A probabilistic Kleene Theorem

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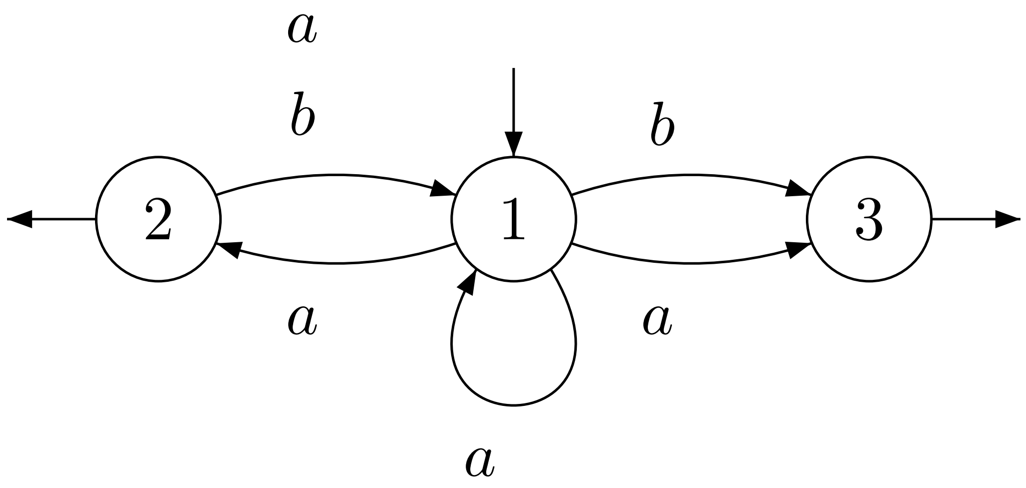
Part of works published at ATVA'12 with Benedikt Bollig, Paul Gastin and Marc Zeitoun

Kleene's Theorem



Finite State Automata

same expressivity

Regular Expressions

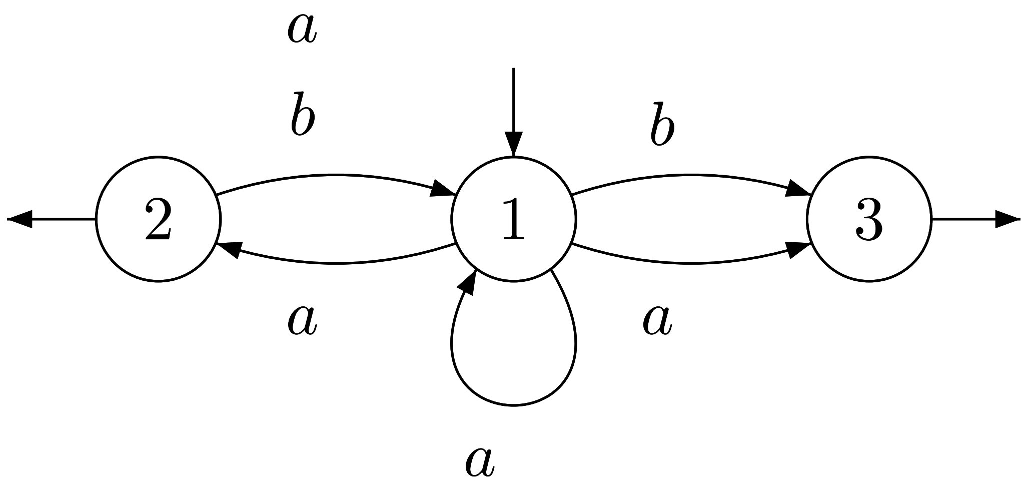
$$E ::= \quad a \mid E+E \mid E \cdot E \mid E^*$$

Motivations

- Theoretically: relate denotational and computational models
- Practically: easier to write specifications using regular expressions vs. easier to check properties (emptiness, inclusion...) with automata
- Goal: translate expressions to automata, as efficiently as possible

Kleene's Theorem

Finite State Automata



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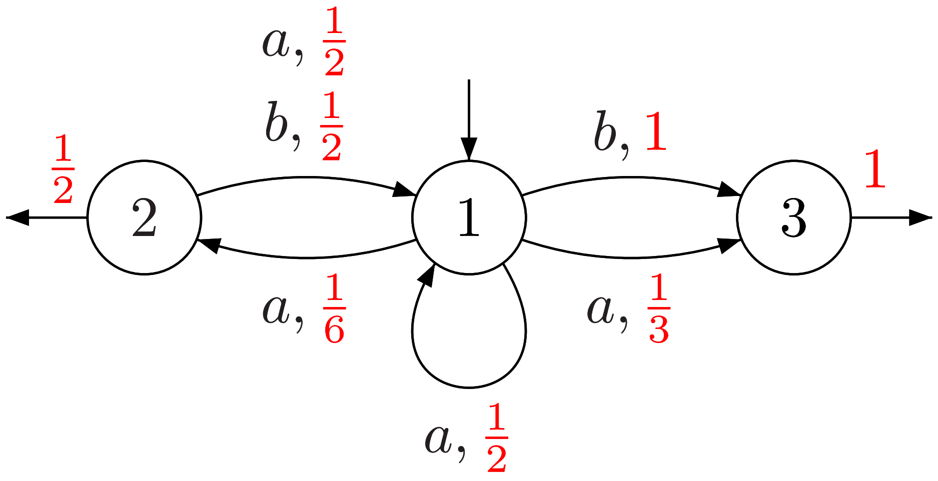
Kleene’s Theorem

Weighted Finite State Automata



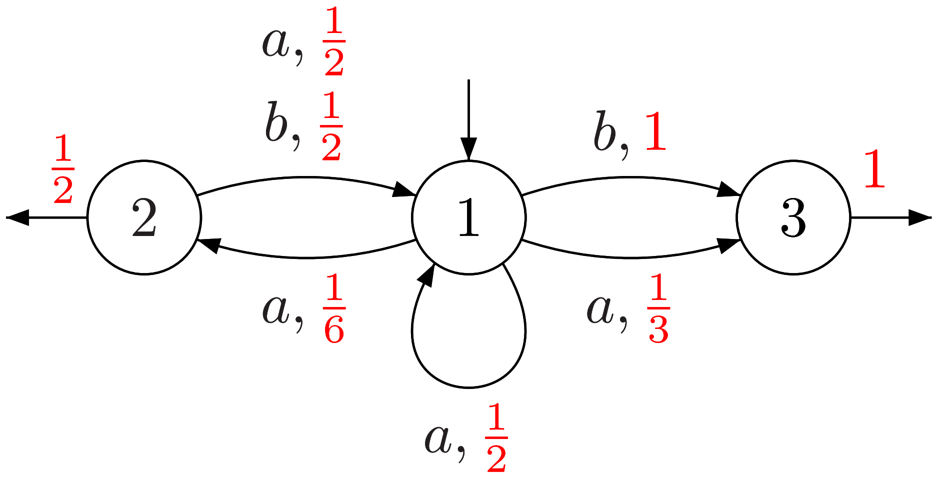
same expressivity

Regular Expressions

E ::= *a* | E+E | E · E | E^*

Kleene's Theorem

Weighted Finite State Automata



$\Sigma^* \to \mathbb{R}$

same expressivity

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Kleene's Theorem

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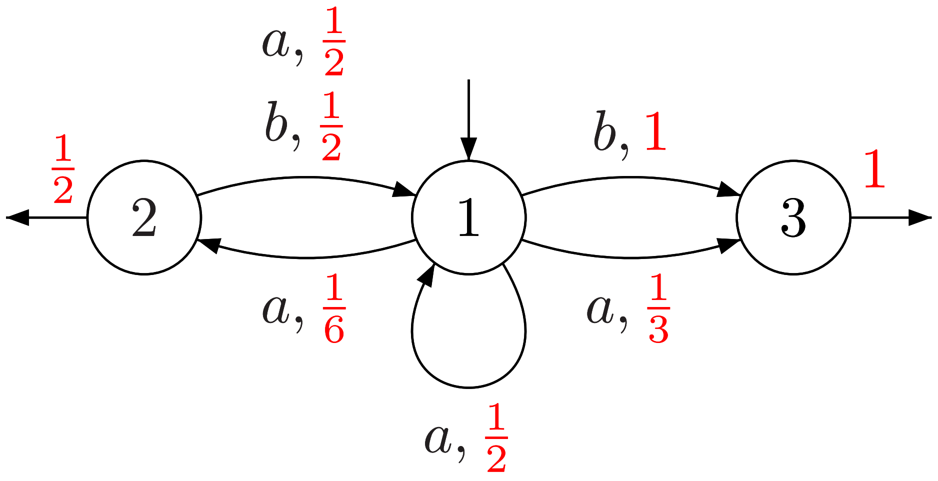

$\Sigma^* \rightarrow \mathbb{R}$

same expressivity

$a \ a \ b$

two runs:
1→2→1→3 of weight 1/6x1x1x1=1/6
and 1→1→1→3 of weight 1/2x1/2x1x1=1/4

Regular Expressions

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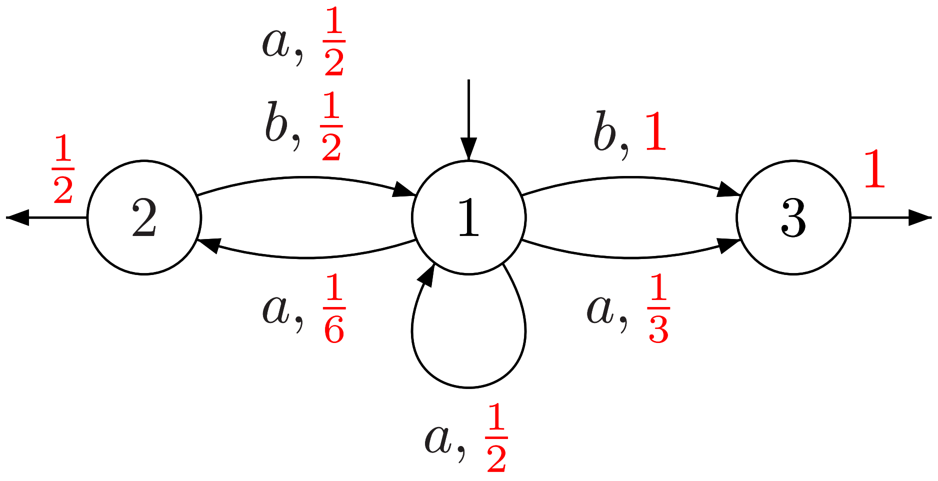
Weighted Finite State Automata



$\Sigma^* \rightarrow \mathbb{R}$

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hence, *a a b* recognized with weight 1/6+1/4=5/12

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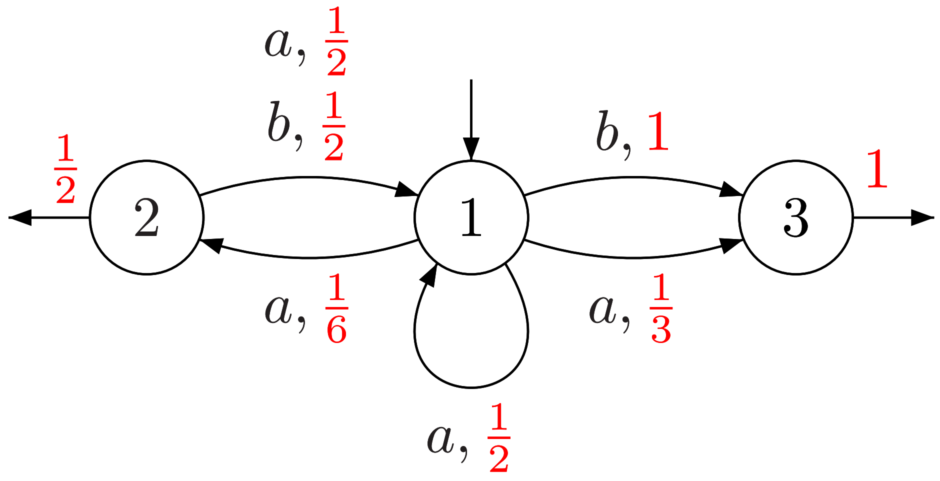
Weighted Finite State Automata



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Weighted Regular Expressions

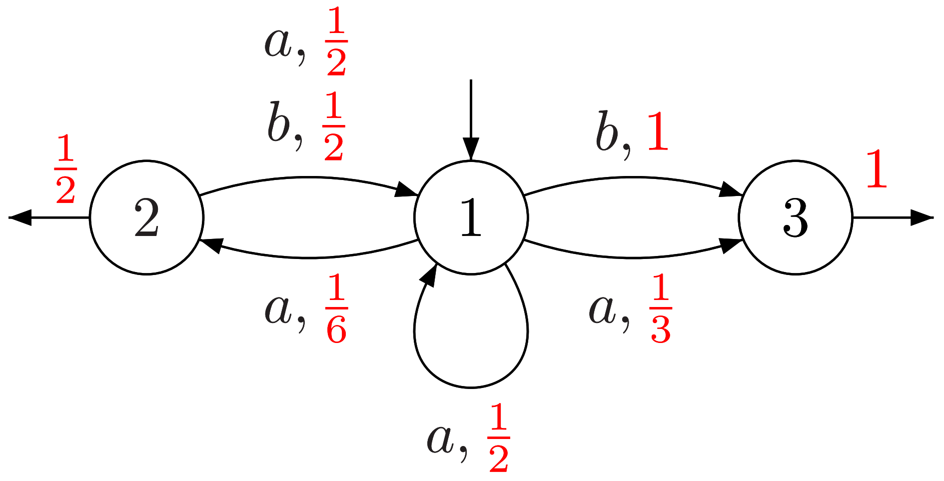
$$E ::= p \mid a \mid E+E \mid E \cdot E \mid E^*$$

E proper

Schützenberger

Kleene's Theorem

Weighted Finite State Automata



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Weighted Regular Expressions

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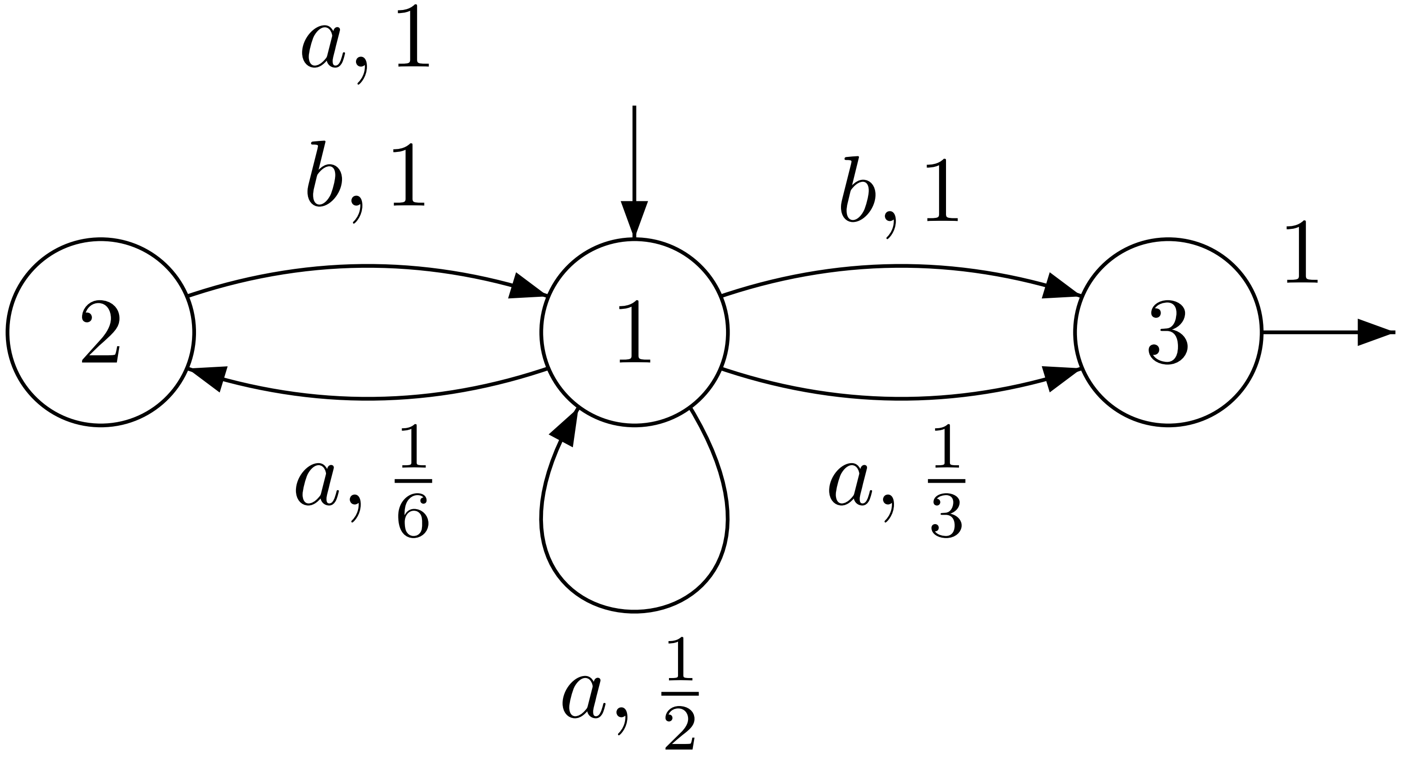
E proper

[1] S. Kleene (1956). Representation of events in nerve nets and finite automata.
[2] M.-P. Schützenberger (1961). On the Definition of a Family of Automata. Information and Control.
For an overview about Weighted Automata, see, e.g., Handbook of Weighted Automata. Editors: Manfred Droste, Werner Kuich, and Heiko Vogler. EATCS Monographs in Theoretical Computer Science. Springer, 2009.

Probabilistic case?



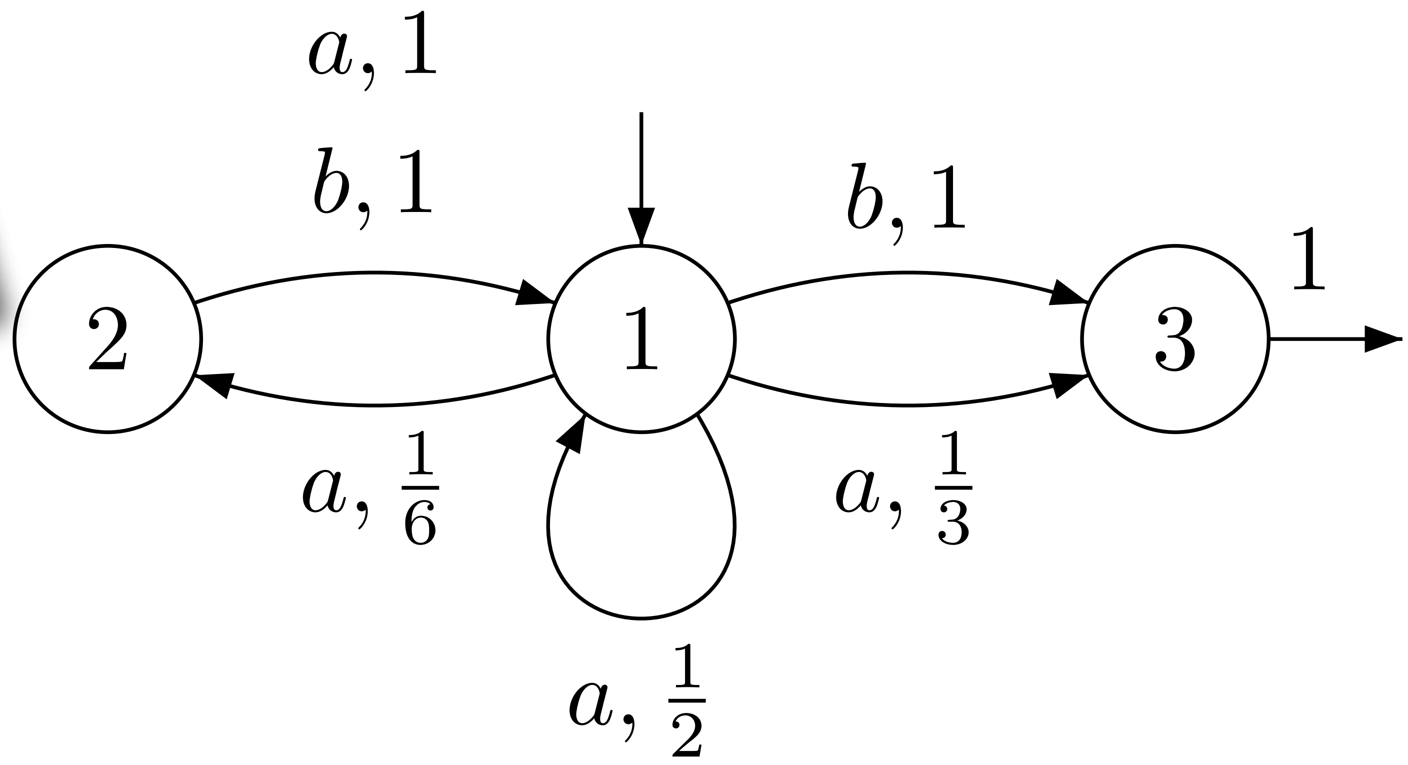
$$\mathcal{A} = (Q, \iota, Acc, \mathbb{P})$$

$$\mathbb{P} : Q \times \Sigma \times Q \to [0, 1]$$

$$Acc(q) + \sum_{q' \in Q} \mathbb{P}(q, a, q') \leq 1 \text{ for all } (q, a) \in Q \times A$$

Probabilistic case?

Reactive Probabilistic
Finite Automata



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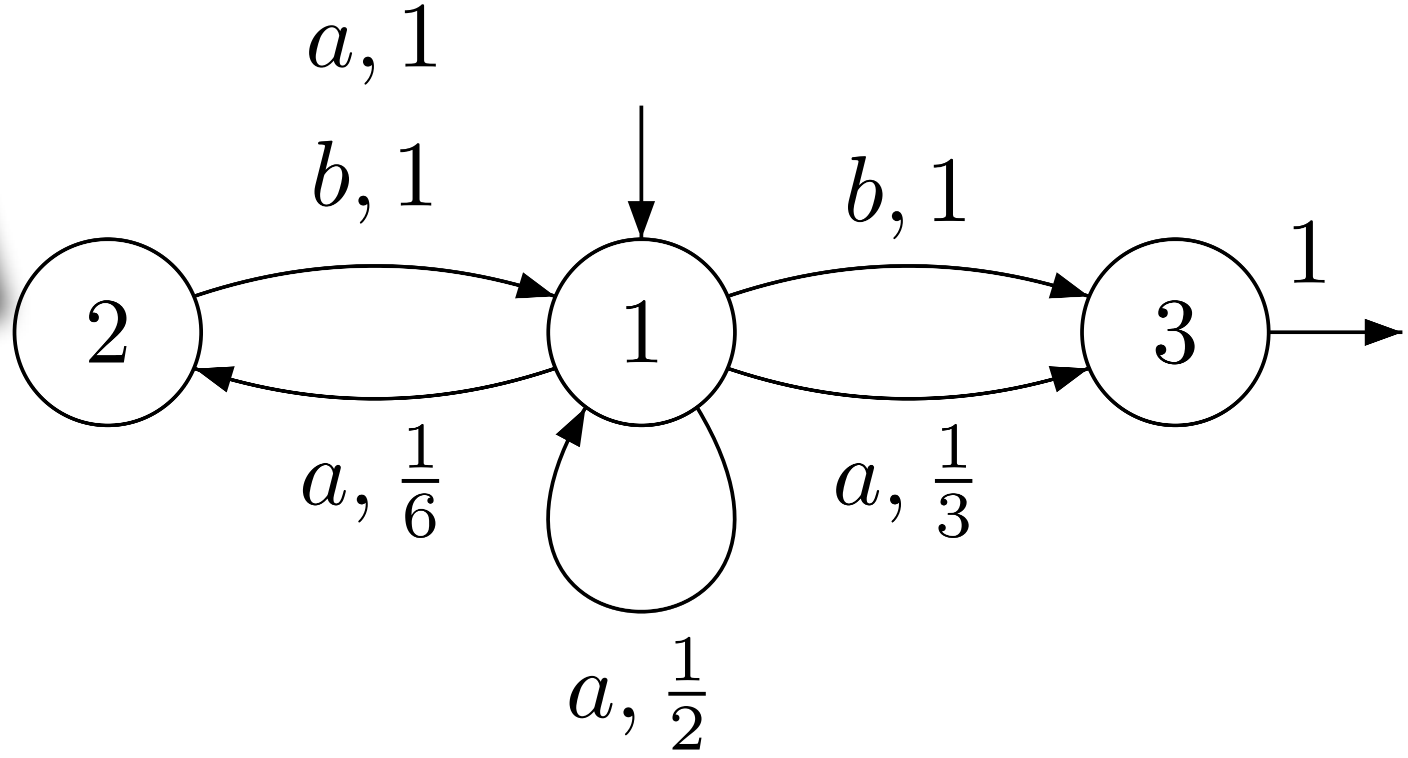
Probabilistic case?

Reactive Probabilistic Finite Automata



$$\mathcal{A} = (Q, \iota, Acc, \mathbb{P})$$

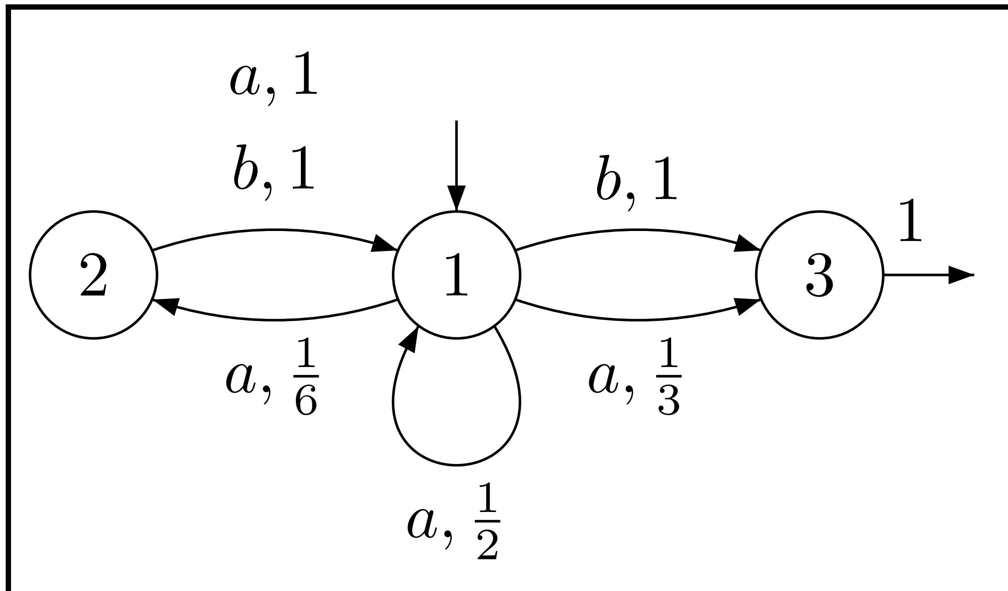
Applying Schützenberger's Theorem over these special Weighted Automata, we obtain regular expressions (proper)

ll $(q, a) \in Q \times A$

$q \in Q$

$$(\frac{1}{6}a(a+b) + \frac{1}{2}a)^*(\frac{1}{3}a + b)$$

What kind of expressions?



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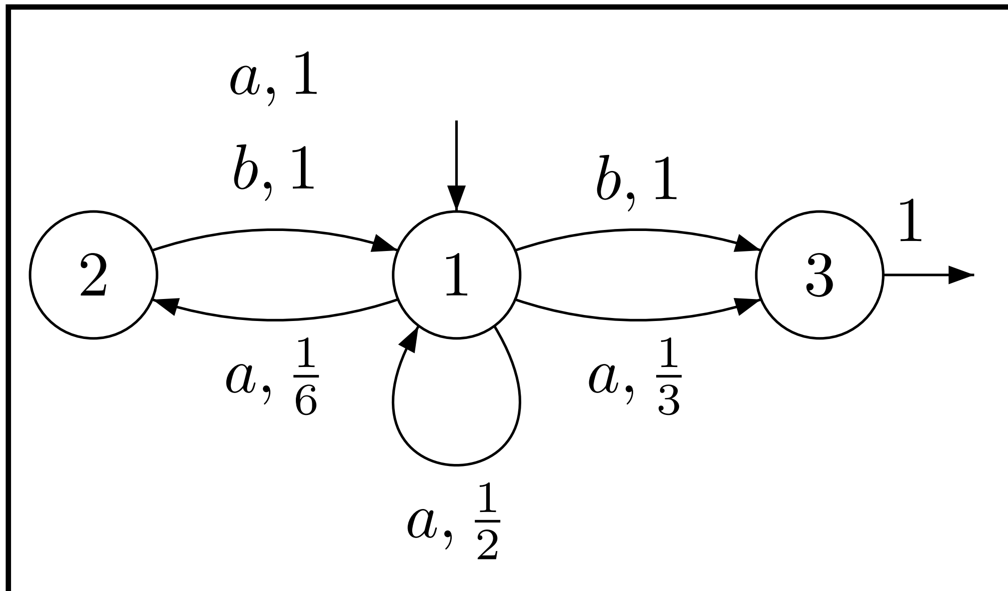

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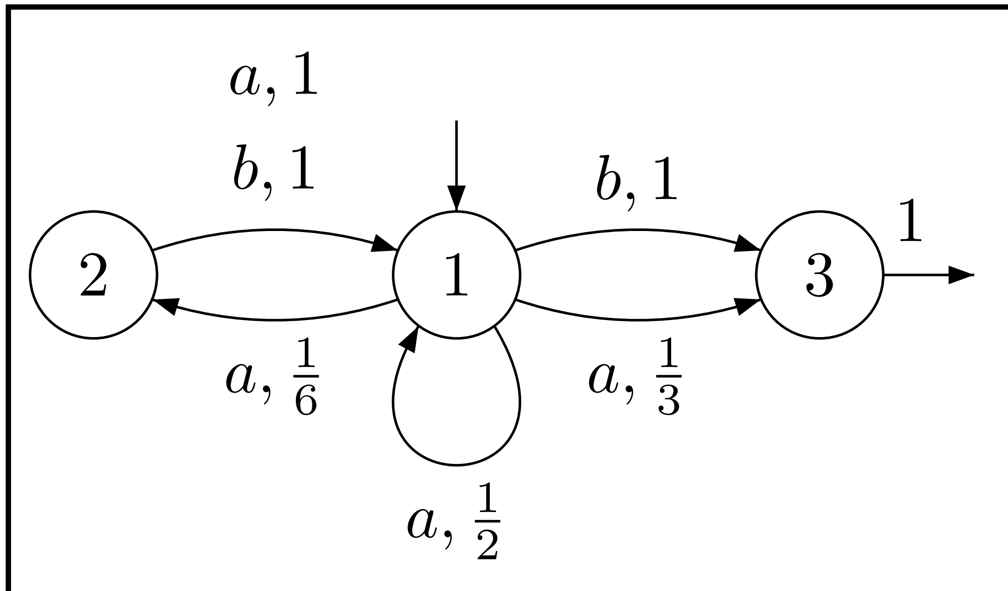
$$\left(\frac{1}{6}a(a+b)+\frac{1}{2}a\right)^*\left(\frac{1}{3}a+b\right)$$ ✓



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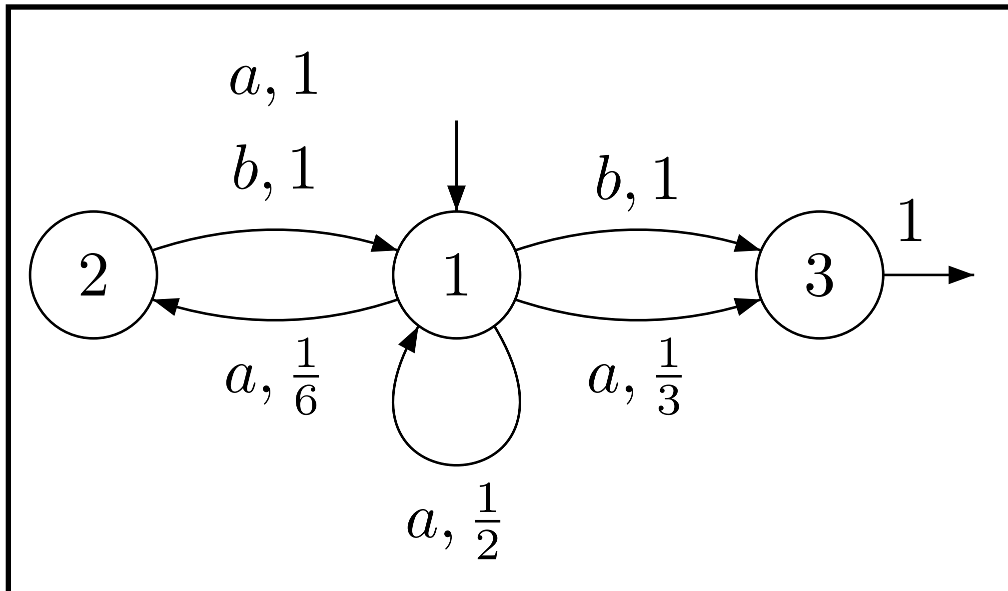

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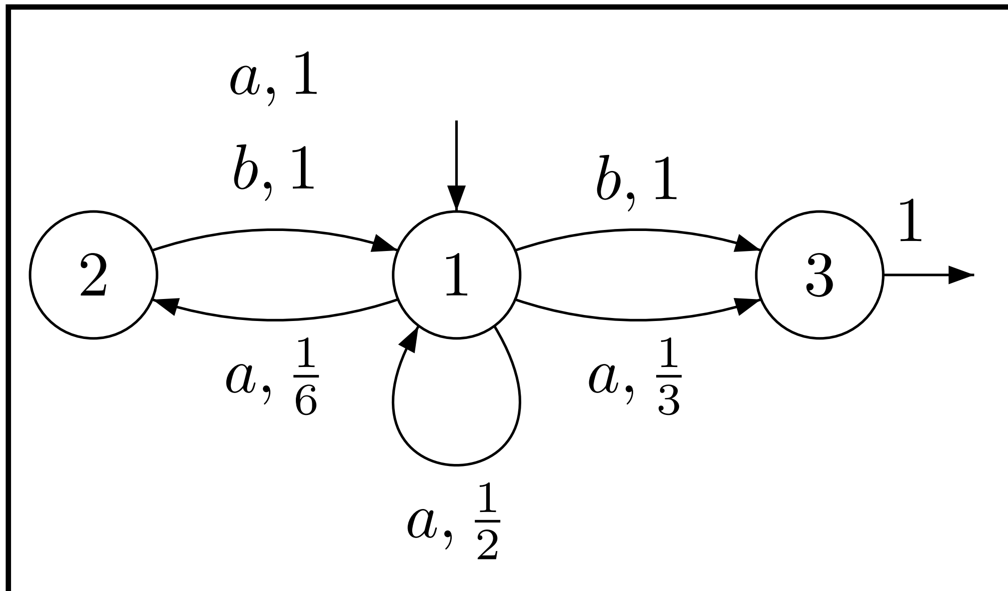

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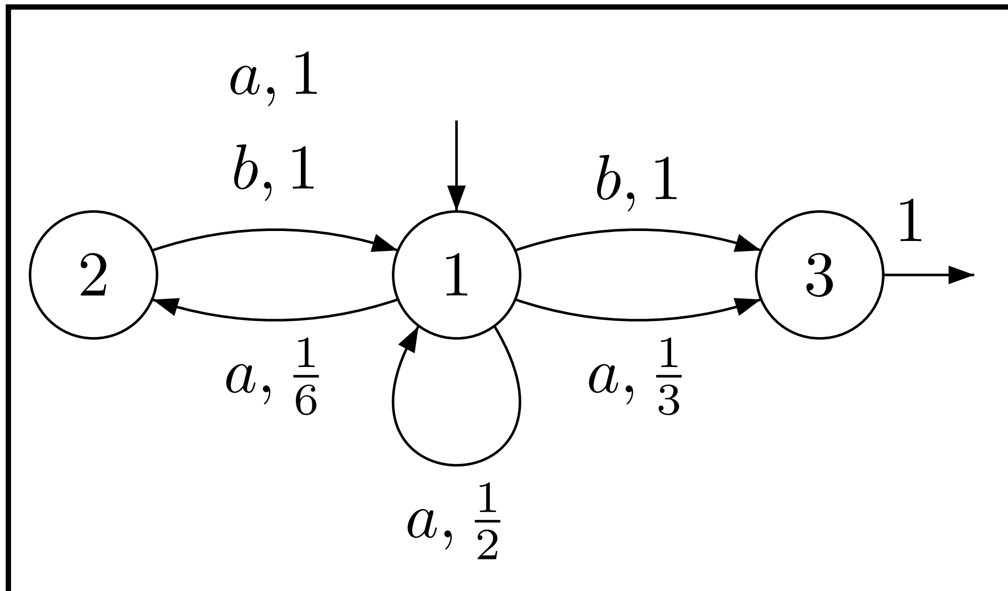

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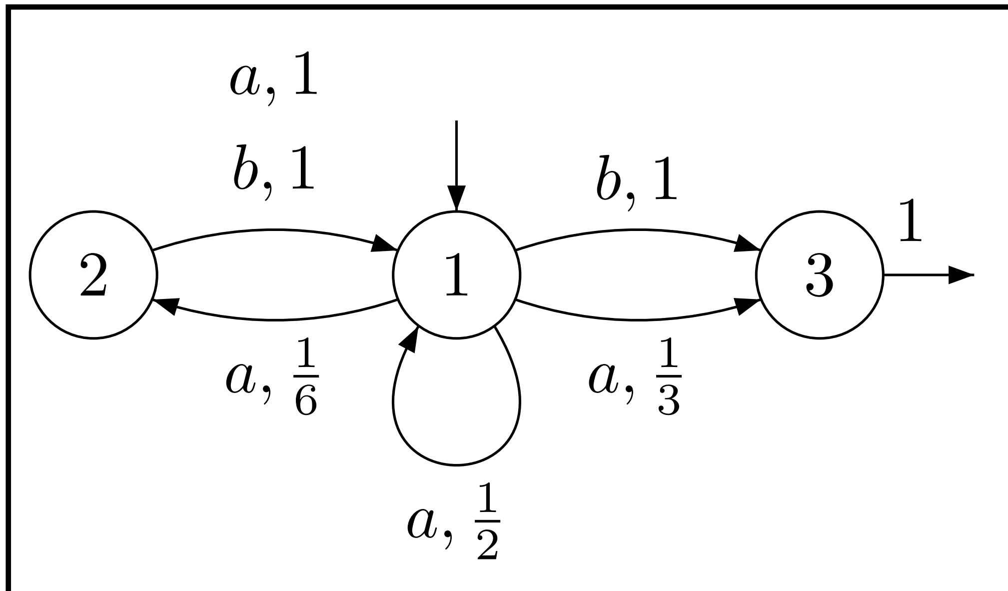

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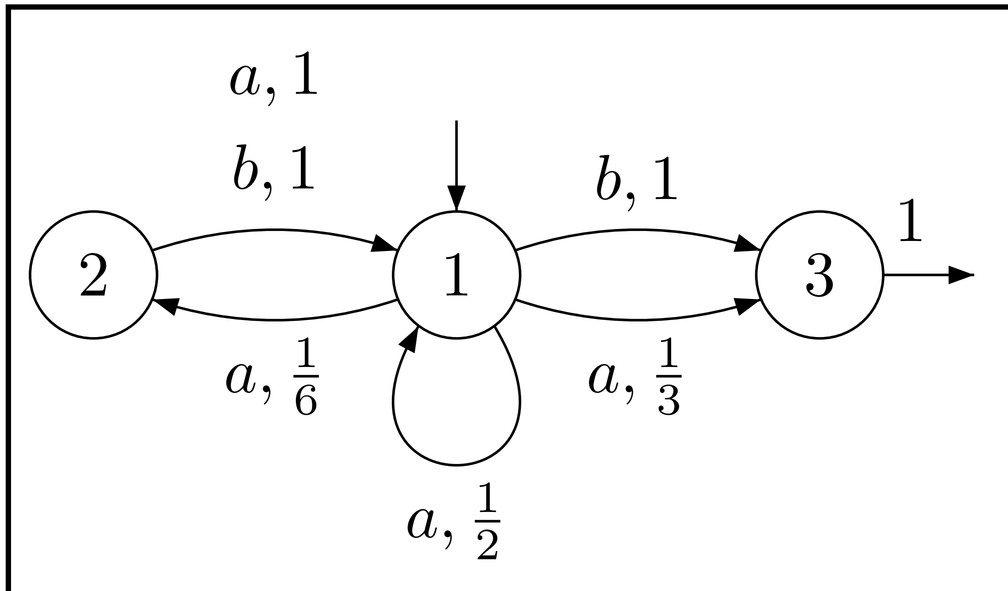
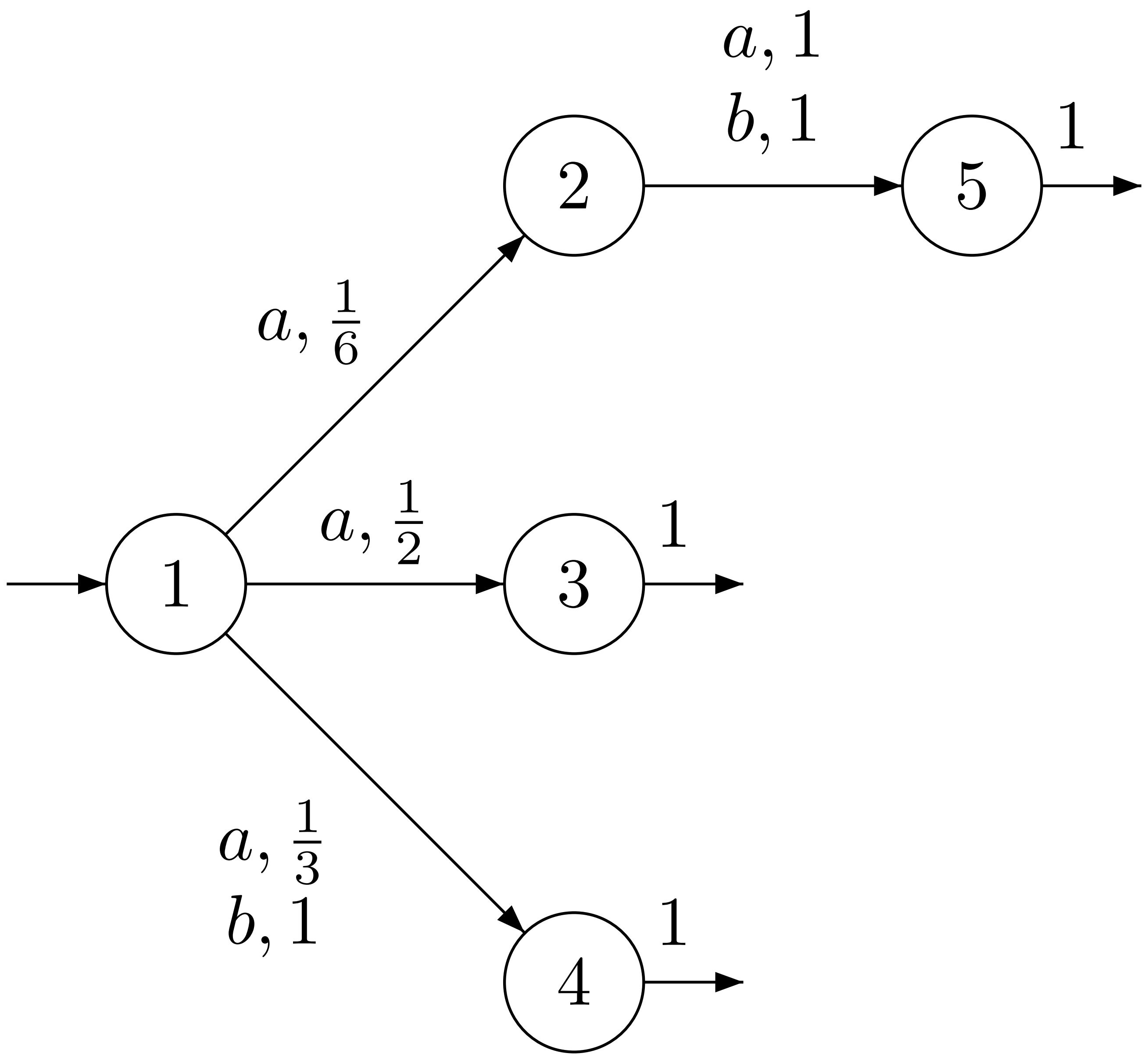
Searching for a natural fragment
of weighted regular expressions
representing probabilistic behaviors



Constructing Probabilistic Expressions

How to iterate?



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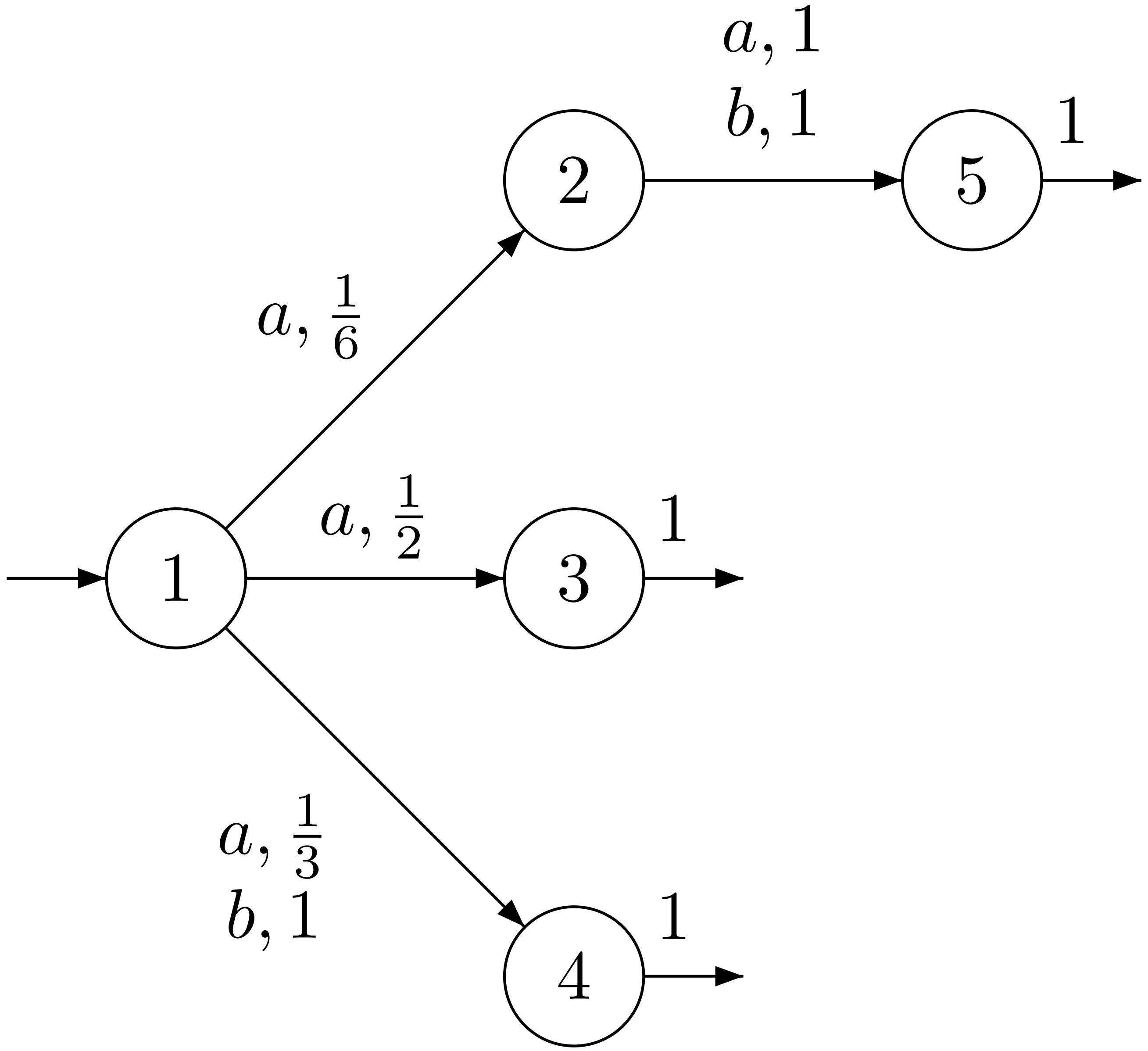

$$\frac{1}{6}a(a+b) + \frac{1}{2}a + (\frac{1}{3}a + b)$$

Constructing Probabilistic Expressions

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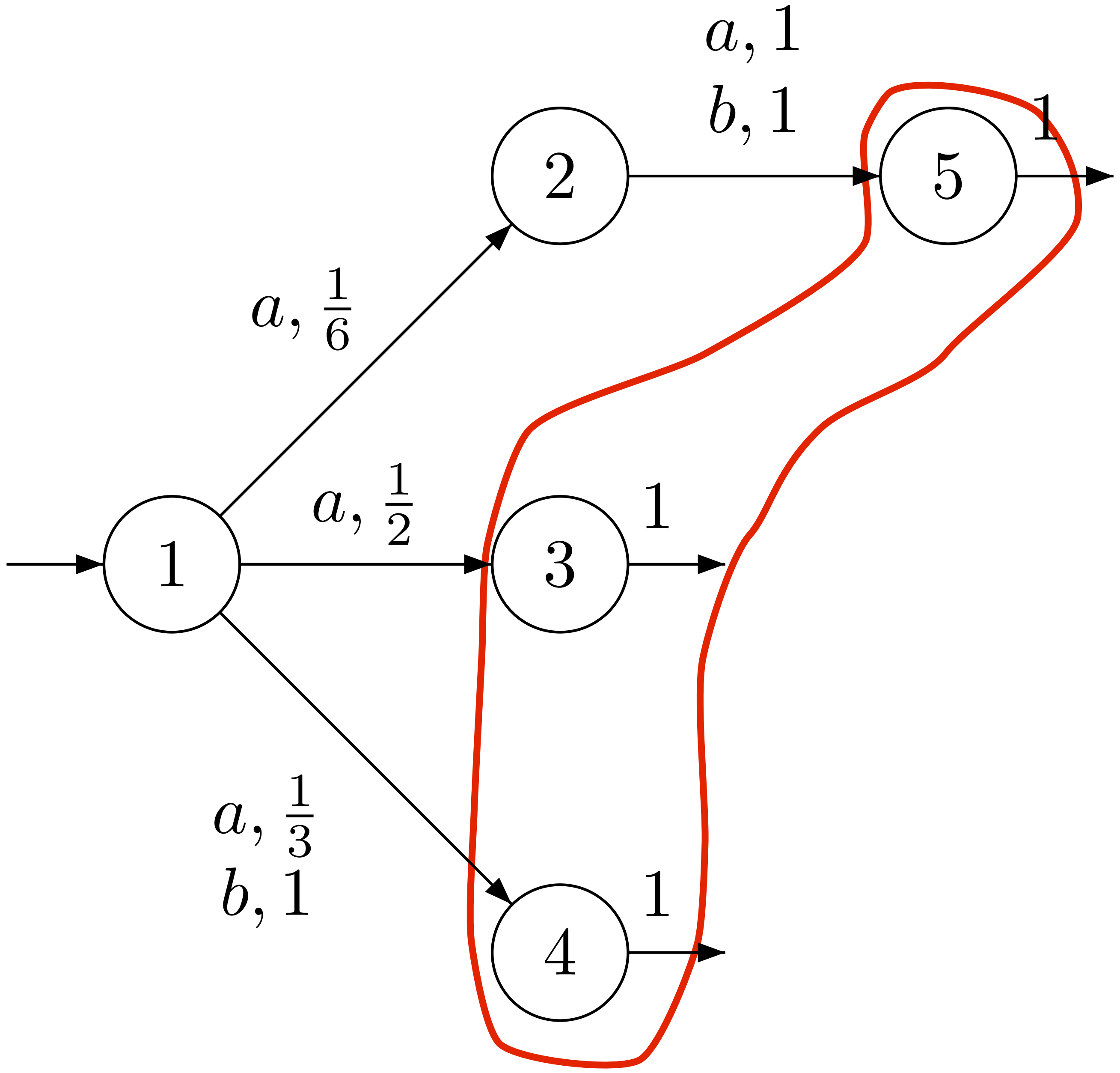

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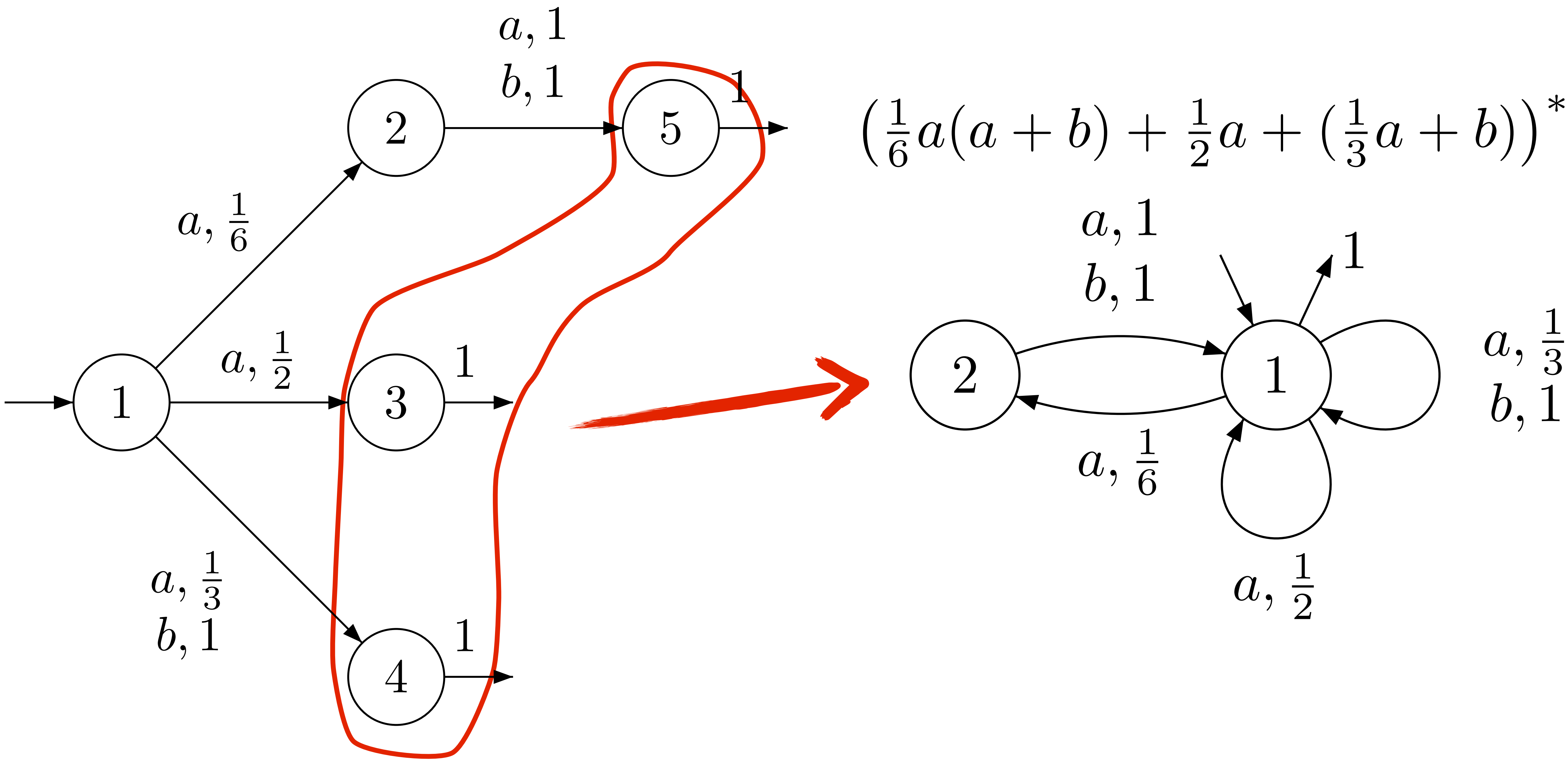

$$\left(\tfrac{1}{6}a(a+b) + \tfrac{1}{2}a + (\tfrac{1}{3}a + b)\right)^*$$

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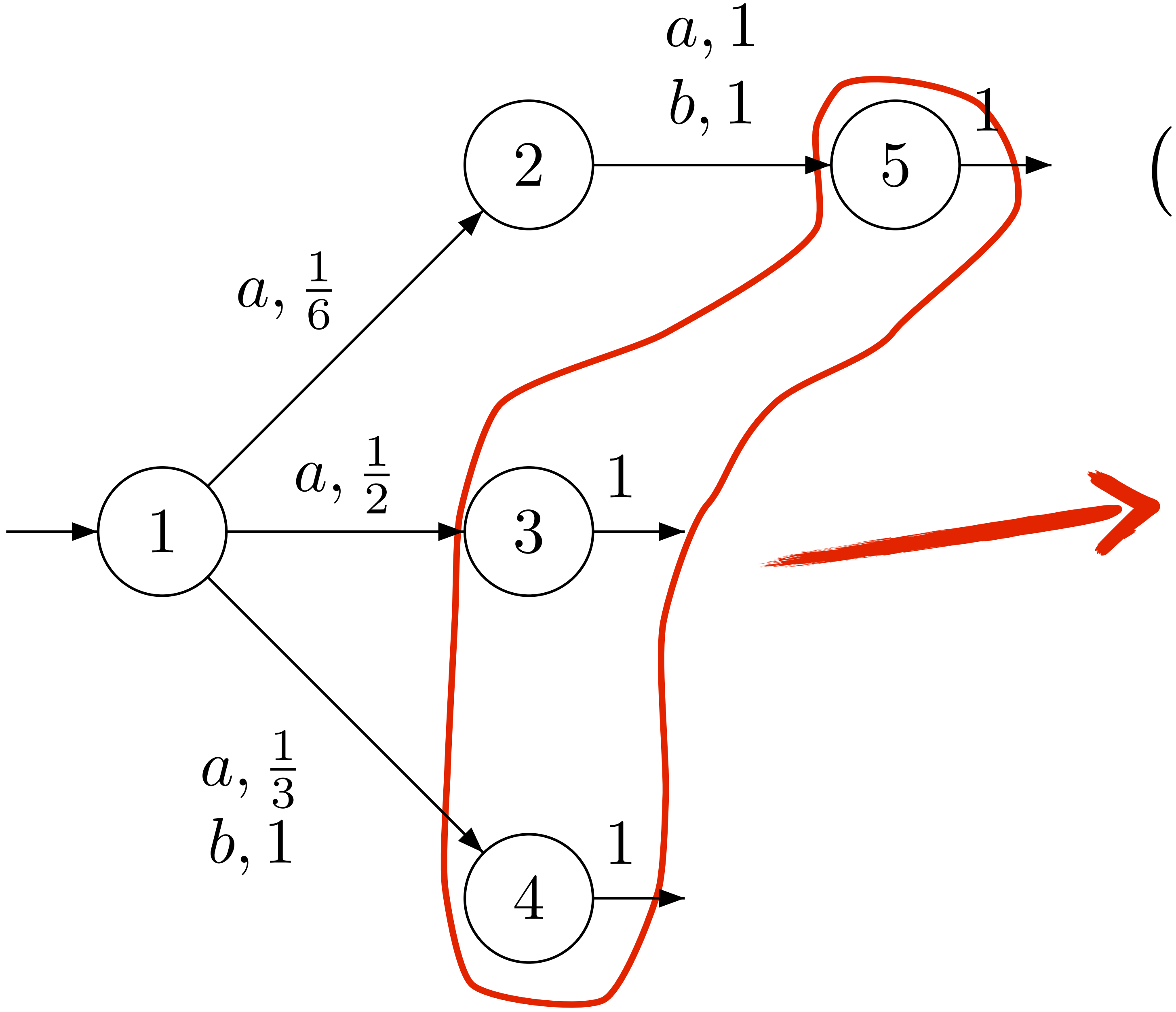

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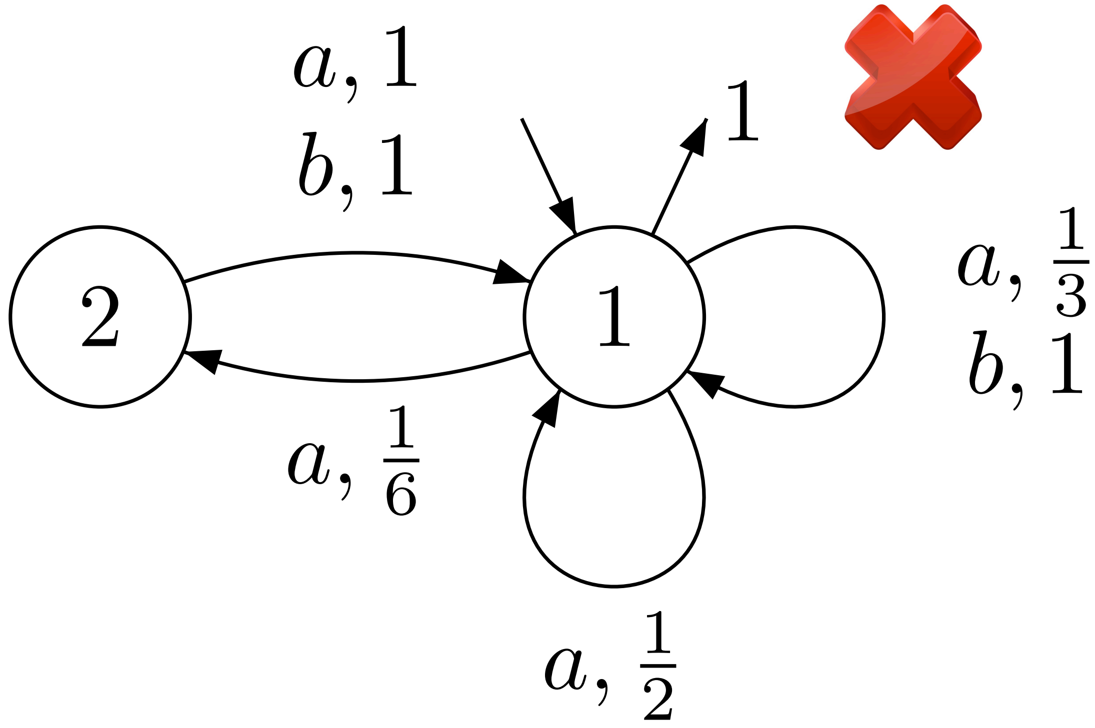
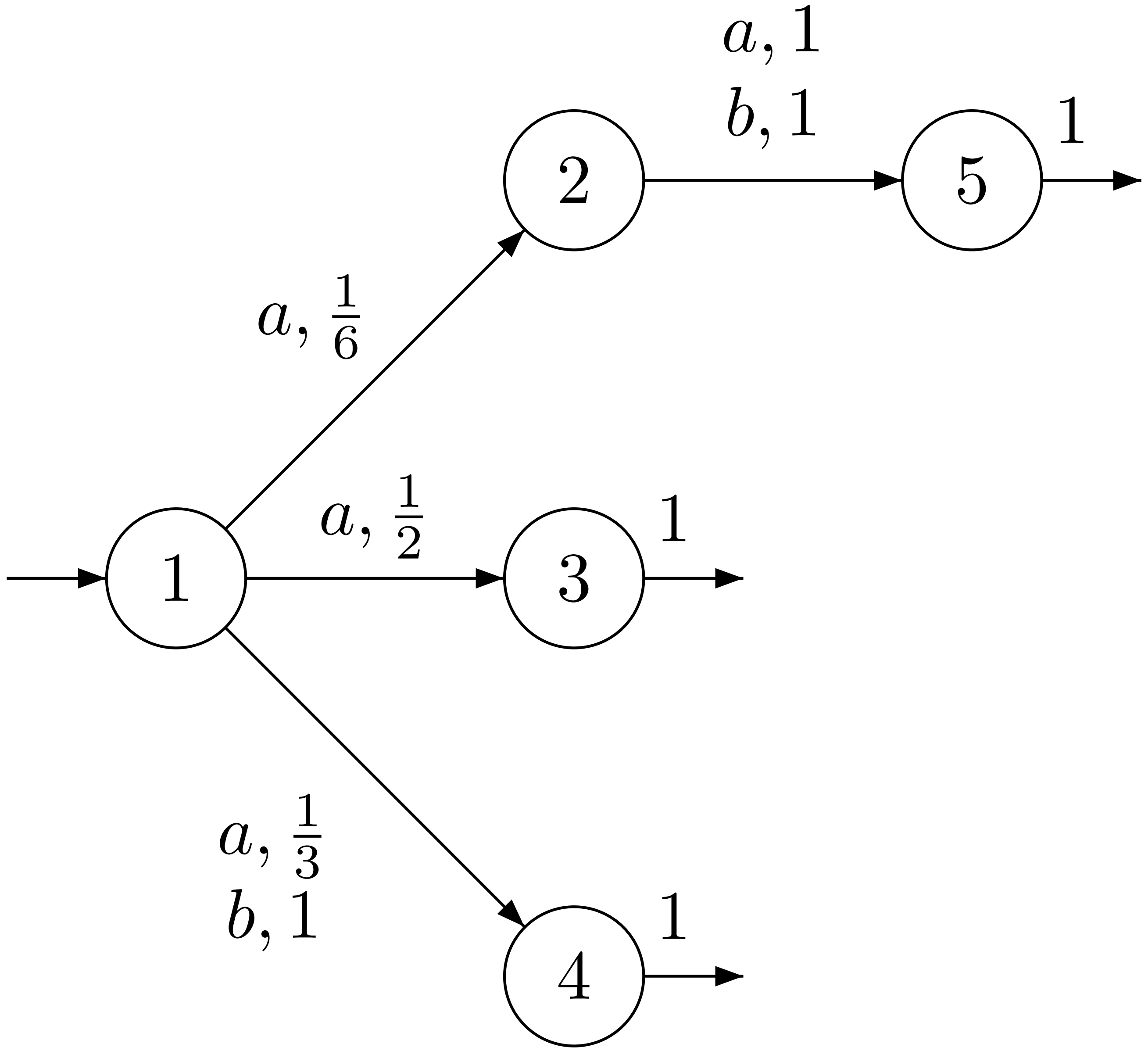


Not a valid Probabilistic Automaton anymore (acceptance condition not fulfilled)

Constructing Probabilistic Expressions

How to iterate?



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Constructing Probabilistic Expressions

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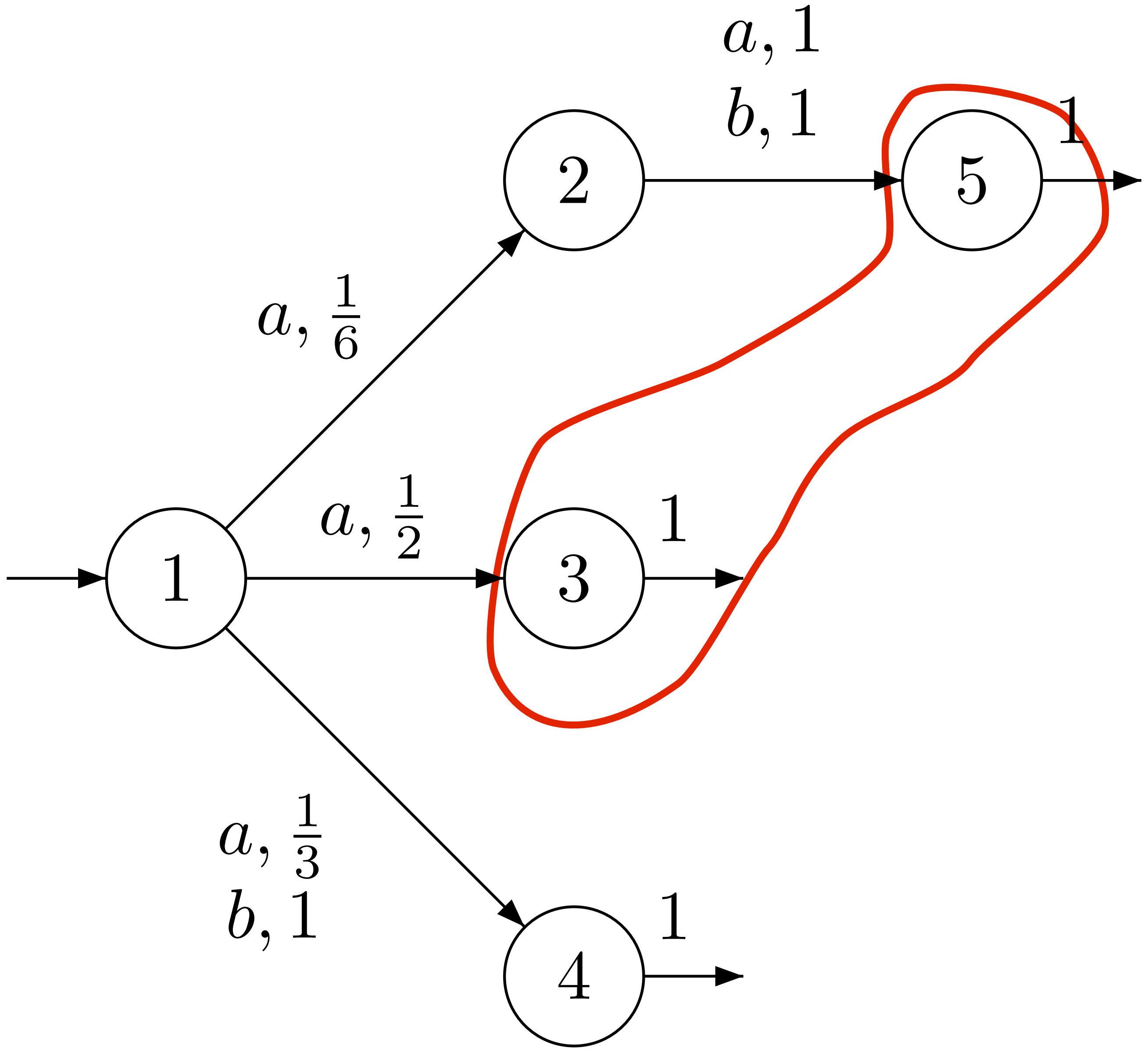

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Constructing Probabilistic Expressions

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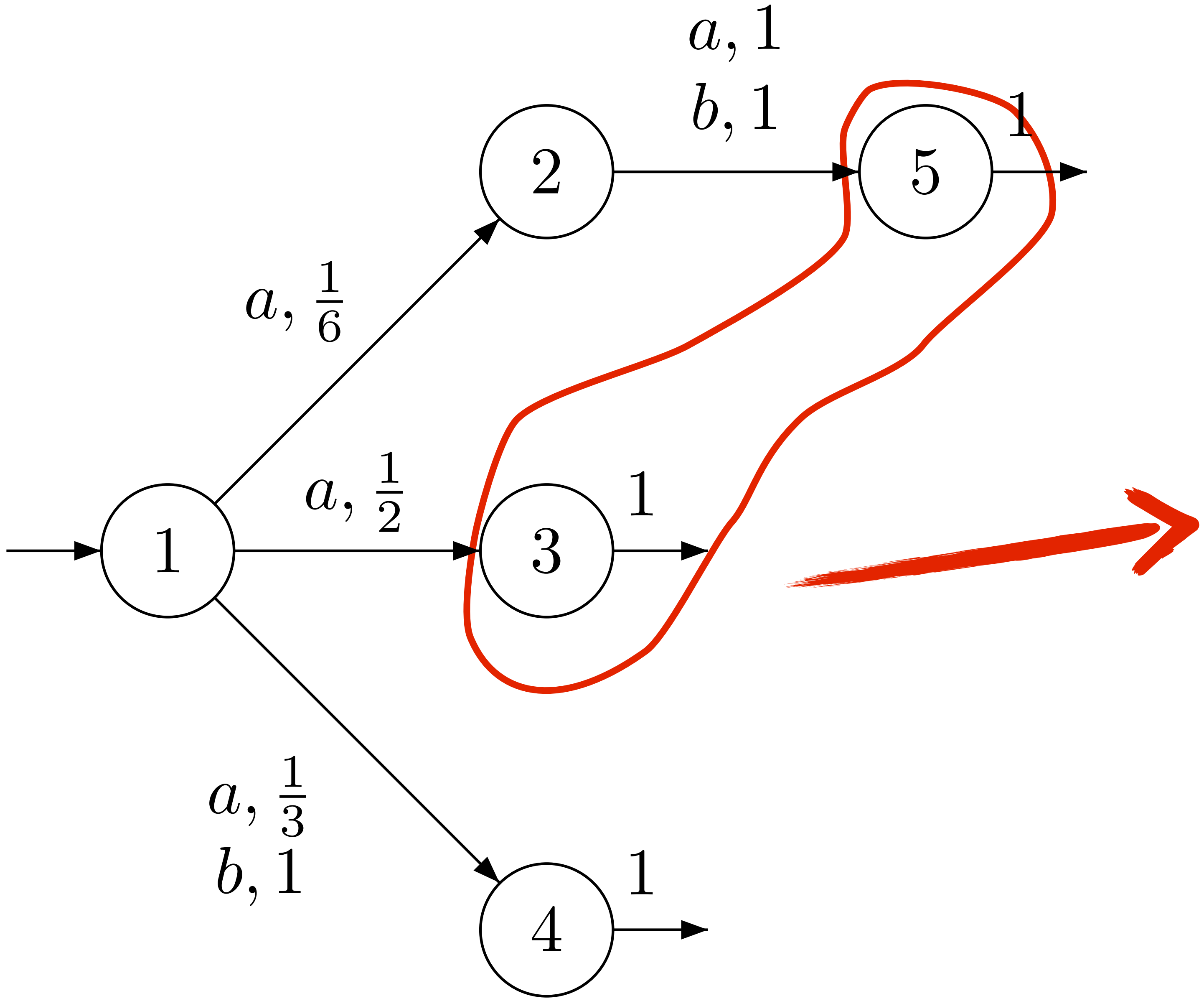
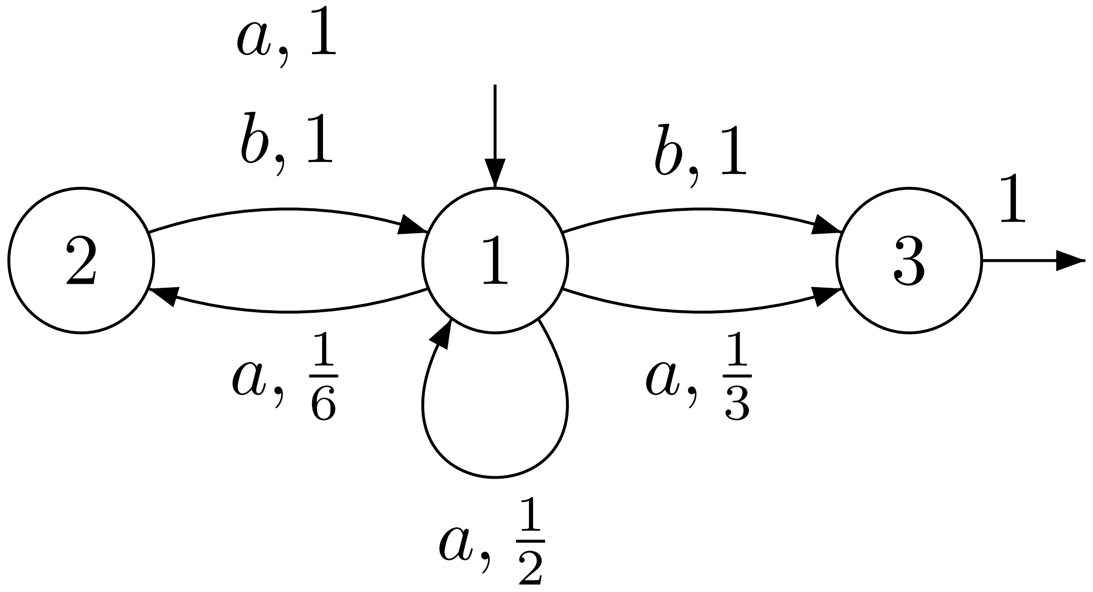


$$(\tfrac{1}{6}a(a+b) + \tfrac{1}{2}a)^*(\tfrac{1}{3}a + b)$$



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Constructing Probabilistic Expressions

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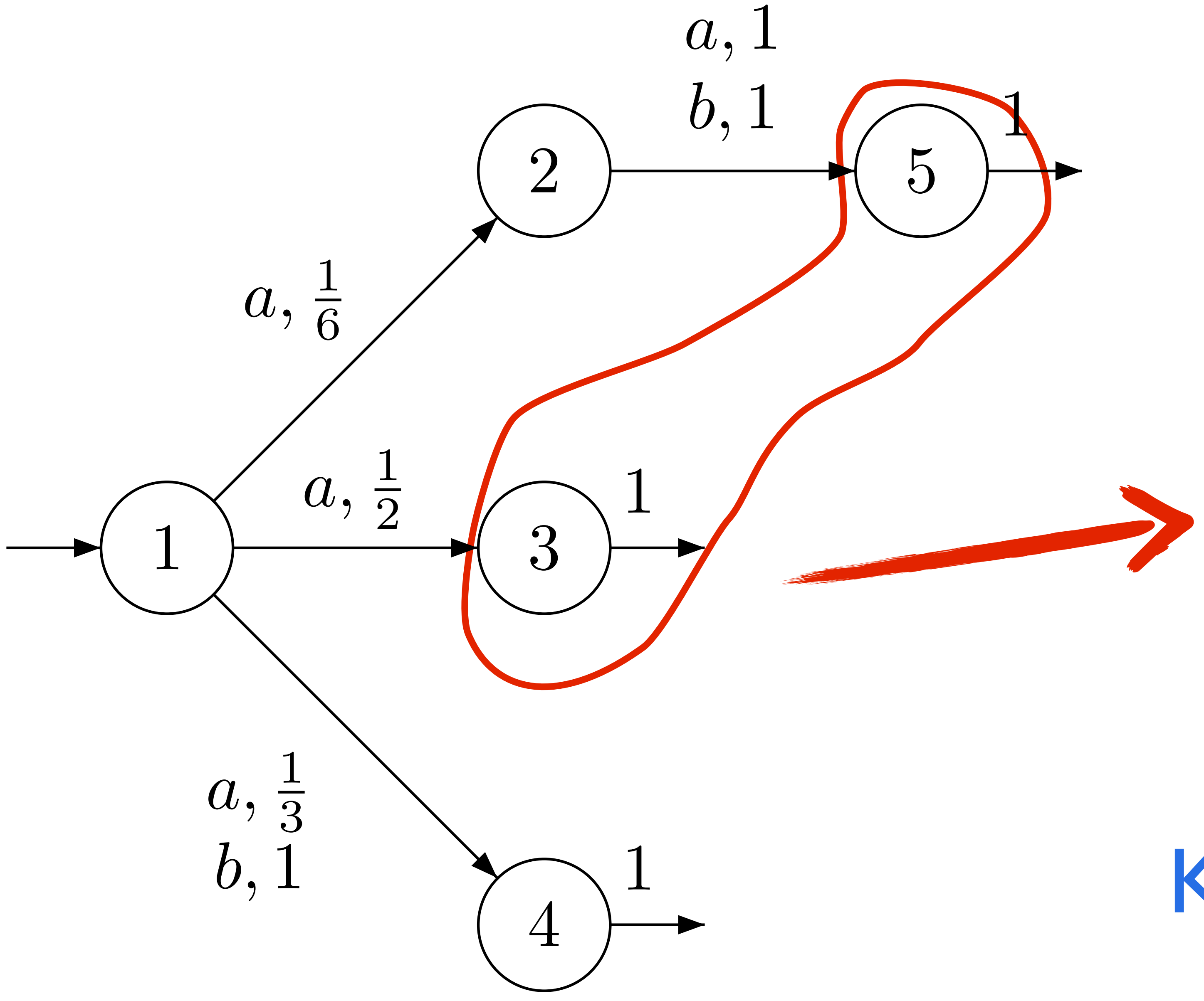

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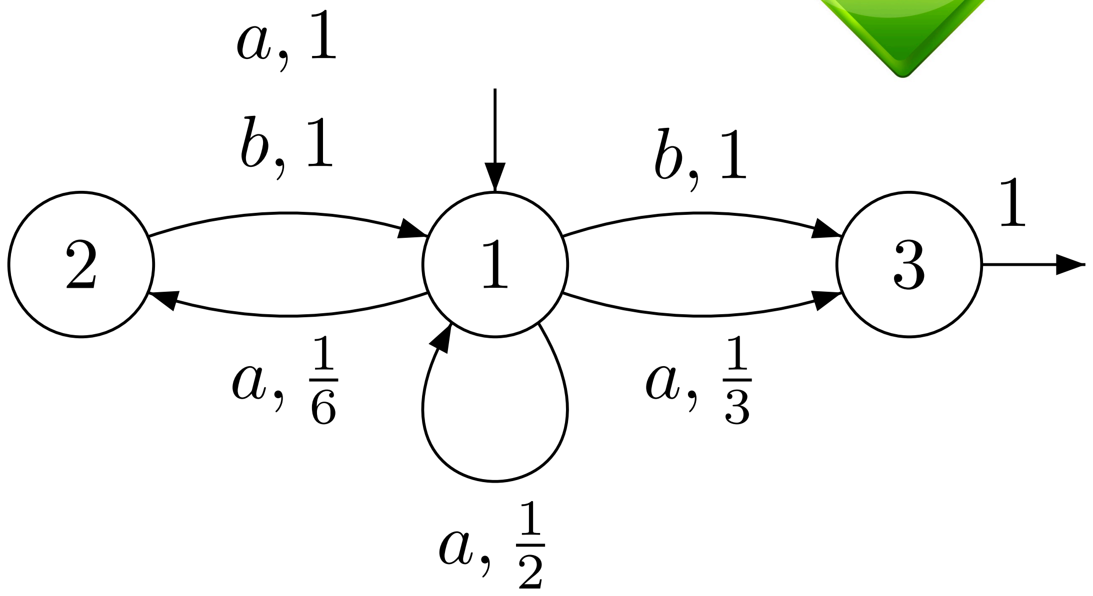
✓



Keep some *branch* for termination of the Probabilistic Automaton

Probabilistic Expressions

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- $a \in A$ and $p \in [0,1]$ are PREs

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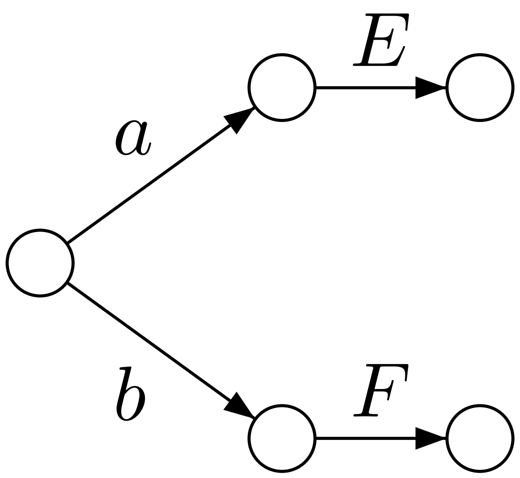

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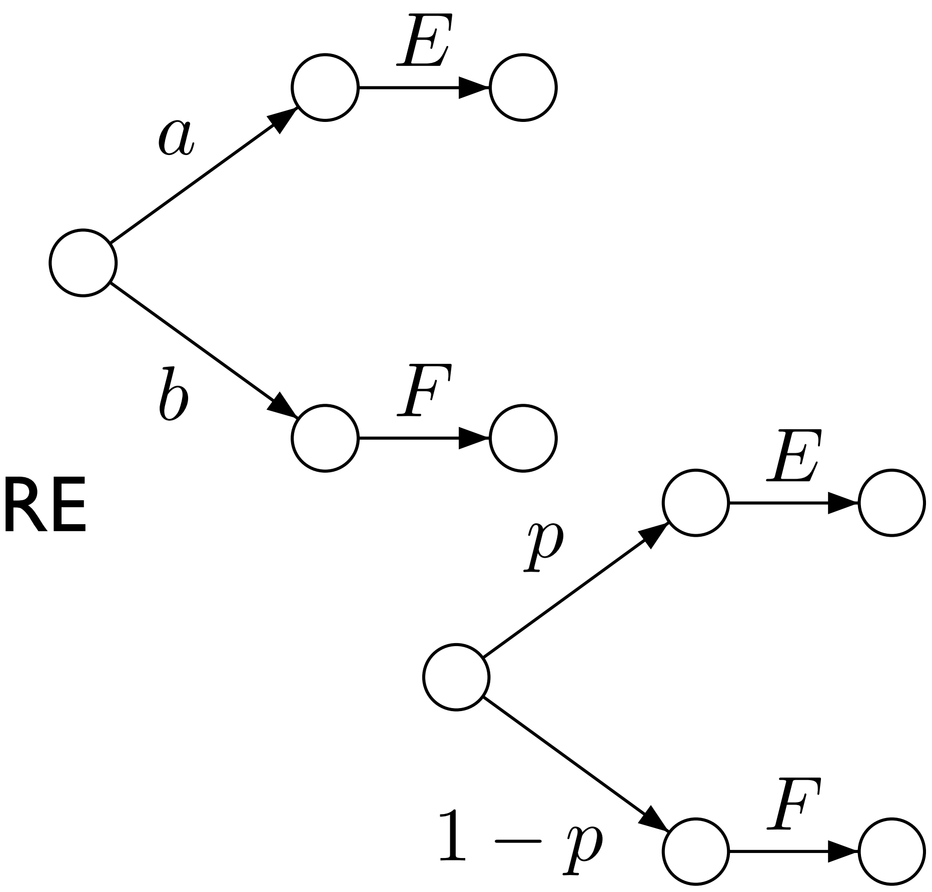

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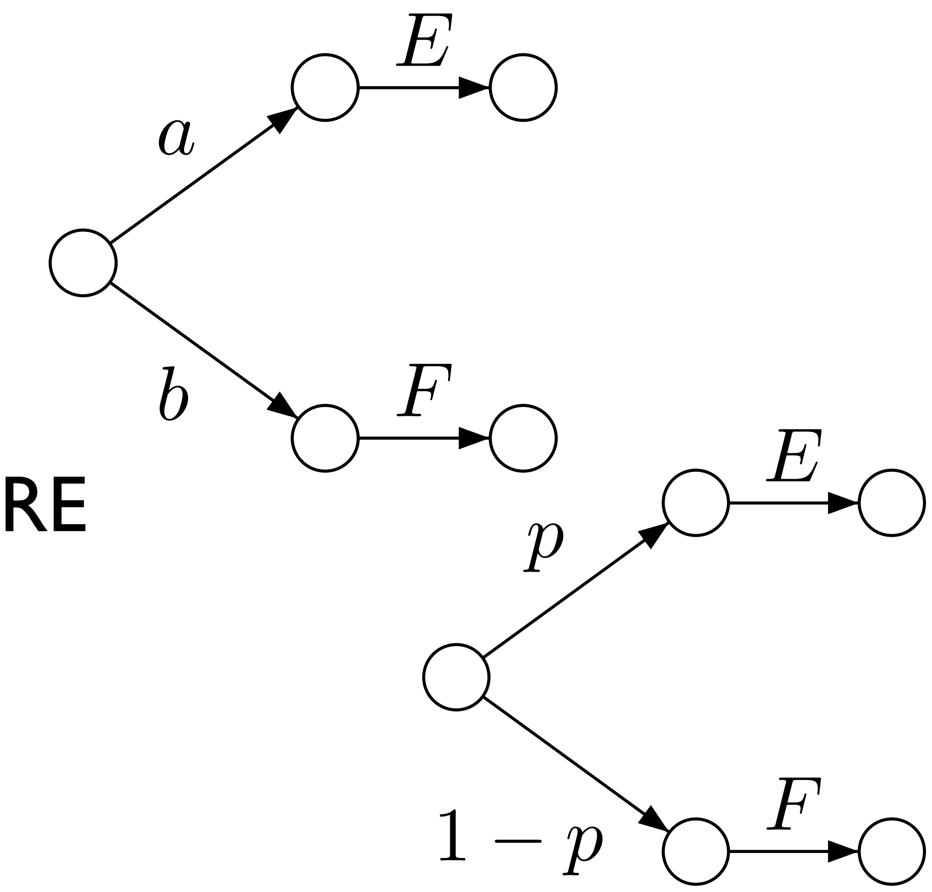

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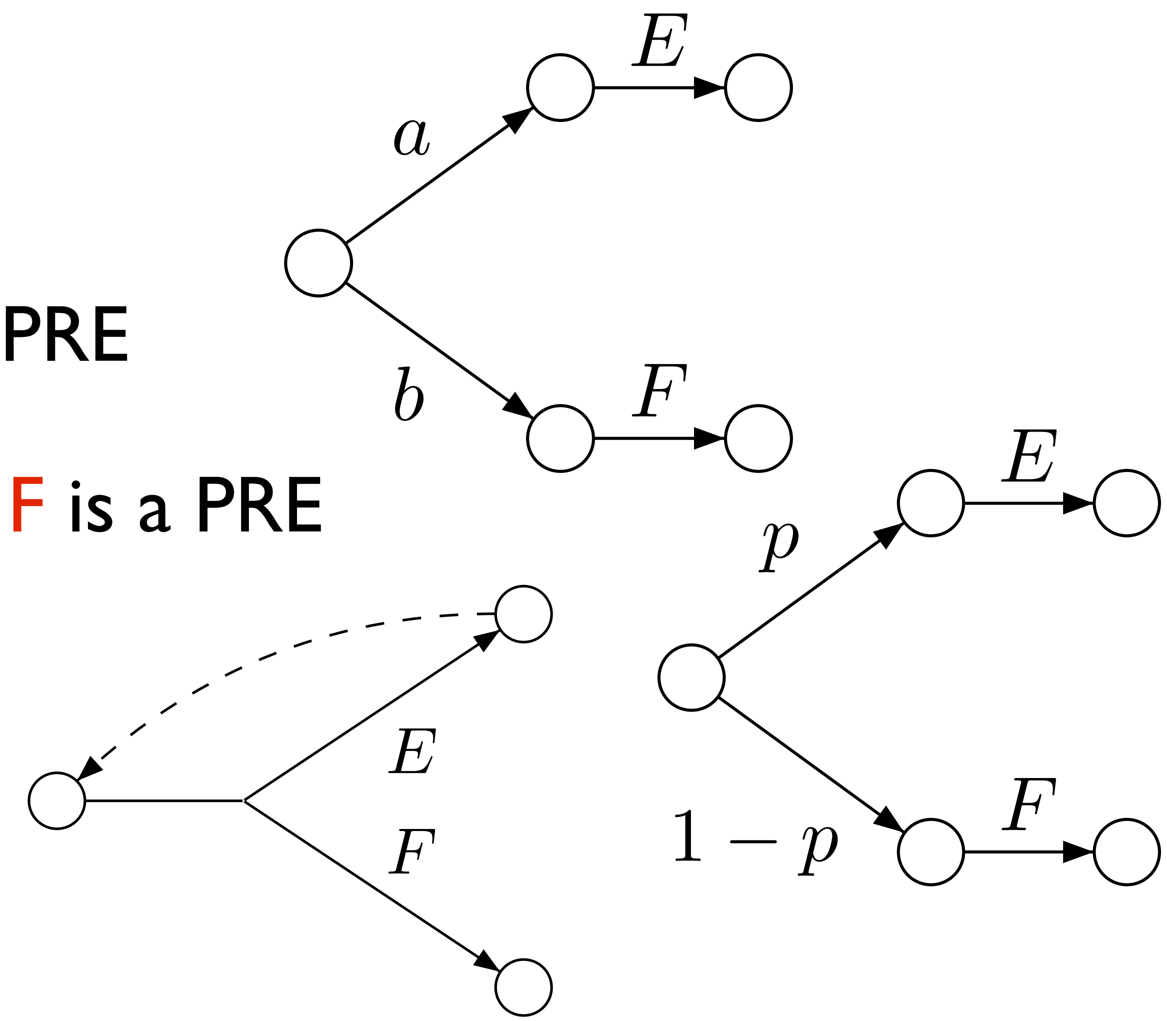

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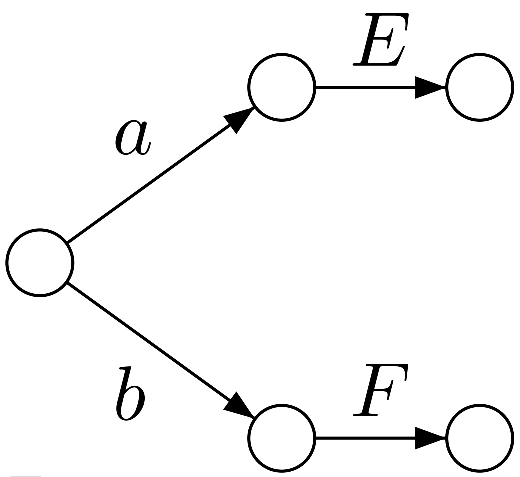

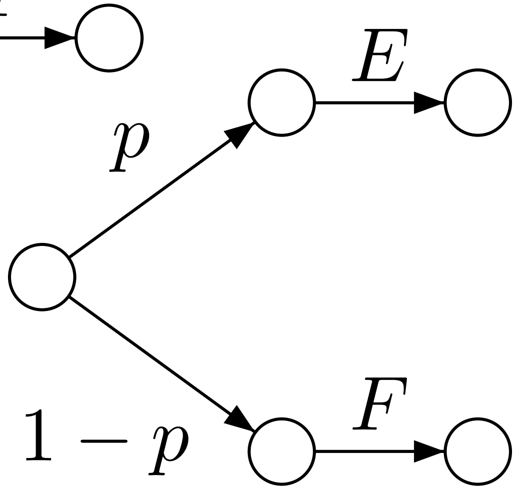

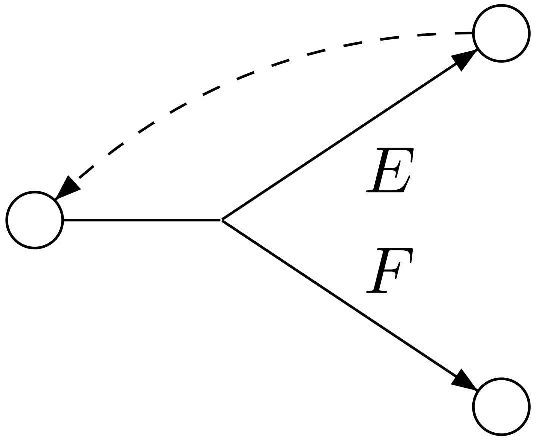

$(a \cdot E)^* \cdot b \cdot F$

$(p \cdot E)^* \cdot (1-p) \cdot F$

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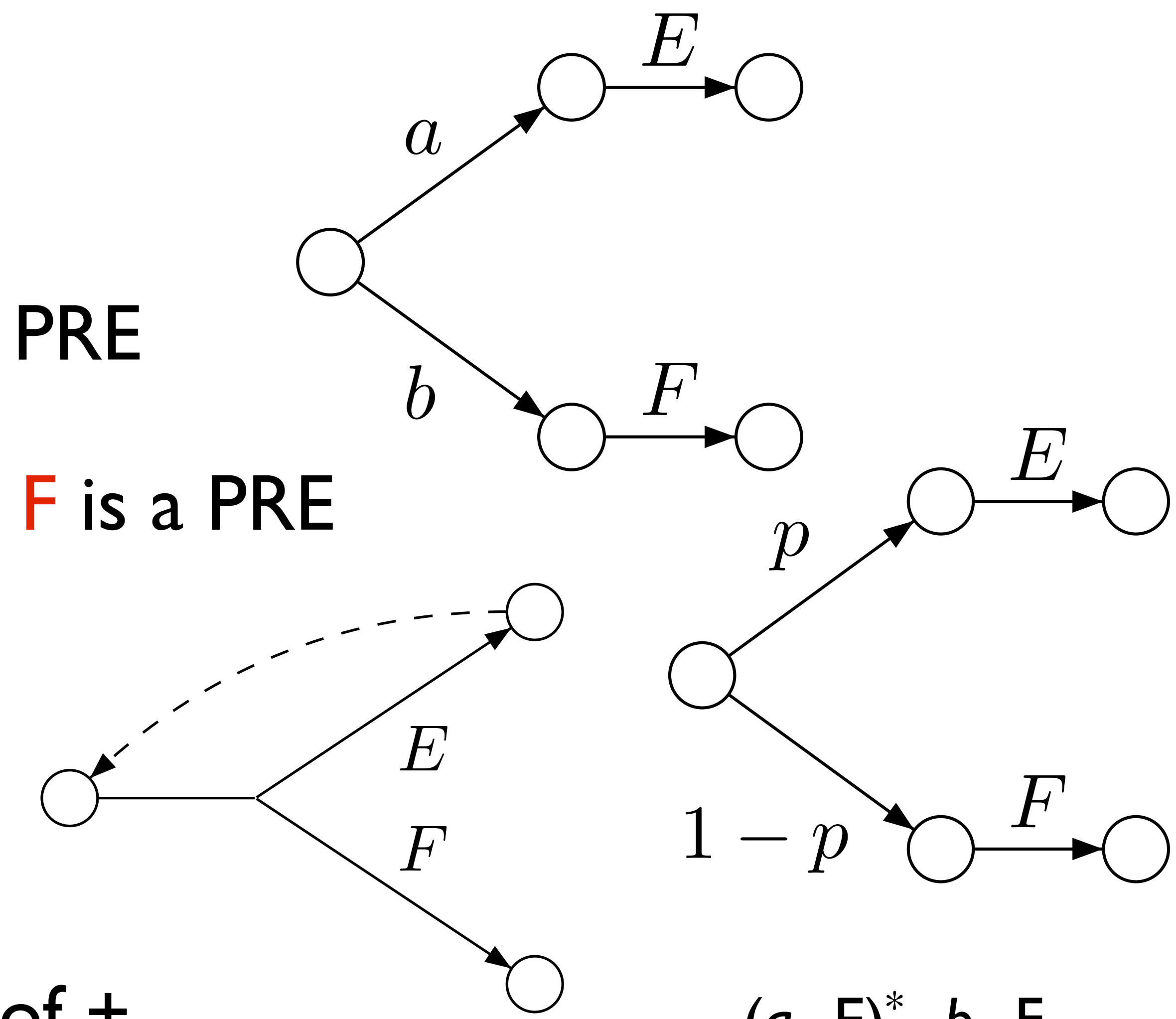
- Closure of PRE under commutativity of +, associativity of + and $\cdot$, distributivity of $\cdot$ over +



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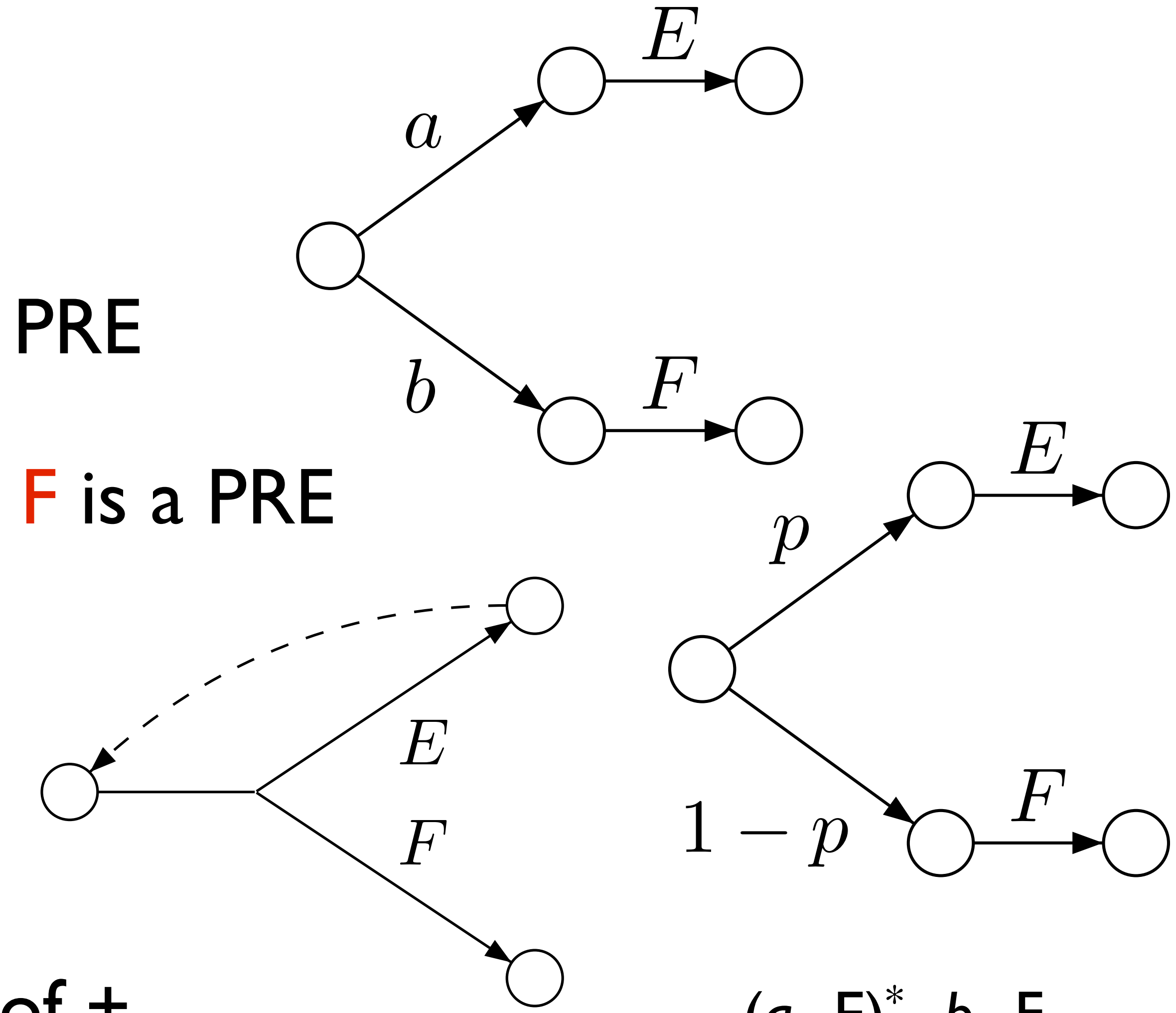

$(a \cdot E)^* \cdot b \cdot F$

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Semantics given as a fragment of regular expressions in complete semirings...

$$\mathbb{P}(E \cdot F, u) = \sum_{u=vw} \mathbb{P}(E, v) \times \mathbb{P}(F, w)$$

Example

$$\begin{tikzcd}
a+b \arrow[d] & & \\
a^*b \arrow[dr] & & \frac{1}{2}a+\frac{1}{2}b \arrow[dl] \\
& a^*b(\frac{1}{2}a+\frac{1}{2}b) \arrow[d] & \\
& a^*b\frac{1}{2}a+a^*b\frac{1}{2}b \arrow[d] & \\
& (a^*b\frac{1}{2}a)^*a^*b\frac{1}{2}b &
\end{tikzcd}$$

Example

Deterministic choice

$$\begin{tikzcd}
a + b \arrow[d] & & \\
a^* b \arrow[dr] & & \frac{1}{2} a + \frac{1}{2} b \arrow[dl] \\
& a^* b \left(\frac{1}{2} a + \frac{1}{2} b \right) \arrow[d] & \\
& a^* b \frac{1}{2} a + a^* b \frac{1}{2} b \arrow[d] & \\
& (a^* b \frac{1}{2} a)^* a^* b \frac{1}{2} b &
\end{tikzcd}$$

Example

Deterministic choice

Star rule

$$\begin{tikzcd}
a + b \arrow[d] & & \\
a^*b \arrow[dr] & & \frac{1}{2}a + \frac{1}{2}b \arrow[dl] \\
& a^*b(\frac{1}{2}a + \frac{1}{2}b) \arrow[d] & \\
& a^*b\frac{1}{2}a + a^*b\frac{1}{2}b \arrow[d] & \\
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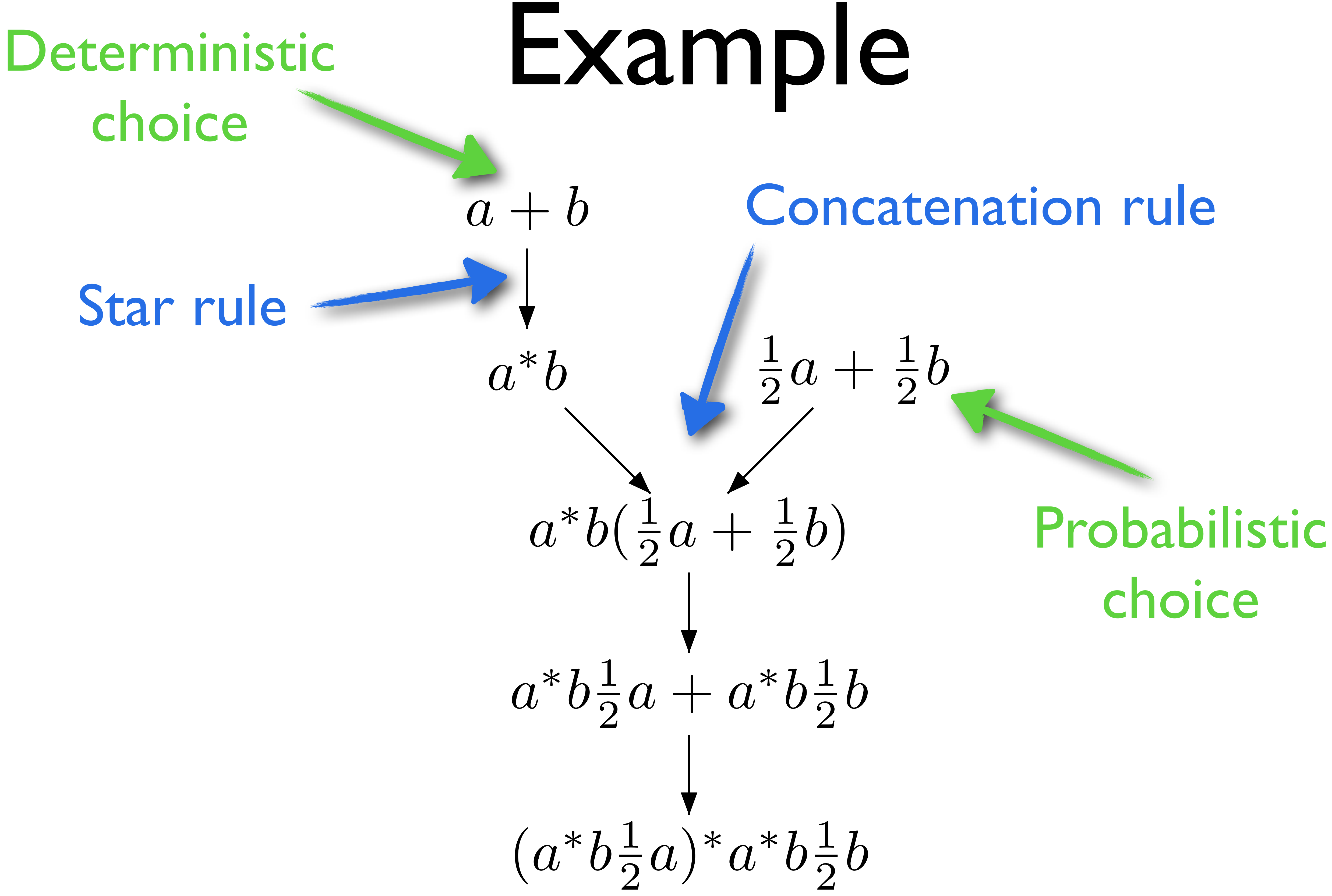
\end{tikzcd}$$

Probabilistic choice

Example



Deterministic choice

Star rule

Concatenation rule

Probabilistic choice

$$\begin{tikzcd}
a+b \arrow[d] & & \\
a^*b \arrow[dr] & & \frac{1}{2}a+\frac{1}{2}b \arrow[dl] \\
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& a^*b\frac{1}{2}a+a^*b\frac{1}{2}b \arrow[d] & \\
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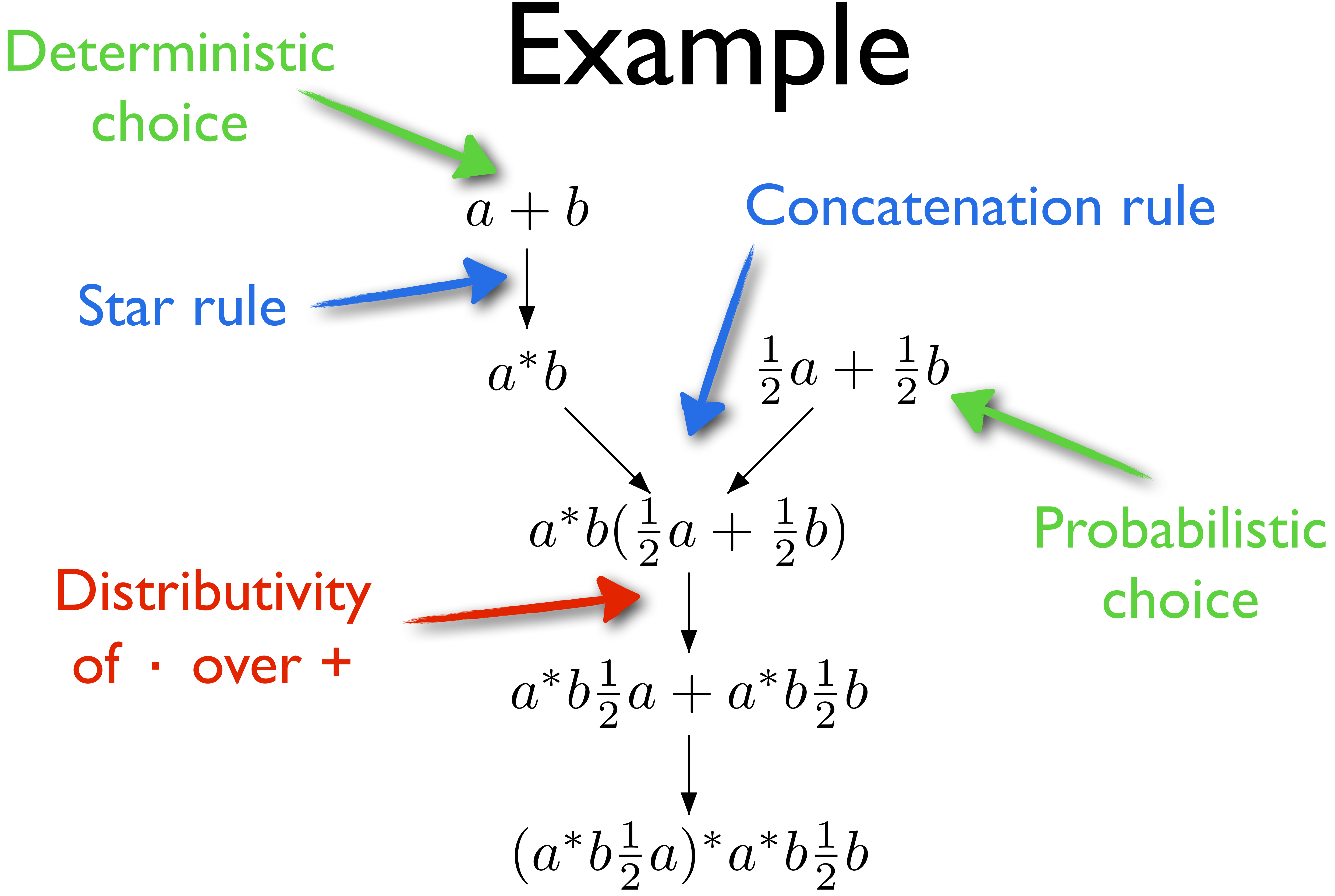
\end{tikzcd}$$

Example



$$\begin{tikzcd}
a + b \arrow[d] & & \\
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\end{tikzcd}$$

Deterministic choice

Star rule

Concatenation rule

Probabilistic choice

Distributivity of · over +

Example

Deterministic choice

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$$\begin{tikzcd}
a + b \arrow[d] & & \\
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Probabilistic choice

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\end{tikzcd}$$

Probabilistic choice

Distributivity of · over +

Star rule

The choice in the star is made *far* from the beginning...

Probabilistic Kleene-Schützenberger Theorem

- Every PRE can be translated into an equivalent Probabilistic automaton.

- Every Probabilistic automaton can be denoted by an equivalent PRE.

From Automata to Expressions

- Usual procedures (Brozozwski-McCluskey, elimination, McNaughton-Yamada...) keeping probabilistic constraints in mind
- Requires to prove some (useful) properties of PREs, e.g., if E+F and G are PREs, then E+F · G is a PRE

[1] J. A. Brzozowski and E. J. McCluskey (1963). Signal Flow Graph Techniques for Sequential Circuit State Diagrams. IEEE Trans. on Electronic Computers 12.

From Expressions to Automata

[1] V. M. Glushkov (1961). The abstract theory of automata. Russian Math. Surveys 16.
[2] G. Berry and R. Sethi (1986). From regular expressions to deterministic automata. Theoretical Computer Science 48.

From Expressions to Automata

- We have to prove closure properties of Probabilistic Automata (for example constructing standard automata, [1,2])

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- Problem: « if $E+F$ is a PRE, then $E^* \cdot F$ is a PRE » is not an *inductive* rule

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- Probabilistic Automata in a normal form: accepting states labelled by subexpressions they are computing

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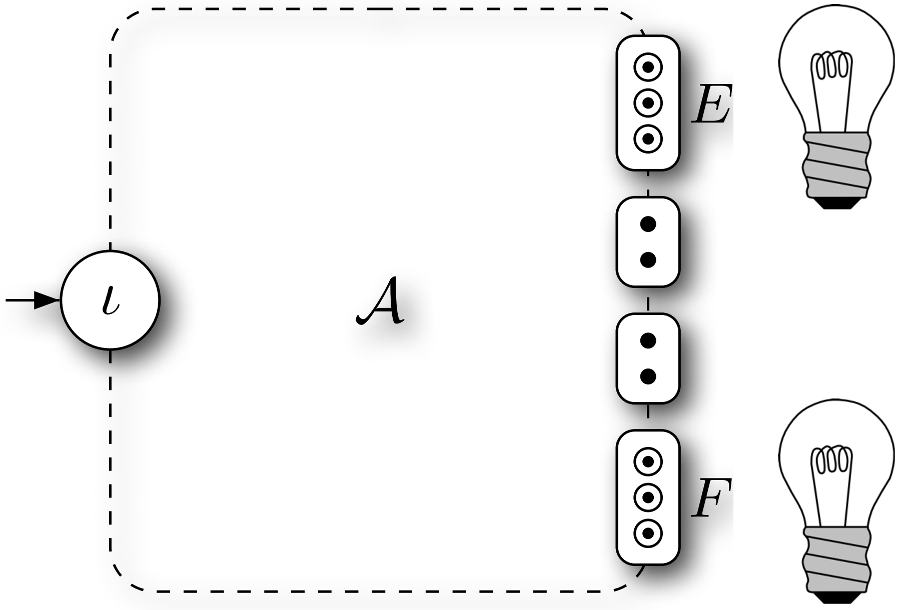

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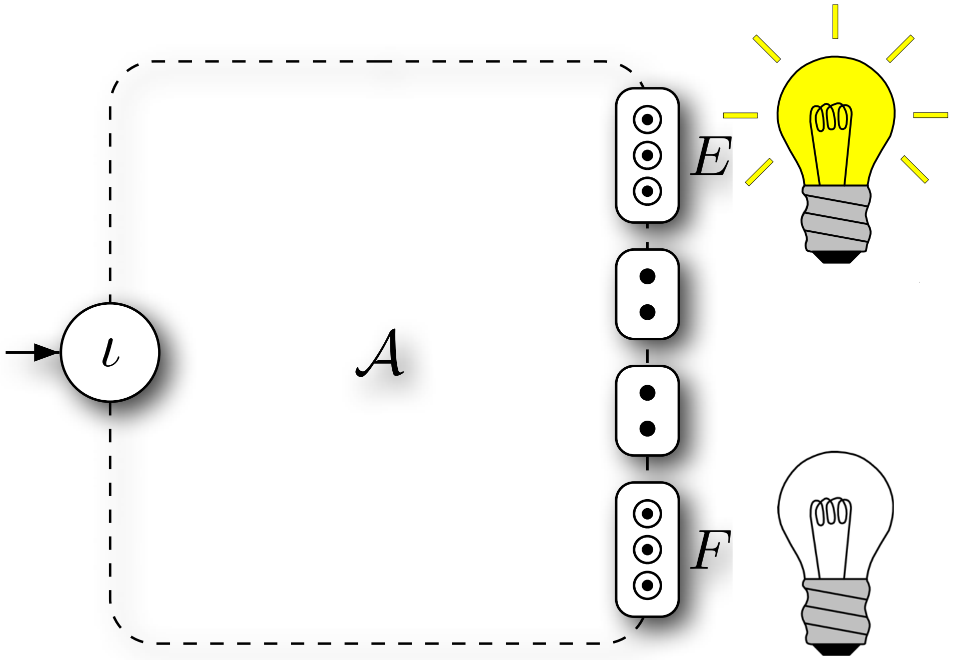

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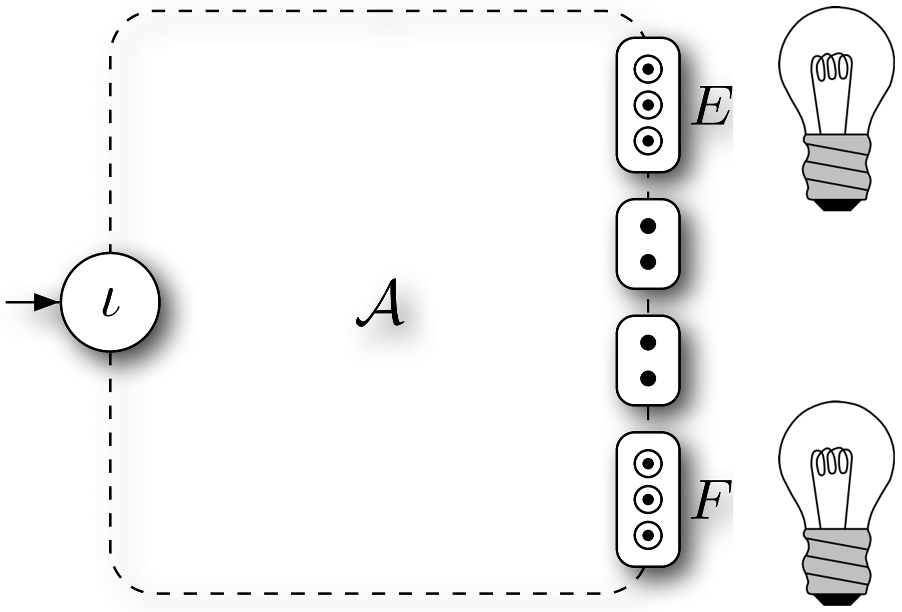

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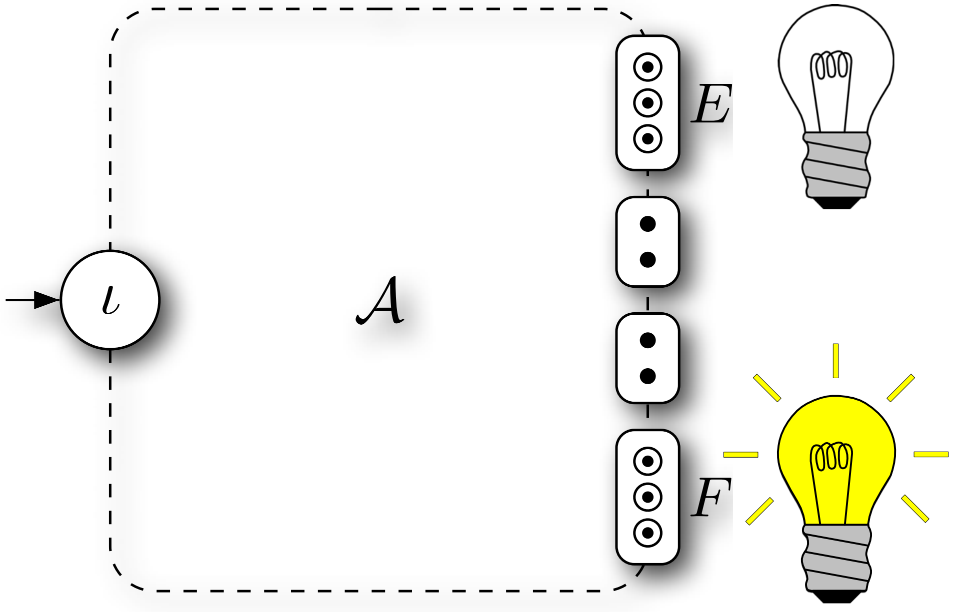

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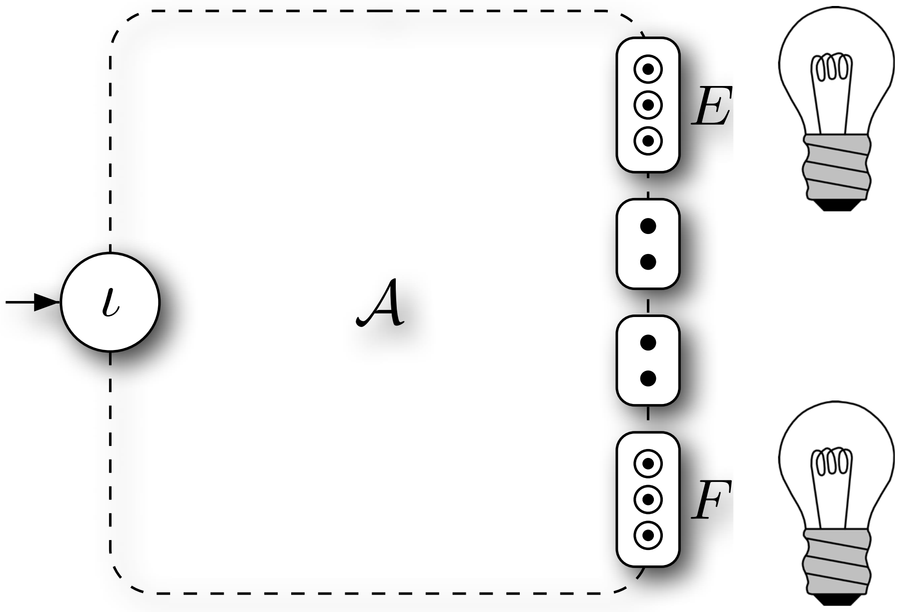

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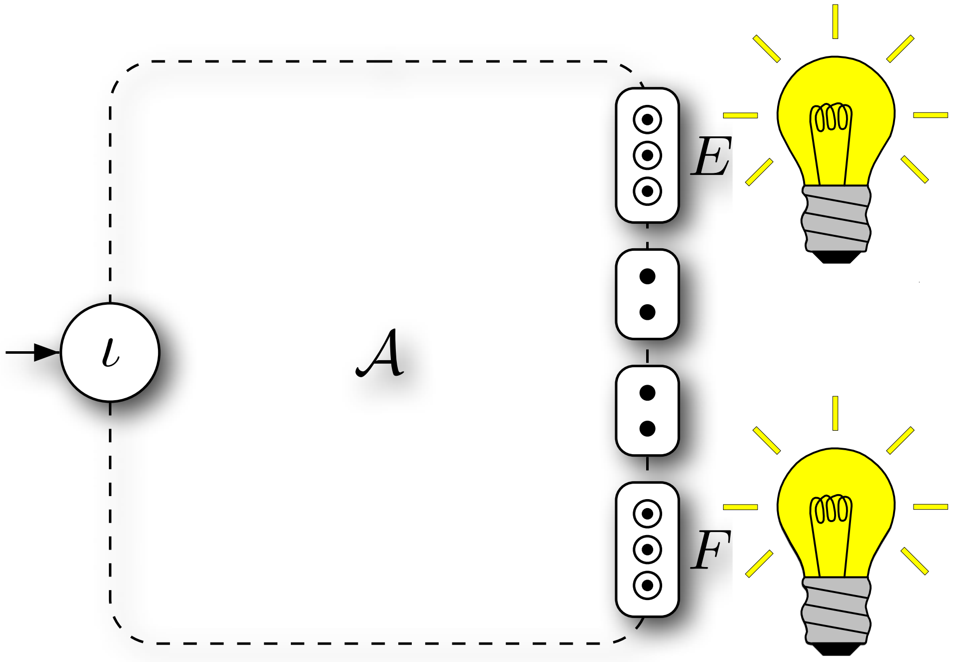

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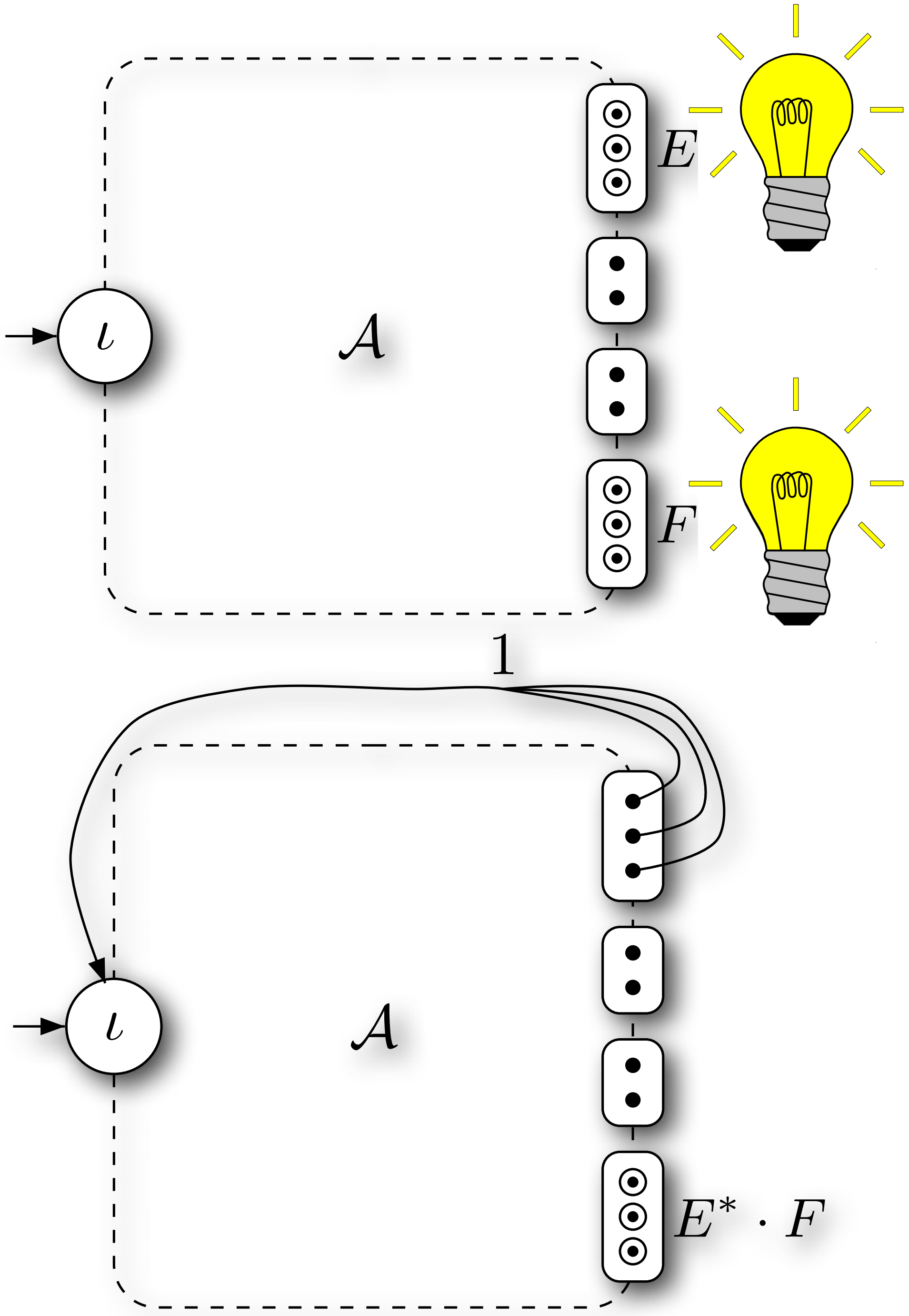

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Corollaries

- Equivalence problem for PREs is decidable: given PREs E and F, does they generate the same semantics? (translation into automata [1])
- Threshold problem for PREs is undecidable: given a PRE E and a threshold *s*, is there a word *w* which is mapped by to a probability greater than *s*? (by reduction to automata [2])

[1] M.-P. Schützenberger (1961). On the Definition of a Family of Automata. Information and Control.
[2] A. Paz. (1971). Introduction to probabilistic automata. Academic Press,

Summary and Future Works

- General Kleene-Schützenberger theorems for Probabilistic models (classical, extended to two-way automata, pebble automata in full paper [1])
- Study of Probabilistic Expressions and their extensions permits us to better understand which behavior Probabilistic Automata can generate
- In [2], we proved that Weighted Automata (with two-way and pebbles) can be evaluated efficiently
- Future work: get logical formalisms generating the same expressivity, and implement quick algorithms to perform translation from PREs to PAs (as there are some for weighted automata, see [2,3] e.g.)

[1] B. Bollig, P. Gastin, B. M. and M. Zeitoun. (2012). A Probabilistic Kleene Theorem. In Proceedings of ATVA'12.

[2] P. Gastin and B. M. (2006). Adding Pebbles to Weighted Automata. In Proceedings of CIAA'12.

[3] C. Allauzen, and M., Mohri, (2006). A Unified Construction of the Glushkov, Follow, and Antimirov Automata. In Proceedings of MFCS'06