Logics for Weighted Automata and Transducers

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Based on joint works with Paul Gastin, Benedikt Bollig and Marc Zeitoun

Software Verification

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Critical Software

- communication systems
- e-commerce
- health databases
- energy production

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Property to be verified

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TO BE VERIFIED

Software Verification

Property to be verified

Is the property verified or not by the software?

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Software Verification

Property to be verified

- May an *error* state be reached?
- Is there a book written by X, rented by Y?
- Does this leader election protocol permit to elect the leader?

erified
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Software Verification

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- May an *error* state be reached?
- Is there a book written by X, rented by Y?
- Does this leader election protocol permit to elect the leader?

From **Boolean** to **Quantitative** Verification

- What is the probability for an *error* state to be reached?
- How many books, written by X, have been rented by Y?
- What is the maximal delay ensuring that this leader election protocol permits the election?

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TO BE VERIFIED

Formal Verification

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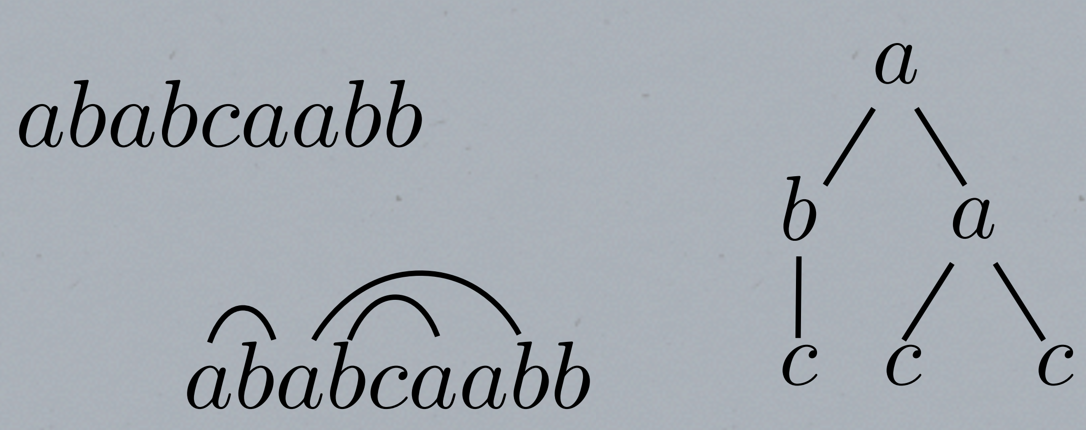
TO BE VERIFIED

Formal Verification

Property to be verified

Is the property verified or not by the model?

Formal Model



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TO BE VERIFIED

Formal Verification

Property to be verified
Formal Specification

$$(a+b)^\star c(ac)^+$$

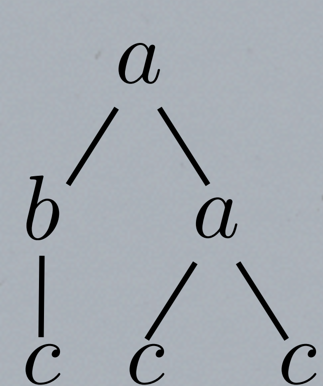
$$\forall x \forall y\,(x<y \Rightarrow \exists z\,(x<z<y))$$

$$\mathsf{F}\,\mathsf{G}\,(p\,\mathsf{U}\,q)$$

Is the property verified or not by the model?

Formal Model

$ababcaabb$



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TO BE VERIFIED

Qualitative/Quantitative

- Qualitative, Boolean: [Büchi'60], [Elgot'61], [Trakhtenbrot'61]

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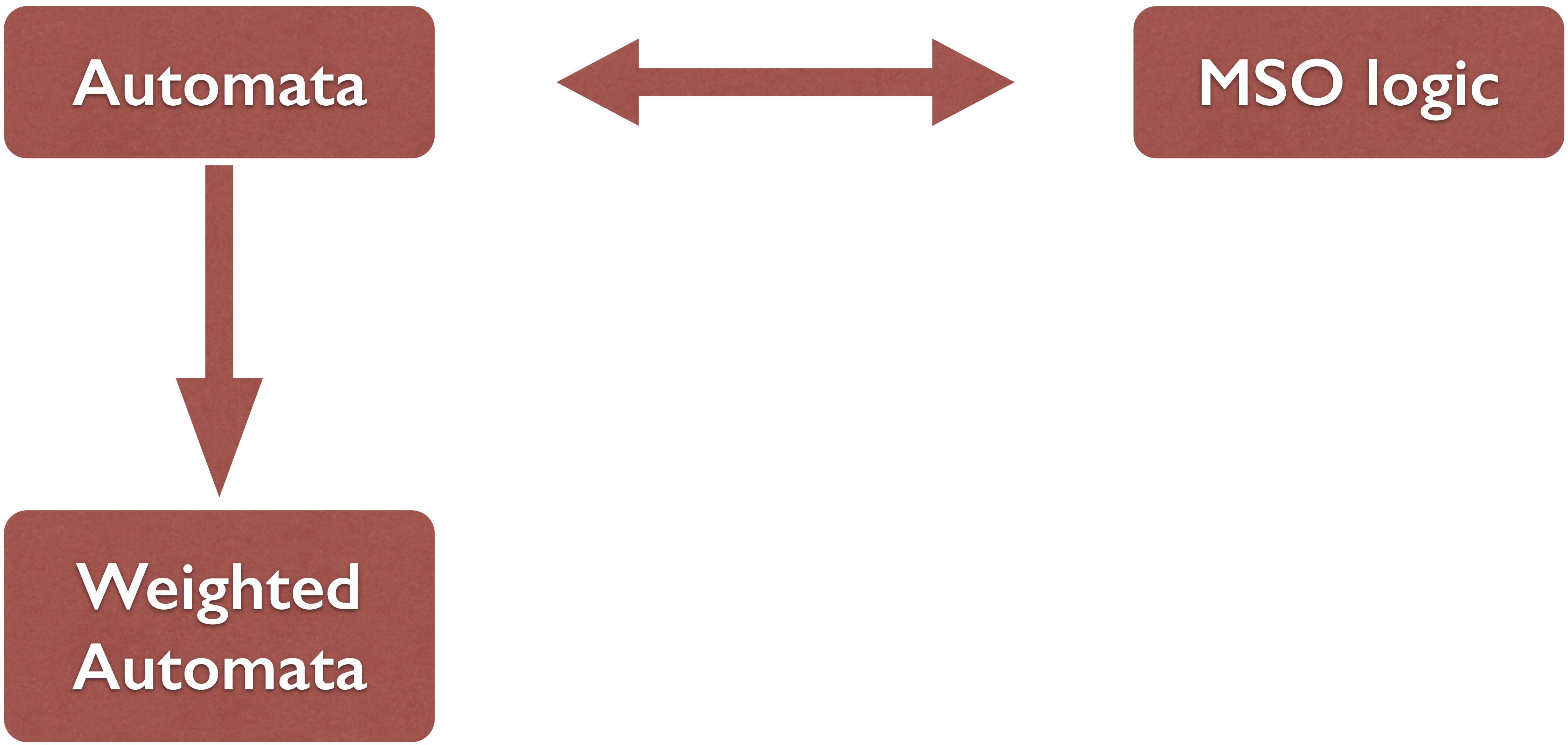

- Quantitative, weights

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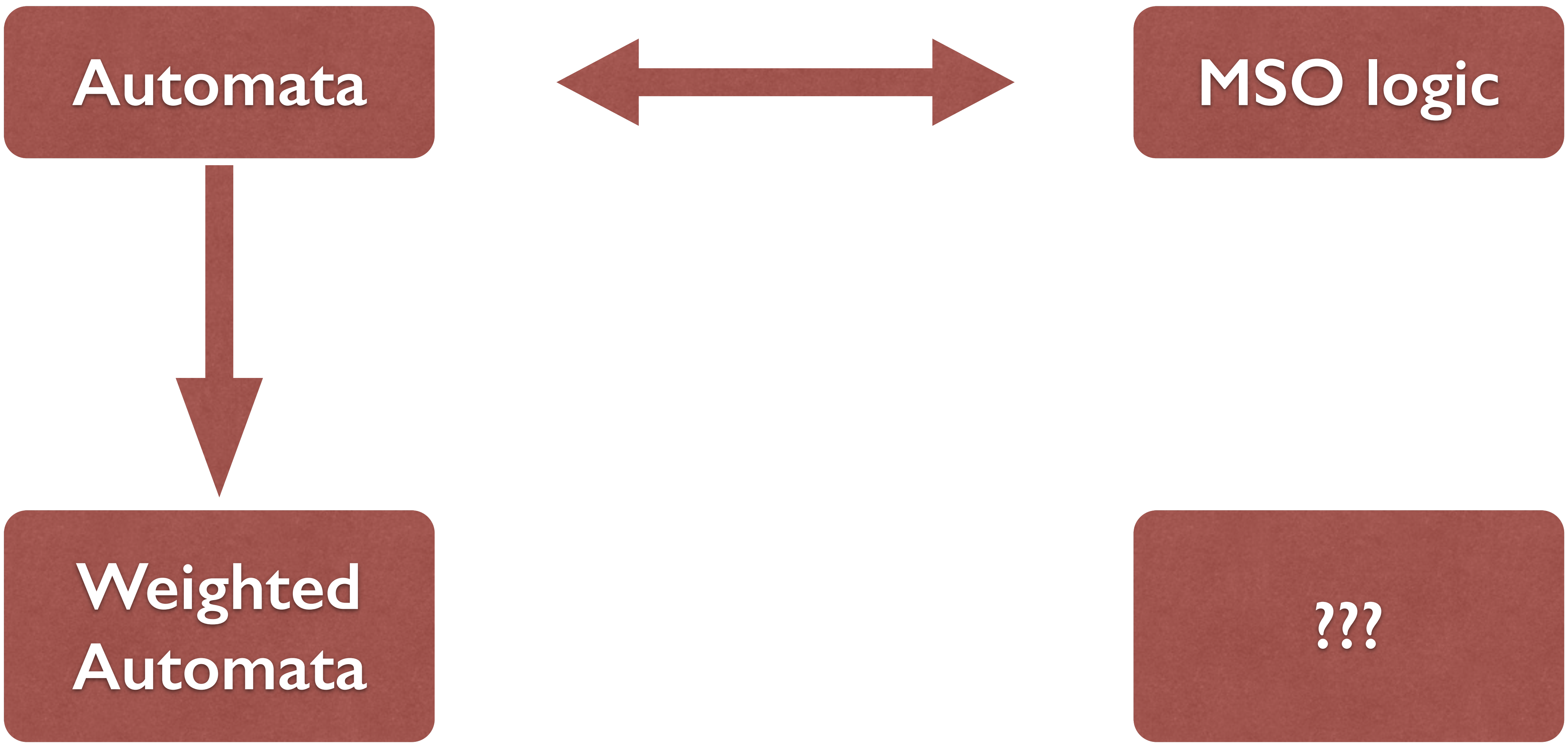

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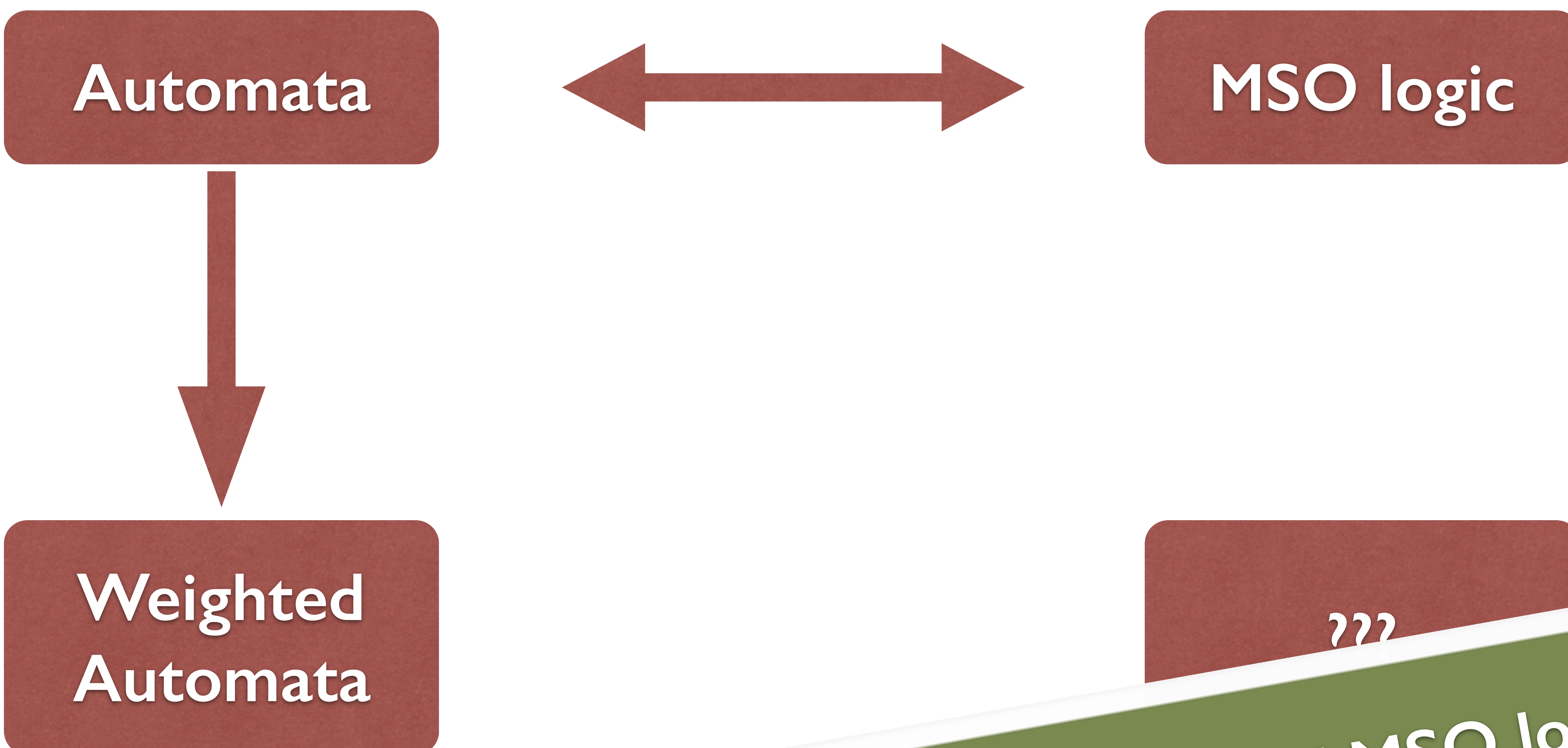

- Quantitative, weights

Find suitable weighted MSO logic

Qualitative/Quantitative

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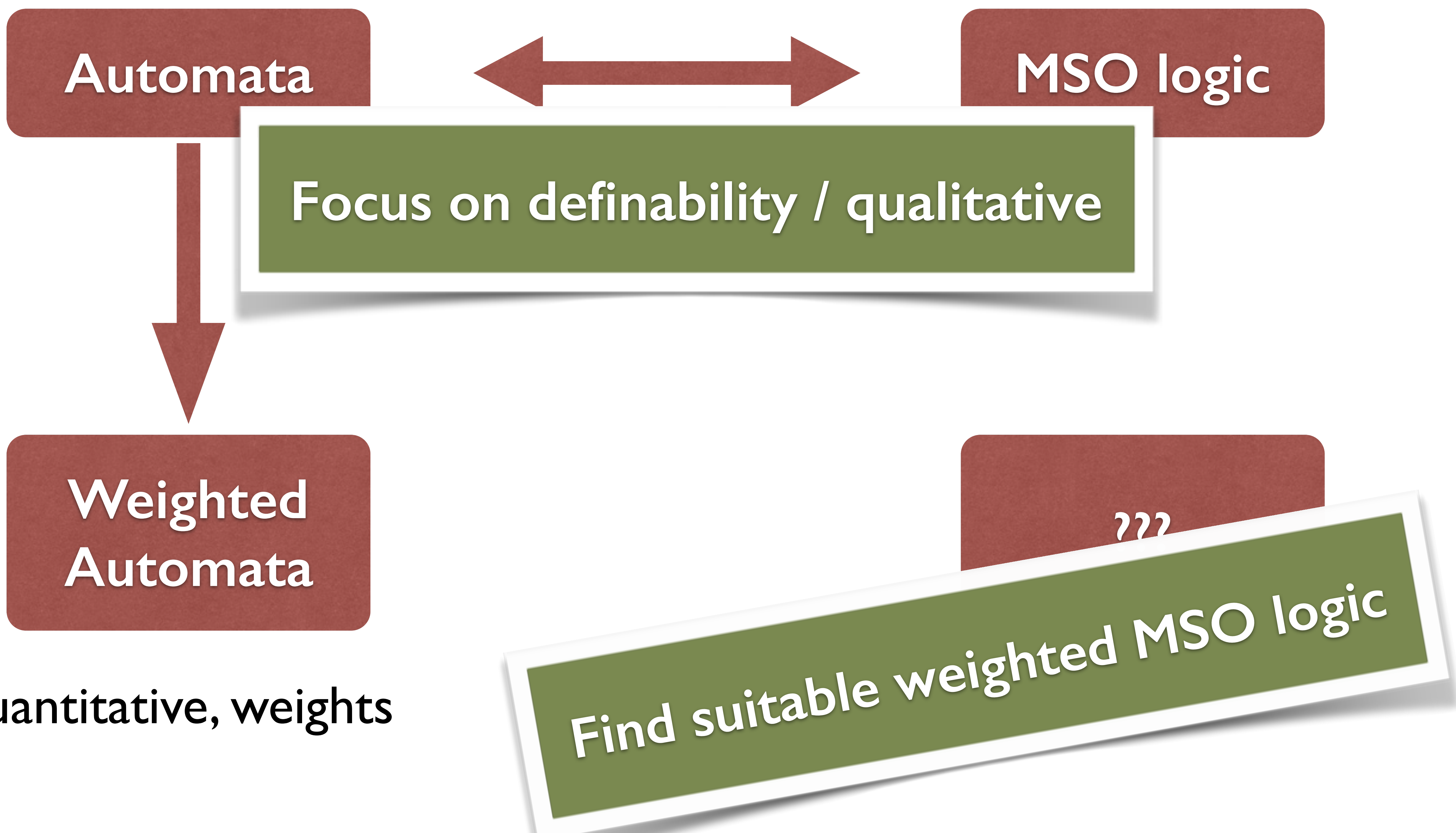

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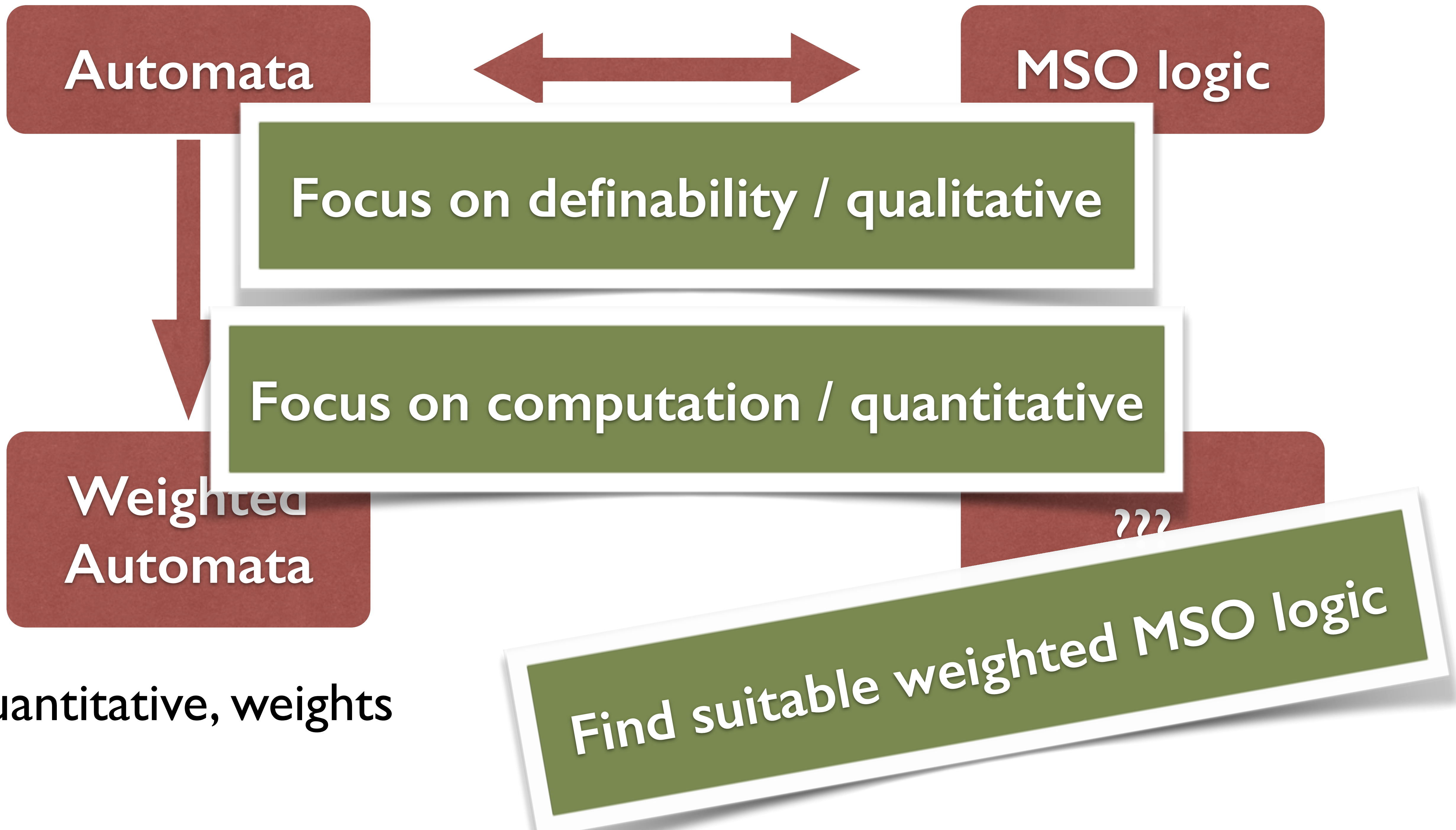
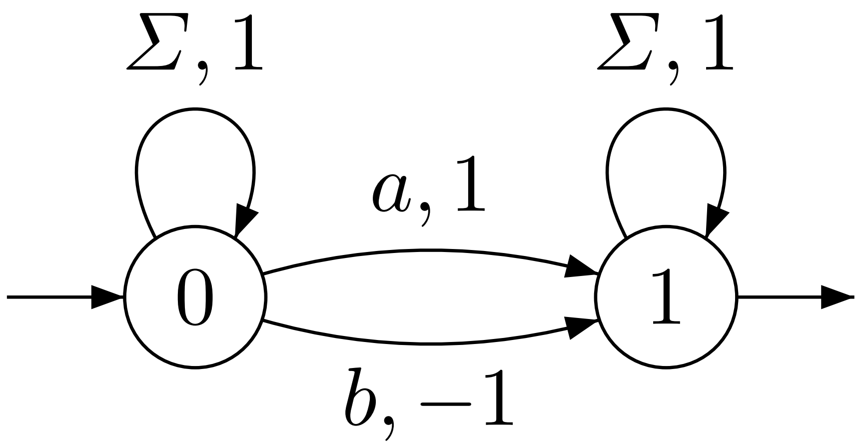
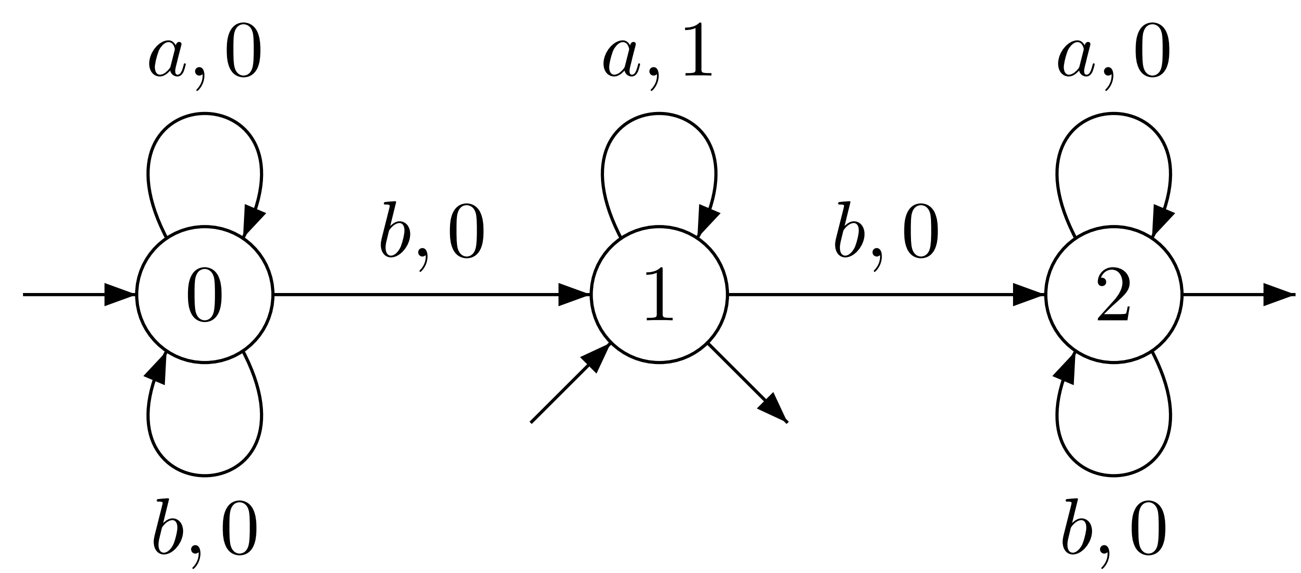


- Quantitative, weights

Weighted Automata





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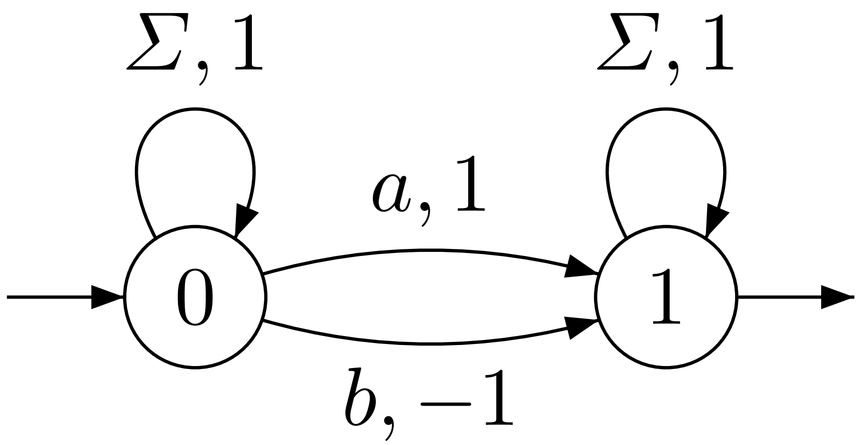
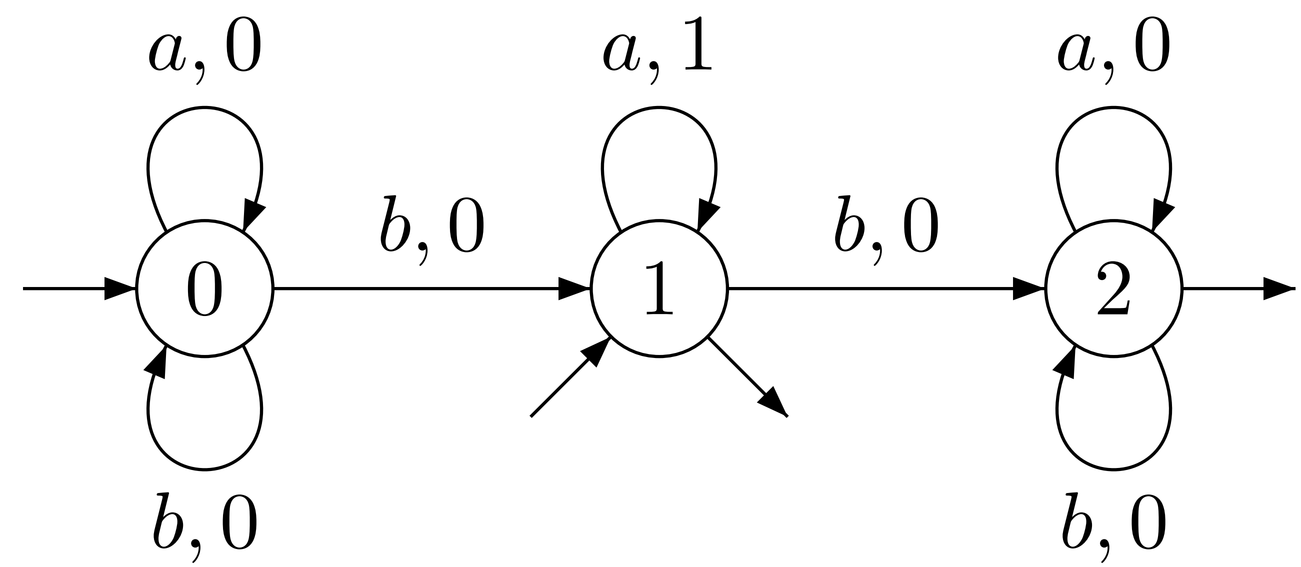
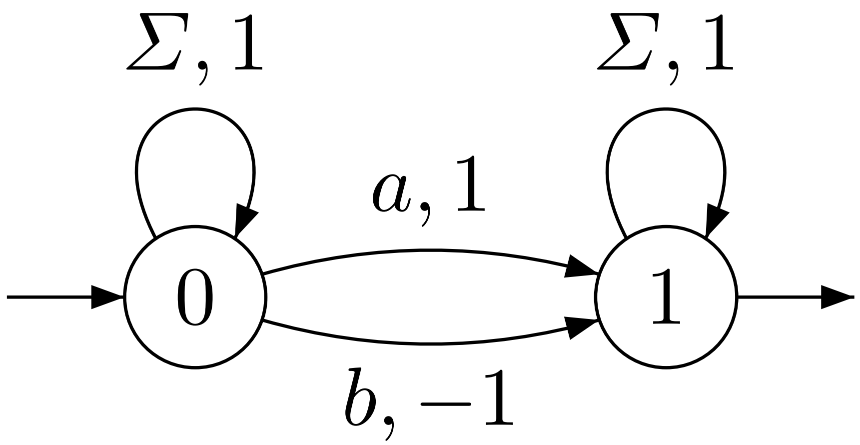
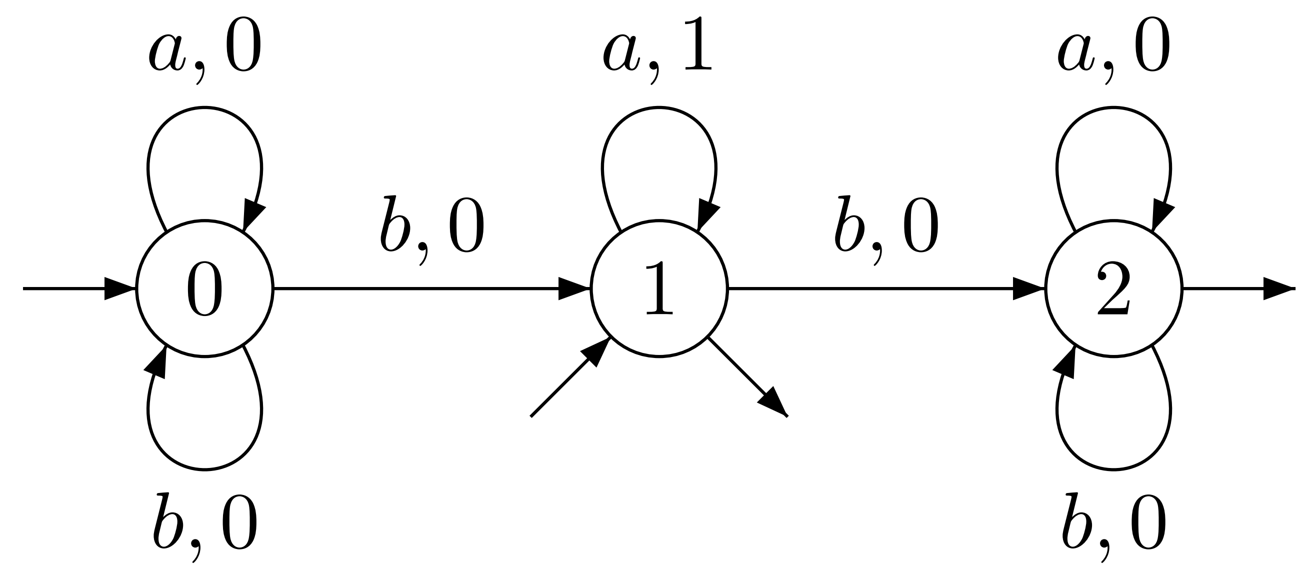




$$(\mathbf{Z}, +, \times, 0, 1)$$

Weighted Automata





$$(\mathbf{Z}, +, \times, 0, 1)$$

$$0 \xrightarrow{a,1} 1 \xrightarrow{b,1} 1 \xrightarrow{a,1} 1$$

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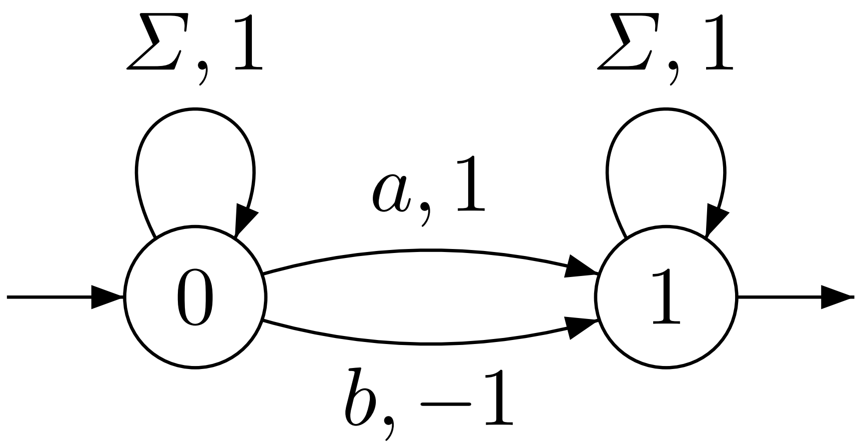
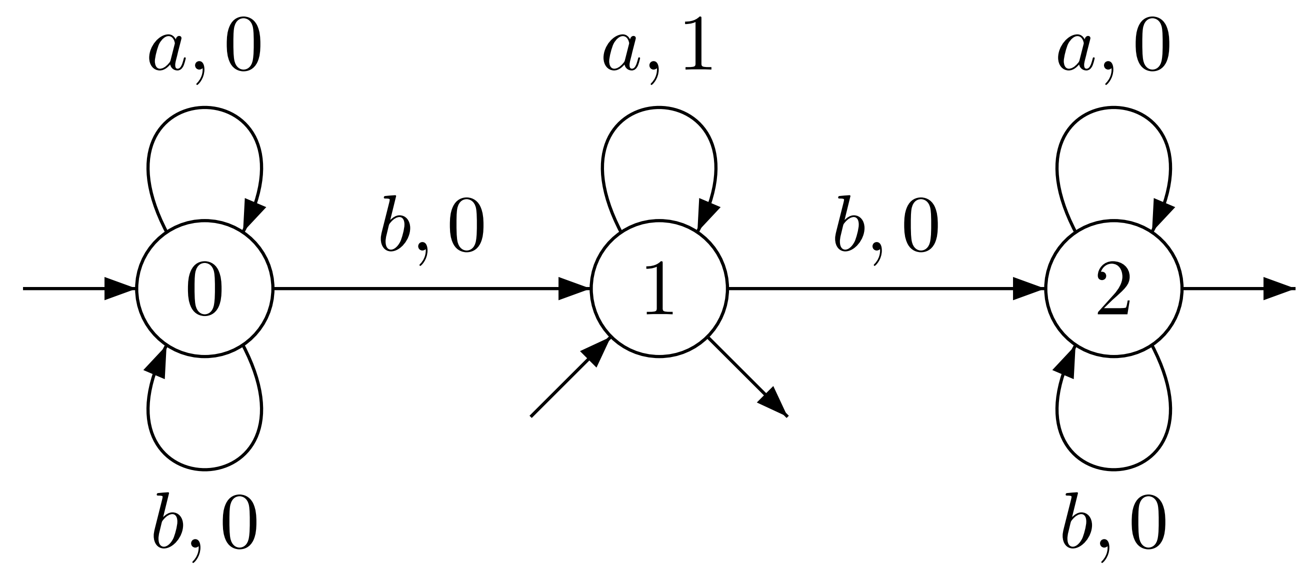
Weighted Automata





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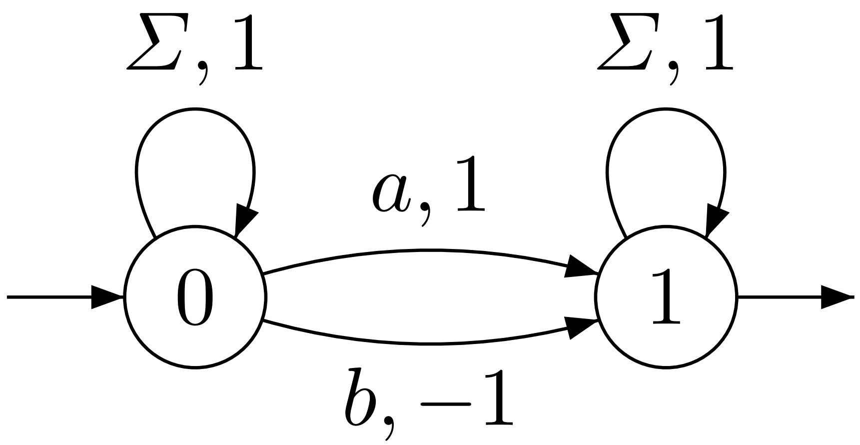

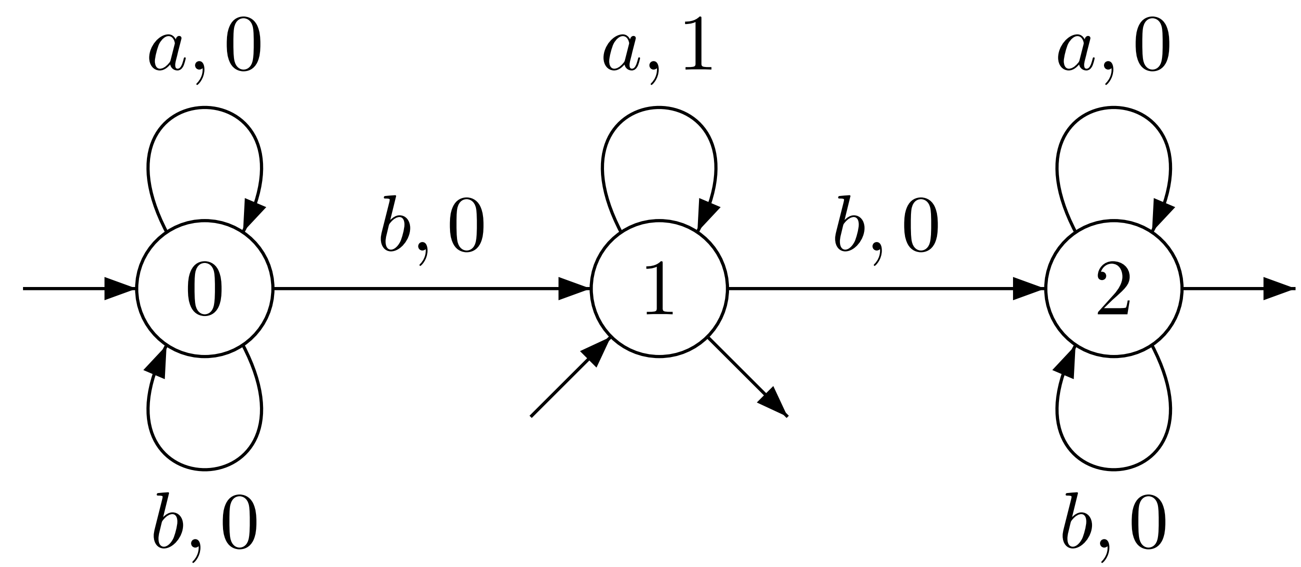

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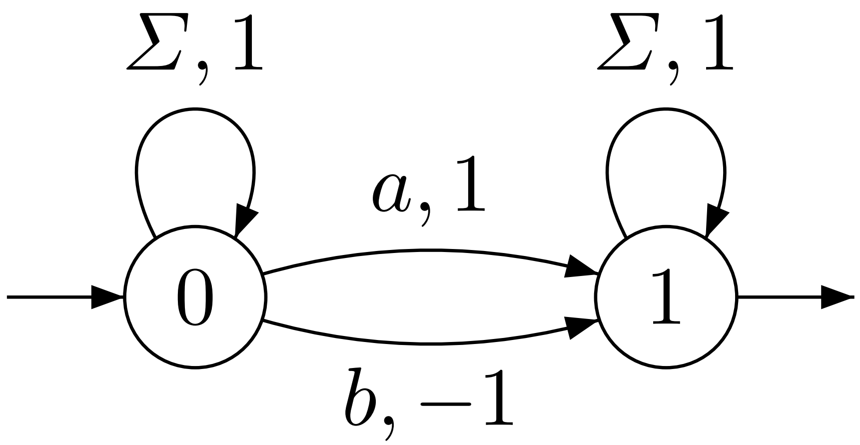

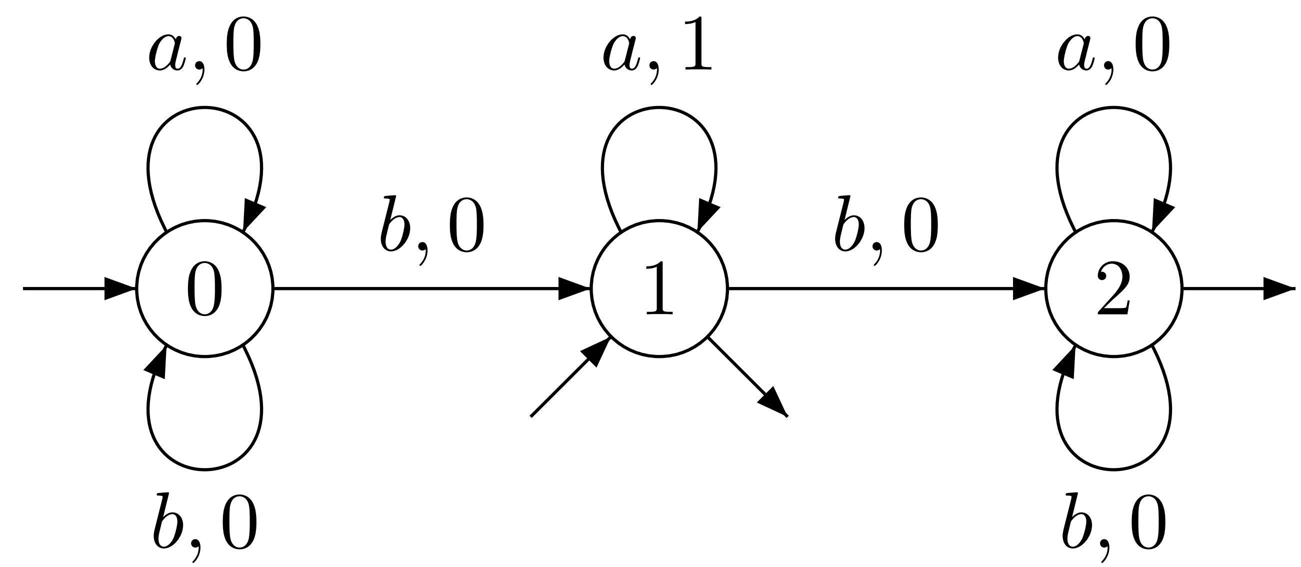

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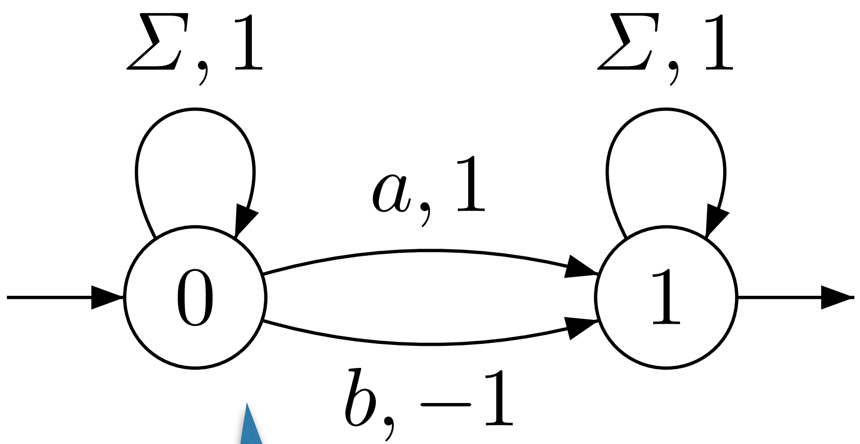
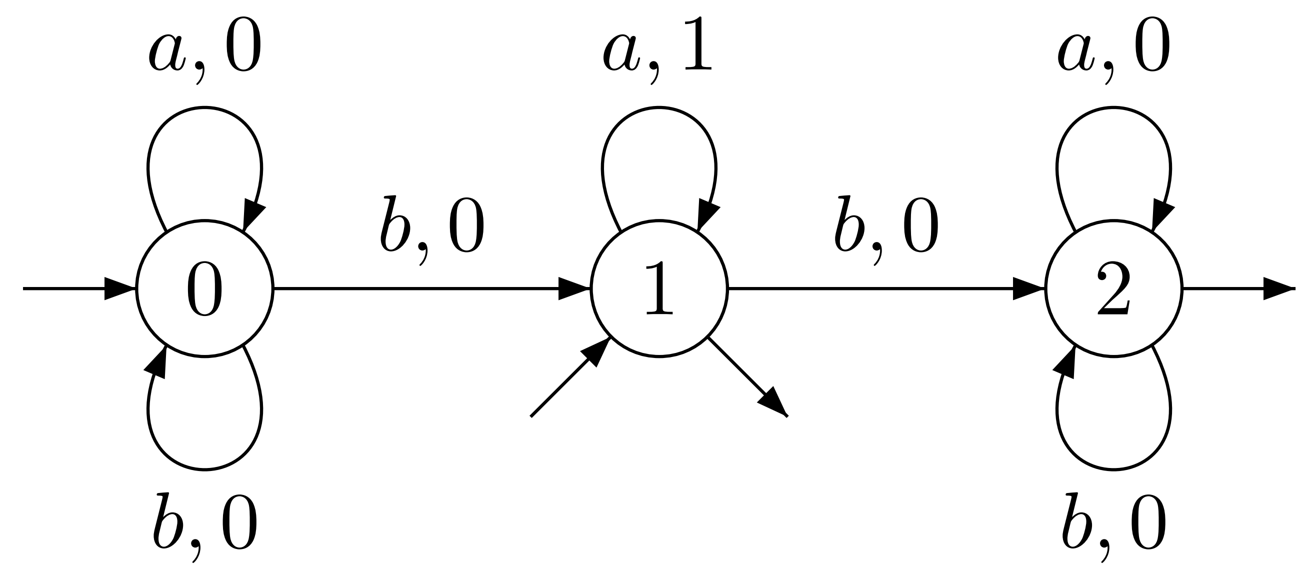
Semantics of *aba*: 1+(-1)+1 = 1

Weighted Automata



$\#_a(w)$ - $\#_b(w)$

$(\mathbf{Z}, +, \times, 0, 1)$



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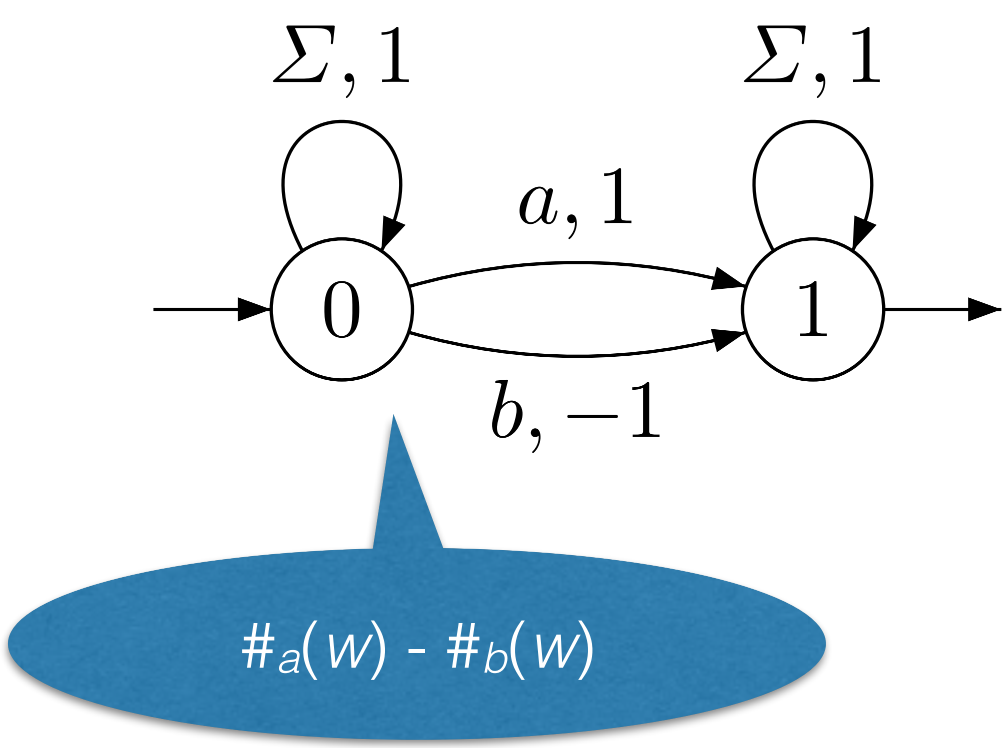

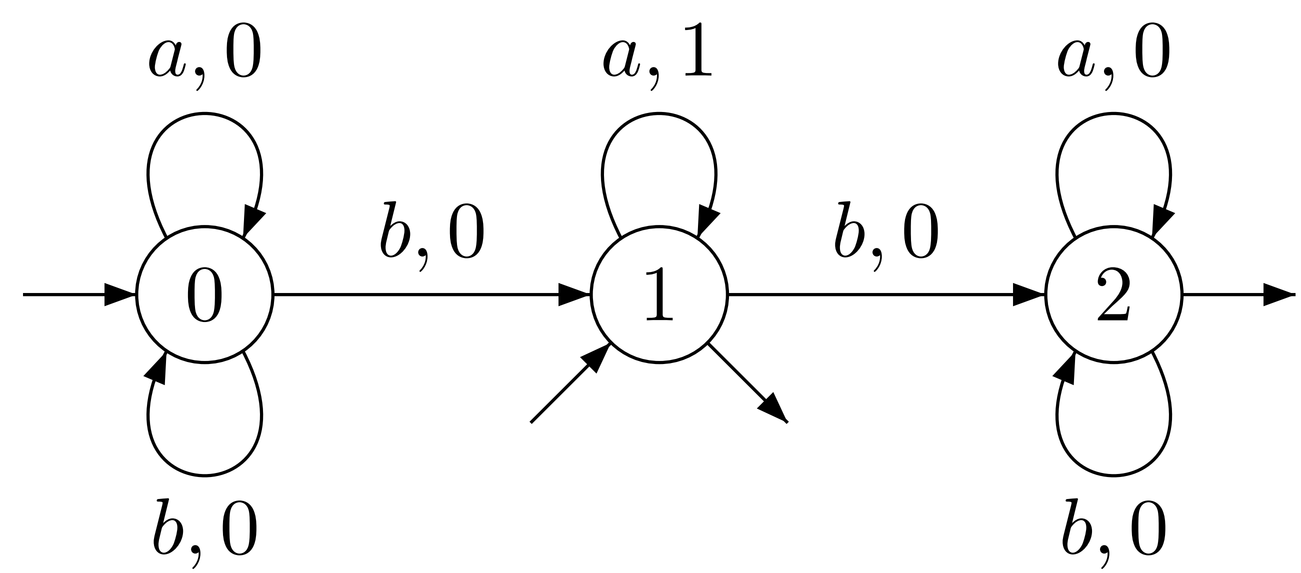

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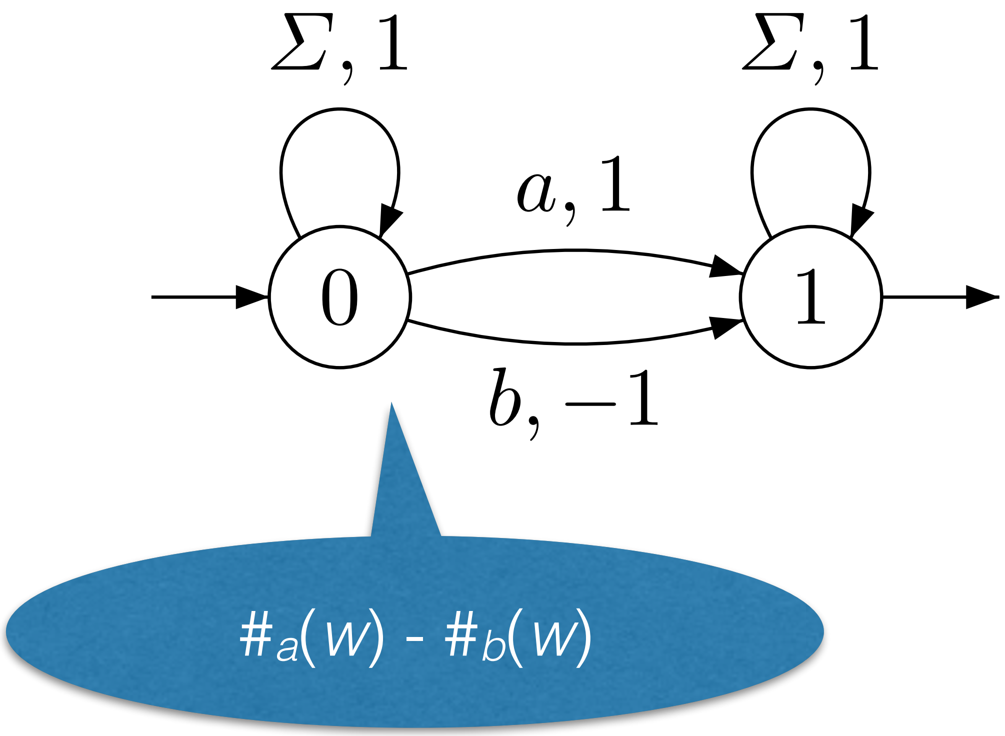

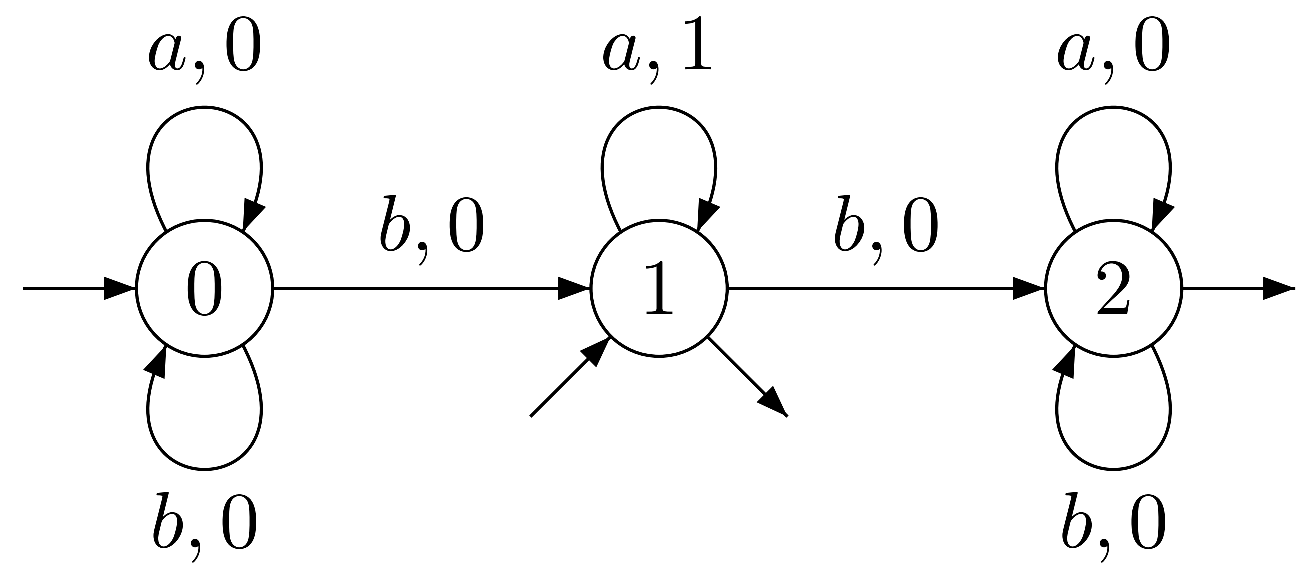

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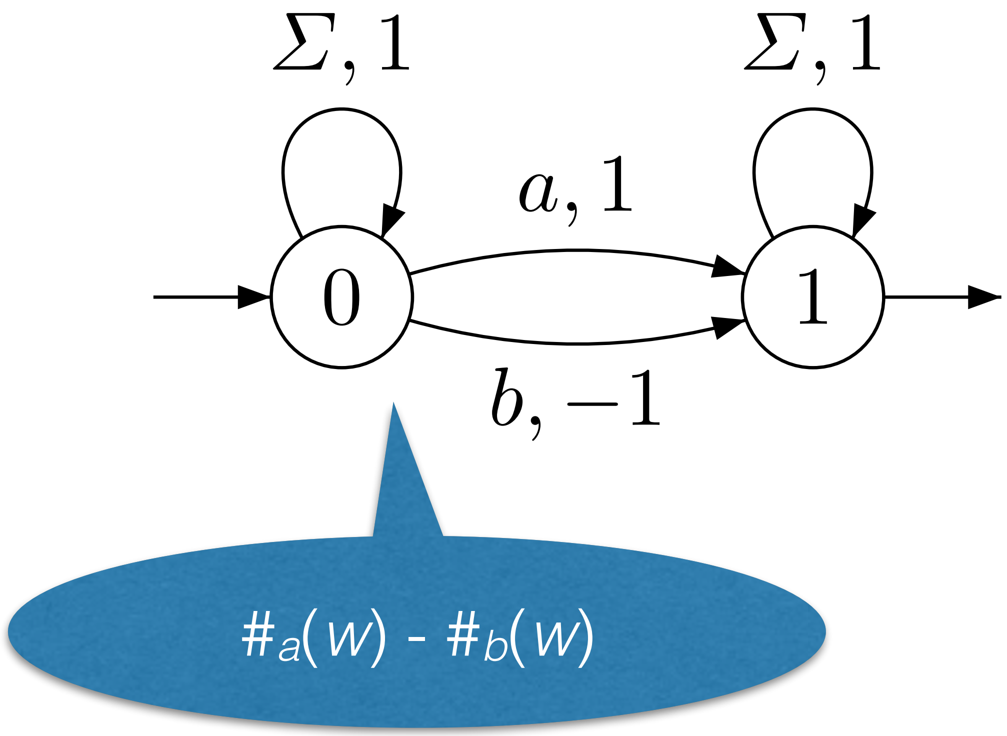
Semantics of *aba*: 1+(-1)+1 = 1

Weighted Automata



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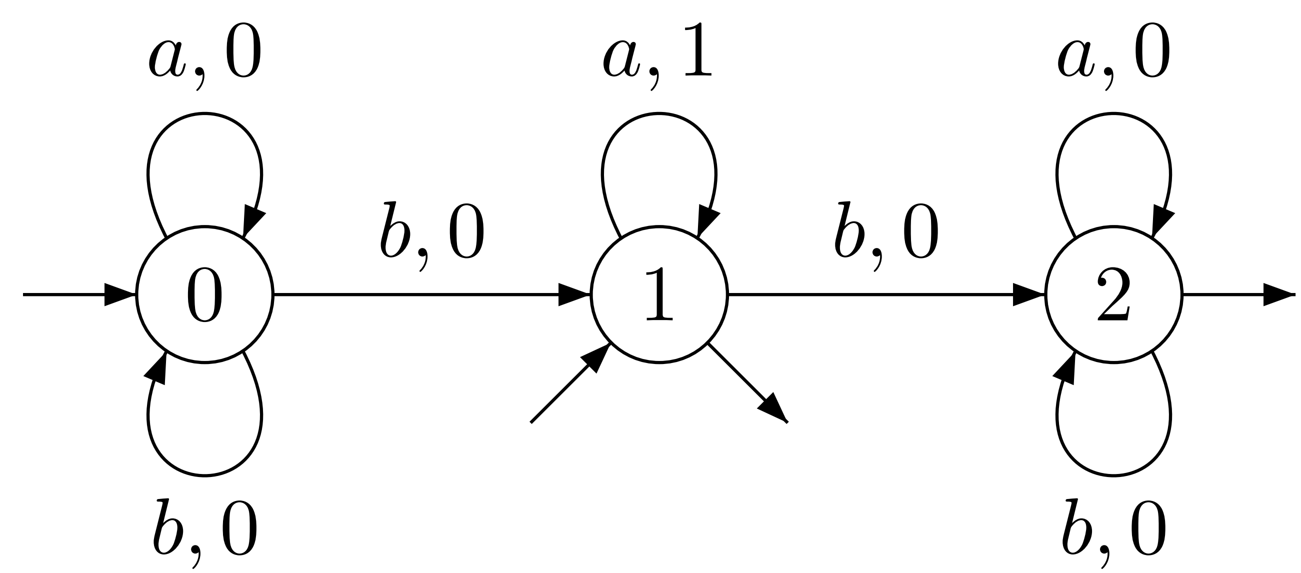

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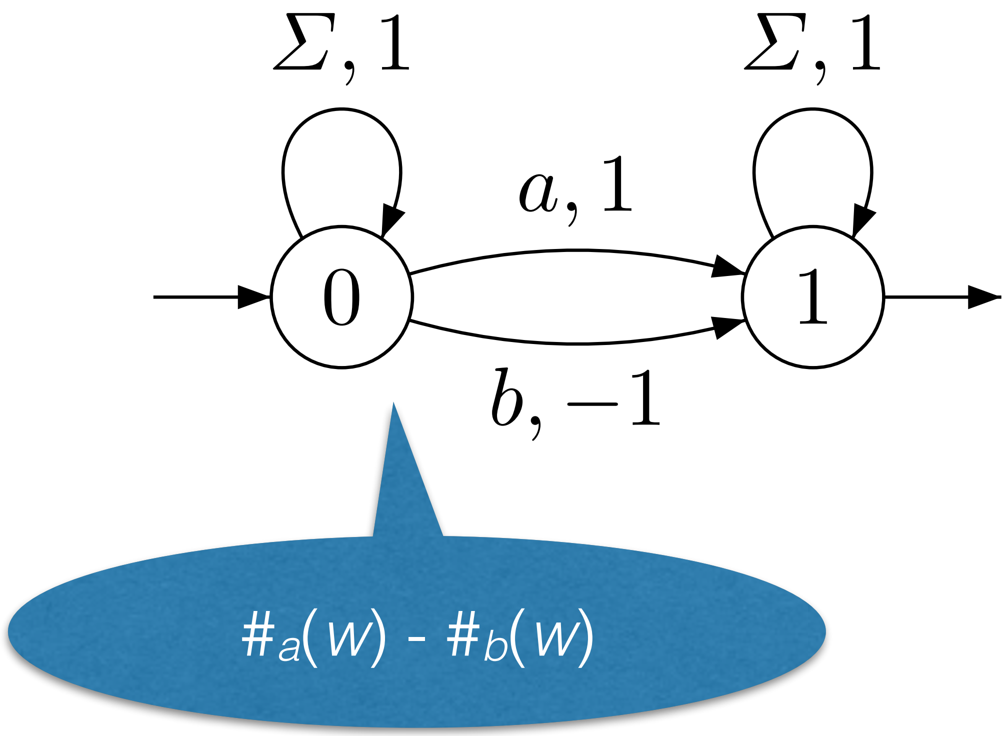

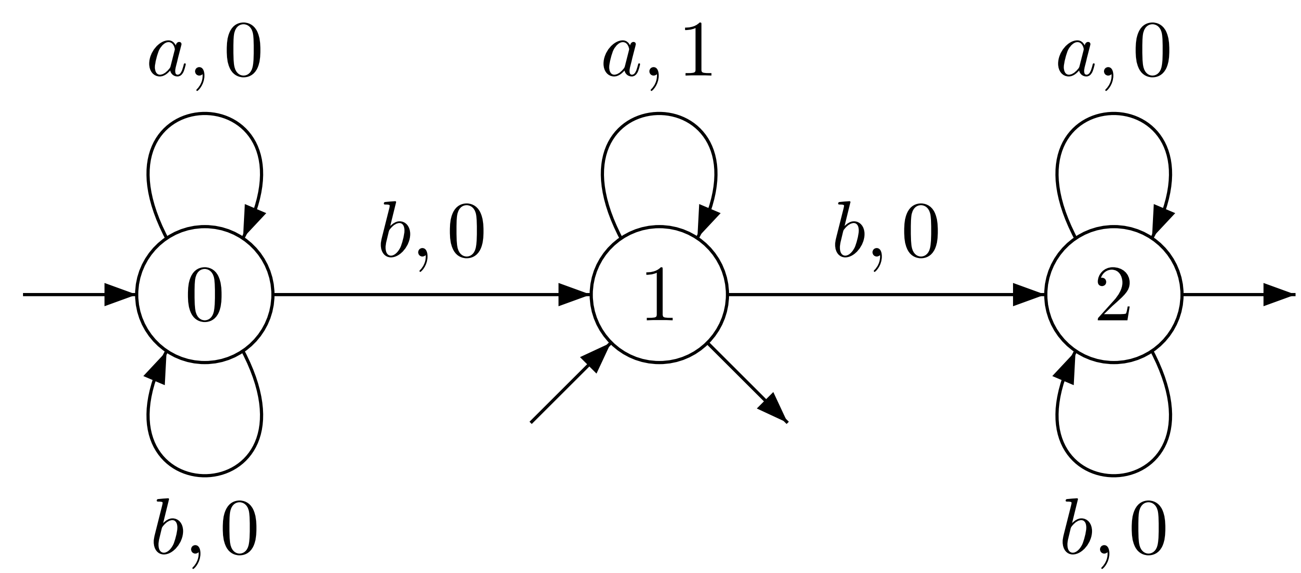

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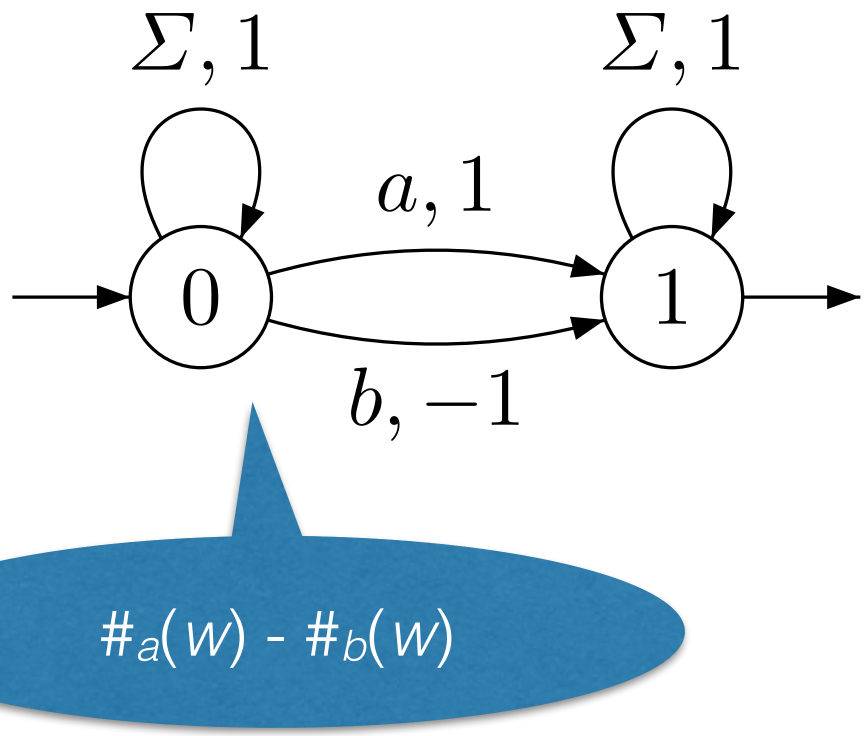
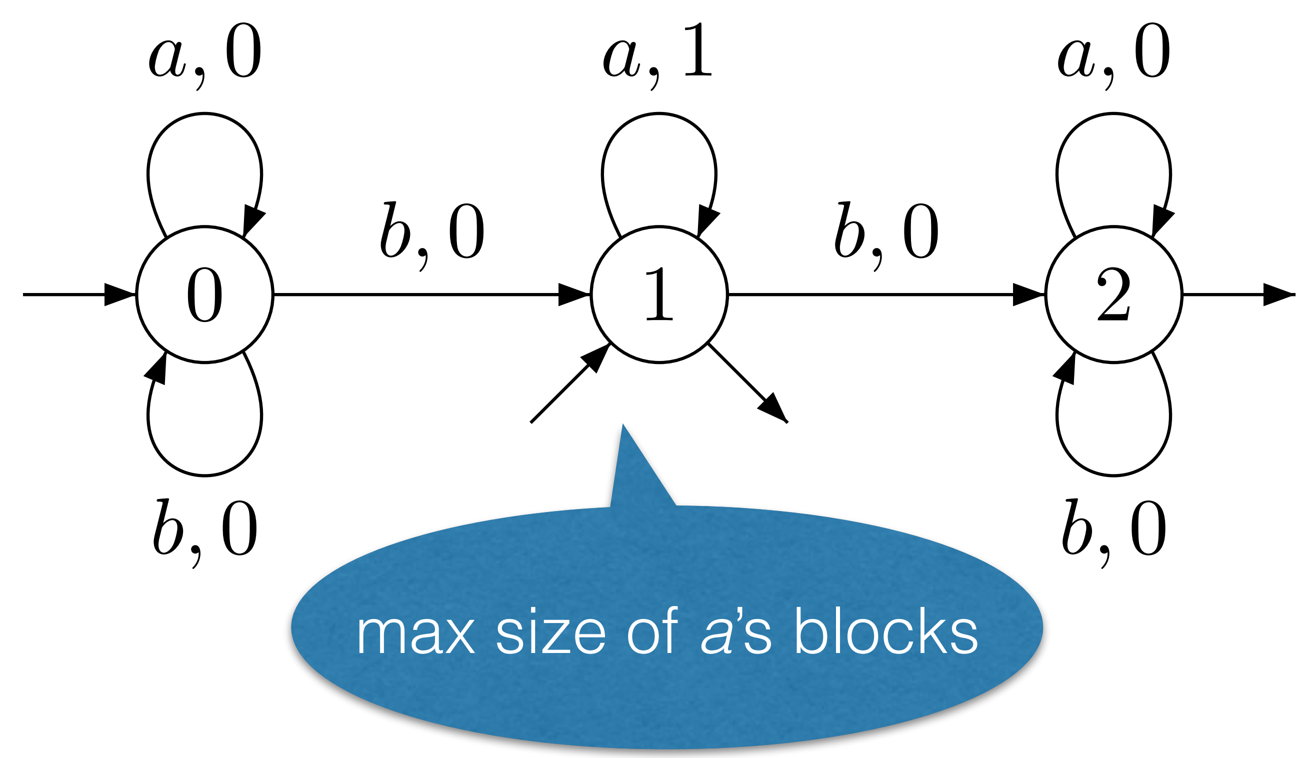
Semantics of *aaba*: max(2,1) = 2

Weighted Automata





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Semantics of *aba*: 1+(-1)+1 = 1

Semantics of *aaba*: max(2,1) = 2

How to Specify Quantitative Properties?

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Weighted Monadic Second Order Logic [Droste&Gastin 05]
generalized to trees [Droste&Vogler 06], infinite words [Droste&Rahonis 07], nested words [Mathissen 10] or pictures [Fichtner 11]

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- Core weighted logic for weighted automata
- Enhancing the logic to handle more properties: FO vs pebbles
- A special case: the transducers

Weighted MSO

$$\varphi ::= s \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg P_a(x) \mid \neg(x \leqslant y) \mid \neg(x \in X)$$
$$\mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi \mid \exists X\, \varphi \mid \forall X\, \varphi$$

[Droste-Gastin, 05]

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Negation restricted to atomic formulae

[Droste-Gastin, 05]

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Arbitrary constants from a semiring

Negation restricted to atomic formulae

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- Semantics in a semiring $\mathbb{S} = (S, +, \times, 0, 1)$
  - Atomic formulae: 0, 1
  - disjunction, existential quantifications: sum
  - conjunction, universal quantifications: product
- Inspired from the boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$

[Droste-Gastin, 05]

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- Examples

$$\varphi_1 = \exists x\, P_a(x)$$

$$[\![\varphi_1]\!](w) = |w|_a$$

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- Examples

$$\varphi_1 = \exists x\, P_a(x)$$

$$\varphi_2 = \forall x\, \exists y\, (y \leqslant x \wedge P_a(y))$$

$$[\![\varphi_1]\!](w) = |w|_a$$

$$[\![\varphi_2]\!](abaab) = 1 \times 1 \times 2 \times 3 \times 3$$

$$[\![\varphi_2]\!](a^n) = n!$$

[Droste-Gastin, 05]

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Too big to be computed by a weighted automaton

[Droste-Gastin, 05]

Weighted MSO

$\varphi ::= s \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg P_a(x) \mid$ ... $) \mid \neg(x \in X)$
$\mid \varphi \vee \varphi \mid \varphi \wedge$ ...

We need to restrict weighted MSO

- Examples

$$\varphi_1 = \exists x\, P_a(x) \qquad \varphi_2 = \forall x\, \exists y\, (y \leqslant x \wedge P_a(y))$$

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Theorem: weighted automata = restricted wMSO

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$$\mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi \mid \exists X\, \varphi \mid \forall X\, \varphi$$

φ almost boolean

Theorem: weighted automata = restricted wMSO

[Droste-Gastin, 05]

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commutativity

φ almost boolean

Theorem: weighted automata = restricted wMSO

[Droste-Gastin, 05]

Core weighted MSO logic

- Boolean fragment

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[Gastin-Monmege, 15]

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- Step formulae $\qquad \Psi ::= s \mid \varphi\,?\,\Psi : \Psi$

[Gastin-Monmege, 15]

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if … then … else …

[Gastin-Monmege, 15]

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$$P_a(x) \,?\, 1 : (P_b(x) \,?\, -1 : 0)$$

[Gastin-Monmege, 15]

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$$[\![\Psi]\!](w, \sigma) = s$$

some value occurring in Ψ

[Gastin-Monmege, 15]

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se …

$$P_a(x) \,?\, 1 : 0$$

P_a

$x \in$ … $X_2 \,?\, s_2 : \cdots (x \in X_{n-1} \,?\, s_{n-1} : s_n) \cdots)$

A step formula takes finitely many values

For each value, the pre-image is MSO-definable

$$[\![\Psi]\!](w, \sigma) = s$$

some value occurring in Ψ

[Gastin-Monmege, 15]

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- core wMSO $\qquad \Phi ::= \mathbf{0} \mid \varphi\,?\,\Phi : \Phi \mid \Phi + \Phi \mid \sum_x \Phi \mid \sum_X \Phi \mid \prod_x \Psi$

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$$\{\!|\prod_x \Psi|\!\}(w,\sigma) = \{\{(\llbracket\Psi\rrbracket(w,\sigma[x \mapsto i]))_i\}\} \in \mathbb{N}\langle R^\star\rangle$$

singleton multiset

[Gastin-Monmege, 15]

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- Semantics
  - $\{\!|\, \mathbf{0} \,|\!\}(w, \sigma) = \varnothing$
  - sums over multisets

$$\{\!|\, \prod_x \Psi \,|\!\}(w, \sigma) = \{\!\{ (\llbracket \Psi \rrbracket (w, \sigma[x \mapsto i]))_i \}\!\} \in \mathbb{N}\langle R^\star \rangle$$

[Gastin-Monmege, 15]

Multisets of weight structures

- A run generates a sequence of weights $\mathrm{wgt}(\rho) = s_1 s_2 \cdots s_n$
- Abstract semantics $\{\!|\mathcal{A}|\!\}(w) = \{\{\mathrm{wgt}(\rho) \mid \rho \text{ run on } w\}\}$

multiset

[Gastin-Monmege, 15][Droste-Perevoshchikov, 14]

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$$\{\!|\mathcal{A}|\!\} \colon \Sigma^\star \to \mathbb{N}\langle R^\star \rangle$$

weights of A

[Gastin-Monmege, 15][Droste-Perevoshchikov, 14]

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Multisets of weight structures

Semiring: sum-product

$$\mathsf{aggr}_{\mathsf{sp}}(A) = \sum \prod A = \sum_{r_1 \cdots r_n \in A} r_1 \times \cdots \times r_n$$

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weights of A

Average value
Discounted value…

- Aggregation $\to S$

[Gastin-Monmege, 15][Droste-Perevoshchikov, 14]

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[Gastin-Monmege, 15][Droste-Perevoshchikov, 14]

Core weighted MSO logic

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$

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Theorem: weighted automata = core wMSO

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Theorem: weighted automata = core wMSO

- Abstract semanti[...]
- Concrete [...] $\to S$

Easy constructive proofs
preservation of the constants
no restriction on core wMSO
no hypotheses on weights

[Gastin-Monmege, 15]

Extensions

More general models than words:
trees, nested words…

More powerful logics:
deciding if a wMSO formula is expressible in core wMSO?

More powerful automata: finding equivalent fragments of wMSO

Weighted FO logic

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \forall X\, \varphi$$
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Weighted FO logic

We can keep Boolean MSO or restrict to FO…

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Reintroduction of the product

Unconditional product quantification

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Reintroduction of the product

Unconditional product quantification

$$\varphi_2 = \forall x\, \exists y\, (y \leqslant x \wedge P_a(y)) \qquad [\![\varphi_2]\!](a^n) = n!$$

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Reintroduction of the product

Unconditional product quantification

$$[\![\prod_x \prod_y 2]\!](w) = 2^{|w|^2}$$

Pebble weighted automata

$I \in \mathbb{S}^Q$

$T \in \mathbb{S}^Q$

$$\mathcal{A} = (Q, A, I, \delta, T)$$

$\delta \colon Q \times \mathrm{Test} \times \mathrm{Move} \times Q \to \mathbb{S}$

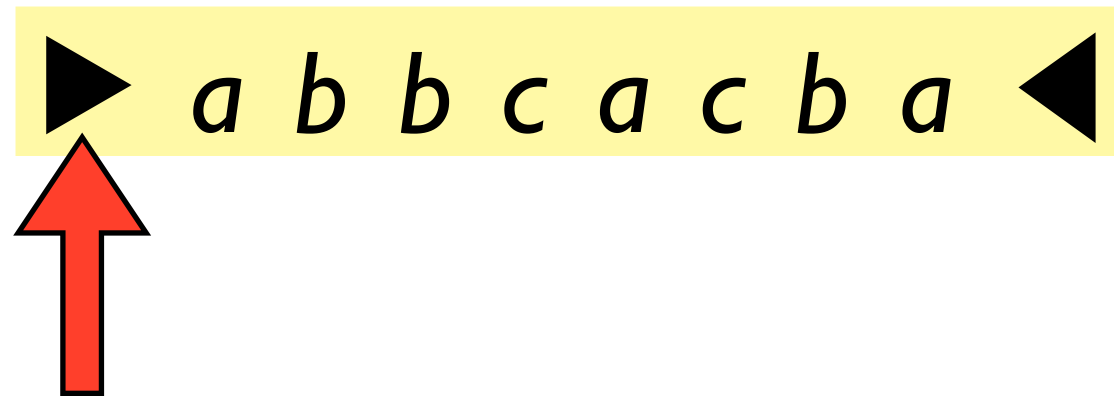
$\mathrm{Move} = \{\rightarrow, \leftarrow, \uparrow\} \cup \{\downarrow_x \mid x \in \mathrm{Peb}\}$

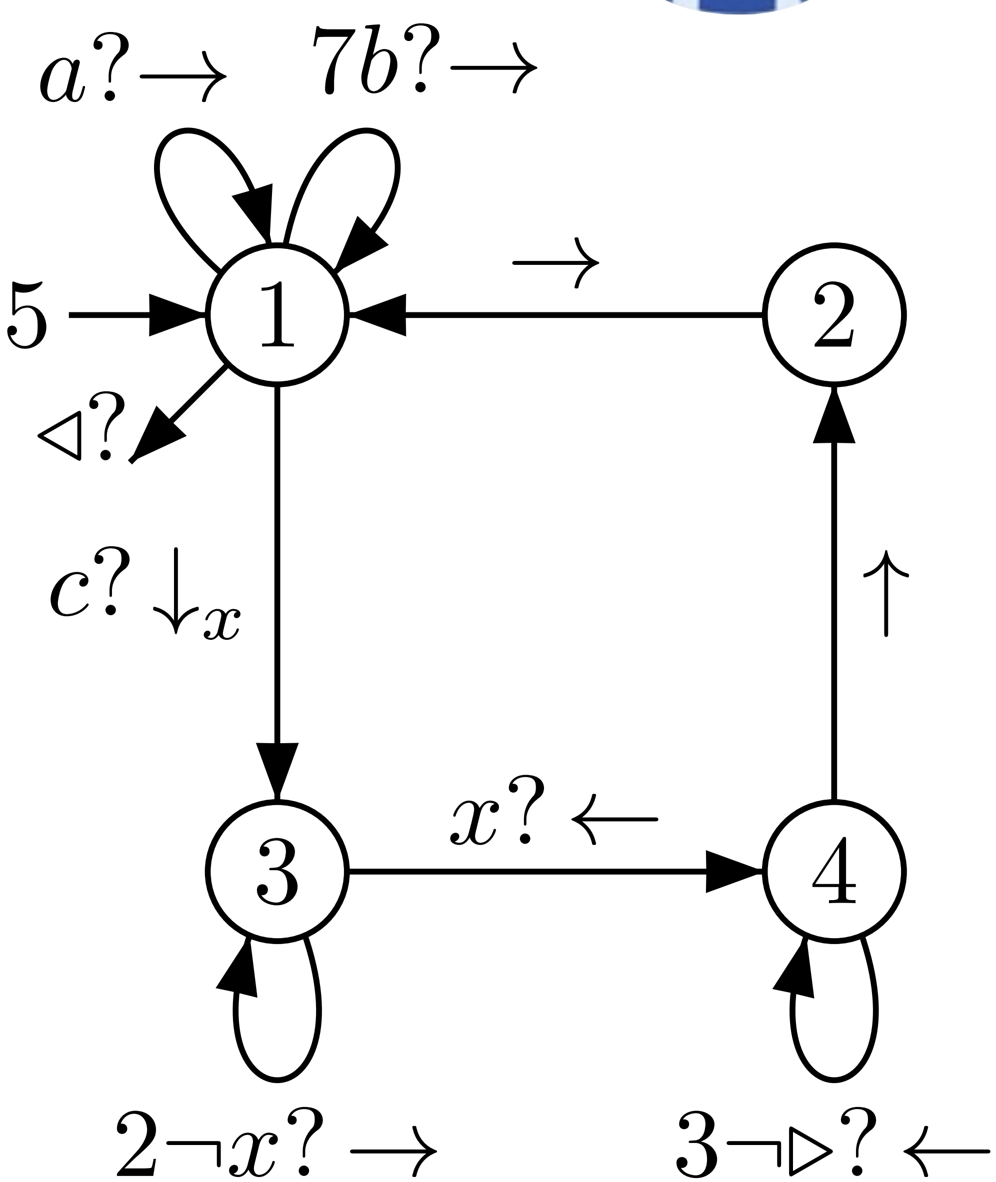
Run as a finite sequence of configurations (W, σ, q, i, π)

with free pebbles $\sigma \colon \mathrm{Peb} \to \mathrm{pos}(W)$

and a stack of currently dropped pebbles $\pi \in (\mathrm{Peb} \times \mathrm{pos}(W))^*$

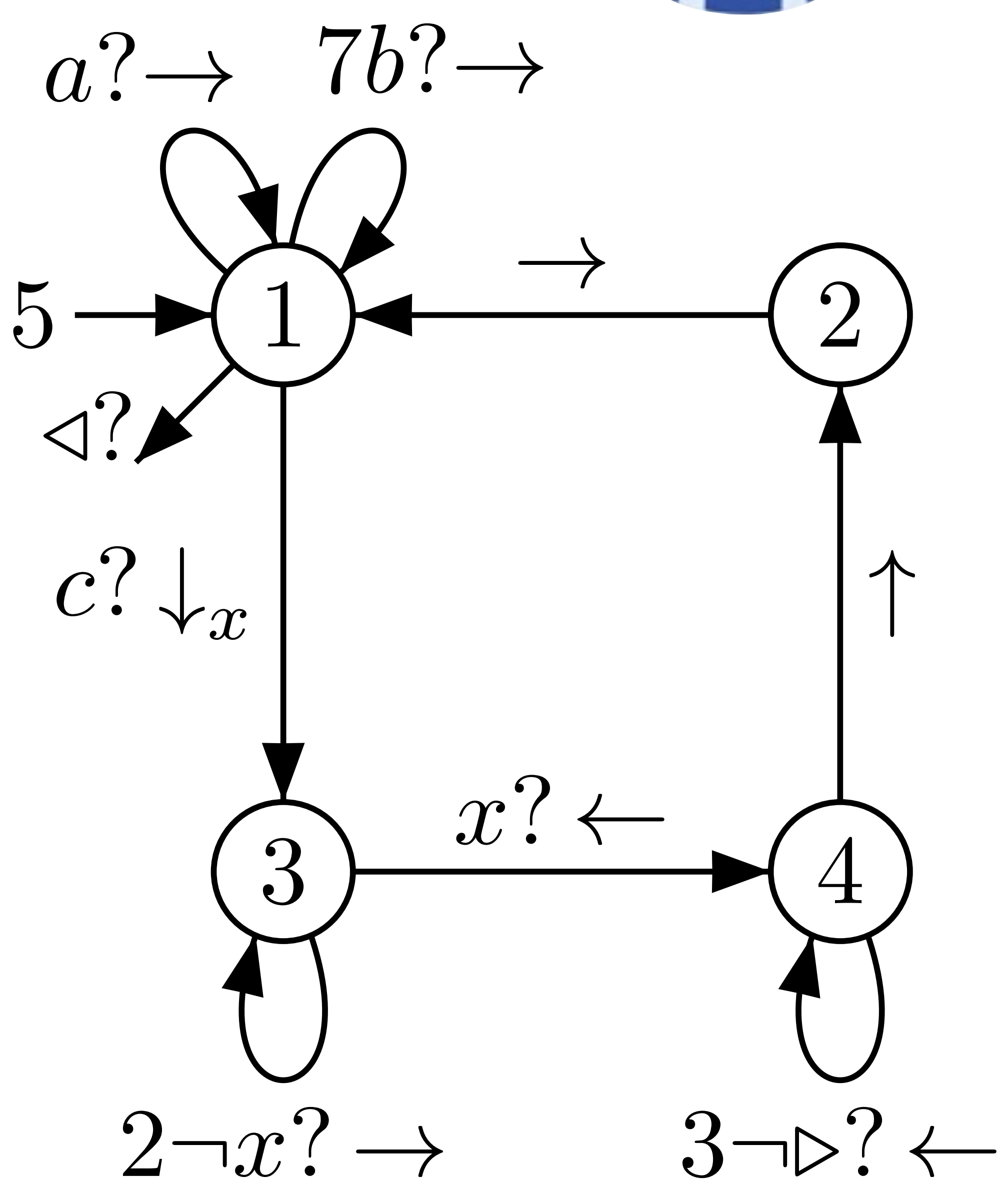
Pebble weighted automata



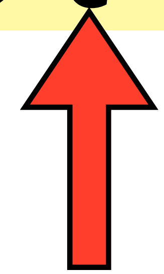


[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

Pebble weighted automata



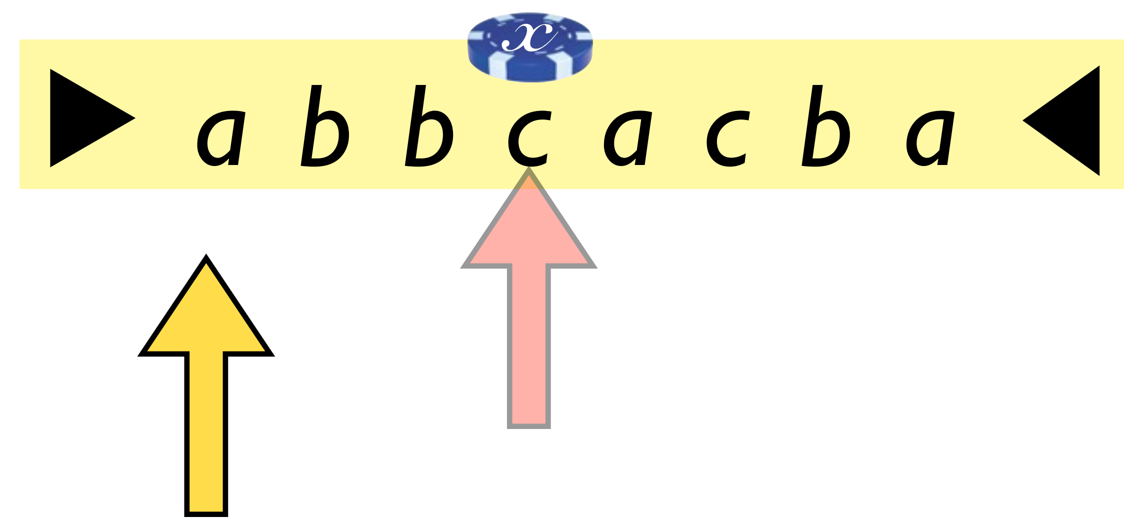
▶ *a b b c a c b a* ◀



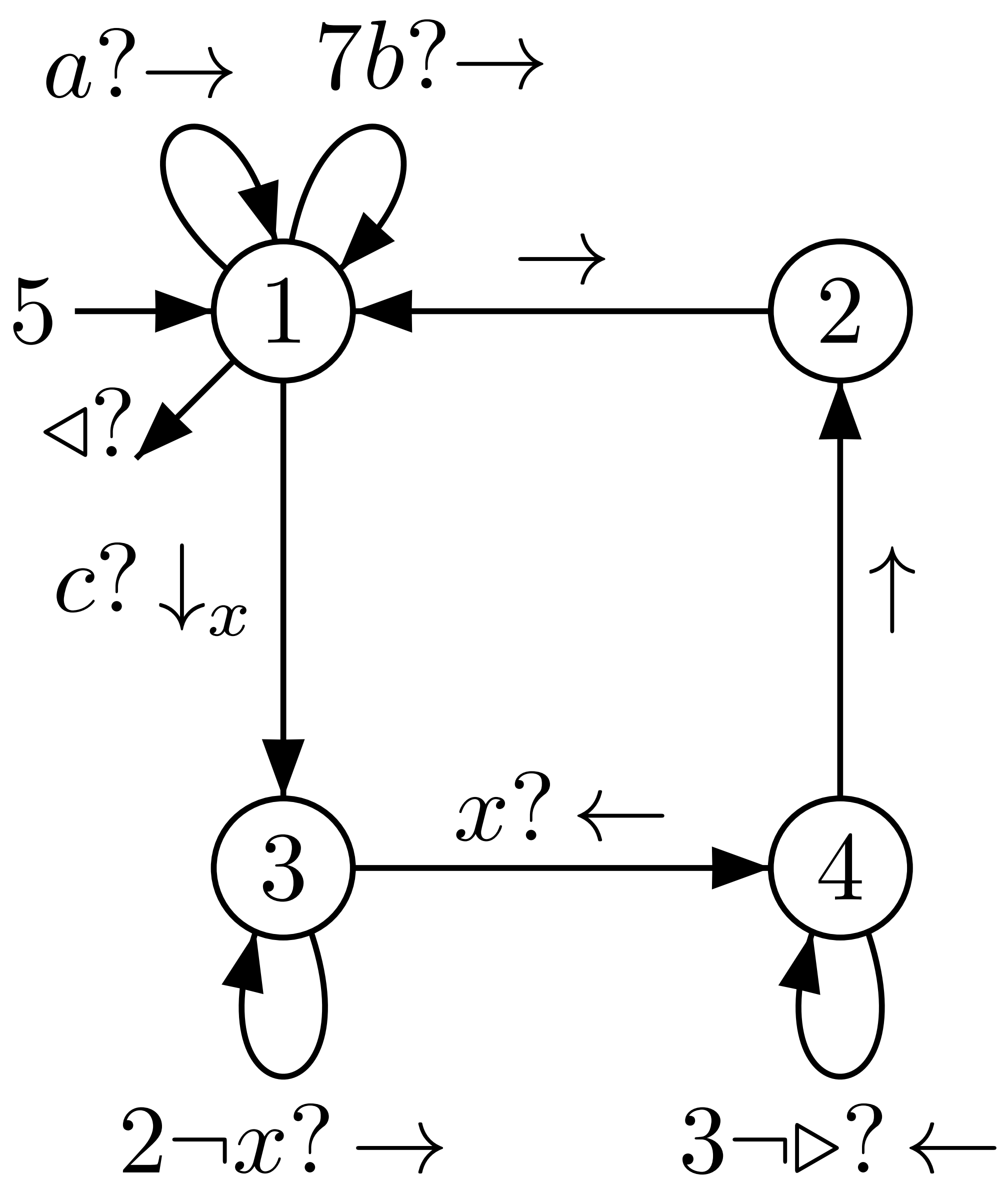
5 x1x7x7

[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

Pebble weighted automata



5 x1x7x7x1



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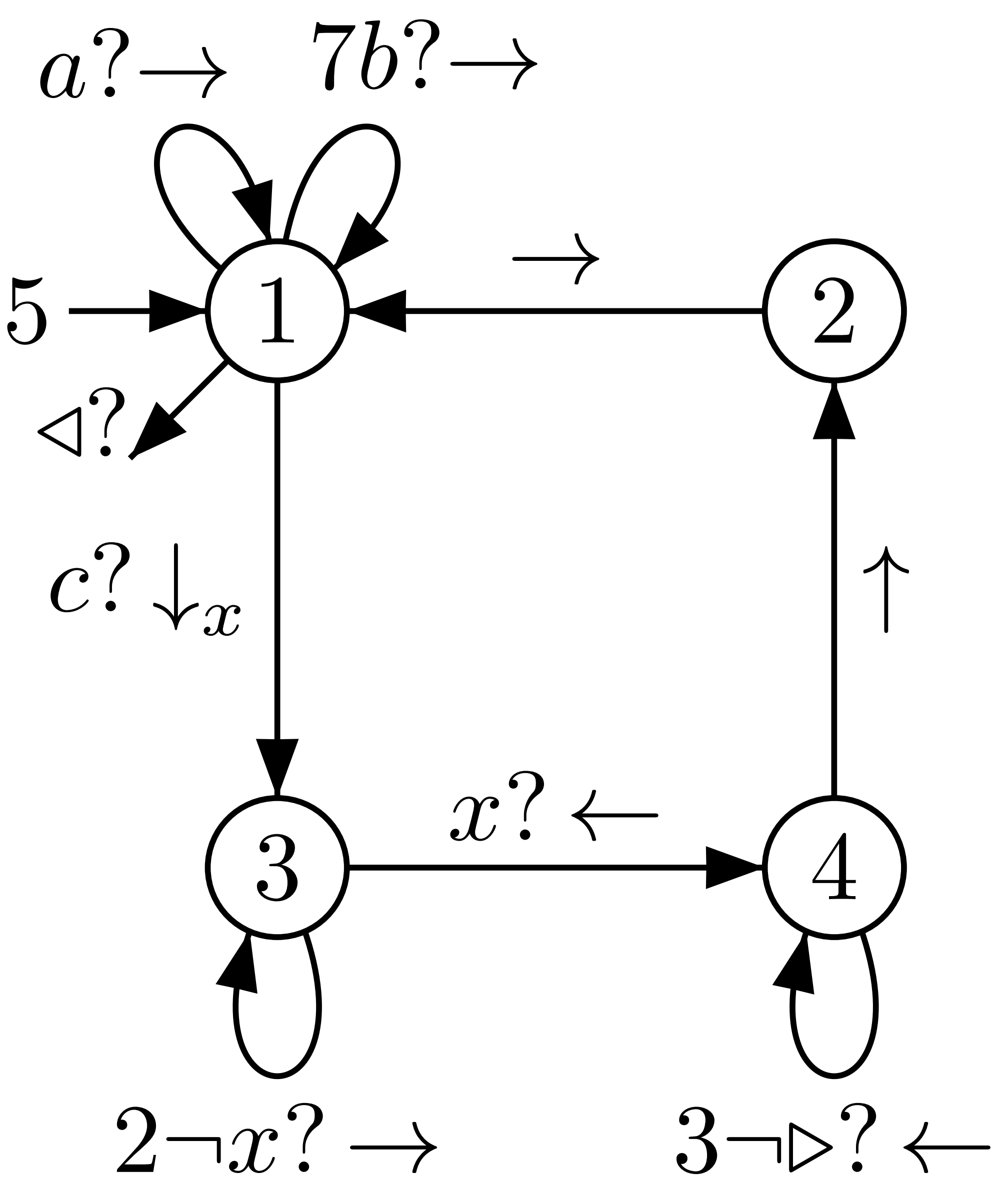

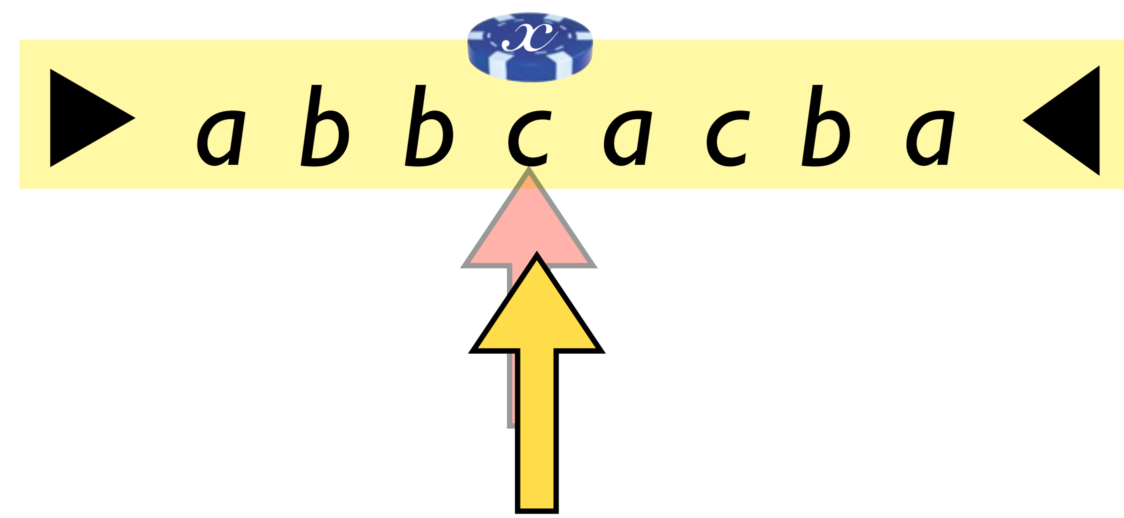

5 x1x7x7x1
x2x2x2

[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

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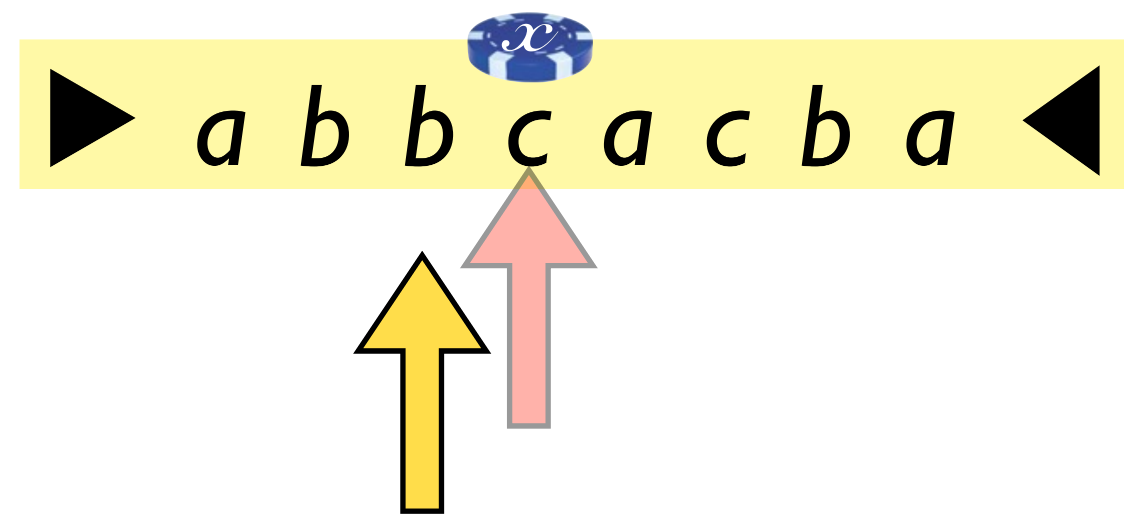

5 x1x7x7x1
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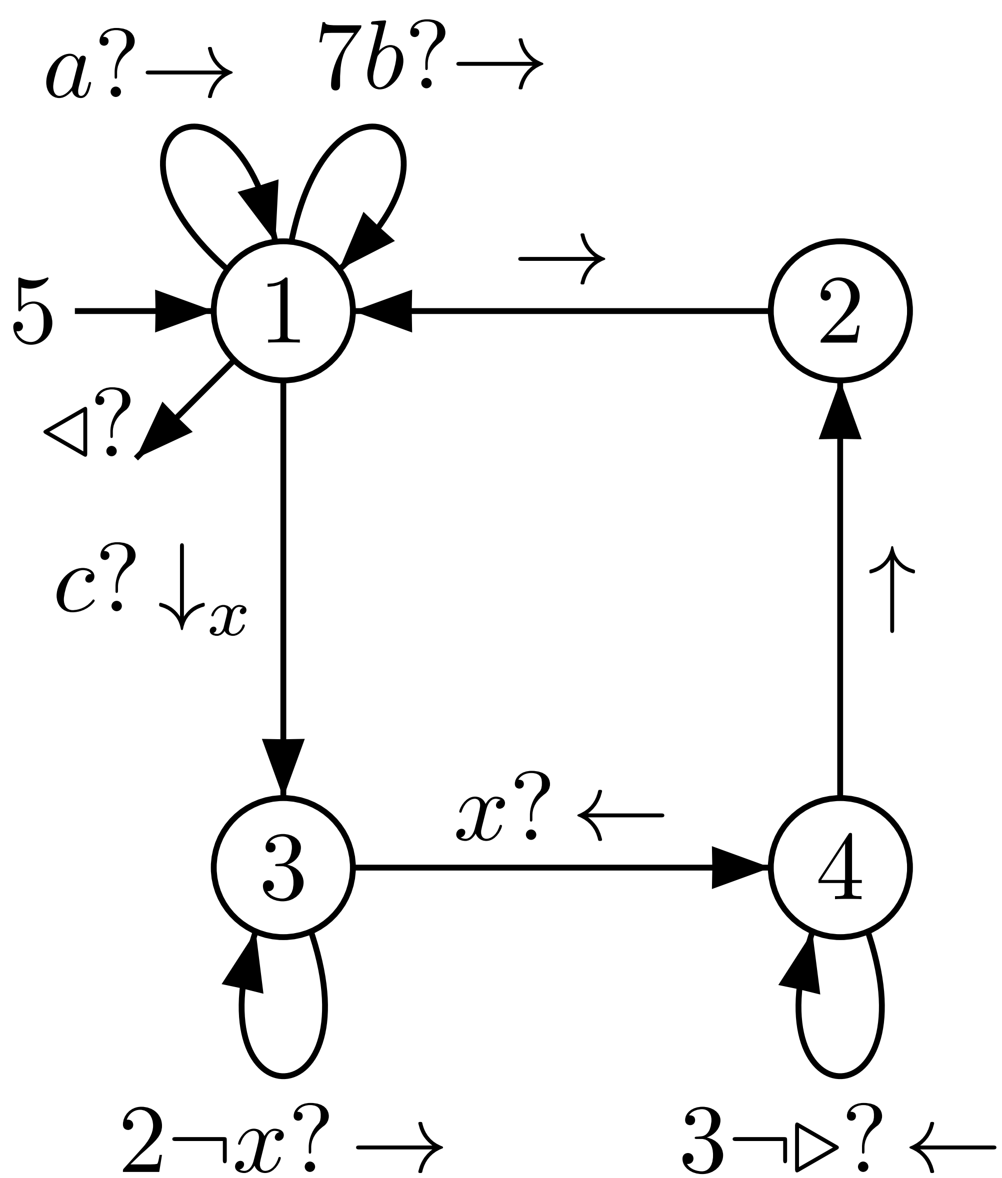

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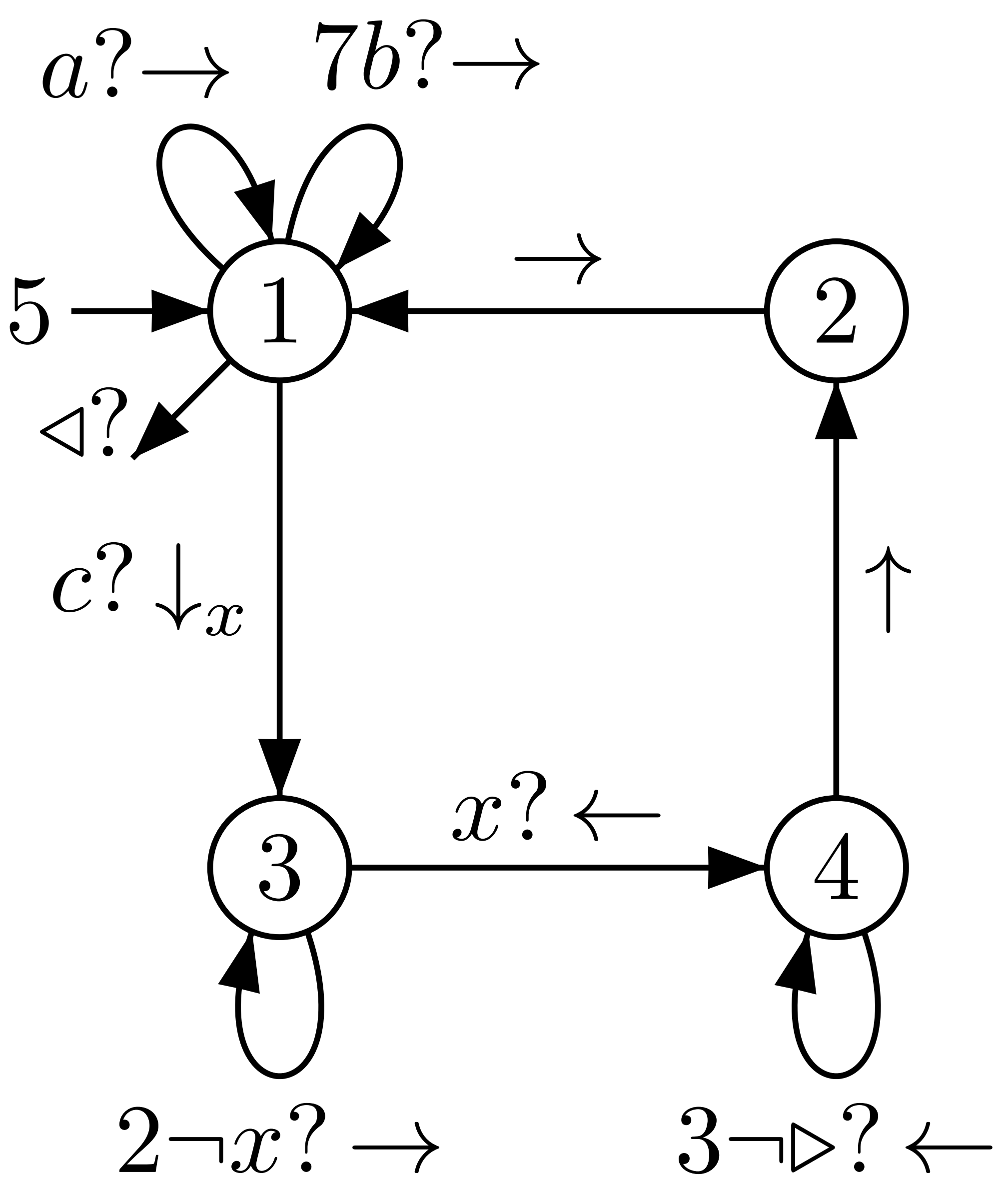
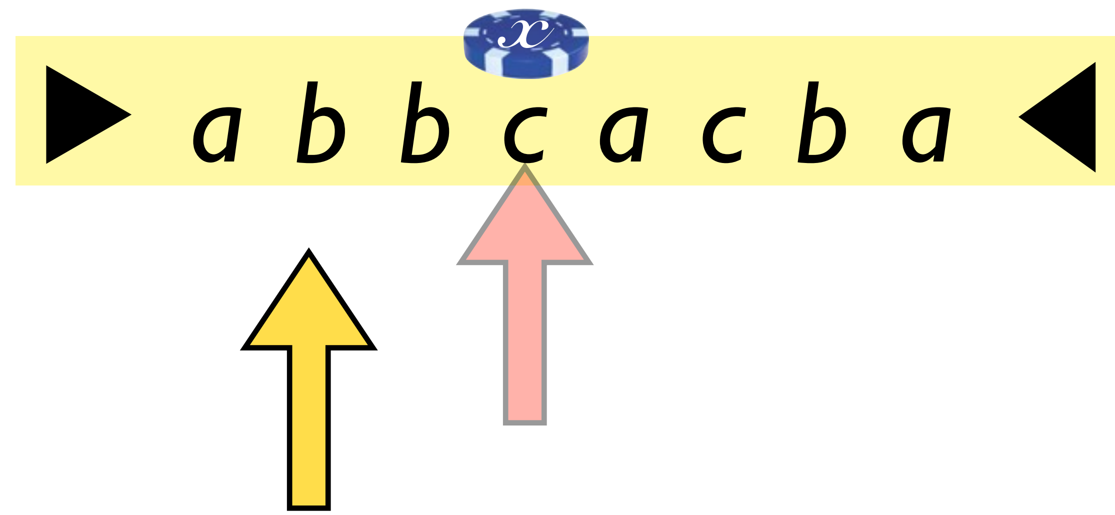
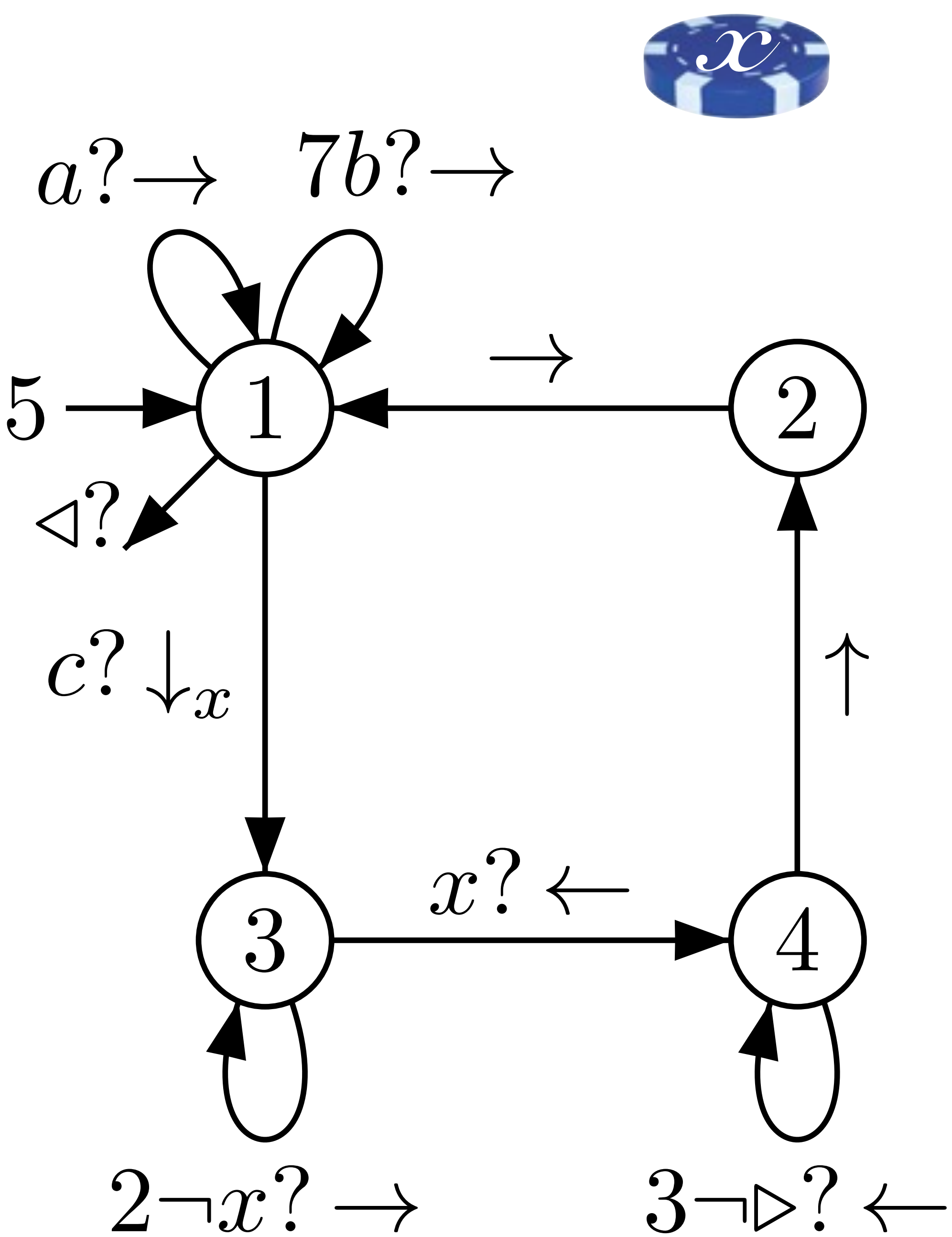


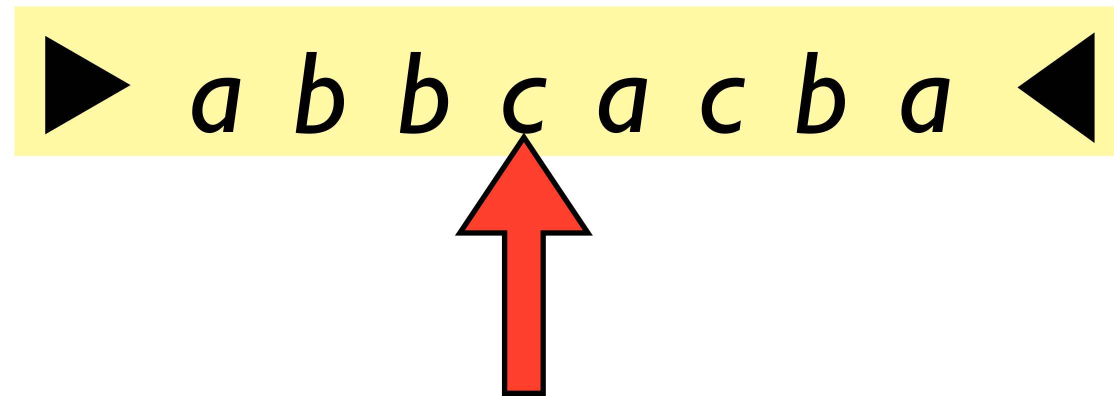


5 x1x7x7x1
x2x2x2x1x3

[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

Pebble weighted automata





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[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

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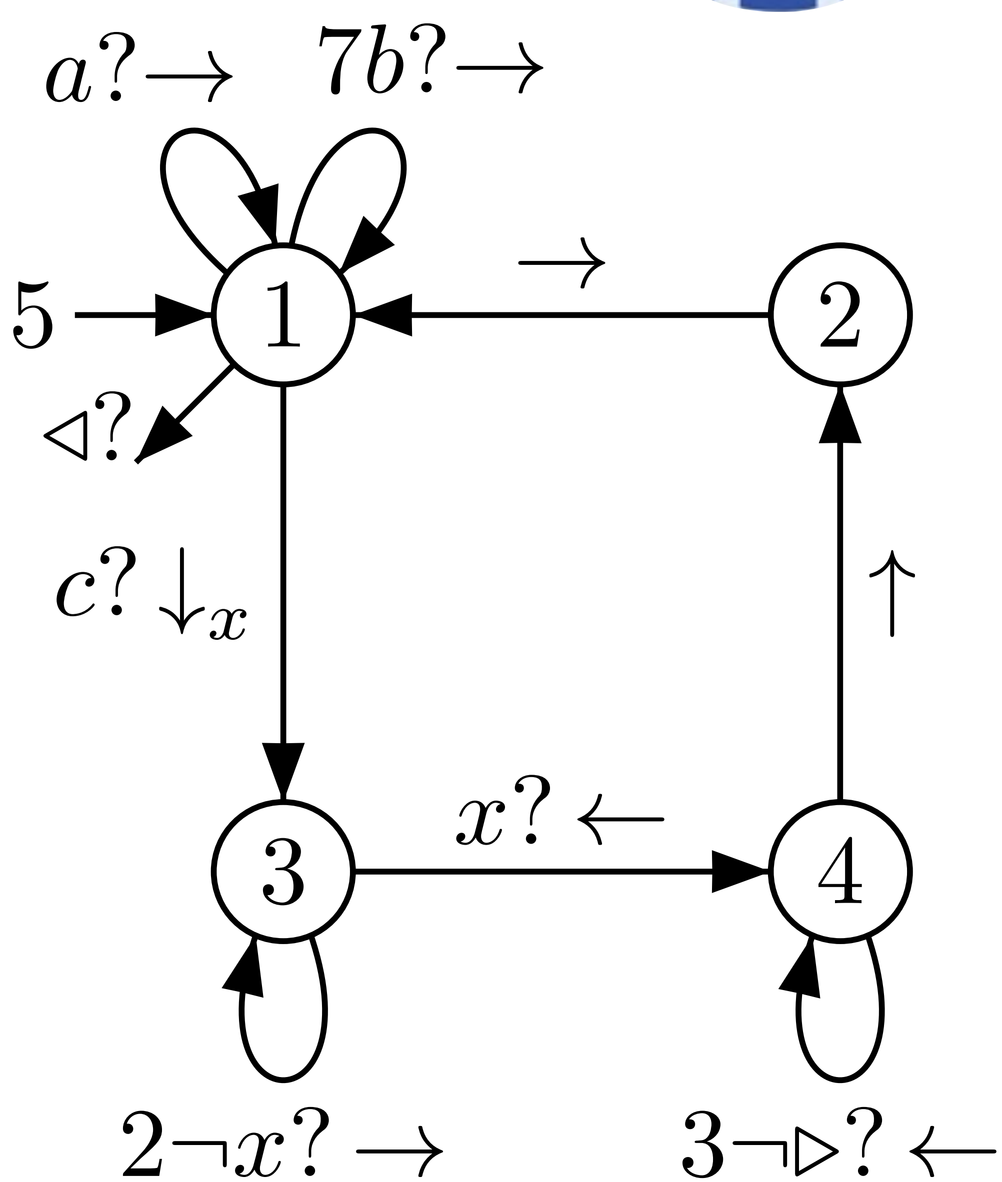

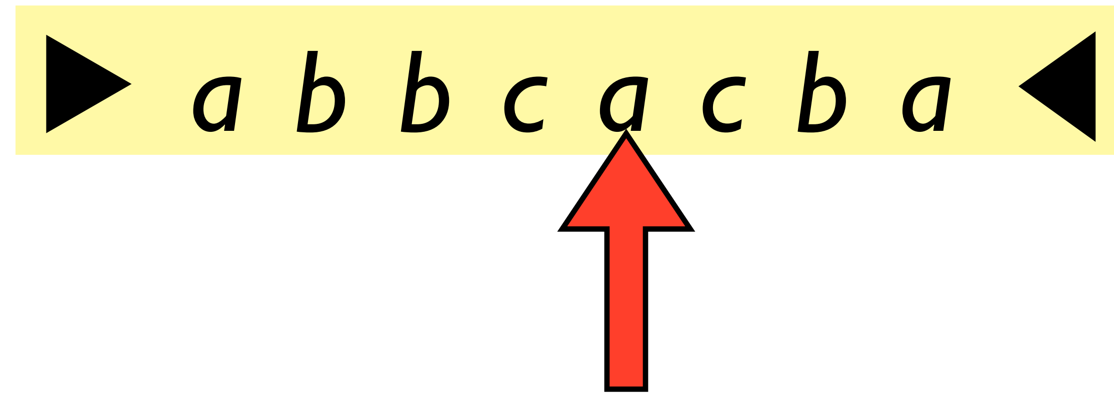

5 x1x7x7x1
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x1

[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

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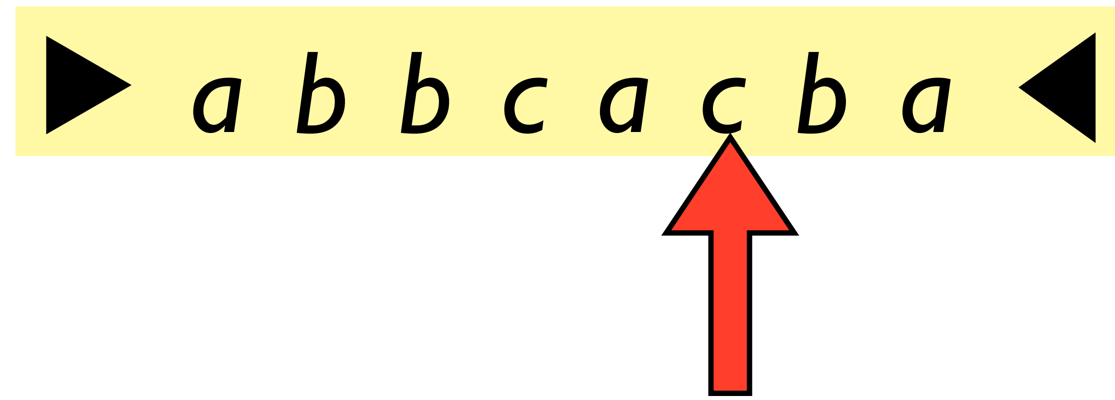

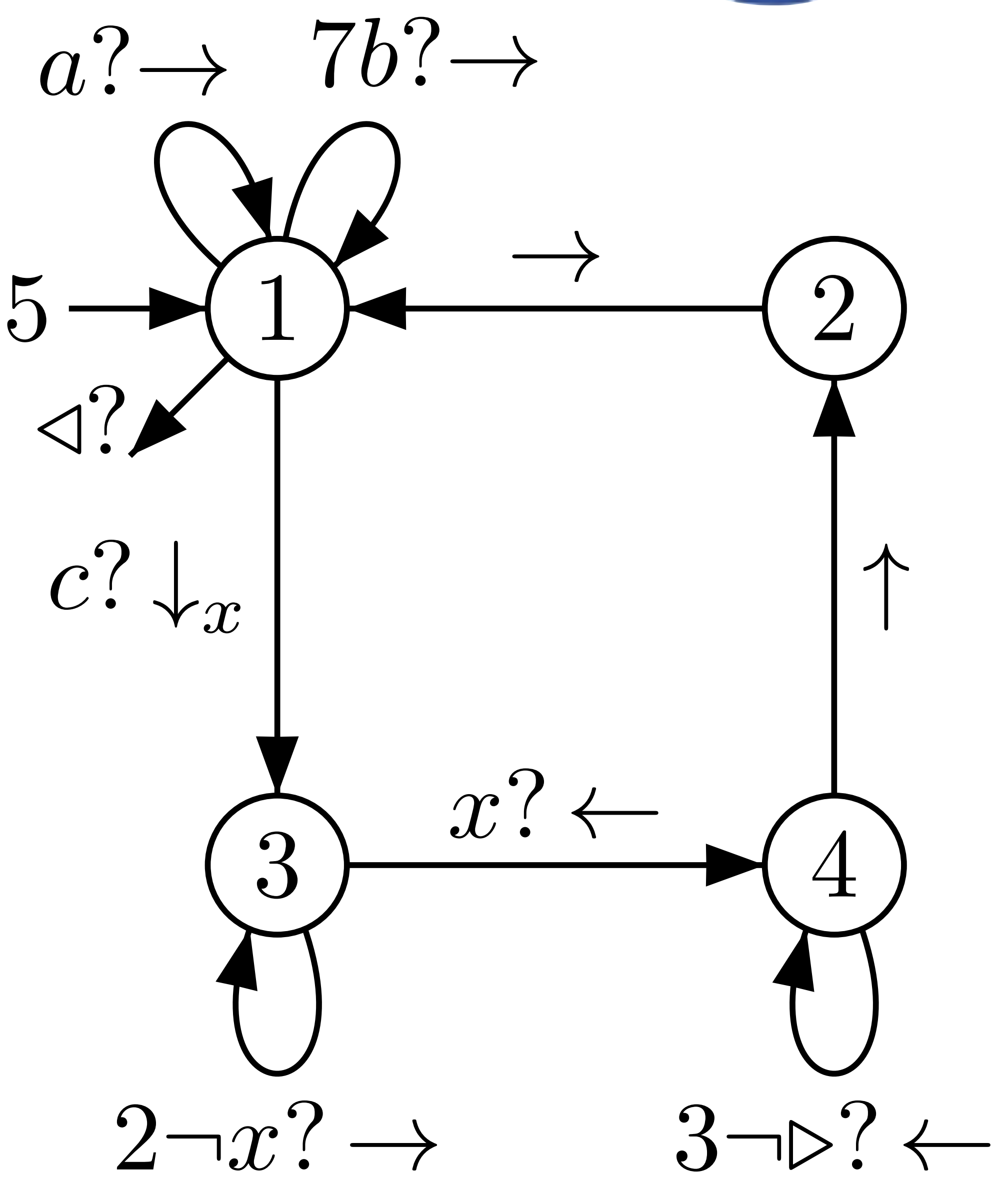

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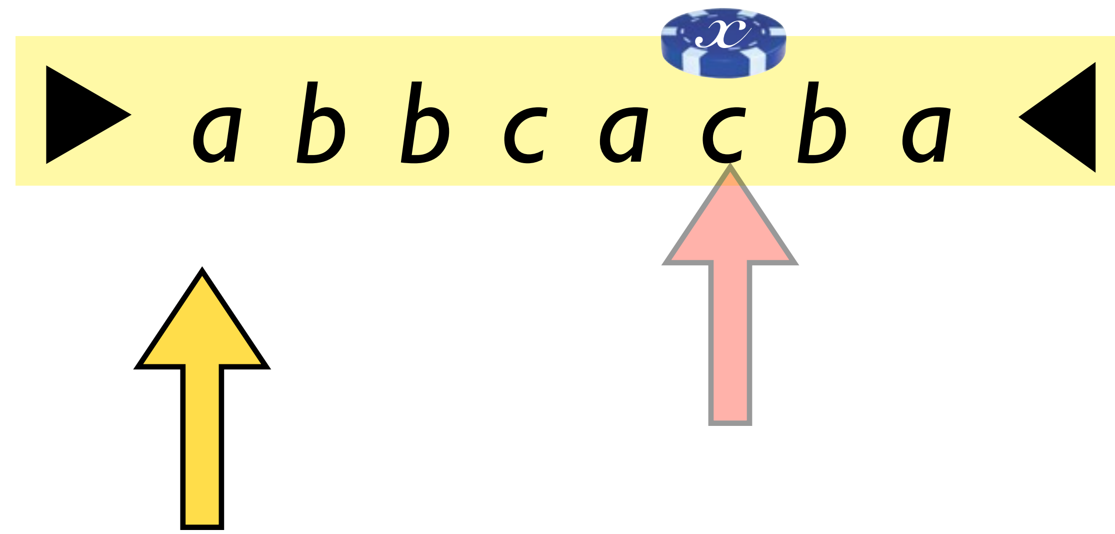
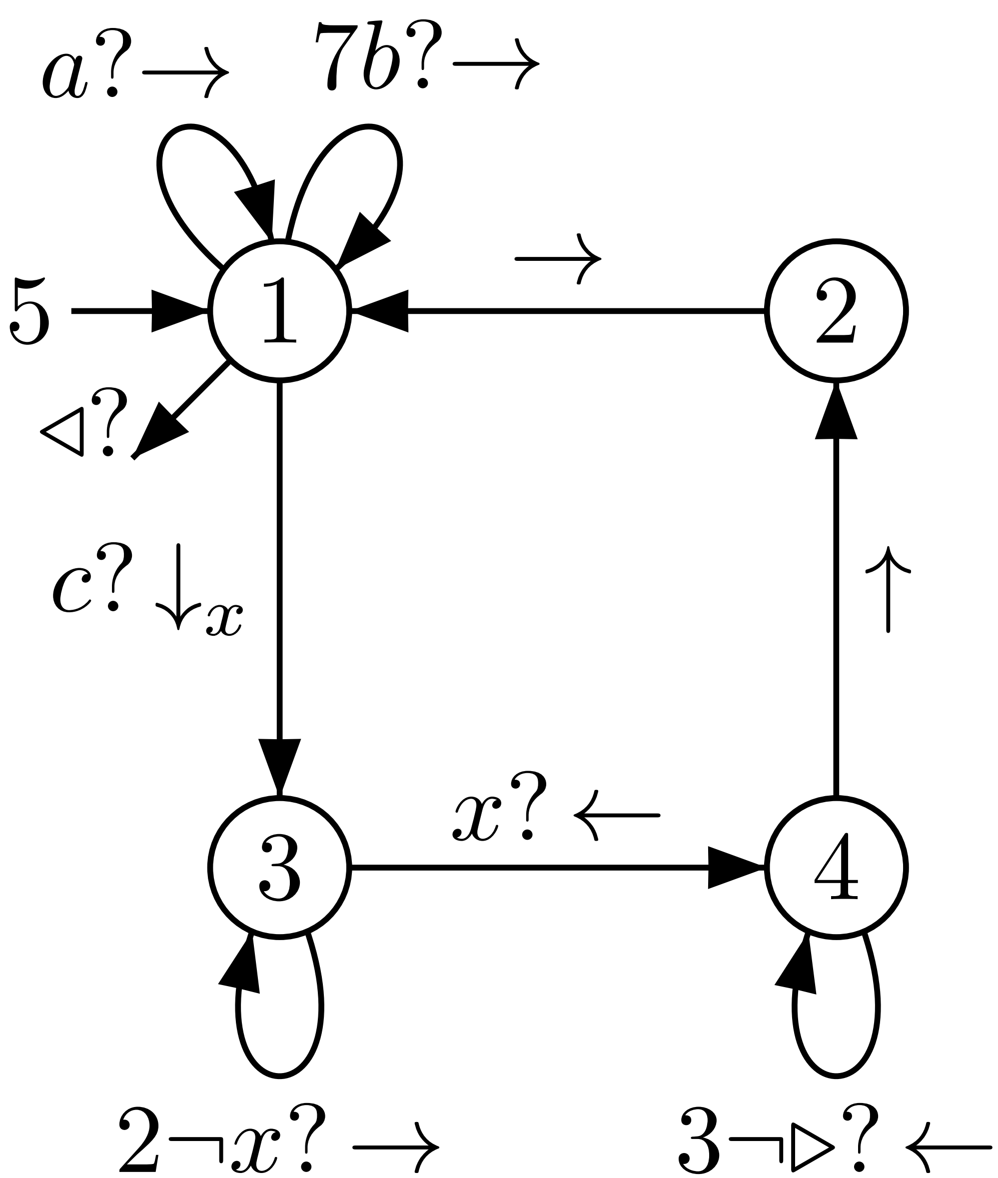
x1x1

[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

Pebble weighted automata





5 x1x7x7x1

x2x2x2x1x3x1

x1x1x1

[Globerman and Harel, 96][Engelfriet and Hoogeboom,99]

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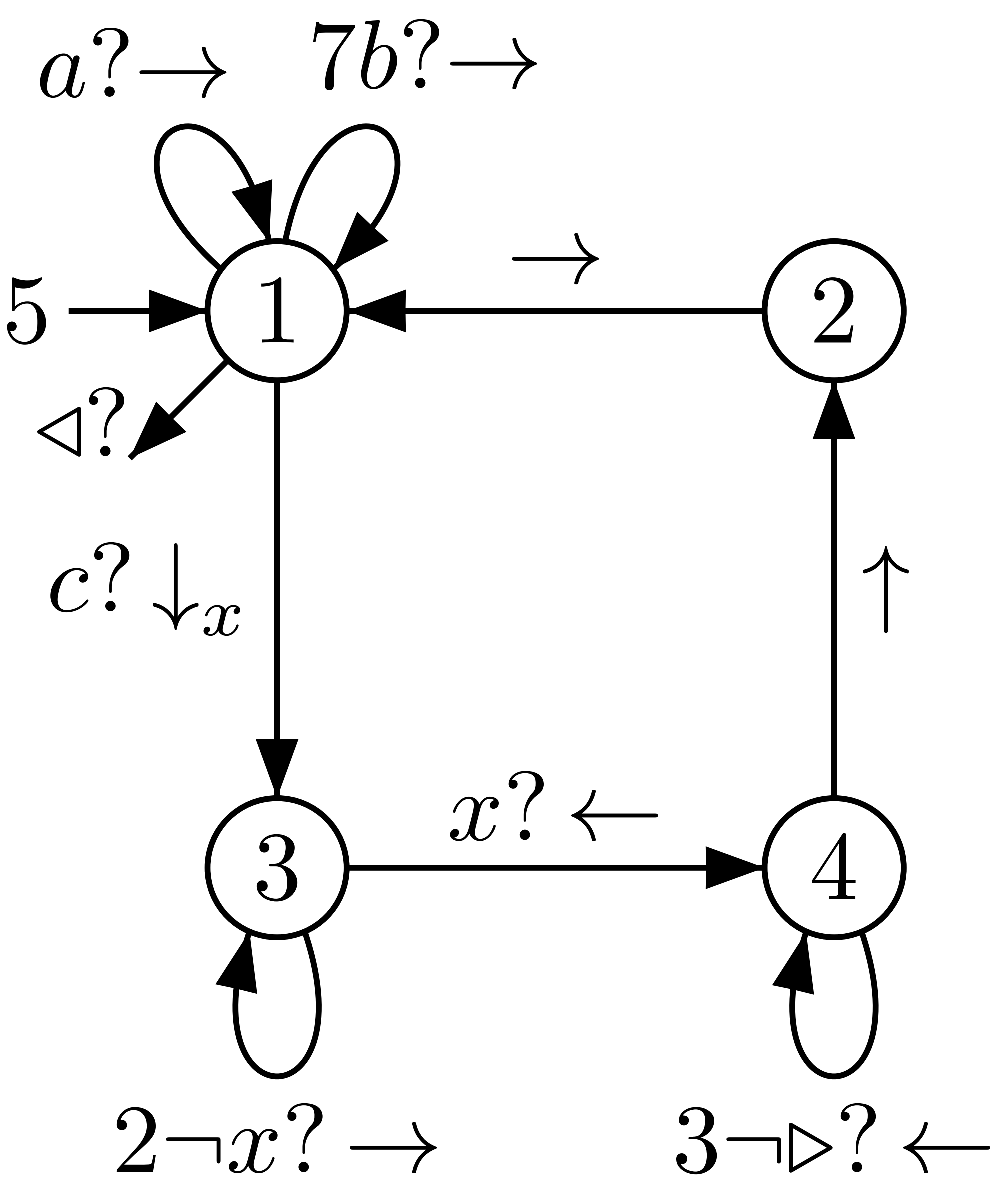

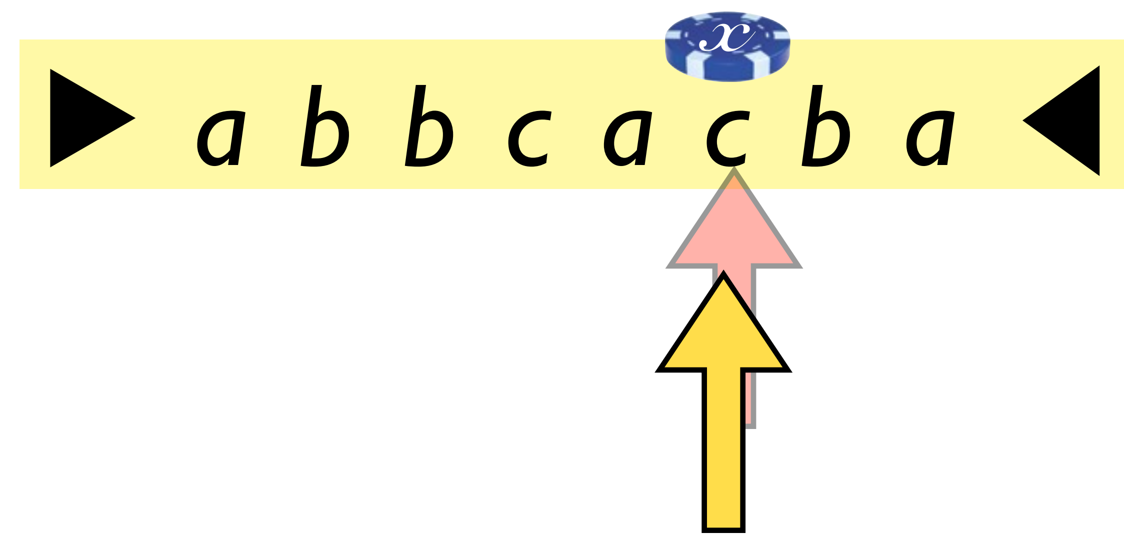

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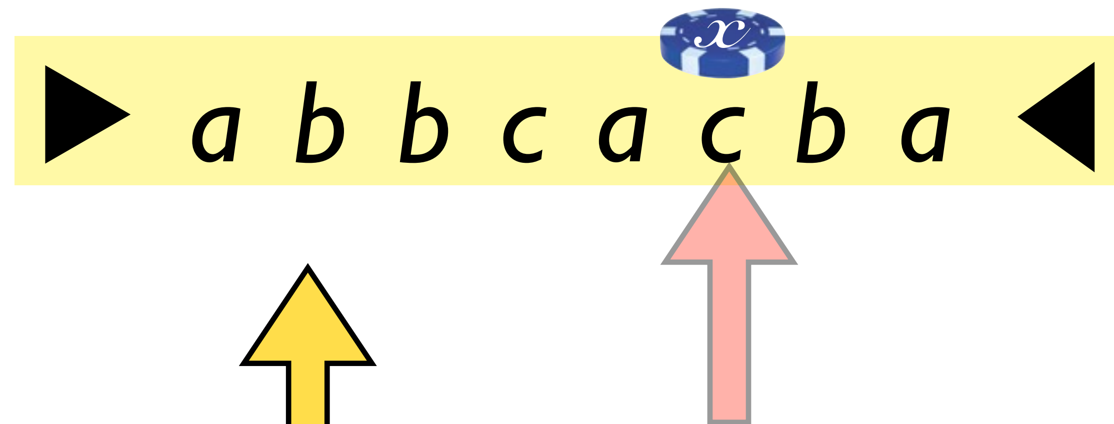

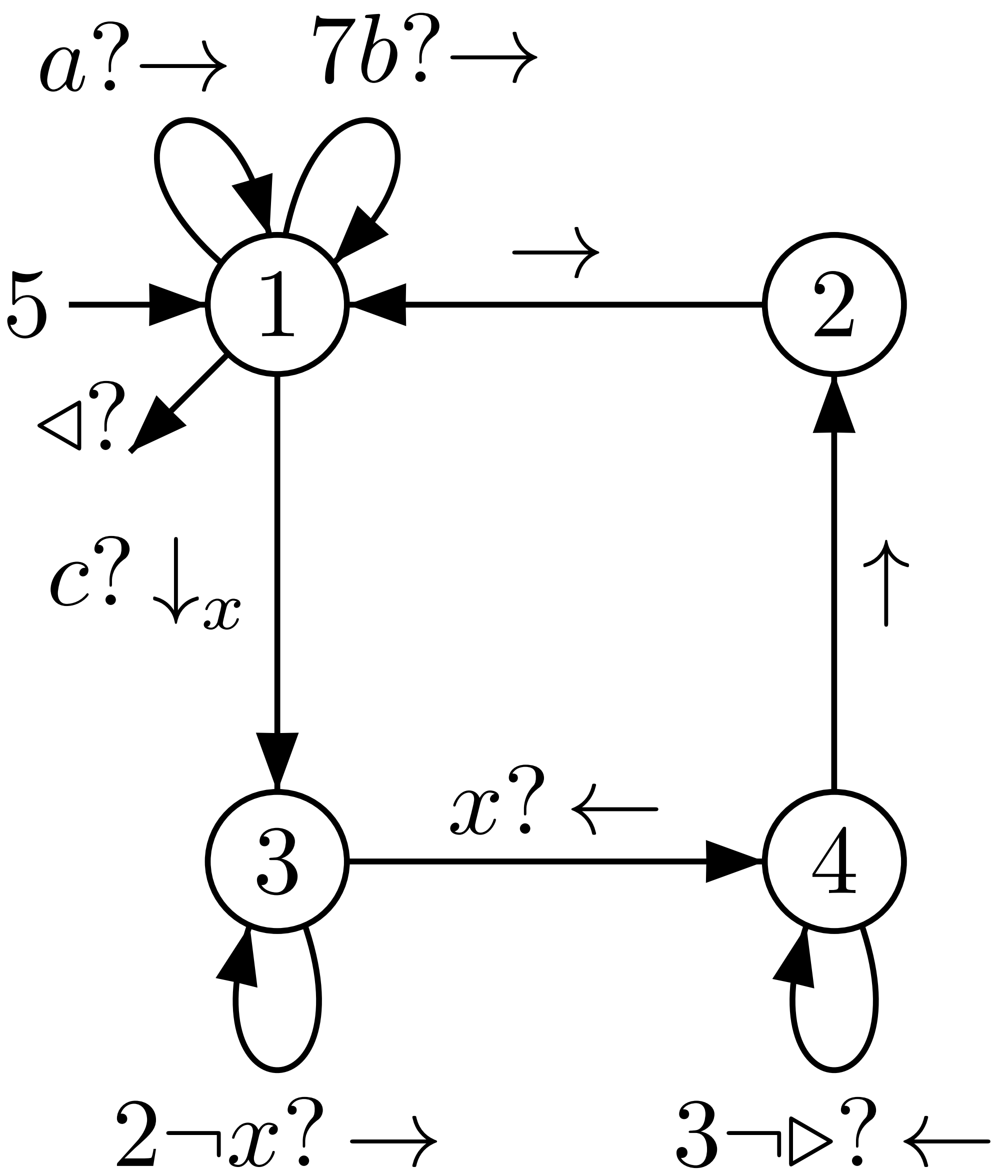

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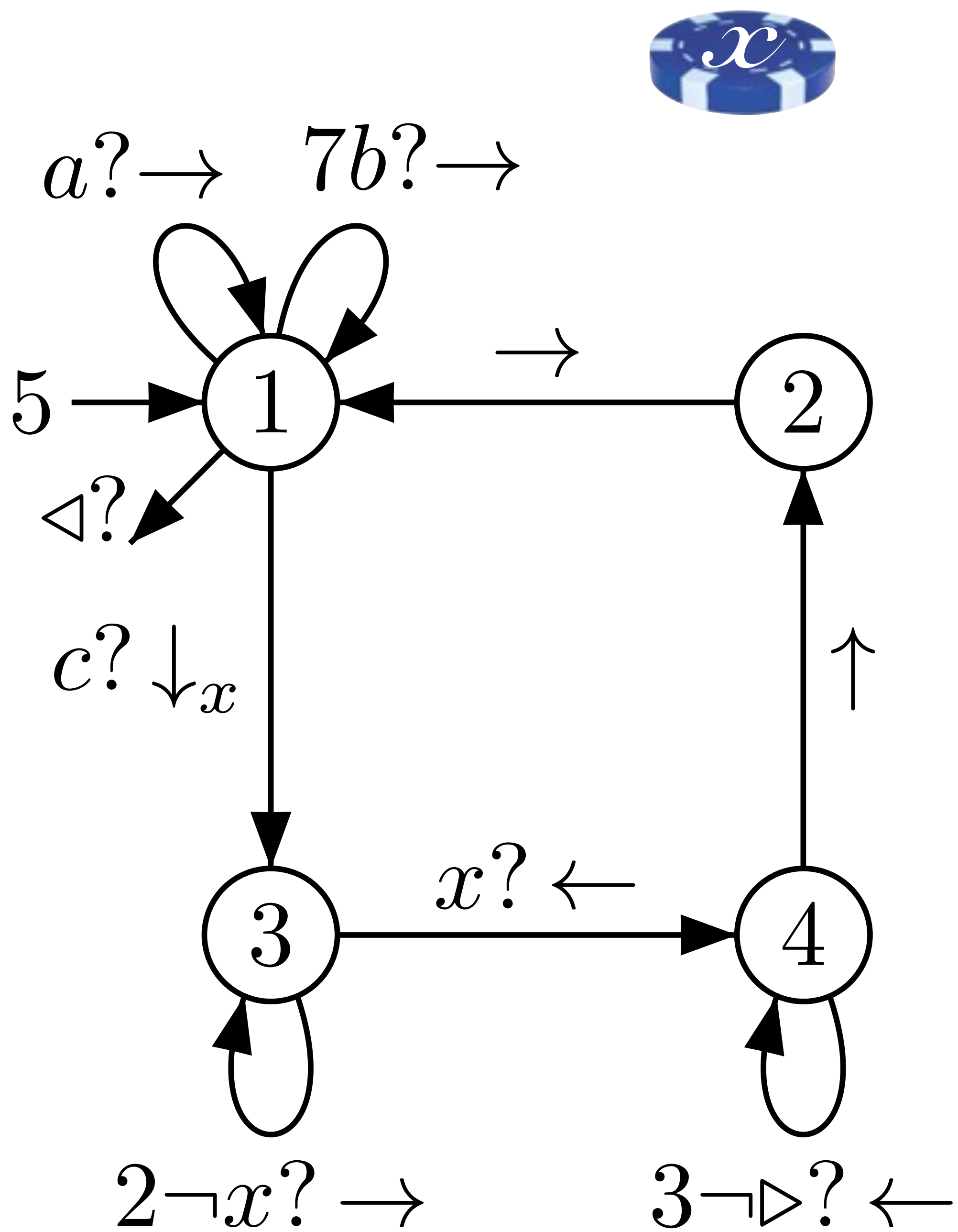

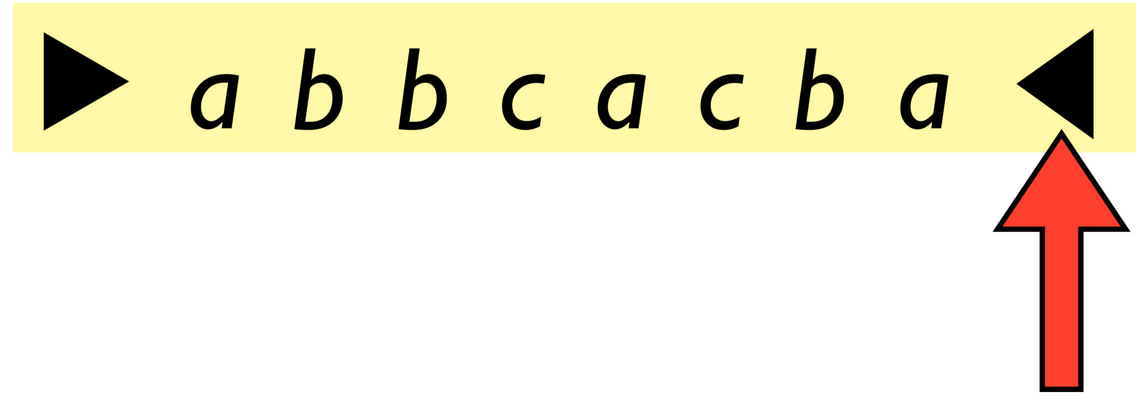

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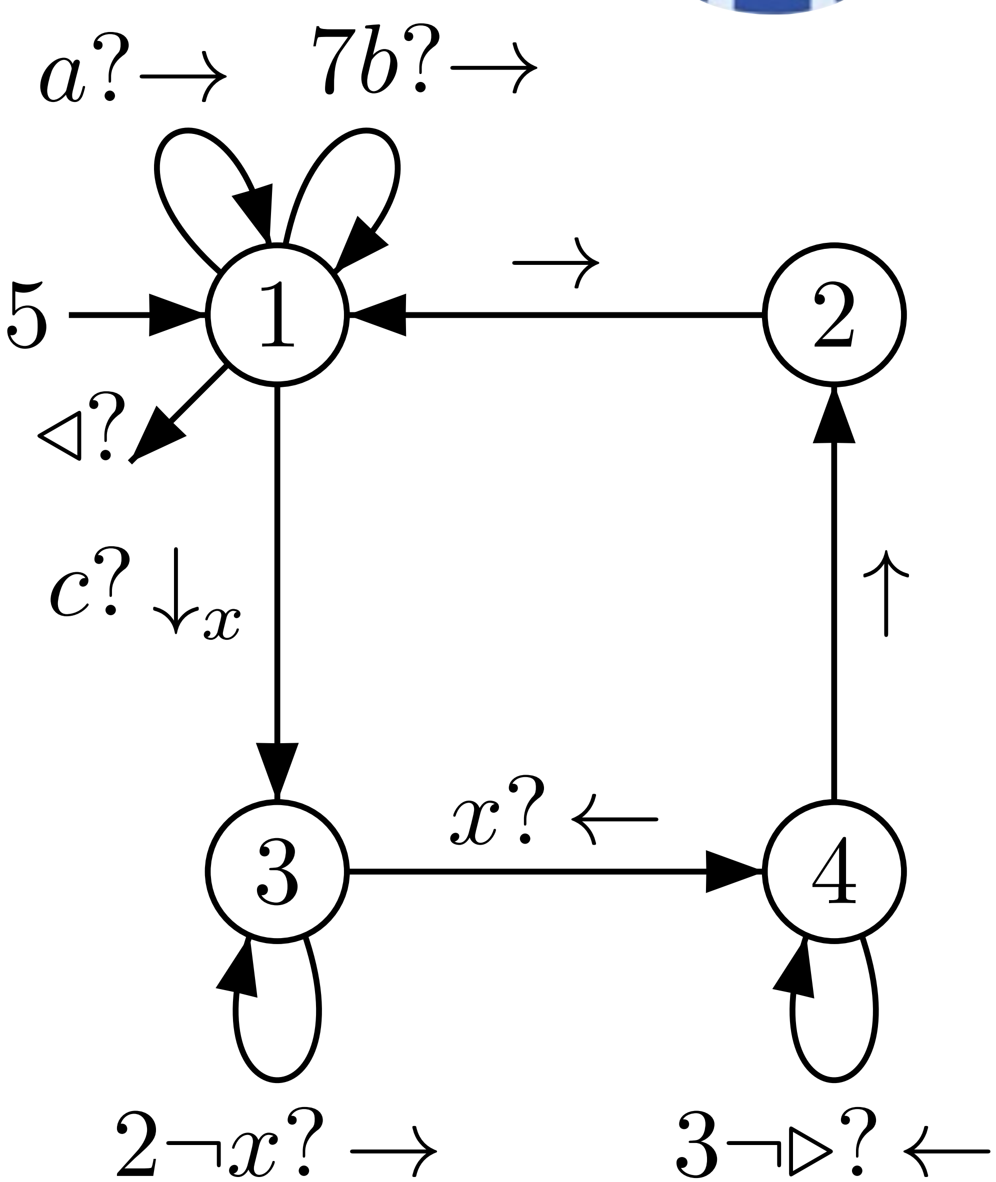

▶ *a b b c a c b a* ◀

Weight of the run: 35 562 240

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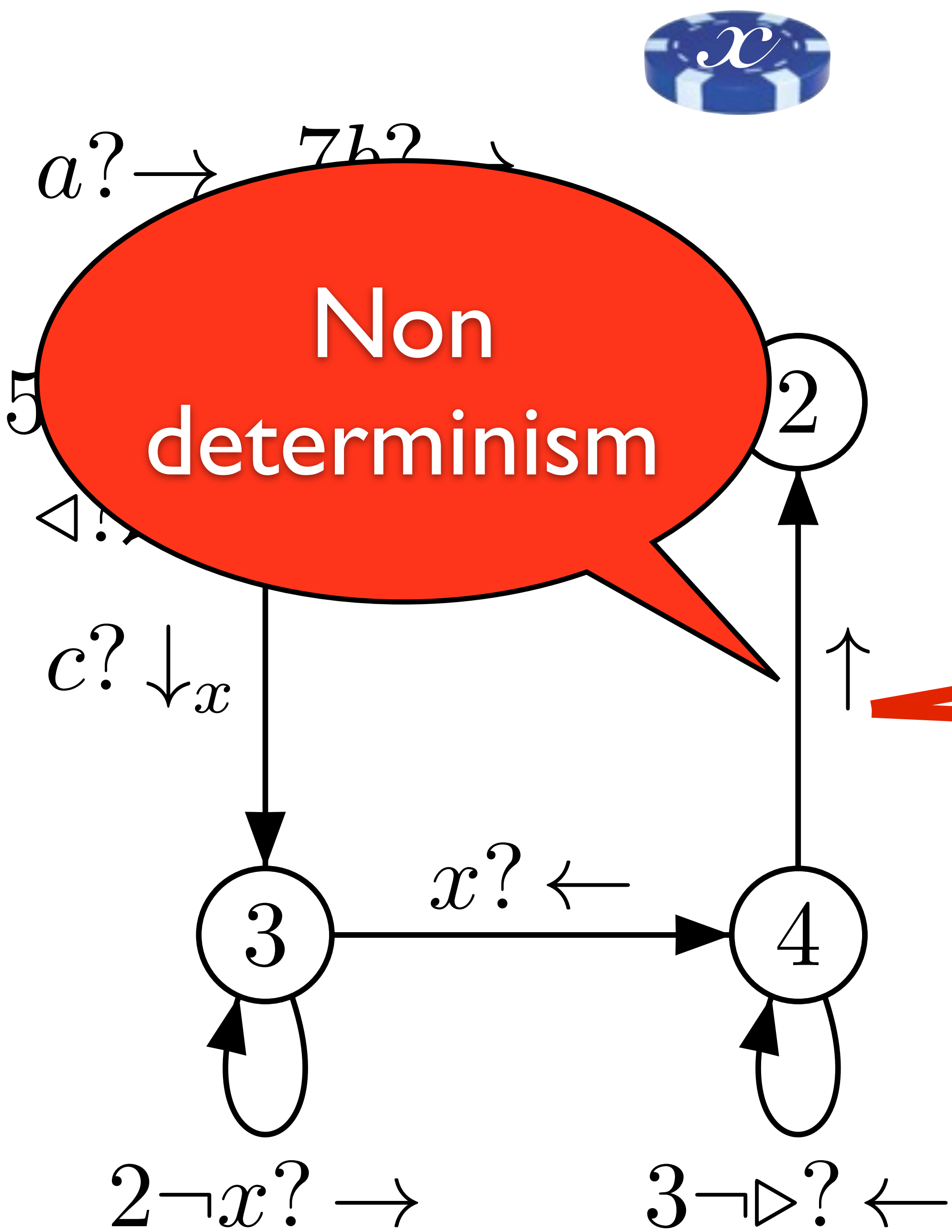

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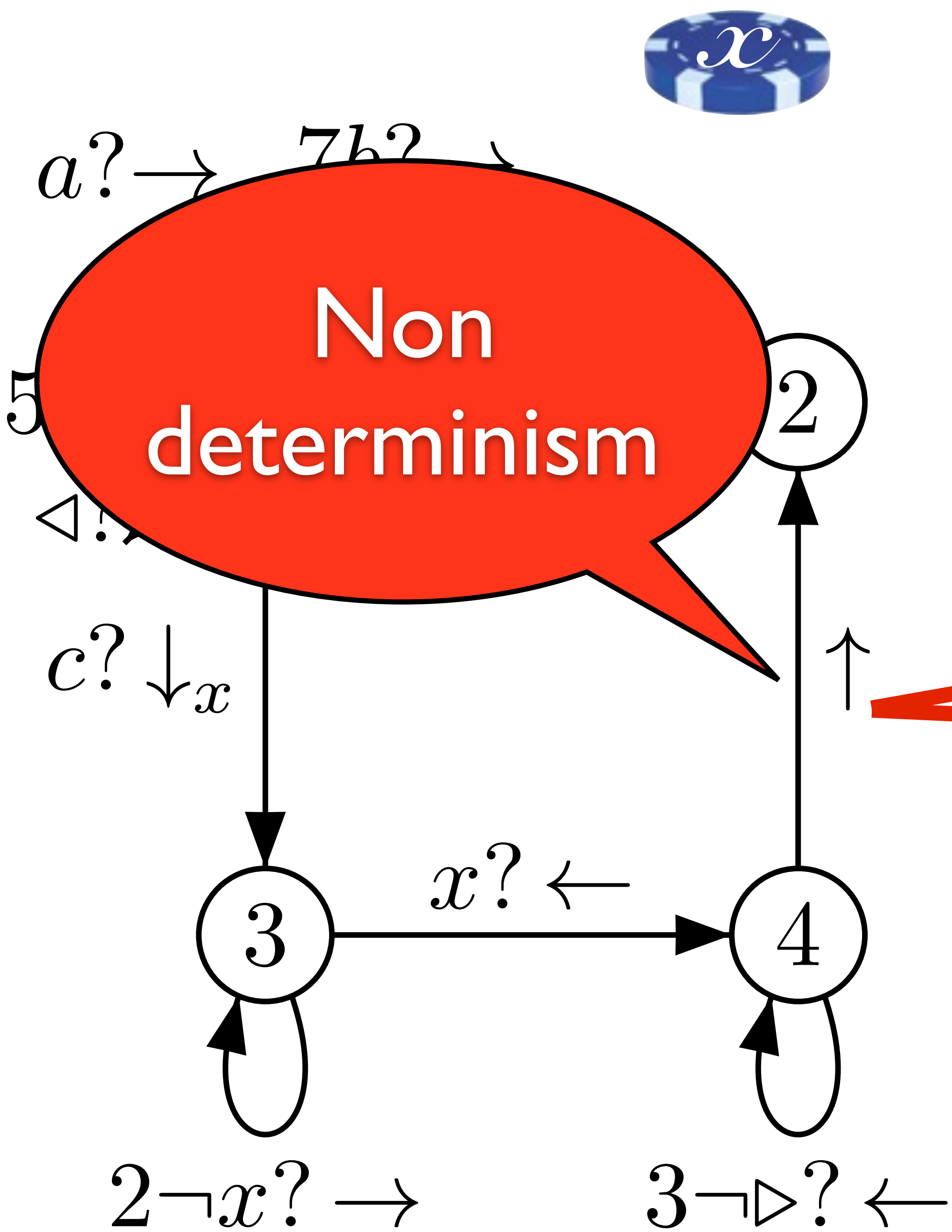

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Sum of the weights of the runs: $5.7^{|w|_b}.\prod_{w_i=c}2^{i-2}(3^i-1)$

Non determinism resolved by sum

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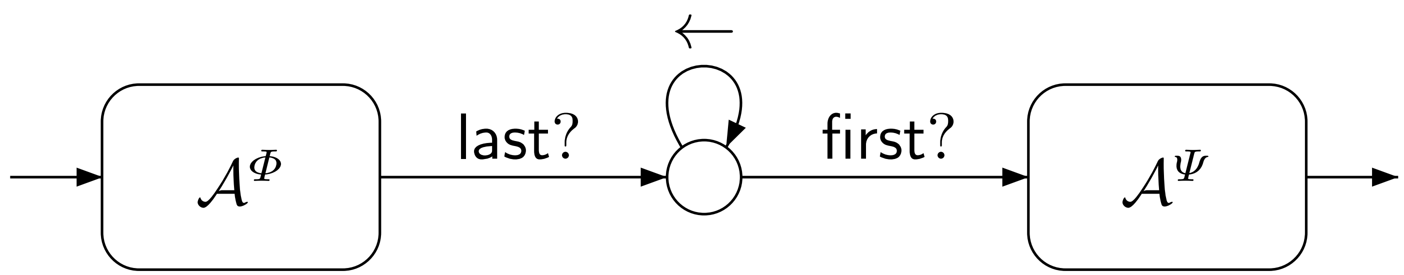
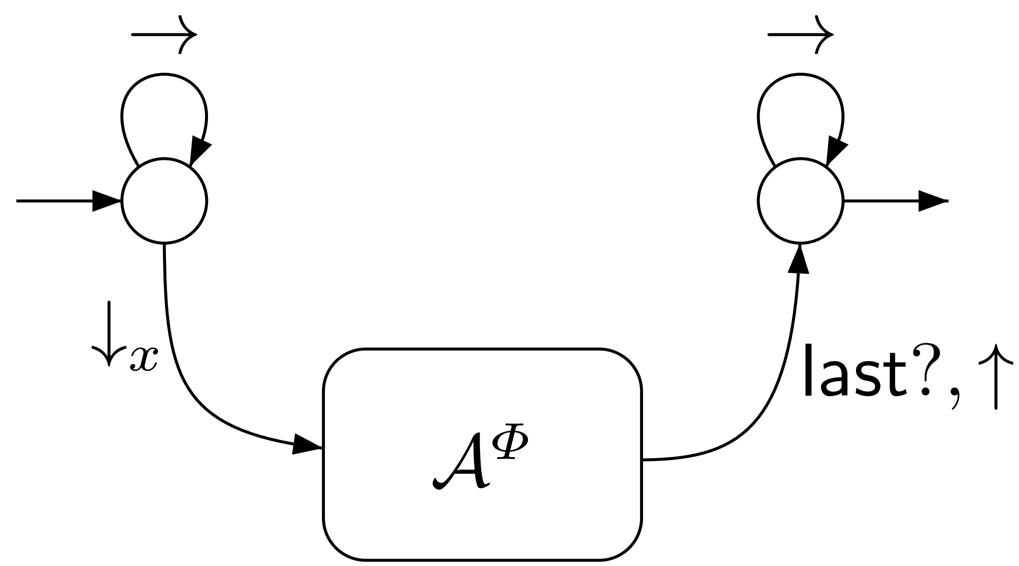
Translating a formula into an automaton

Sum by disjoint union of automata

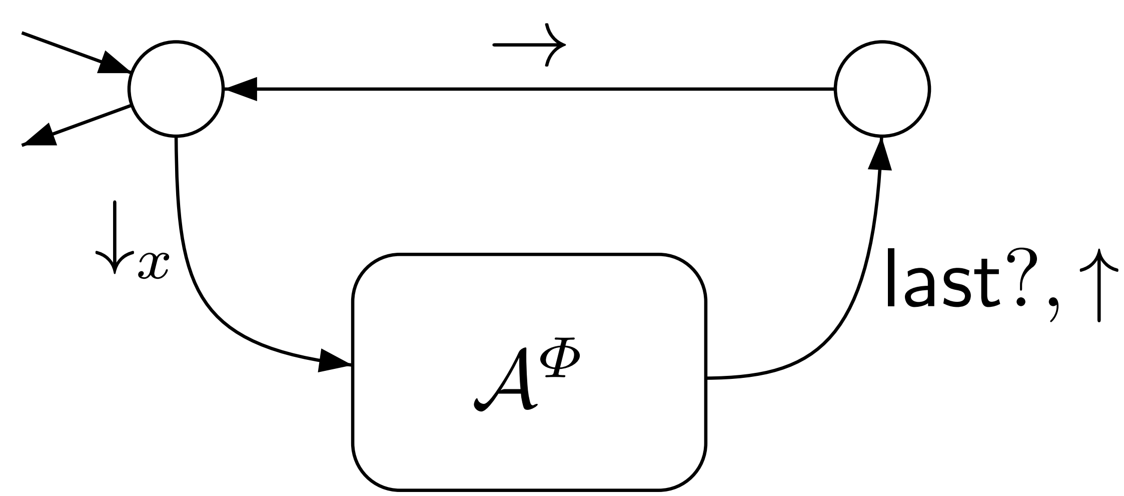
Product:



Sum quantification:



Product quantification:



Translating a formula into an automaton

Challenging for the *Boolean part*:

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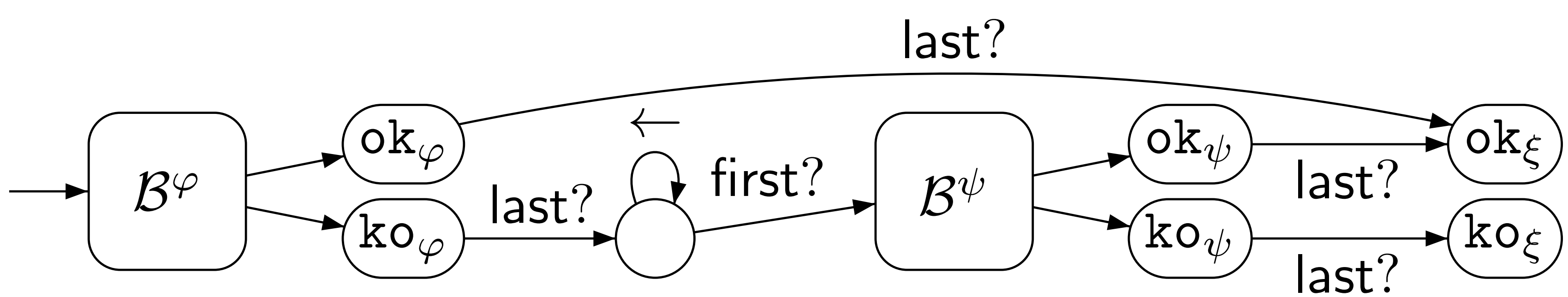
Disjunction/conjunction

$\xi = \varphi \vee \psi$



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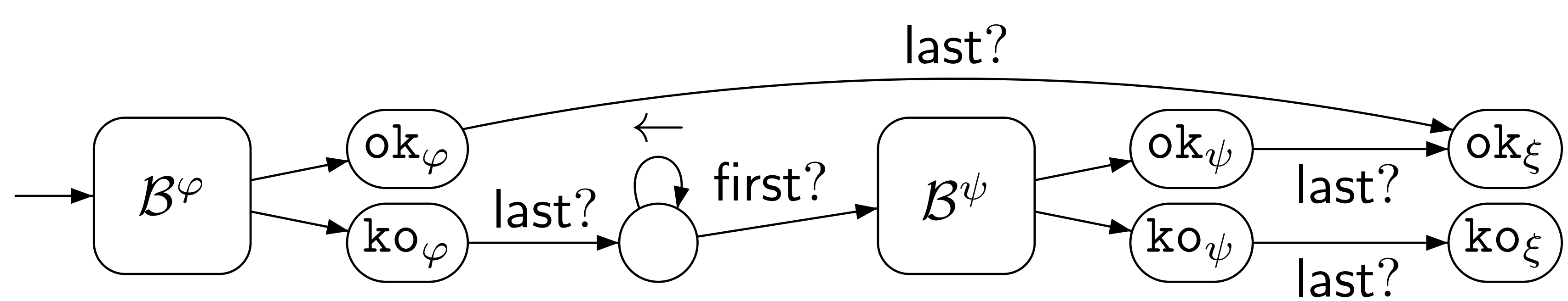
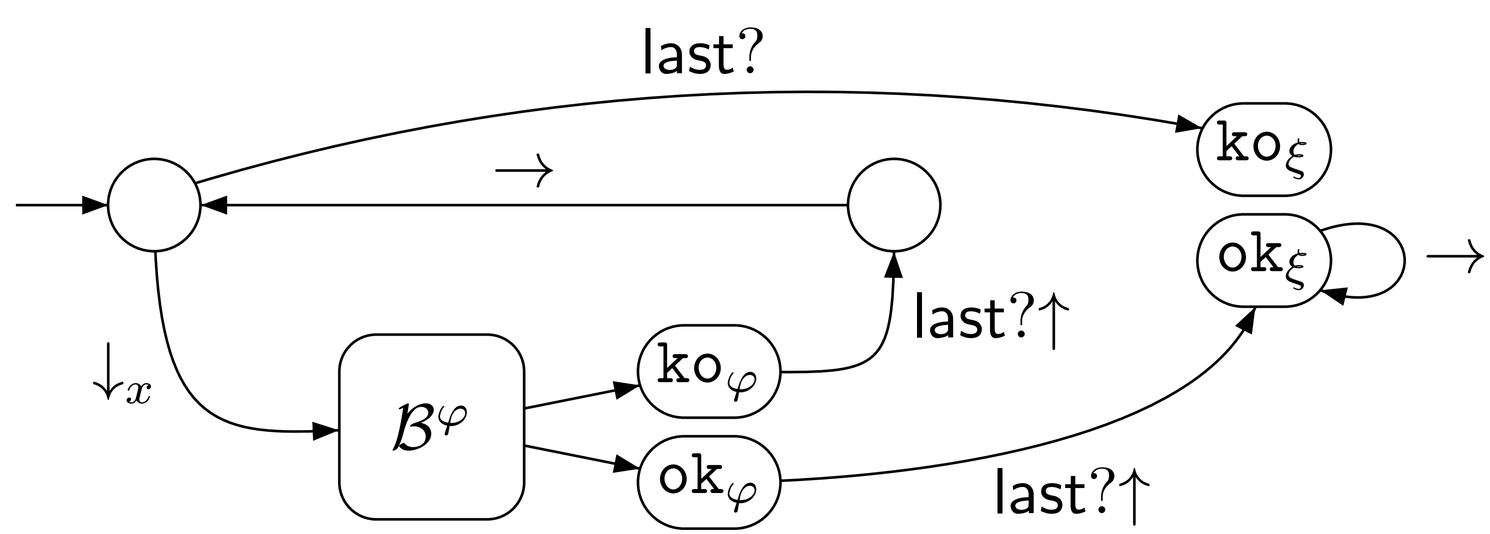


Existential/Universal quantifications

$\xi = \exists x\ \varphi$



Logic equivalent to PWA?

- Weighted FO misses a counting capability…
- Solution: weighted transitive closure operation

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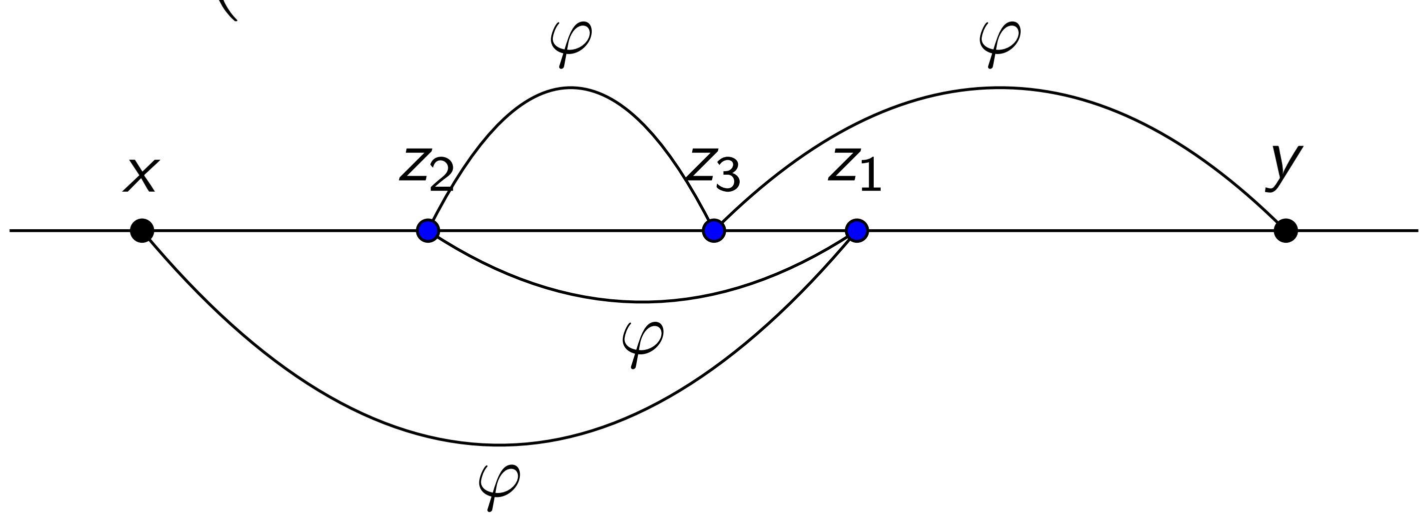
$$\varphi^n(x,y) = \exists z_0 \cdots \exists z_n \Big(x = z_0 \wedge z_n = y \wedge \mathsf{diff}(z_0,\ldots,z_n) \wedge \big[\textstyle\bigwedge_{1\le\ell\le n} \varphi(z_{\ell-1}, z_\ell)\big]\Big)$$



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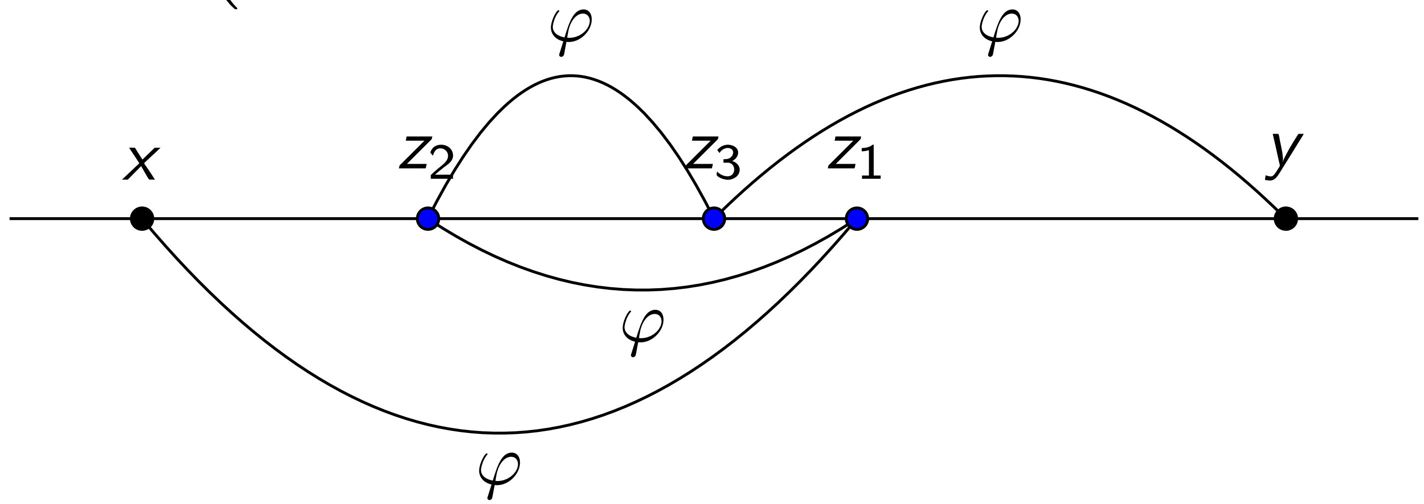
$$\mathrm{TC}_{xy}\varphi = \bigvee_{n\ge 1} \varphi^n$$



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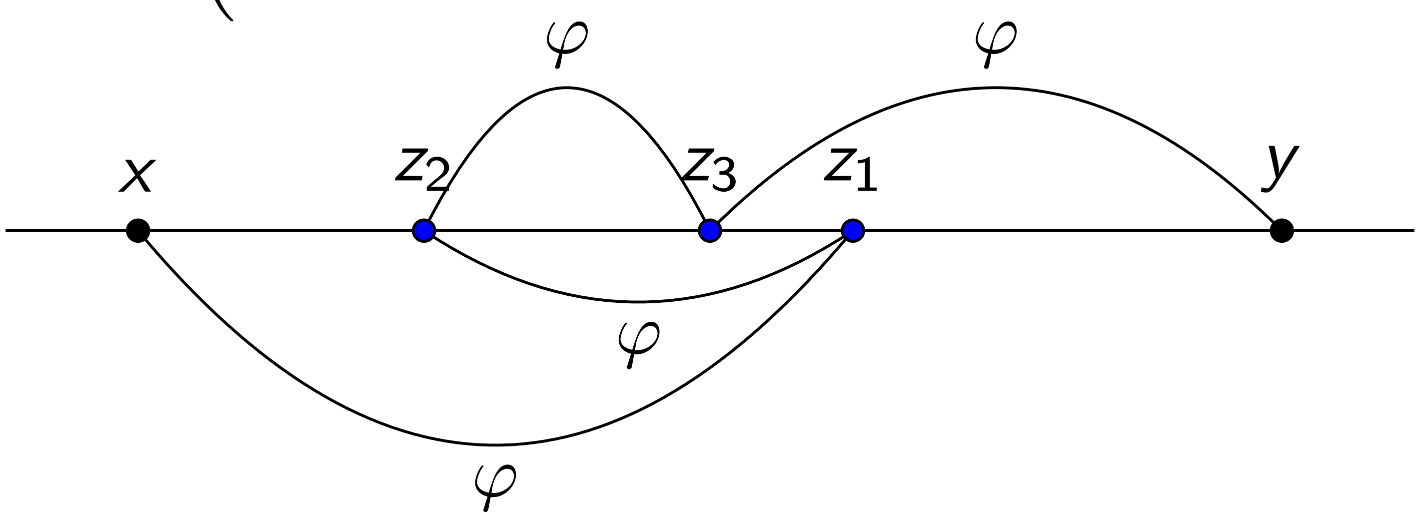
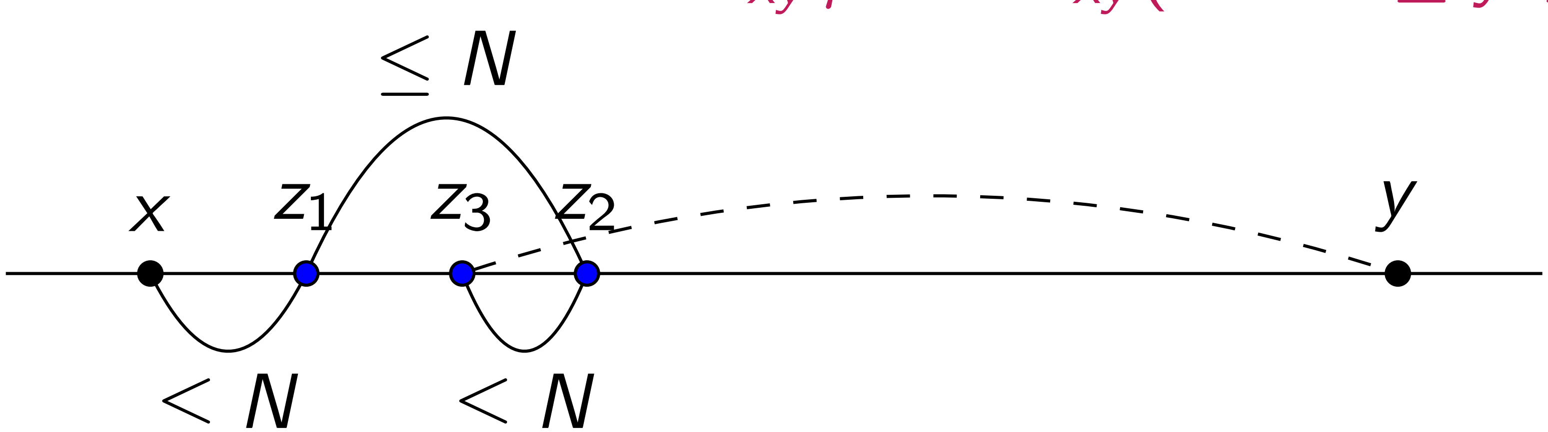


Bounded transitive closure : $N\text{-}\mathrm{TC}_{xy}\varphi = \mathrm{TC}_{xy}(x - N \leq y \leq x + N \wedge \varphi)$



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φ φ

TC

Theorem: PWA = wFO + bounded-TC

φ

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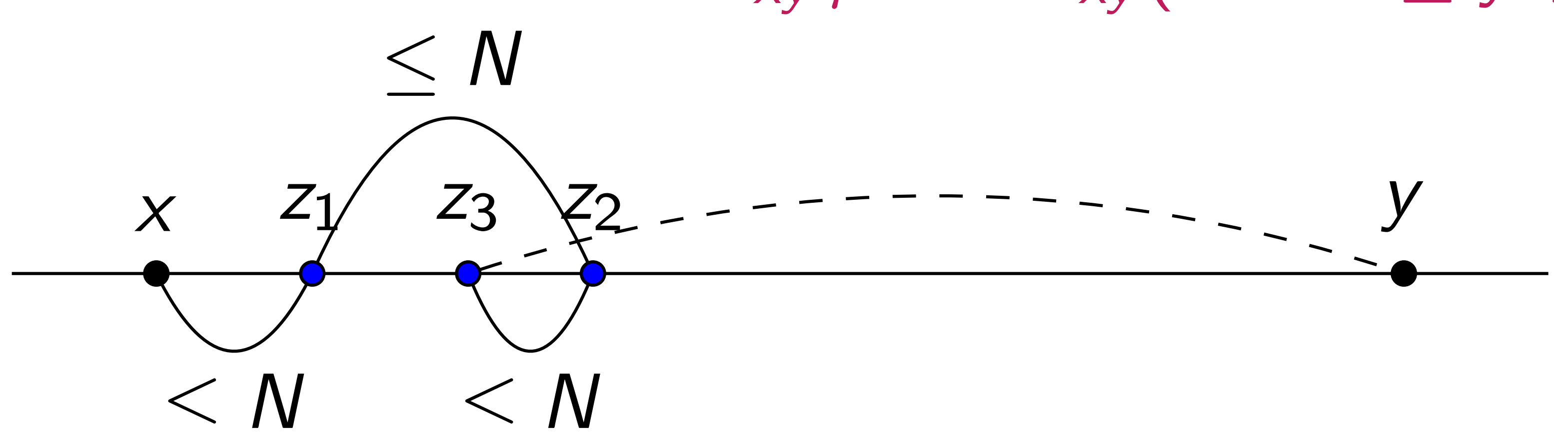
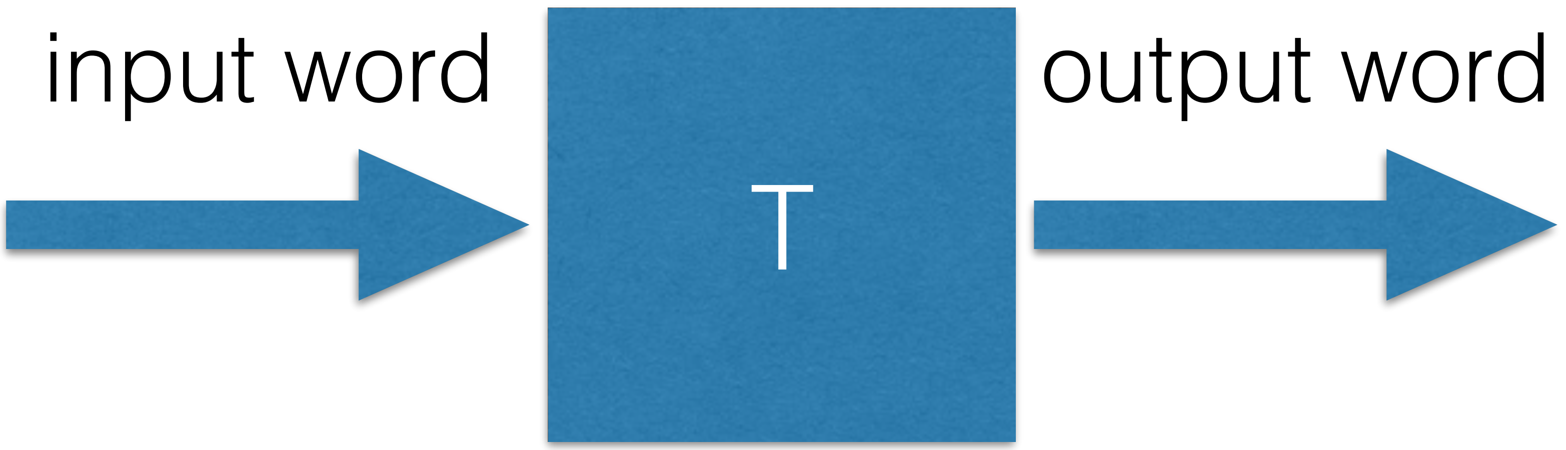


Application to transductions



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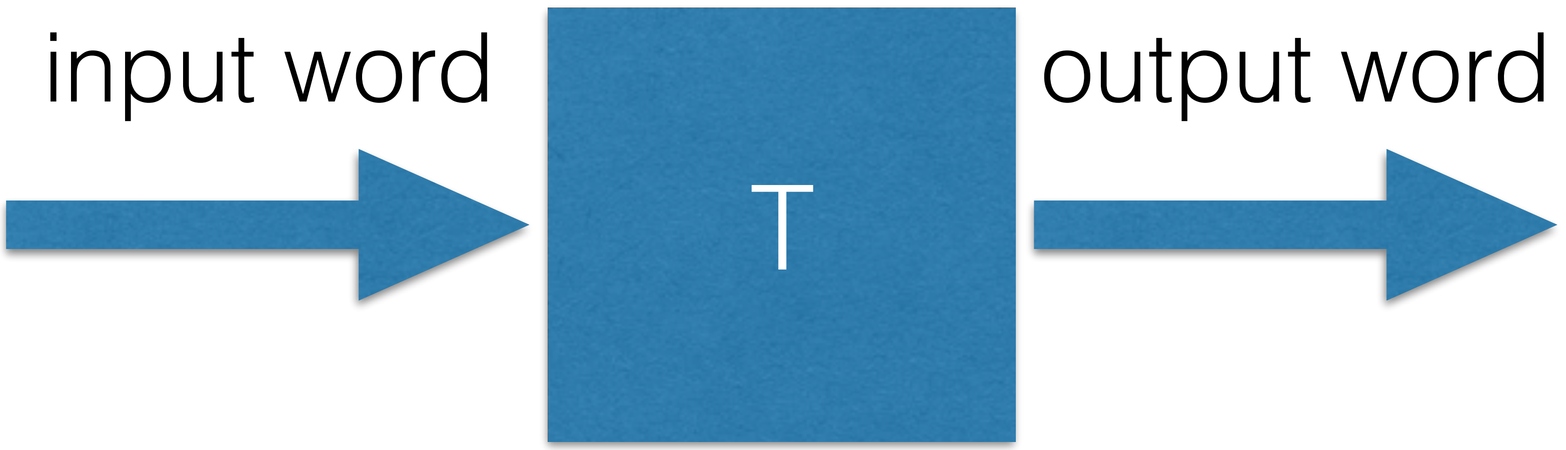

Pattern
matching/replacement

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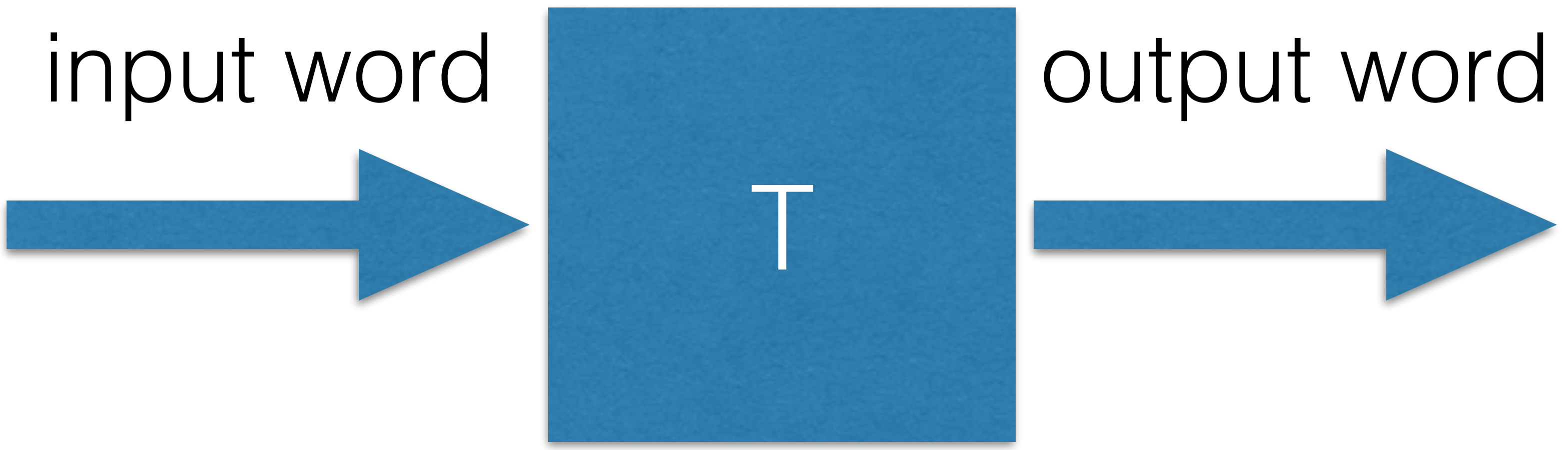

Pattern
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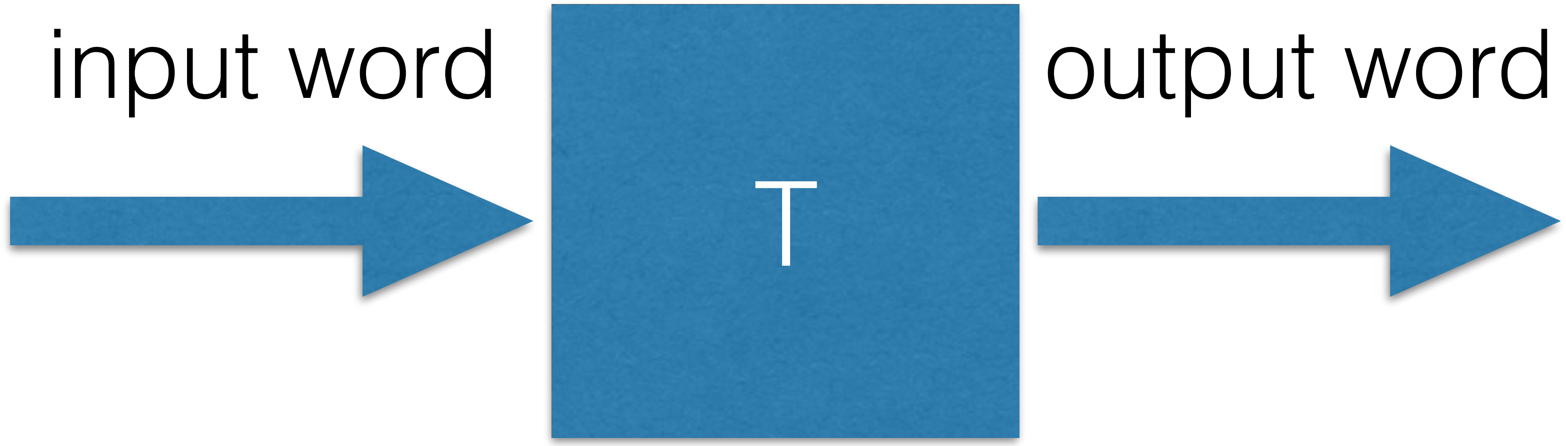

Pattern matching/replacement

Tree/Graph rewriting

Update of XML databases

Existing models over words

Functions {

- Two-way Deterministic Finite-State Transducers
- Functional One-way Finite-State Transducers
- MSOT (à la Courcelle)
- Copyless Streaming String Transducers (Alur et al)

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→ only finite valued relations…

Transduction as weights

- Desire: weight transitions with words… Difficult to equip A* with a semiring structure: how to combine several accepting runs?

- Works for deterministic or unambiguous automata: functional transducers

- For relations: semiring of languages

$$(2^{A^*}, \cup, \cdot, \varnothing, \{\varepsilon\})$$

Examples

$$\prod_x (P_x(a)?\{aa\} : (P_x(b)?\{bb\} : \varnothing))$$

Examples

$\prod_x (P_x(a)?\{aa\} : (P_x(b)?\{bb\} : \varnothing))$ *aba* -> *aabbaa*

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$$\sum_y P_y(\star)?\left[\prod_x (x = y)?\{insert\} : (P_x(\star)?\{\varepsilon\} : (P_x(a)?\{a\} : (P_x(b)?\{b\})))\right]$$

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*a*b*a* -> {*ainsertba*,*abinserta*}

Relation

Examples

$$\prod_x P_x(a)?\{a\}:(P_x(b)?\{\varepsilon\}) \times \prod_x P_x(a)?\{\varepsilon\}:(P_x(b):\{c\})$$

Examples

$$\prod_x P_x(a)?\{a\}:(P_x(b)?\{\varepsilon\})\times\prod_x P_x(a)?\{\varepsilon\}:(P_x(b):\{c\})$$

ababbaabb -> *aaaaccccc*

Examples

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Not comp. by 1-way Func Transducer

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aba -> {*ε,a,b,ab,ba,aa,aba*}

Examples

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$$\prod_x P_x(a)?A^* a A^* :(P_x(b)?A^* b A^*)$$

Examples

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$$\prod_x P_x(a)?\{a,\varepsilon\}:(P_x(b)?\{b,\varepsilon\})$$

aba

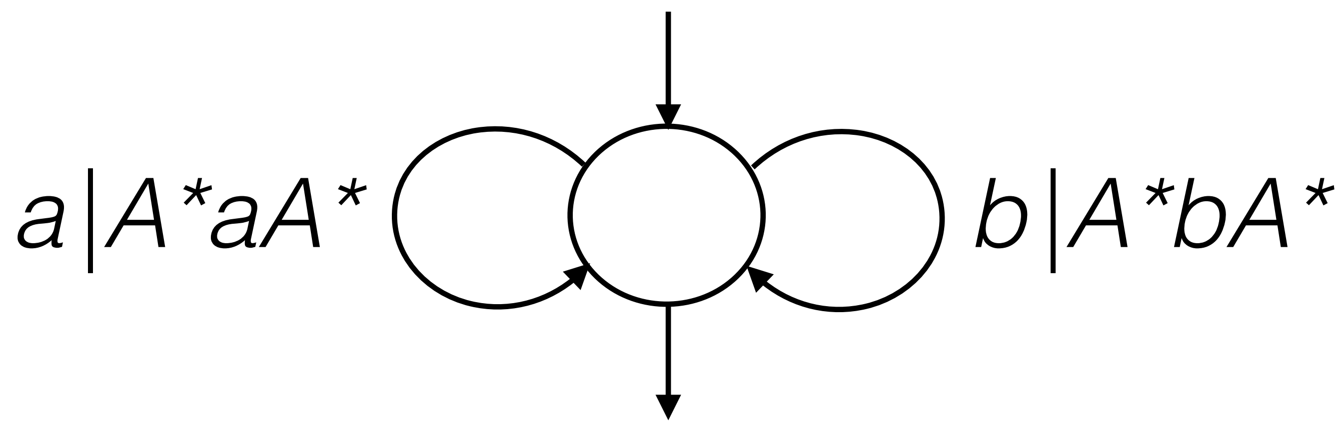
Infinitely-valued relation

$$\prod_x P_x(a)?A^*aA^*:(P_x(b)?A^*bA^*)$$

aba -> *A*aA*bA*aA**

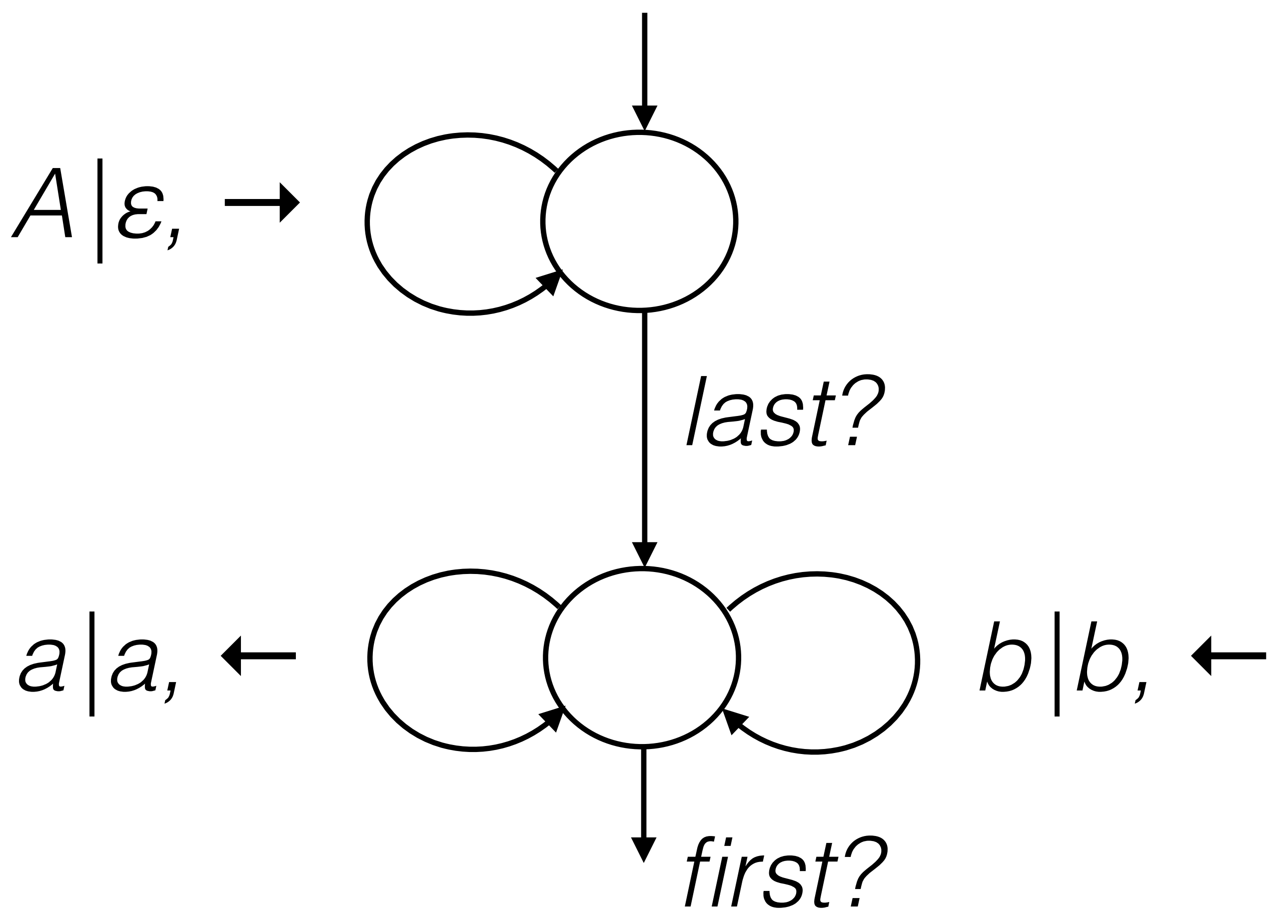
Transducers

$$\prod_x P_x(a)?\,A^*aA^* : (P_x(b)?\,A^*bA^*)$$



Infinite-valued, but <u>deterministic</u>

Reverse?



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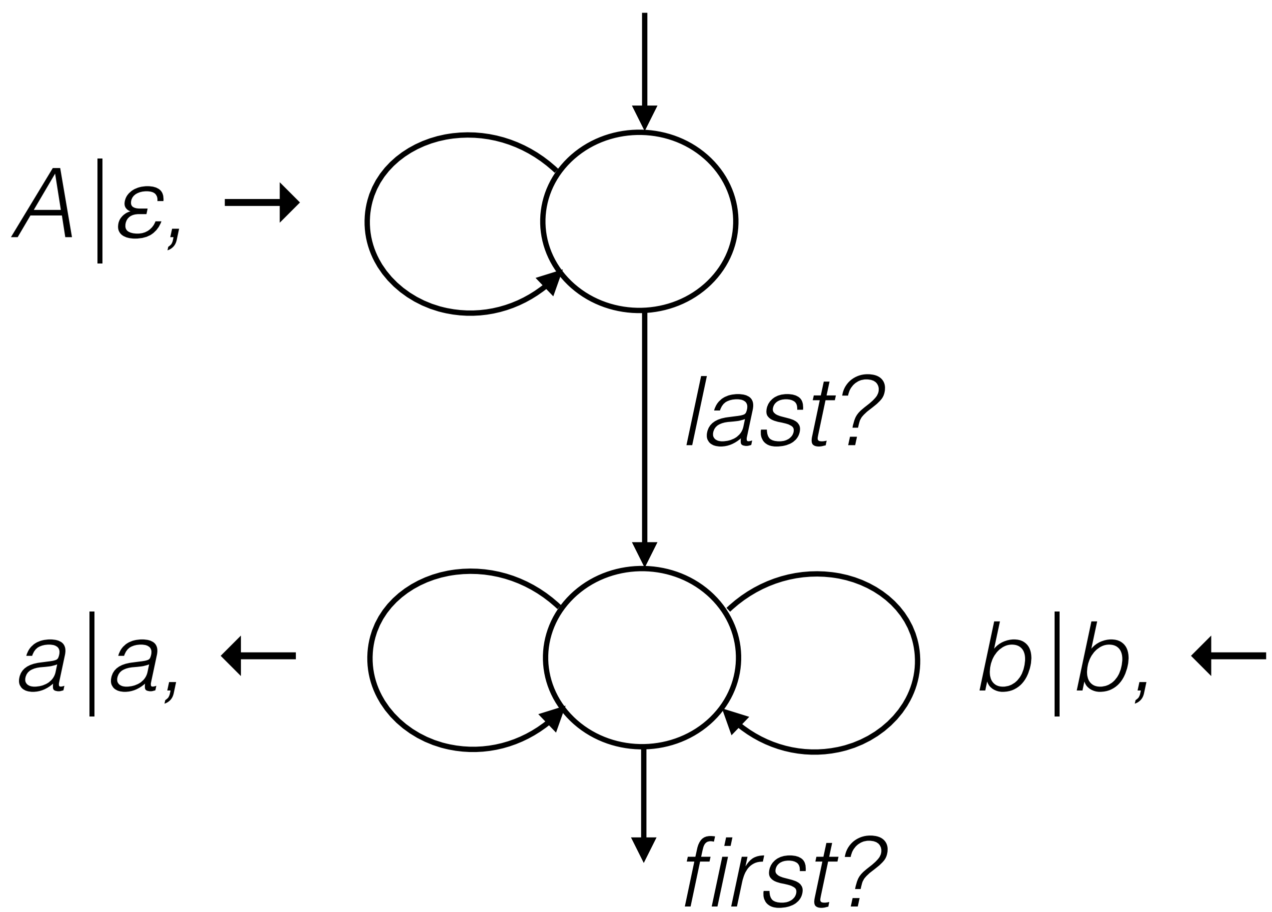

Impossible in FO…
… because of order of interpretation of product

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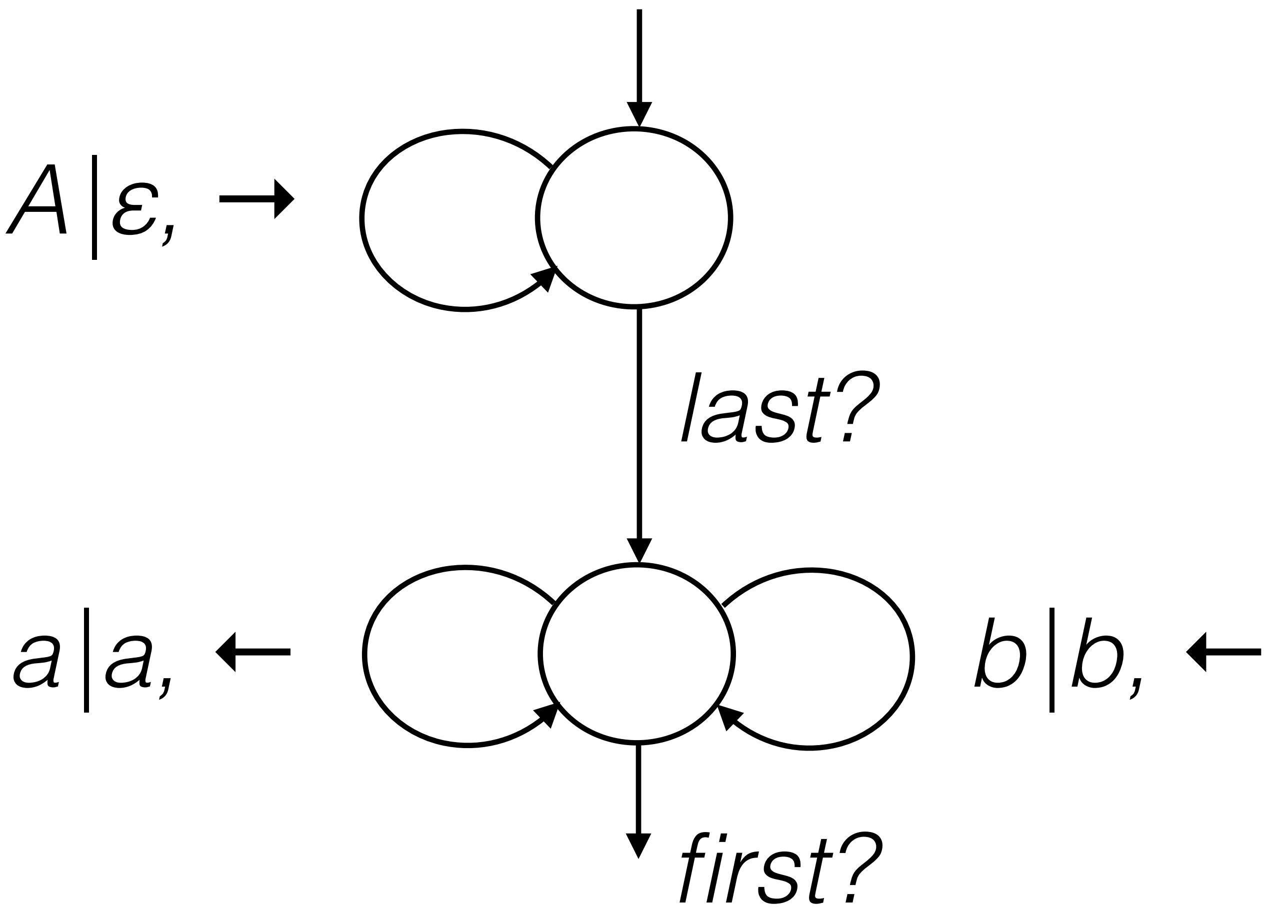

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Solution: in this non-commutative setting,
add right-to-left products

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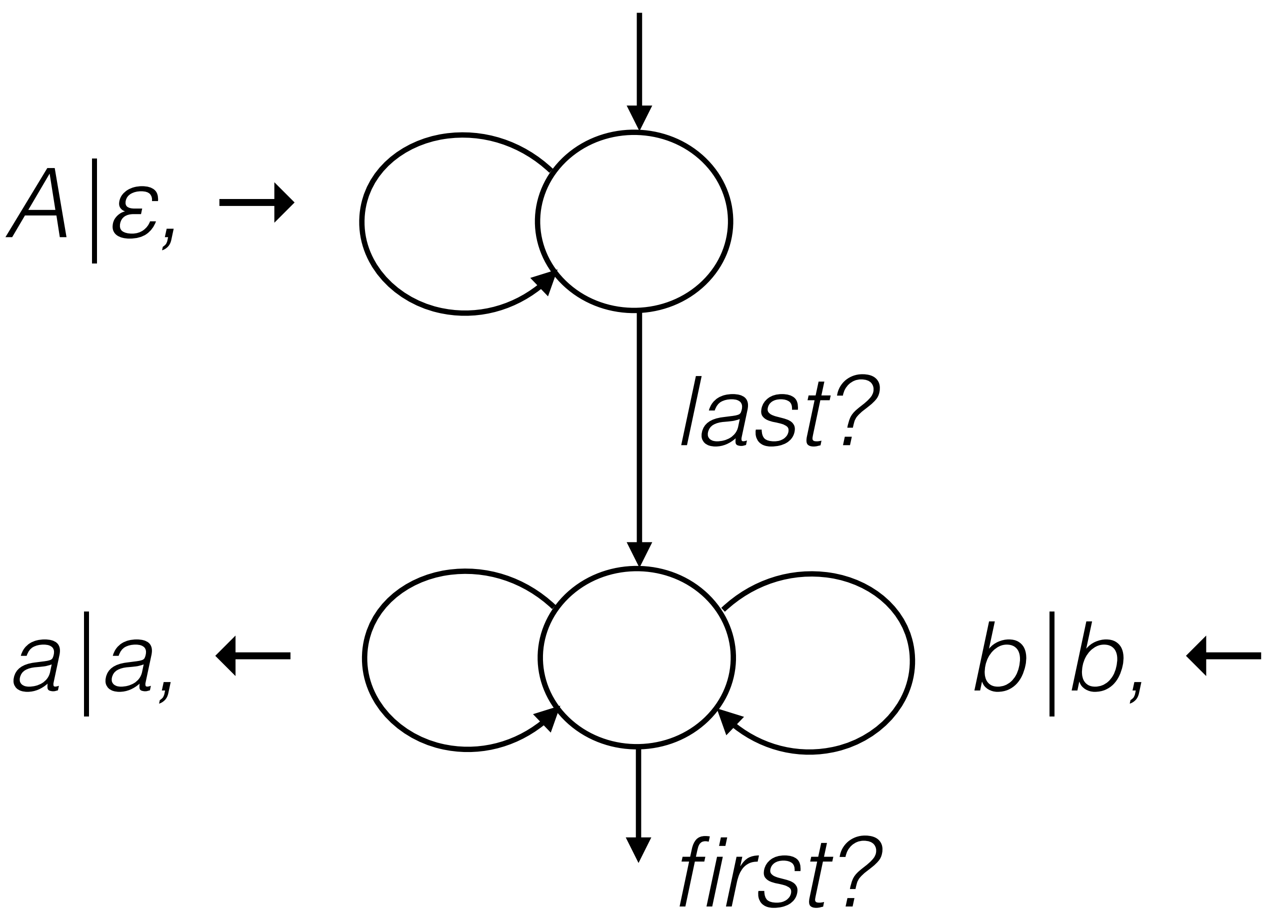

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Transitive closure

$$\varphi ::= \top \mid P_a(x) \mid x \leqslant y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \forall x\, \varphi$$

$$\Phi ::= L \mid \varphi ? \Phi : \Phi \mid \Phi + \Phi \mid \Phi \times \Phi \mid \sum_x \Phi \mid \prod_x \Phi \mid N\text{-}TC_{x,y} \Phi$$

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Regular language

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Regular language

Able to define right-to-left product

$$\coprod_x \Phi(x) \coloneqq [1\text{-}TC_{x,y}(y = x - 1 \,?\, \Phi(x))](\mathit{last}, \mathit{first}) \times \Phi(\mathit{first})$$

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Able to define right-to-left product

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Theorem: Pebble Transducers = FO + bounded-TC

with regular language productions

linear transformation from logic to transducers

Algorithmic questions

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Theorem: Evaluation of FO + bounded-TC with complexity $O(|\text{formula}| \times |\text{input}|^{\#\text{variables}})$

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Thank you!