Frobenius structure in ($*-$)autonomous categories

Séminaire LDP

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Motivations

Theorem (Kruml and Paseka 2008, Santocanale 2020)
Let L be a complete lattice. The following are equivalent:

- L is a completely distributive lattice.
- The quantale $[L, L]_\vee$ of join-preserving endomaps of L is a Frobenius quantale.

Theorem (Raney 1960, Higgs and Rowe 1989)
The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattice.

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Conjecture

Let A be an object of an autonomous category (symmetric monoidal closed). The following are equivalent:

- A is nuclear.
- The object $[A, A]$ of endomorphisms of A is a Frobenius structure.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattices.

Preprints available:

For details and many more beautiful properties

- About unitless Frobenius quantale (first part of the talk):
  `https://hal-amu.archives-ouvertes.fr/LIS-LAB/hal-03661651v1`
  (Currently being reviewed by ACS)

- About Frobenius structure (second part of the talk):
  `https://hal.archives-ouvertes.fr/hal-03739197/`
  (Accepted by CSL 2023)

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Quantales

Definition

A *quantale* $(Q, \star)$ is a complete lattice Q with an associative law

$$\star : Q \times Q \to Q$$

which distributes over the sup on both variables:

$$(\bigvee_{i \in I} x_i) \star y = \bigvee_{i \in I} (x_i \star y) \quad \text{and} \qquad x \star (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \star y_i).$$

Remark

- A quantale is a semigroup in the category **SLatt**.
- A quantale is a posetal monoidal bi-closed category (without unit).

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Recall on adjoint

Adjoint theorem for lattices

Let L and M be two complete lattices. A map $f : L \to M$ is join-preserving iff there exist a meet-preserving map $\rho(f) : M \to L$ such that

$$\frac{f(x) \leq y}{x \leq \rho(f)(y)\,.}$$

Remark

A meet-preserving map $g : M \to L$ is a join-preserving map $g : M^{\mathrm{op}} \to L^{\mathrm{op}}$. The adjoint operation, ρ, is a natural isomorphim $\hom(L, M) \cong \hom(M^{\mathrm{op}}, L^{\mathrm{op}})$.

Implications of a quantale

The maps $(x \star -) : Q \to Q$ and $(- \star y) : Q \to Q$ are sup-preserving.
They both have a right adjoint written $(x \backslash -)$ and $(-/y)$:

$$\frac{\dfrac{x \star y \leq z}{y \leq x \backslash z}}{x \leq z/y\,.}$$

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Important examples of quantales

Examples

- A Heyting algebra is a commutative quantale with $\star = \wedge$:
$$\frac{x \wedge y \leq z}{y \leq x \Rightarrow z\,.}$$
- Let $(S, *)$ be a semigroup, $(P(S), \star)$ is the free quantale over S with:
$$X \star Y = \{xy \mid x \in X, y \in Y\}$$
$$X \backslash Y = \{s \in S \mid x * s \in Y, \text{for all } x \in X\}$$
$$Y / X = \{s \in S \mid s * x \in Y, \text{for all } x \in X\}$$
- The set of endomorphisms $([L, L], \circ)$ over L in **SLatt**. The joins are calculated point-wise (ie : $(\bigvee_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x)$). We have:
$$f \circ (\bigvee_{i \in I} g_i)(x) = f(\bigvee_{i \in I} g_i(x)) = \bigvee_{i \in I} f(g_i(x)) = \bigvee_{i \in I} (f \circ g_i)(x)\,,$$
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Linear logic

It is well known that every symmetric monoidal closed category is a model of proofs of multiplicative intuitionist linear logic.

Proposition

A quantale $(Q, \star)$ is a model of provability of non-commutative intuitionist multiplicative and additive linear logic.

Indeed

As usual,

- The connective $\otimes$ is interpreted by the operation $\star$;
- The two implications $\multimap$ and $\multimapinv$ by $\backslash$ and $/$;
- The two additive connective & and $\oplus$ by the inf $\wedge$ and the sup $\vee$.

What about linear negation?
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Frobenius quantales as usually defined

Usual definition

A *Frobenius quantale* is a tuple $(Q, \star, 0)$ with 0 a dualizing element. That is, we have, for all x in Q,

$$(0/x)\backslash 0 = x = 0/(x\backslash 0) \qquad \text{(dualizing element).}$$

We write ${}^{\perp}(-) := (-\backslash 0) : Q \to Q^{\mathrm{op}}$ and $(-)^{\perp} := (0/-) : Q \to Q^{\mathrm{op}}$

Remark

With this definition, a Frobenius quantale is always unital with unit $0\backslash 0 = 0/0$.

Another remark

In a Frobenius quantale $(Q, \star, 0)$ we have for every $x, y \in Q$,

$$x \leq y^{\perp} \text{ iff } y \leq {}^{\perp}x \qquad \text{(Galois connection)},$$

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Unitless Frobenius quantales

Definition

A *unitless Frobenius quantale* is a tuple $(Q, \star, {}^{\perp}(-), (-)^{\perp})$ with ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ inverse maps such that for all $x, y \in Q$, we have

$$x \backslash {}^{\perp} y = x^{\perp} / y \qquad \text{(Serre pair)}\,,$$

$$\text{or equivalently: } \forall x, y, z, \qquad x \star z \leq {}^{\perp} y \quad \text{iff} \quad z \star y \leq x^{\perp} \qquad \text{(shift relation)}\,.$$

Remark

- In a quantale $(Q, \star)$, if 0 is dualizing then $0/0 = 0\backslash 0$ is the unit of $(Q, \star)$.
- If $(Q, \star, {}^{\perp}(-), (-)^{\perp})$ is a Frobenius quantale with a unit 1 then ${}^{\perp}1 = 1^{\perp}$ is a dualizing element.

Proposition

1. There exist non unital Frobenius quantales.
2. There is no extension which preserves the two negations from a unitless quantale to a unital one.

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Sketch of the proof of the last statement

Sketch of the proofs

1. One can easily adapt the standard Chu construction of a $*$-autonomous category from a monoidal category. In the case of a quantale $(Q, \star)$, the unitless Frobenius quantale $\mathsf{Chu}(Q)$ has a unit iff Q has one. $\square$

2. Let $(Q_0, \star_0, {}^{\perp}(-)_0, (-)^{\perp}_0)$ and $(Q_1, \star_1, {}^{\perp}(-)_1, (-)^{\perp}_1)$ be Frobenius quantales with $i : Q_0 \hookrightarrow Q_1$ an embedding of Q_0 which preserves the two negations. We can suppose that $Q_0 \subset Q_1$ is closed under joins, meets, multiplication and implications. Indeed, we have:

$$x\backslash y = ({}^{\perp}y \star x)^{\perp} \qquad y/x = {}^{\perp}(x \star y^{\perp}) \qquad \bigwedge_{i \in I} x_i = {}^{\perp}(\bigvee_{i \in I} {x_i}^{\perp})$$

We set

$$u := \bigwedge_{x \in Q_0} x\backslash x$$

one can check that we always have $x \star u \leq x$ and $u \star x \leq x$. If Q_1 has a unit, then we also have $x \leq u \star x$ and $x \leq x \star u$. $\square$

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What about interesting examples?

Where do we find unitless Frobenius quantales ?

Recall on usual phase semantic (Girard 1987)

Let M be a commutative monoid and $0 \in P(M)$, then the set of *facts*

$$P(M)_j = \{A \in P(M) \mid (A\backslash 0)\backslash 0 = A\}$$

is a Frobenius quantale.

Note that $j : A \mapsto (A\backslash 0)\backslash 0$ is a nucleus (we will recall the definition and basic properties on the next slide).

Our goal is to generalize this construction for unitless Frobenius quantales of the form $P(S)_j$ with S a a semigroup and j a nucleus.

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Phase quantales

Recall on nuclei

A *nucleus* on a quantale Q is a map $j : Q \to Q$ such that for all $x, y \in Q$:

$$(j \circ j)(x) = jx \qquad x \leq j(x) \qquad j(x) \star j(y) \leq j(x \star y)\,.$$

The set of fixed points $Q_j = \{x \in Q \mid j(x) = x\}$ is a quantale with

$$\bigvee_{i \in I}^{j} x_i = j(\bigvee_{i \in I} x_i) \qquad \text{and} \qquad x \star_j y = j(x \star y).$$

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Caracterization in term of relations

We look at Galois connections (l, r) on $P(S)$ such that

$$l \circ r = r \circ l \qquad \text{and} \qquad x \backslash l(y) = r(x)/y\,.$$

Proposition

If (l, r) is a Galois connection respecting the equations above, then $j = l \circ r = r \circ l$ is a nucleus and $(P(S)_j, \star_j, l, r)$ is a Frobenius quantale.

Galois connections on $P(X)$ are in bijection with relations on X:
From a relation $R \subset X \times X$, we set $l, r : P(X) \to P(X)^{\mathrm{op}}$ by

$$l(A) = \{x \in X \mid xRa \ \forall a \in A\} \qquad r(A) = \{x \in X \mid aRx \ \forall a \in A\}$$

Proposition

A galois connection (l, r) on $P(S)$ respects the equations above iff the corresponding relation has the following properties:

$$\forall x \in S, \exists Y_x \subset S, \forall z \in S, \quad xRz \quad \text{iff} \quad zRy, \forall y \in Y_x \qquad \text{weakly-symmetric 1}$$

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$$\forall x, y, z \in S, \quad x * yRz \quad \text{iff} \quad xRy * z, \qquad \text{associativity wrt the multiplication}$$

Caracterization in term of relations

We look at Galois connections (l, r) on $P(S)$ such that

$$l \circ r = r \circ l \qquad \text{and} \qquad x \backslash l(y) = r(x)/y\,.$$

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If (l, r) is a Galois connection respecting the equations above, then $j = l \circ r = r \circ l$ is a nucleus and $(P(S)_j, \star_j, l, r)$ is a Frobenius quantale.

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Examples from the phase quantale construction

Examples

- Let A be a Frobenius algebra over the field K. That is, a K-algebra with a symmetric pairing $\langle -,- \rangle : A \otimes A \to K$ such that $\langle x * y, z \rangle = \langle x, y * z \rangle$. Then define $xRy \quad \text{iff} \quad \langle x, y \rangle = 0$.

- For the C^*-algebra M_n, we use the pairing $\langle A, B \rangle = \mathrm{tr}(B^* A)$. The quantale $P(M_n)_j$ is the set of closed linear subspace of M_n

- Let H be a Hilbert space and $B_1(H)$ the algebra of trace-class operator (ie operators on H such that $\sum_{e \in \epsilon} \langle |f| e, e \rangle < \infty$). With the same construction we show that closed lineear subspace of $B_1(H)$ is a Frobenius quantale which does not have a unit if H is of infinite dimension.

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The unitless Frobenius quantale of tight maps

Let L be a complete lattice.
$[L, L]_\vee$ and $[L, L]_\wedge$, are the set of sup/inf-preserving endomaps of L.

Definition

For a map $f : L \longrightarrow L$, we define the two Raney's transforms:

$$f^\vee(x) := \bigvee_{x \not\leq t} f(t) \quad \text{and} \quad f^\wedge(x) := \bigwedge_{t \not\leq x} f(t)\,.$$

We write $[L, L]_\vee^t = \{f : L \to L \mid f^{\wedge\vee} = f\}$ the set of *tight maps*.

Remark

They are defined for every map. But if we restrict them, we have

$$(-)^\vee : [L, L]_\wedge \longrightarrow [L, L]_\vee \qquad (-)^\wedge : [L, L]_\vee \longrightarrow [L, L]_\wedge \qquad (-)^\vee \dashv (-)^\wedge$$

$[L, L]_\vee^t$ is the image of $(-)^\vee : L^* \otimes L \cong [L, L]_\wedge \longrightarrow [L, L]_\vee$.

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$$\begin{tikzcd}
L^* \otimes L \cong [L,L]_\wedge \arrow[rr, "(-)^\vee"] \arrow[dr, two heads] & & {[L,L]} \\
& {[L,L]^t_\vee} \arrow[ur, hook] &
\end{tikzcd}$$

Results on tight maps

Proposition (LS and CL)

For every complete lattice L, $([L,L]^t_\vee, \circ, (-)^\perp, (-)^\perp)$ is a Frobenius quantale with $f^\perp = l(f^\wedge)$.

Theorem

Let L be a complete lattice. The following are equivalent:

1. The lattice L is completely distributive;
2. $[L,L]^t_\vee = [L,L]$ (Raney, 1960);
3. L is a nuclear object of **SLatt** (Higgs Rowe 1989);
4. There is a unique sup-preserving map $0 : L \to L$ such that $([L,L], \circ, 0)$ is a Frobenius quantale. (Kruml Paseka 2008, Santocanale 2020);
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What about a categorical study of the last theorem?

Well that's the rest of the talk !

First let's define a unitless Frobenius quantale in a monoidal category !

Next

Dual pair

For an object A of a $*$-autonomous category, we have the two equivalences:

$$\frac{A \otimes X \longrightarrow 0}{X \longrightarrow A^*} \qquad \frac{X \otimes A^* \longrightarrow 0}{X \longrightarrow A^{**} \cong A\,.}$$

Definition

A map $\epsilon : A \otimes B \longrightarrow 0$ in $\mathcal{V}$ is said to be a *dual pairing* (w.r.t. the object 0) if the two induced natural transformations are isomorphims.

$$\hom(X, B) \longrightarrow \hom(A \otimes X, 0)\,, \qquad \hom(X, A) \longrightarrow \hom(X \otimes B, 0)\,.$$

Example

- In a $*$-autonomous category, $(A, A^*, ev_{A,0})$ is a dual pair.
- In **Hilb**, H and $\overline{H}$ is a dual pair with pairing $\langle -, - \rangle : H \otimes \overline{H} \to \mathbb{C}$ the linear extension of the inner product of H.

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Some properties of dual pairs

Proposition

Let (A, B) be a dual pair in a symmetric monoidal closed category.

1. (B, A) is also a dual pair.
2. We have $A \cong B^*$.
3. A is a reflexive object (*i.e* $A \cong A^{**}$).
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- In **SLatt**, $(L, L^{\mathrm{op}}, \epsilon)$, $\epsilon(x, y) = \bot$ if $x \leq y$, and $\epsilon(x, y) = \top$ otherwise.
- In **Coh**, $X^{\mathrm{op}} \cong X^*$ so (X, X^{op}) is also a dual pair.
- In a $*$-autonomous category, $A^* \otimes A \cong [A, A]^*$ so $(A^* \otimes A, [A, A], \epsilon)$ is a dual pair with $\epsilon := ev \circ \sigma \circ ev$.

$$A^* \otimes A \otimes [A, A] \xrightarrow{A^* \otimes ev_{A,A}} A^* \otimes A \xrightarrow{\sigma_{A^*,A}} A \otimes A^* \xrightarrow{ev_{A,0}} 0$$

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Usual adjunction between lattices

For a join preserving map $f : L \to M$, the right adjoint to it $\widetilde{f} : M^{\mathrm{op}} \to L^{\mathrm{op}}$ is the only map s.t:

$$f(x) \leq y \qquad \text{iff} \qquad x \leq \widetilde{f}(y)$$

$$\begin{tikzcd}
L \otimes M^{\mathrm{op}} \arrow[r, "f \otimes M^{\mathrm{op}}"] \arrow[d, "L \otimes \widetilde{f}"'] & M \otimes M^{\mathrm{op}} \arrow[d, "\epsilon_M"] \\
L \otimes L^{\mathrm{op}} \arrow[r, "\epsilon_L"] & 0\,.
\end{tikzcd}$$

Adjoints in dual pair

Let (A_0, B_0), (A_1, B_1) be two dual pairs. For every morphism $f : A_0 \longrightarrow A_1$ we define $\widetilde{f} : B_1 \longrightarrow B_0$ by transposing:

$$\frac{\dfrac{A_0 \longrightarrow A_1}{A_0 \otimes B_1 \longrightarrow 0}}{B_1 \longrightarrow B_0}$$

$$\begin{tikzcd}
A_0 \otimes B_1 \arrow[r, "f \otimes B_1"] \arrow[d, "A_0 \otimes \widetilde{f}"'] & A_1 \otimes B_1 \arrow[d, "\epsilon_1"] \\
A_0 \otimes B_0 \arrow[r, "\epsilon_0"] & 0\,.
\end{tikzcd}$$

Definition
We say that (f, g) is an adjoint pair if $g = \widetilde{f}$.

Adjoints in dual pair

Let (A_0, B_0), (A_1, B_1) be two dual pairs. For every morphism $f : A_0 \longrightarrow A_1$ we define $\widetilde{f} : B_1 \longrightarrow B_0$ by transposing:

$$\frac{A_0 \longrightarrow A_1}{\dfrac{A_0 \otimes B_1 \longrightarrow 0}{B_1 \longrightarrow B_0}}$$

$$\begin{tikzcd}
A_0 \otimes B_1 \arrow[r, "f \otimes B_1"] \arrow[d, "A_0 \otimes \widetilde{f}"'] & A_1 \otimes B_1 \arrow[d, "\epsilon_1"] \\
A_0 \otimes B_0 \arrow[r, "\epsilon_0"] & 0\,.
\end{tikzcd}$$

Definition

We say that (f, g) is an adjoint pair if $g = \widetilde{f}$.

Next

The category of semigroups over a monoidal category

Objects of $\mathbf{Sem}_{\mathcal{C}}$: pairs (A, μ_A) such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "A \otimes \mu_A"] \arrow[d, "\mu_A \otimes A"'] & A \otimes A \arrow[d, "\mu_A"] \\
A \otimes A \arrow[r, "\mu_A"'] & A.
\end{tikzcd}$$

Morphisms of $\mathbf{Sem}_{\mathcal{C}}$: arrows $f : A_0 \longrightarrow A_1$ such that

$$\begin{tikzcd}
A_0 \otimes A_1 \arrow[r, "f \otimes f"] \arrow[d, "\mu_{A_0}"'] & A_1 \otimes A_1 \arrow[d, "\mu_{A_1}"] \\
A_0 \arrow[r, "f"'] & A_1.
\end{tikzcd}$$

Quantales

Definition

A *quantale* $(Q, \star)$ is a semigroup in the category **SLatt**.

Remark

In a quantale, $(x \star -) : Q \to Q$ and $(- \star y) : Q \to Q$ both have a right adjoint, the left and right implications:

$$x \star y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / y$$

We have

$$-/- : Q \otimes Q^{\mathrm{op}} \longrightarrow Q^{\mathrm{op}} \qquad \text{and} \qquad -\backslash- : Q^{\mathrm{op}} \otimes Q \longrightarrow Q^{\mathrm{op}}$$

$$z/(y \star x) = (z/y)/x \qquad \text{and} \qquad (x \star y)\backslash z = x \backslash (y \backslash z)$$

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Implications in a quantale

$x \star y \leq z \quad \text{iff} \quad x \leq z/y$

$$\begin{tikzcd} Q \otimes Q \otimes Q^{\mathrm{op}} \arrow[r, "Q \otimes -/-"] \arrow[d, "\star \otimes Q^{\mathrm{op}}"'] & Q \otimes Q^{\mathrm{op}} \arrow[d, "\epsilon_Q"] \\ Q \otimes Q^{\mathrm{op}} \arrow[r, "\epsilon_Q"] & 0\,. \end{tikzcd}$$

$z \geq x \star y \quad \text{iff} \quad x \backslash z \geq y$

$$\begin{tikzcd} Q^{\mathrm{op}} \otimes Q \otimes Q \arrow[r, "Q^{\mathrm{op}} \otimes \star"] \arrow[d, "- \backslash - \otimes Q"'] & Q^{\mathrm{op}} \otimes Q \arrow[r, "\sigma"] & Q \otimes Q^{\mathrm{op}} \arrow[d, "\epsilon_Q"] \\ Q^{\mathrm{op}} \otimes Q \arrow[r, "\sigma"] & Q \otimes Q^{\mathrm{op}} \arrow[r, "\epsilon_Q"] & 0\,. \end{tikzcd}$$

Implications as actions

Let (A, B) be a dual pair such that (A, μ_A) is a semigroup.
We define $\alpha^\ell_A : A \otimes B \to B$ and $\alpha^\rho_A : B \otimes A \to B$ as the only morphisms such that

$$\begin{tikzcd}
A \otimes A \otimes B \arrow[r, "A\otimes\alpha^\ell_A"] \arrow[d, "\mu_A\otimes B"] & A \otimes B \arrow[d, "\epsilon"] \\
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B \otimes A \otimes A \arrow[r, "B\otimes\mu_A"] \arrow[d, "\alpha^\rho_A\otimes A"] & B \otimes A \arrow[r, "\sigma"] & A \otimes B \arrow[d, "\epsilon"] \\
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\quad
\begin{tikzcd}
B \otimes A \otimes A \arrow[r, "B\otimes\mu_A"] \arrow[d, "\alpha_A^\rho \otimes A"] & B \otimes A \arrow[r, "\sigma"] & A \otimes B \arrow[d, "\epsilon"] \\
B \otimes A \arrow[r, "\sigma"] & A \otimes B \arrow[r, "\epsilon"] & 0 .
\end{tikzcd}$$

Defined that way, α_A^ρ and α_A^ℓ are indeed actions, *i.e*

$$\begin{tikzcd}
A \otimes A \otimes X \arrow[r, shift left, "\mu_A \otimes X"] \arrow[r, shift right, "A\otimes\alpha^\ell"'] & A \otimes X \arrow[r, "\alpha^\ell"] & X.
\end{tikzcd}$$

Next

The case of Frobenius quantales

In a Frobenius quantale $(Q, \star, {}^{\perp}(-), (-)^{\perp})$, we have

- $(Q, Q^{\mathrm{op}}, \epsilon)$ is a dual pair;
- $(Q, \star)$ is a semigroup;
- ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ and $x \leq {}^{\perp}y$ iff $y \leq x^{\perp}$;

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Q \otimes Q \arrow[r, "A \otimes (-)^{\perp}"] \arrow[d, "{}^{\perp}(-) \otimes A"'] & Q \otimes Q^{\mathrm{op}} \arrow[d, "\alpha^{\ell}_{A}"] \\
Q^{\mathrm{op}} \otimes Q \arrow[r, "\alpha^{\rho}_{A}"'] & Q^{\mathrm{op}} .
\end{tikzcd}$$

Frobenius structures

Definition

A *Frobenius structure* is a tuple $(A, B, \epsilon, \mu_A, l, r)$ where

- (A, B, ϵ) is a dual pair;
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- $l, r : A \longrightarrow B$ and (l, r) is an invertible adjoint pair

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Co-multiplication

In a quantale, we can define two comultiplications

$$x \oplus_{\perp} y := {}^{\perp}(y^{\perp} \star x^{\perp}) \qquad\qquad x \,{}_{\perp}\!\oplus y := ({}^{\perp}y \star {}^{\perp}x)^{\perp} .$$

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The diagram on the left commutes iff the diagram on the right does,

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defining a multiplication on B.

Lemma

1. (B, μ_B) is a semigroup ;
2. l and r are semigroup morphisms from (A, μ_A) to (B, μ_B).
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Frobenius structure and associative bracketed semigroups

Proposition

For a Frobenius structure $(A, B, \epsilon, \mu_A, l, r)$, we can define

$$\pi^l_A := \epsilon \circ (A \otimes l) : A \otimes A \to 0\,,$$

We have :

- (A, μ_A, π^l_A) is an associative bracketed semigroup;
- π^l_A is a dual pairing.

Conversely, from an associative bracketed semigroup (A, μ_A, π_A) for which π_A is a dual pairing, one obtains a Frobenius structure.

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Previous work on Frobenius structure

Various work have been done such:

- Lawvere 1969: Frobenius monad;
- Kock 2003: Monoid and comonoid in a monoidal category (same tensor);
- Street 2004: Pseudo-monoid with a pairing $A \otimes A \to I$ making A his own bidual;
- Egger 2010: Monoid and comonoid on a linear distributive category (two different tensor).

Next

Nuclearity

From here, $\mathcal{C}$ is symmetric monoidal closed and $0 = I$.

Definition

For every object A of $\mathcal{C}$, there exists a canonical arrow

$$\mathtt{mix}_A : A^* \otimes A \longrightarrow [A, A].$$

An object A is *nuclear* if $\mathtt{mix}_A$ is an isomorphism.

Example

- In k-**Vect** they are the vector spaces of finite dimension.
- In a commutative unital quantale $(Q, \star, 1)$, they are the invertible elements.
- In **Coh** they are necessarily the trivial coherent space.
- In **HypCoh** there is no nuclear object.

Theorem (Raney 1960, Higgs and Rowe 1989)

The nuclear objects of **SLatt** are exactly the completely distributive lattices.

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Adjunction and Nuclearity

Definition

For $\eta : I \to B \otimes A$, and $\epsilon : A \otimes B \to I$, (A, B, ϵ, η) is an *adjunction* if

$$
\begin{tikzcd}
A \otimes B \otimes A \arrow[d, "\epsilon \otimes A"'] & A \otimes I \arrow[l, "A \otimes \eta"'] \arrow[d, "\rho_A"] \\
I \otimes A \arrow[r, "\ell_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes B \arrow[r, "\eta \otimes B"] \arrow[d, "\ell_B"'] & B \otimes A \otimes B \arrow[d, "B \otimes \epsilon"] \\
B & B \otimes I \arrow[l, "\rho_B"'] .
\end{tikzcd}
$$

Proposition

An object is nuclear iff there exist a (right or left) adjoint to it.

Recall : nuclearity and Frobenius quantale

Theorem Kruml and Paseka 2008, Santocanale 2020)

Let L be a complete lattice. The following are equivalent:

- L is a completely distributive lattice.
- The set of endomorphisms of L is a Frobenius quantale.

The first implication is actually a corollary of a more general result.

Theorem (LS and CL)

Let L be a complete lattice. The image of the Raney's transform $(-)^{\vee} : [L, L]_{\wedge} \to [L, L]$ can always be endowed with a Frobenius quantale structure.

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$$\begin{tikzcd}
L^* \otimes L \cong [L, L]_{\wedge} \arrow[rr, "(-)^{\vee}"] \arrow[dr, two heads] & & {[L, L]} \\
& {[L, L]^t_{\vee}} \arrow[ur, hook] &
\end{tikzcd}$$

Next

From Nuclearity to Frobenius structure

Theorem (LS and CL)

In a symmetric monoidal closed category, if A is nuclear then $[A, A]$ can be endowed with a Frobenius structure.

Sketch of the proof

- We verify that if $\mathtt{mix}$ is invertible, then $(A^* \otimes A, [A, A], \epsilon, \mu_{A^* \otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure.
- As $A^* \otimes A$ is isomorphic to $[A, A]^*$ and Frobenius structures are closed under iso, we obtain the desired theorem.

It has already been noticed that

Theorem (Street 2004)

If X has a (right or left) adjoint X^* and $X \cong X^{**}$, then $X^* \otimes X$ is a Frobenius pseudo-monoid.

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Result

Theorem (LS and CL)

Let $\mathcal{C}$ be a $*$-autonomous category such that **Sem**$_{\mathcal{C}}$ has an epi-mono factorization system and A an object of $\mathcal{C}$.
The image of $\mathtt{mix}_A$ can always be endowed with a Frobenius structure.

$$\begin{tikzcd}
A^* \otimes A \arrow[rr, "\mathtt{mix}_A"] \arrow[dr, two heads] & & {[A, A]} \\
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\end{tikzcd}$$

Corollary

If A is nuclear then $[A, A]$ can always be endowed with a Frobenius structure.

Result

Theorem (LS and CL)

Let $\mathcal{C}$ be a $*$-autonomous category such that **Sem**$_{\mathcal{C}}$ has an epi-mono factorization system and A an object of $\mathcal{C}$.
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From Frobenius structure to nuclearity

Conjecture

Let $([A,A],[A,A]^*,\mu,r,I)$ be a Frobenius structure in an autonomous category. Then A is a nuclear object.

We actually need to add a technical hypothesis.

Sketch of a proof

We use the caracterisation of nuclearity with adjoints. So we want:

$$\eta : I \longrightarrow A^* \otimes A \qquad\qquad \epsilon : A \otimes A^* \longrightarrow I$$

such that (A, A^*, ϵ, η) is an adjunction.

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- We identify $[A, A]^*$ with $A^* \otimes A$. Suppose $([A, A], A^* \otimes A, ev, \mu, r, l)$ is a Frobenius structure.

- $[A, A]$ is a monoid. As $r : [A, A] \to A^* \otimes A$ is an iso, $A^* \otimes A$ is also a monoid with unit $\eta : I \to A^* \otimes A$.

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Its multiplication is given by

$$
\begin{tikzcd}
A^* \otimes A \otimes A^* \otimes A \arrow[r, "A^*\otimes A\otimes l^{-1}"] \arrow[d, "r^{-1}\otimes A^*\otimes A"'] \arrow[dr, "\mu_{A^*\otimes A}"] & A^* \otimes A \otimes [A, A] \arrow[d, "A^*\otimes ev"] \\
{[A, A] \otimes A^* \otimes A} \arrow[r, "\mu_{A,A,0}\otimes A"'] & A^* \otimes A\,.
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From Frobenius structure to nuclearity

That is, we have a diagram of the shape

$$\begin{tikzcd} A^* \otimes A \otimes A^* \otimes A \arrow[d, bend right=80, "A^* \otimes g"'] \arrow[d, bend left=80, "h \otimes A"] \\ A^* \otimes A \end{tikzcd}$$

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\end{tikzcd}$$

This map actually exits if we ask I to embed into A as a retract, *i.e* if there exists $p : I \to A$ and $c : A \to I$ such that $c \circ p = \mathrm{id}_I$.

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$$\begin{tikzcd}
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Definition
If for every object A in $\mathcal{C}$, I embeds into A as a retract, $\mathcal{C}$ is *pseudoaffine*.

Examples

- **SLatt**
- k-**Vect**
- **Coh**
- **HypCoh**

Theorem (LS and CL)
If $\mathcal{C}$ is pseudoaffine and $([A, A], [A, A]^*, ev, \mu, r, I)$ is a Frobenius structure then A is a nuclear object.

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Discussing condition of the last theorem

Let $(P, \leq)$ be a poset (the base category) and F an endofunctor of **Rel**. Then one can form the category P_F-**Set**:

- Objects: maps $A : FX \to P$;
- Arrows $A \to B$: relations FR with $R \in P(X \times Y)$ such that $xFRy$ implies $A(x) \leq B(y)$.

Theorem (Schalk and De Paiva 2004)

If $(Q, \star, 1)$ is a unital commutative Frobenius quantale, the category Q_F-**Set** is a $*$-autonomous category.

Examples

Of course one can construct many nice $*$-autonomous categories.
Among them, **Coh** and **HypCoh** are subcategories of 3_Δ-**Set** and $3_{P_{\text{fin}}}$-**Set** where 3 is a quantale over the 3 element chain (cf. Schalk and de Paiva 2004 for the multiplication).

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To study nuclearity, we take $F = \mathtt{id}_{\mathbf{Rel}}$ and ask that $1 = 0$ in Q which implies that $I = 0$ in Q-**Set**.

Lemmas

A Q-**Set** A is nuclear if the image of A is included in the invertible element of Q, ie if for all $x, y \in X$,

$$A(x)\backslash A(y) = A(x)^{\perp} \star A(y)\,.$$

A Frobenius structure on $[A, A]$ in Q-**Set** is given by a pair of inverse map (f, g) over the underlyng set X such that for all $x, y \in X$:

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Theorem(LS and CL)

In Q-**Set**, the statement that $[A, A]$ endows a Frobenius structure is equivalent to A being nuclear if one of the following conditions is true:

- The Frobenius quantale Q has no infinite chain;
- The underlyng set X is finite;
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And in general?

We found no reason why it should be true in general.
After some time we were able to construct an infinite quantale Q such that a Frobenius structure on $[A, A]$ is not nuclear !

Counterexample

It is just the infinite chain $\mathbb{Z}$ with ∞ and $-\infty$ and another unit between -1 and 0.
Then X could be $\mathbb{Z}$, A the inclusion and f and g the antecedent and successor (cf drawing).

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Conclusion

Results

- A new definition of Frobenius quantale which does not involve unit and its study;
- A definition of Frobenius structures in autonomous categories;
- Generalisation of the double negation construction;
- Proof of our conjecture up to a technical (but quite natural) hypothesis.

What we will do next

- Connect with linear logic semantic;
- Study the logic of pseudoaffine category;
- Understand "how much" we need $*$-autonomous categories;
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Thank you!

References

D. A.Higgs et K. A. Rowe (1989)

Nuclearity in the category of complete semilattices, *Journal of Pure and Applied Algebra*, Volume 57, Issue 1, 1989, Pages 67-78

R. Street (2004)

Frobenius monads and pseudomonoids, *Journal of Mathematical Physics*, Vol. 45, 2004, pp 3930-3948

J.M. Egger (2010)

The Frobenius relations meet linear distributivity, *Theory and Applications of Categories*, Vol. 24, 2010, No. 2, pp 25-38

P-A. Melliès (2013)

Dialogue categories and Frobenius monoids *Lecture Notes in Computer Science*, vol 7860

References

David Kruml and Jan Paseka (2008)
Algebraic and Categorical Aspects of Quantales, *Handbook of Algebra*, Vol. 5, pp 323-362

P. Eklund, J. Gutiérrez Garcia, U. Höhle et J. Kortelainen (2018)
Semigroups in complete lattices, *Springer*, 2018

L. Santocanale (2020)
Dualizing sup-preserving endomaps of a complete lattice, *ACT 2020*

L. Santocanale (2020)
The involutive quantaloid of completely distributive lattices, *RAMICS 2020*