q belongs to the interior of the simple polygon $\text{conv}(T)$. Suppose by way of contradiction that q is dominated by some point $p' \in P$. By Lemma 1 of [15] the distance function d_2 on P is convex. This means that for any point $t \in T$, as p varies along the geodesic $\gamma(p',q)$, $d_2(t,p)$ is a convex function of p. Since q belongs to the interior of $\text{conv}(T)$, one can select a point $p \in \gamma(p',q) \cap \text{conv}(T)$ which still dominates q and is visible from q (i.e., $[p,q] \subseteq P$). Denote by q' the first intersection of the boundary of $\text{conv}(T)$ with the ray with origin p which passes via the point q. By the definition of q', we infer that $q \in [p,q'] = \gamma(p,q')$. Pick any point $t \in T$. By second part of Lemma 1 of [15], $d_2(t,q) < \max\{d_2(t,q'), d_2(t,p)\}$. Since $d_2(t,p) \leq d_2(t,q)$ by the choice of p, we obtain that $d_2(t,q) < d_2(t,q')$. Since this inequality holds for all points of T, q and p both dominate the boundary point q', a contradiction with $q' \in \mathcal{P}_{d_2}(T)$.

G. Toussaint [18] presented an $O(m+n\log m)$-time algorithm for constructing the geodesic convex hull of an n-point set T of a simple polygon P with m sides. Together with Proposition 1 this shows that $\mathcal{P}_{d_2}(T)$ can be constructed within the same time bounds.

3 Simple rectilinear polygons

In this section, P is a simple rectilinear polygon (i.e., a simple polygon having all edges axis–parallel) with m edges endowed with the geodesic d_1-metric.. A *rectilinear path* is a polygonal chain consisting of axis–parallel segments lying inside P. The length of a rectilinear path in the d_1-metric equals the sum of the lengths of its constituent segments. For two points $x,y \in P$, the *geodesic d_1-distance* $d_1(x,y)$ is the length of the minimum length rectilinear path (i.e., rectilinear geodesic) connecting x and y. An axis–parallel segment c is a *cut segment* of P if it connects two edges of P and lies entirely in P. One basic property of resulting metric space (P,d_1) is that its axis-parallel cuts and the two subpolygons defined by such cuts are convex and gated [4]. A subset M of a metric space (X,d) is called *gated* [19] provided every point $v \in X$ admits a *gate* in M, i.e. a point $g(v) \in M$ such that

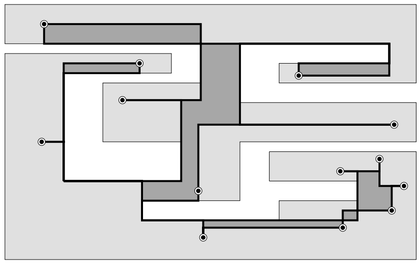

Figure 2: Example of $\mathcal{P}_{d_1}(T)$

$g(v) \in I(v,u)$ for all $u \in M$.

3.1 Characterization

We extend the characterization of [2] to Pareto envelopes $\mathcal{P}_{d_1}(T)$ in rectilinear polygons:

Proposition 2 $\mathcal{P}_{d_1}(T) = \cap_{i=1}^n(\cup_{j=1}^n I(t_i,t_j))$.

Proof. One direction of the proof is obvious: if p is not Pareto, then $p \notin \cup_{j=1}^n I(t_i,t_j)$. To prove the converse, let $p \in \mathcal{P}_{d_1}(T)$ but $p \notin \cup_{j=1}^n I(t_i,t_j)$ for some t_i. Let $c_v = [q', q'']$ and $c_h = [p', p'']$ be the maximal vertical and horizontal cuts which pass through the point p. Denote by P_1, P_2, P_3 and P_4 the subpolygons of P defined by these cuts. Let $P_1 \cap P_3 = P_2 \cap P_4 = \{p\}$ and $t_i \in P_1$. Obviously, $P_1, \ldots, P_4$ are gated. Note that p is the gate in P_1 of any point of P_3. As $p \notin \cup_{j=1}^n I(t_i,t_j)$, we conclude that $P_3 \cap T = \emptyset$. Set $P_{j,k} := P_j \cup P_k$, where $j,k \in \{1,2,3,4\}$ and $j \neq k$. Note that the four subpolygons $P_{j,j+1(\text{mod} j)}$ are gated sets of P as intersection of gated sets. We distinguish two cases: (i) p is the gate of t_i in one of the cuts c_h or c_v, say the first, and (ii) the gates q and z of t_i in c_v and c_h are different from p.

First, consider the case (i). Since p is the gate of t_i in c_h, obviously it is also the gate of t_i in $P_{3,4}$. From the choice of p and t_i we conclude that $P_{3,4} \cap T = \emptyset$. Let $g_1, \ldots, g_n$ be the gates of $t_1, \ldots, t_n$ of T in c_v. First, suppose that these gates are all different from p. Then all $g_1, \ldots, g_n$ belong to the segment $[q', p] \subset c_v$ which separates P_1 and P_2. Let g_k be the closest to p such gate. Then $g_k \in I(p, g_j)$ and, since $g_j \in I(p, t_j)$, we infer that $g_k \in \cap_{j=1}^n I(p,t_j)$, thus g_k dominates p, contradiction that p is Pareto. Now assume that p is the gate of some point $t_j \neq t_i$ in c_v. If $t_j \in P_2$, then p is the gate of t_j in $P_{1,4}$, contrary to $p \notin I(t_i,t_j)$. Thus $t_j \in P_1$. Let u and w be the gates of t_i and t_j in c_v and c_h. Pick some rectilinear geodesics $\gamma(t_i,u), \gamma(t_j,w)$, and $\gamma(t_i,t_j)$ between the pairs t_i,u; t_j,w, and t_i,t_j, respectively. Since $p \notin I(t_i,t_j)$, $\gamma(t_i,t_j)$ cannot share common points with both segments $[u,p]$ and $[p,w]$. Let $\gamma(t_i,t_j) \cap [u,p] = \emptyset$. Let u' be a closest to u point of $\gamma(t_i,u) \cap \gamma(t_i,t_j)$. Necessarily $u' \neq u$. Let w' be a closest to w point of $\gamma(t_j,w) \cap \gamma(t_i,t_j)$. Since P is a simple polygon, the region of the plane bounded by $[u,p], [p,w]$, the part of $\gamma(t_i,u)$ between u,u', the part of $\gamma(t_i,t_j)$ between u', w', and the part of $\gamma(t_j,w)$ between w', w, is contained in P. Let $[u'', u]$ be the last link in the subpath of $\gamma(t_i,u)$ between u' and u. Then for some $\delta > 0$, the segment $[v', v'']$ belongs to P, where $v' \in [u'', u], v'' \in [p,w]$ and $d(u,v') = d(p,v'') = \delta$. This contradicts that p is the gate of t_i in c_h.

Now, consider case (ii). Let u be the furthest from t_i point of $I(t_i,q) \cap I(t_i,z)$. Pick rectilinear geodesics $\gamma(u,q)$ and $\gamma(u,z)$ between u,q and u,z. Let $[q',q]$ and $[z',z]$ be the last links of these paths. Let q'' be the point of c_h with the same x-coordinate as q'. Let z'' be the point of c_v with the same y-coordinate as z'. Since P is a simple polygon, the region between $[q,p], [z,p]$ and $\gamma(u,q), \gamma(u,z)$ belongs to P. Moreover, since $q,z \in I(p,t_i)$ and $I(p,t_i)$ is convex, this region necessarily belongs to $I(p,t_i)$. In particular, both rectangles $R' = [q',q,p,q'']$ and $R'' = [z',z,p,z'']$ belong to $I(p,t_i)$. As we already stated, all points of T are outside P_3. Let g_v be the closest to p gate in c_v of a point of $T \cap P_2$, while g_h be the closest to p gate in c_h of a point of $T \cap P_4$. Since $p \notin \cup_{j=1}^n I(t_i,t_j)$, we conclude that g_v and g_h are different from p. Let $0 < \delta < \min\{d(p,g_v), d(p,g_h), d(z',z), d(q',q)\}$. Consider a point $p' \in R' \cap R''$ whose coordinates differ by δ from those of p. Since $p' \in I(p,t_i)$, we obtain that $d(p',t_i) = d(p,t_i) - 2\delta$. For any other t_j we have $d(p',t_j) \leq d(p,t_j)$. This contradicts that p is Pareto. □

A subset S of P is *ortho-convex* if the intersection of S with any axis–parallel cut of P is connected.

Lemma 2 *$\mathcal{P}_{d_1}(T)$ is a closed ortho-convex set of P.*

3.2 The algorithm

Now, we describe the algorithm for constructing the Pareto envelope $\mathcal{P}_{d_1}(T)$ for a set T of n points in a simple rectilinear polygon P with m vertices. In the sequel, we will refer to points of T as *terminals*. The algorithm uses Chazelle's algorithm for computing all vertex-edge visible pairs of a simple polygon [3] and the optimal point-location methods [8, 12]. Using Chazelle's algorithm, we derive a decomposition of the polygon P into rectangles, employing only horizontal cuts which pass through the vertices of P. Using the optimal point-location methods [8, 12] we compute in $O(n \log m)$ total time which rectangles of the decomposition contain the terminals (notice that the induced subdivision is monotone, hence the point-location structure can be built in linear time). At the next step, we sort by y all terminals from each rectangle. With these sorted lists, we refine the initial subdivision by dividing each rectangle containing terminals with the horizontal cuts passing via terminals. The dual graph of this decomposition $\mathcal{D}$ is a tree $\mathcal{T}$:

the nodes of a tree are the rectangles of $\mathcal{D}$, and two nodes in $\mathcal{T}$ are adjacent iff the corresponding rectangles are bounded by the common cut. We suppose that $\mathcal{T}$ is rooted at some rectangle. Any cut c of our subdivision divides the polygon P into two subpolygons P'_c and P''_c which correspond to two subtrees $\mathcal{T}'_c$ and $\mathcal{T}''_c$ of $\mathcal{T}$. It can be easily shown that if $P''_c \cap T = \emptyset$ (in this case we say that P''_c is T-empty), then $\mathcal{P}_{d_1}(T)$ is contained in $P'_c \cup c$ (any point of P''_c is dominated by its gate in c). By proceeding the tree $\mathcal{T}$, in linear time we can remove all T-empty subpolygons and their corresponding subtrees. We will denote the resulting polygon, subdivision, and tree also by $P, \mathcal{D}$, and $\mathcal{T}$. The resulting decomposition $\mathcal{D}$ and its tree $\mathcal{T}$ can be constructed in time $O(m + n(\log n + \log m))$. If all terminals are vertices of P, then we avoid the application of point-location methods and ranking of terminals, requiring only $O(n + m)$ time.

Given a non-root rectangle R, we denote by e'_R and e''_R the horizontal sides of R, so that e'_R separates R from to the root of $\mathcal{T}$. The set of gates of all terminals in R can be partitioned into the subset G'_R of gates located on e'_R and the subset G''_R of gates located on e''_R. Let $g'_l(R), g'_r(R)$ be the leftmost and the rightmost points from G'_R and let $g''_l(R)$, $g''_r(R)$ be the leftmost and the rightmost points from G''_R. In the full version, we show how to compute the four extremal gates $g'_l(R)$, $g'_r(R)$ $g''_l(R)$ and $g''_r(R)$ for all rectangles $R \in \mathcal{D}$ in total linear time by using an upward and a downward traversal of $\mathcal{T}$.

For each rectangle $R \in \mathcal{D}$, given the quadruplet of gates $Q_R = \{g'_l(R), g'_r(R), g''_l(R), g''_r(R)\}$, at the next step we compute the Pareto envelope $\mathcal{P}_{d_1}(Q_R)$ of Q_R. It consists of a *box* B_R having its horizontal sides on the sides of R and two horizontal segments which are incident either to two points of the quadruplet lying on the same horizontal side of R or to two opposite points lying on different horizontal sides of R (one or both these segments can be degenerated). In general, these segments do not necessarily belong to the final Pareto envelope $\mathcal{P}_{d_1}(T)$. On the other hand, as we will show below, B_R minus its horizontal sides is exactly the set $\mathcal{P}_{d_1}(T) \cap R^0$, where $R^0 := R \setminus (e'_R \cup e''_R)$ (clearly, the horizontal sides of B_R belong to $\mathcal{P}_{d_1}(T)$ as well because $\mathcal{P}_{d_1}(T)$ is closed). Now, if we consider any horizontal cut c, then we show that $\mathcal{P}_{d_1}(T) \cap c$ is the smallest segment $s_c \subseteq c$ spanned by the terminals and/or the horizontal sides of all boxes B_R located on c. Clearly, having at hand the four gates of each rectangle, the sets B_R and s_c can be determined in $O(n + m)$ time. To conclude, it remains to prove the correctness of two last steps of the algorithm. This follows from the following two lemmata whose proof is given in the full version.

Lemma 3 $\mathcal{P}_{d_1}(T) \cap R^0 = B_R \cap R^0$.

Lemma 4 $\mathcal{P}_{d_1}(T) \cap c = s_c$.

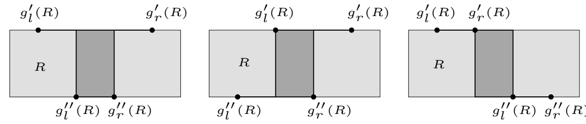

Figure 3: $\mathcal{P}_{d_1}(Q_R)$

Summarizing the results of this section, we obtain our main result:

Theorem 5 *The Pareto envelope of n terminals located in a simple rectilinear polygon P with m edges can be constructed in time $O(n + m(\log n + \log m))$ ($O(n + m)$ if all terminals are vertices of P).*

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