Optimal Reachability in Divergent Weighted Timed Games

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Motivation: quantitative aspects of real-time synthesis

$$\boxed{\text{Environment}} \quad \| \quad \boxed{\text{Controller??}} \quad \models \quad \text{Spec}$$

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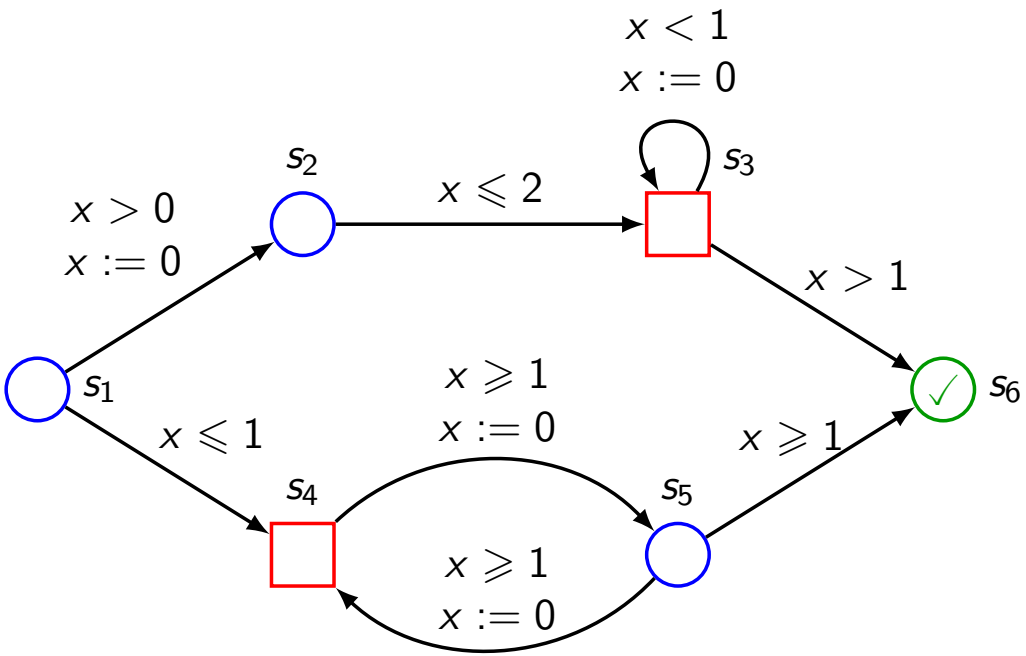
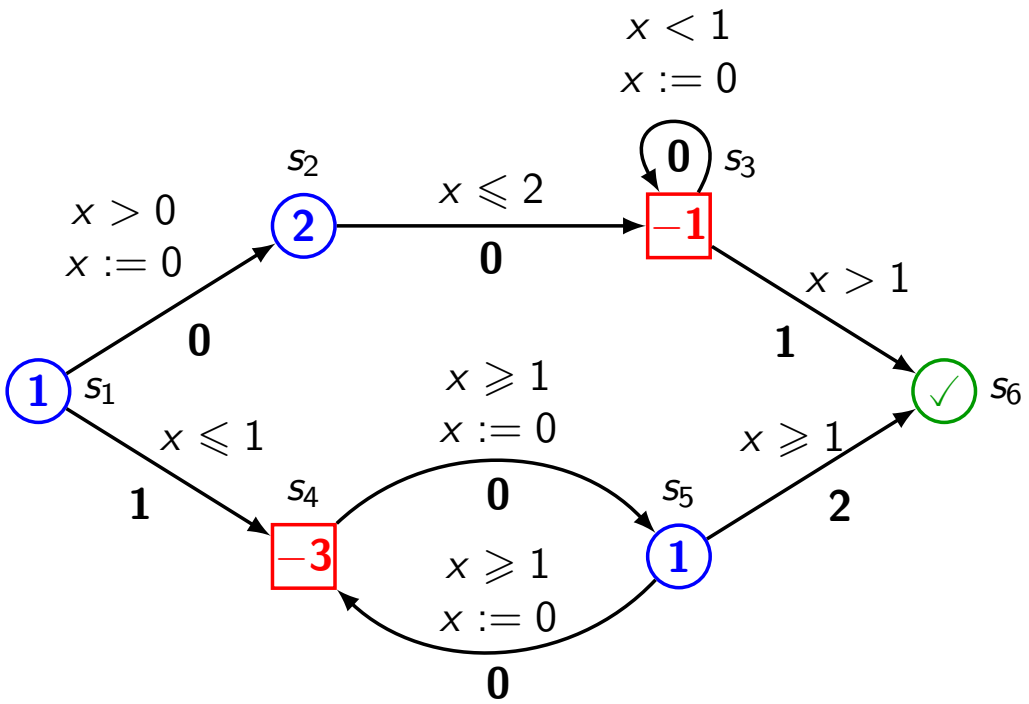
Main motivation since my postdoc here: **negative weights**
$\Longrightarrow$ model, e.g., production/consumption of resources

Weighted timed games



Timed automaton
with state partition between
2 players
+ reachability objective

Weighted timed games



Timed automaton
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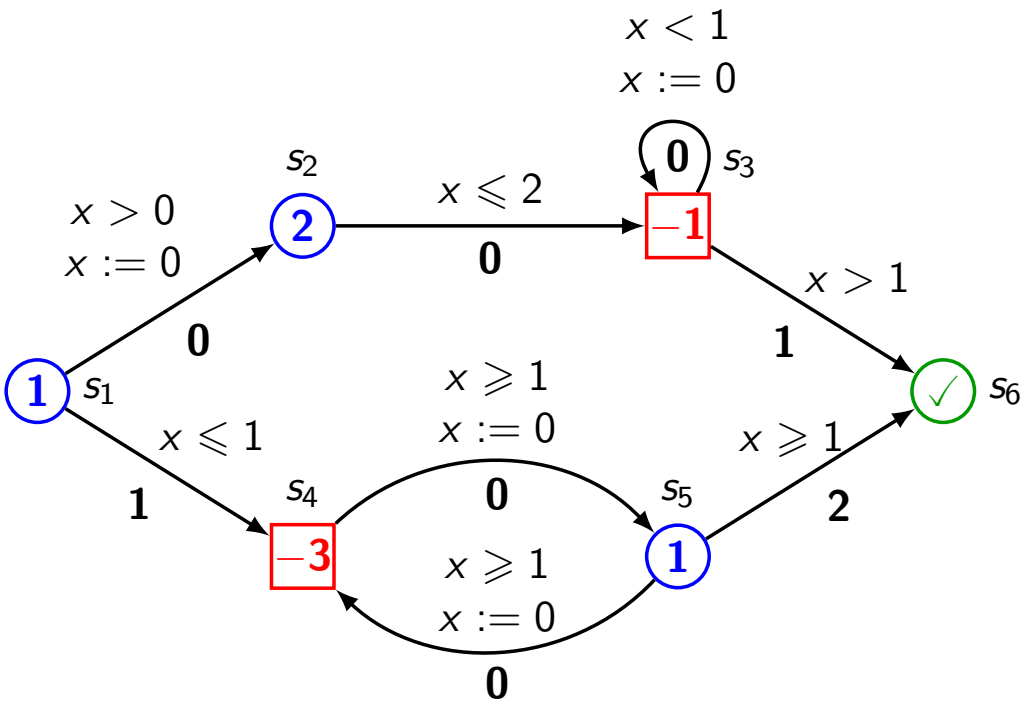

$(s_1, 0)$

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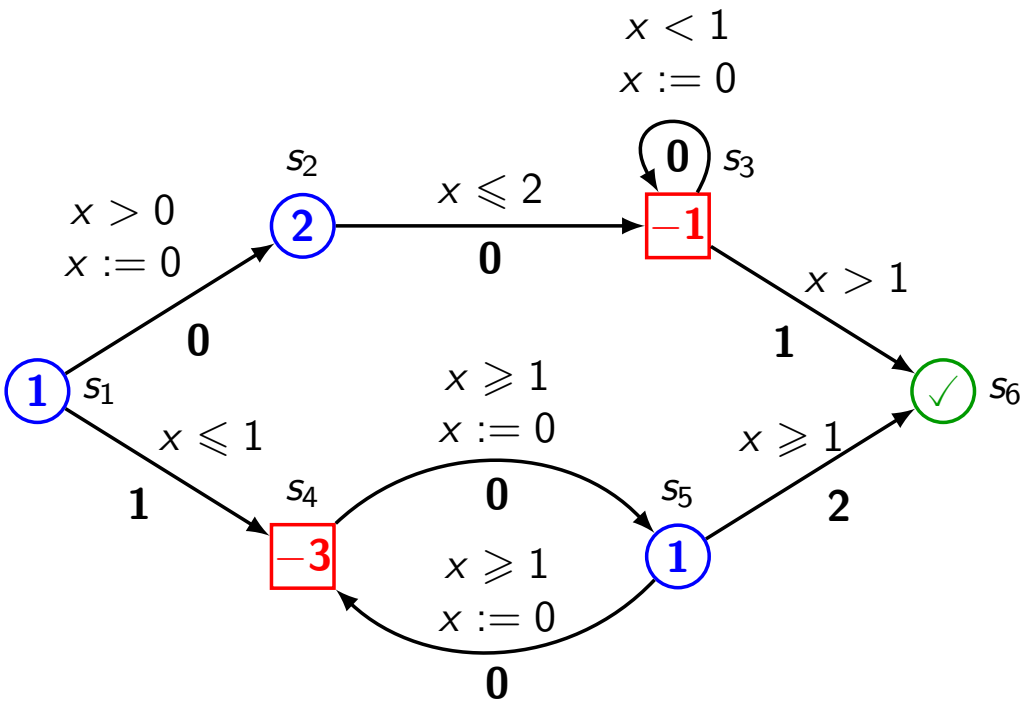

$(s_1, 0) \xrightarrow{0.4, \searrow} (s_4, 0.4)$

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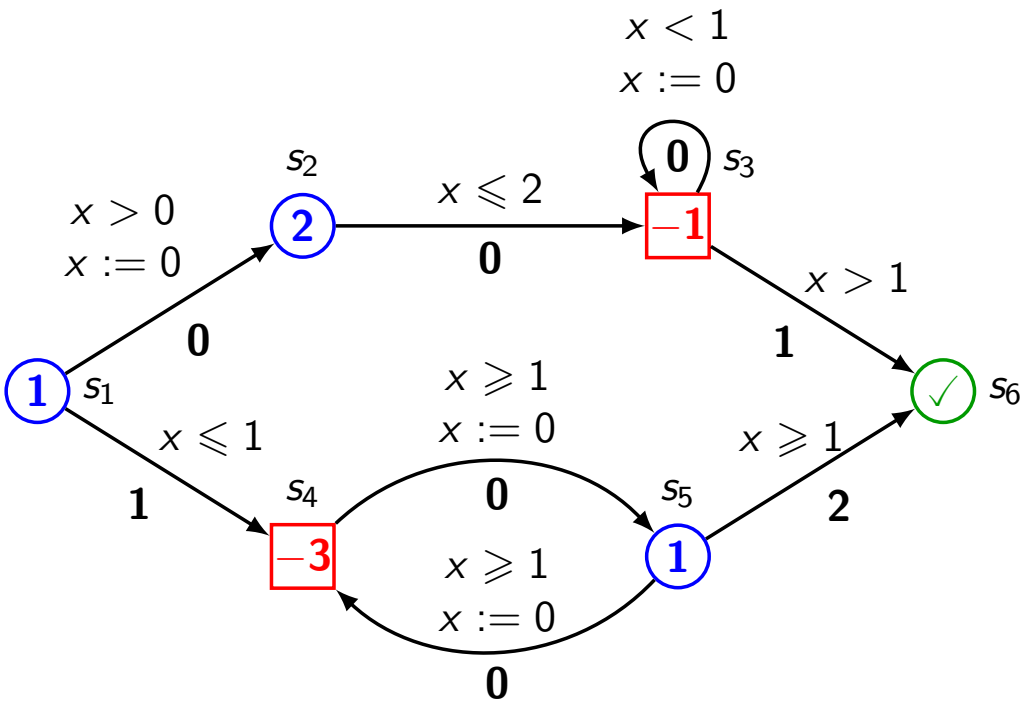

$$(s_1, 0) \xrightarrow{0.4, \searrow} (s_4, 0.4) \xrightarrow{0.6, \rightarrow} (s_5, 0)$$

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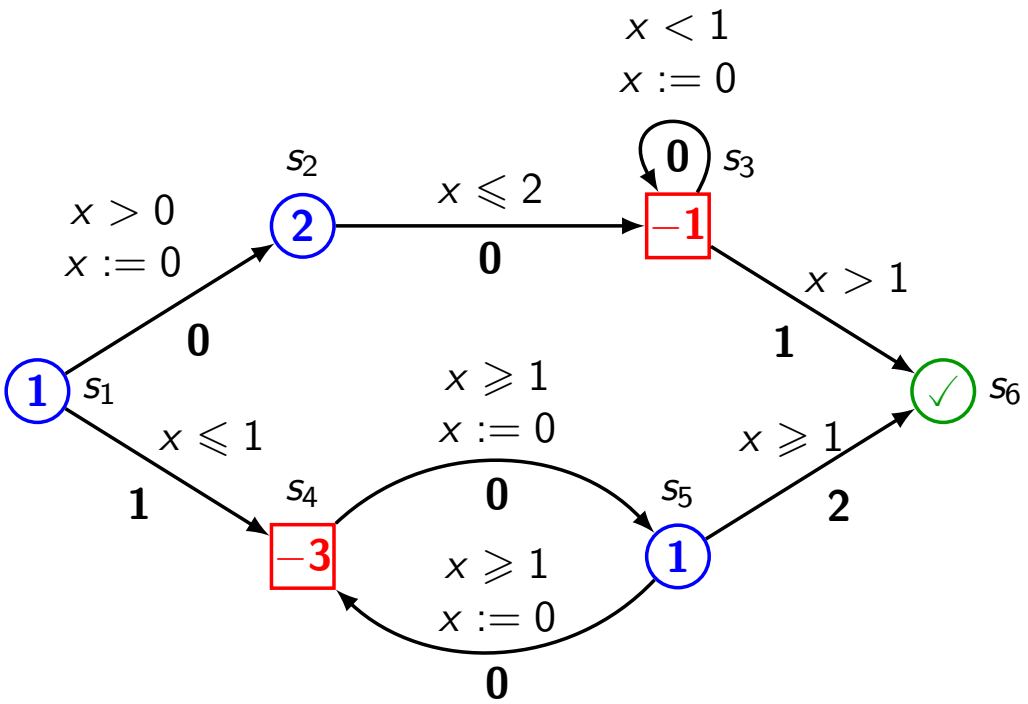

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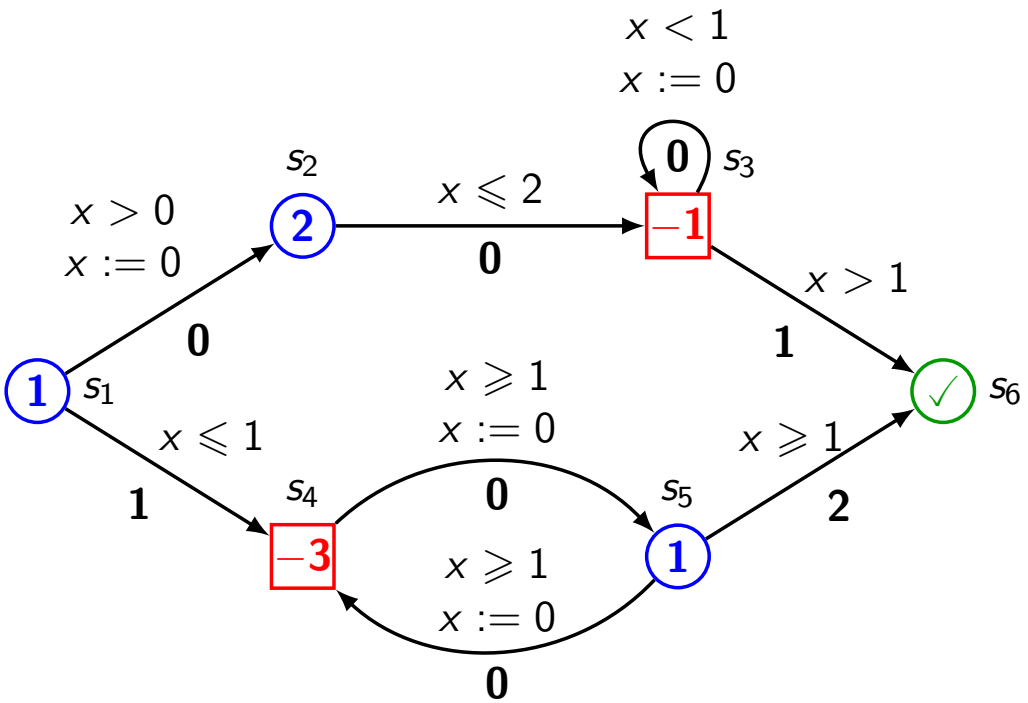
Weighted timed games



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$$(s_1,0) \xrightarrow[\mathbf{1}\times 0.4+\mathbf{1}]{0.4,\searrow} (s_4,0.4) \xrightarrow[-\mathbf{3}\times 0.6+\mathbf{0}]{0.6,\rightarrow} (s_5,0) \xrightarrow[+\mathbf{1}\times 1.5+\mathbf{0}]{1.5,\leftarrow} (s_4,0) \xrightarrow[-\mathbf{3}\times 1.1+\mathbf{0}]{1.1,\rightarrow} (s_5,0) \xrightarrow[+\mathbf{1}\times 2+\mathbf{2}]{2,\nearrow} (\checkmark,2) \qquad = 1.8$$

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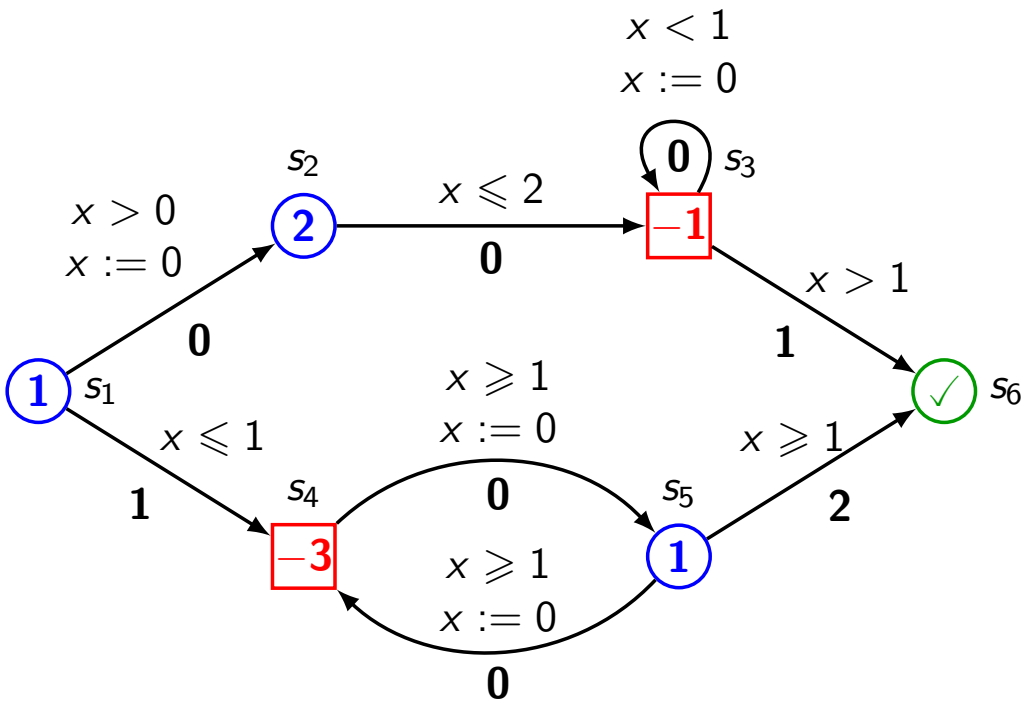

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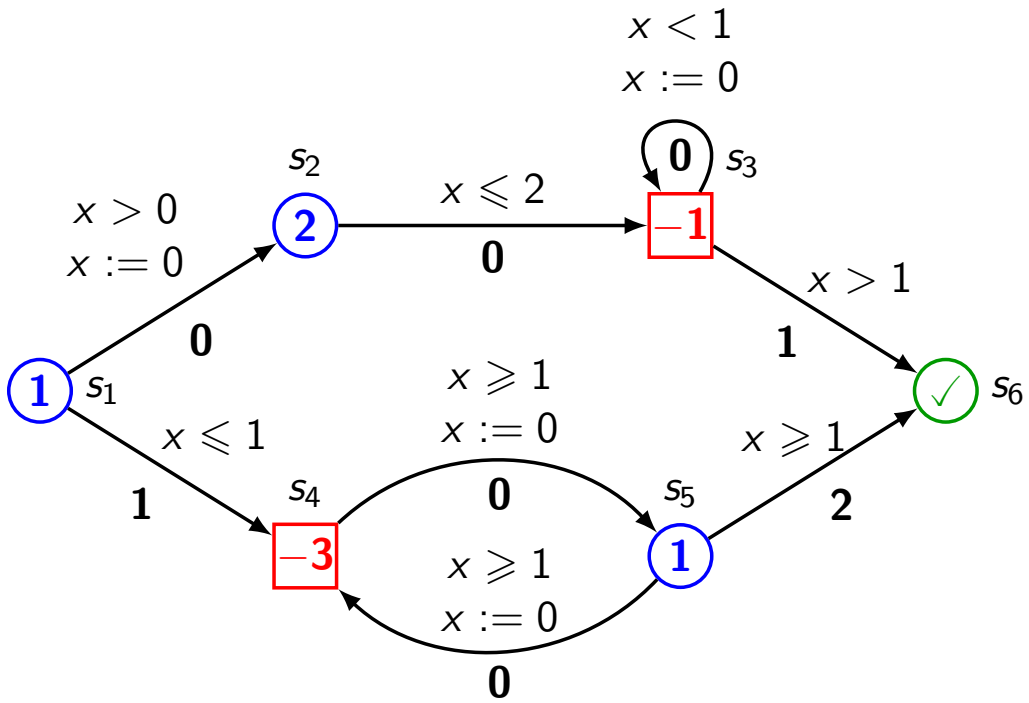
$$(s_1,0) \xrightarrow[\mathbf{1}\times 0.2+\mathbf{0}]{0.2,\nearrow} (s_2,0) \xrightarrow[+\mathbf{2}\times 0.9+\mathbf{0}]{0.9,\rightarrow} (s_3,0.9) \xrightarrow[-\mathbf{1}\times 0.2+\mathbf{0}]{0.2,\circlearrowright} (s_3,0) \xrightarrow[-\mathbf{1}\times 0.9+\mathbf{0}]{0.9,\circlearrowright} (s_3,0) \quad \cdots \quad \cdots \qquad = +\infty$$

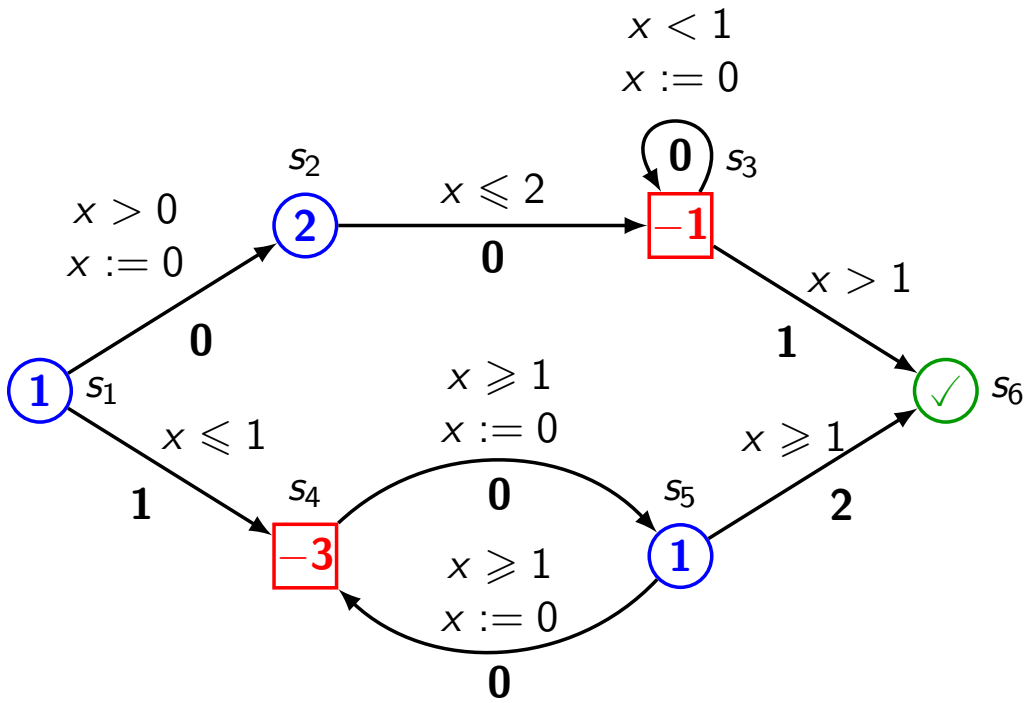
Weight of an execution : $\begin{cases} +\infty & \text{if } \checkmark \text{ not reached} \\ \text{total weight until } \checkmark & \text{otherwise} \end{cases}$

Strategies and objectives



Strategy for a player: map finite executions to a delay and a transition

Strategies and objectives



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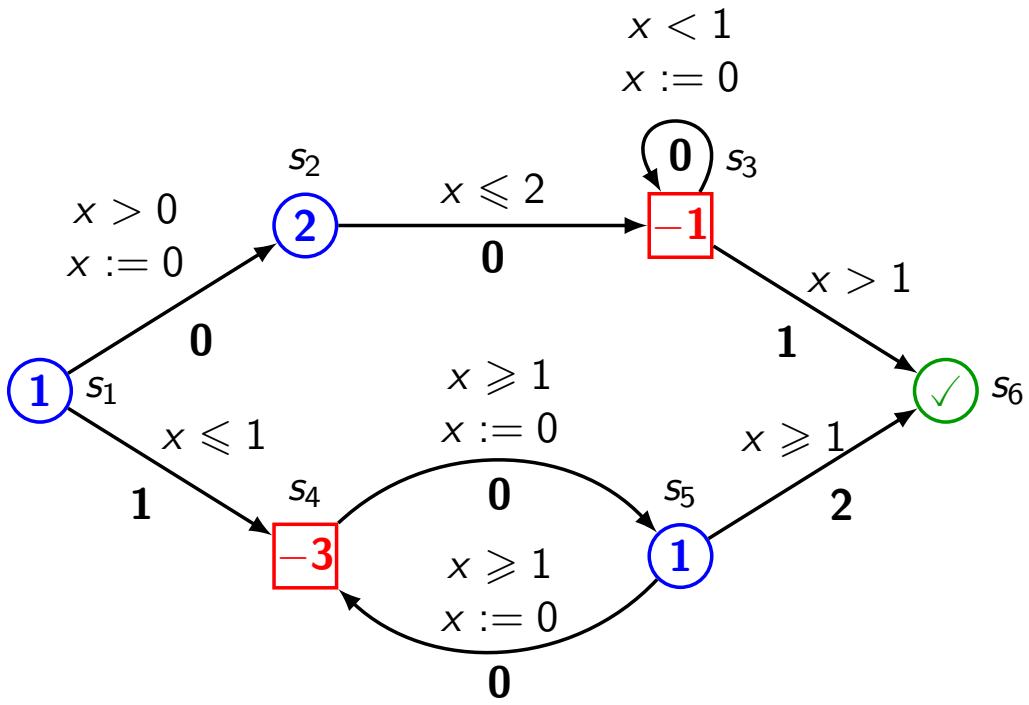

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Main object of interest:
$\mathrm{Val}(\ell, \nu) = \inf_{\sigma_{\mathrm{Min}} \in \mathrm{Strat}^{\mathrm{Min}}} \sup_{\sigma_{\mathrm{Max}} \in \mathrm{Strat}^{\mathrm{Max}}} \mathrm{Weight}(\mathrm{Exec}(\ell, \nu, \sigma_{\mathrm{Min}}, \sigma_{\mathrm{Max}})) \in \overline{\mathbb{R}}$
What weight can players guarantee? Find (almost) optimal strategy?

The real motivation!

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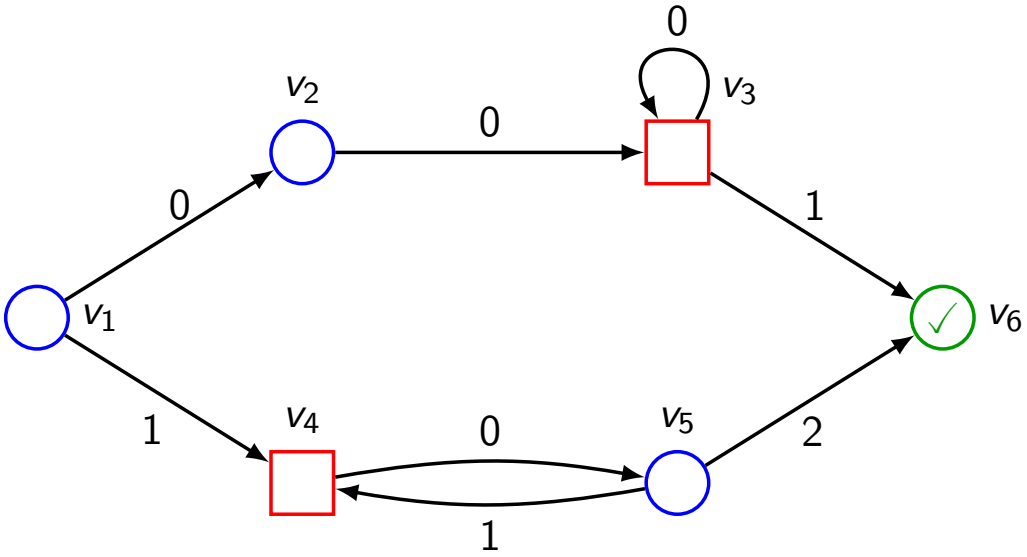
Missing: decidable fragment of games without constraints on number of clocks, and with negative weights

A special case first: Weighted (untimed) games



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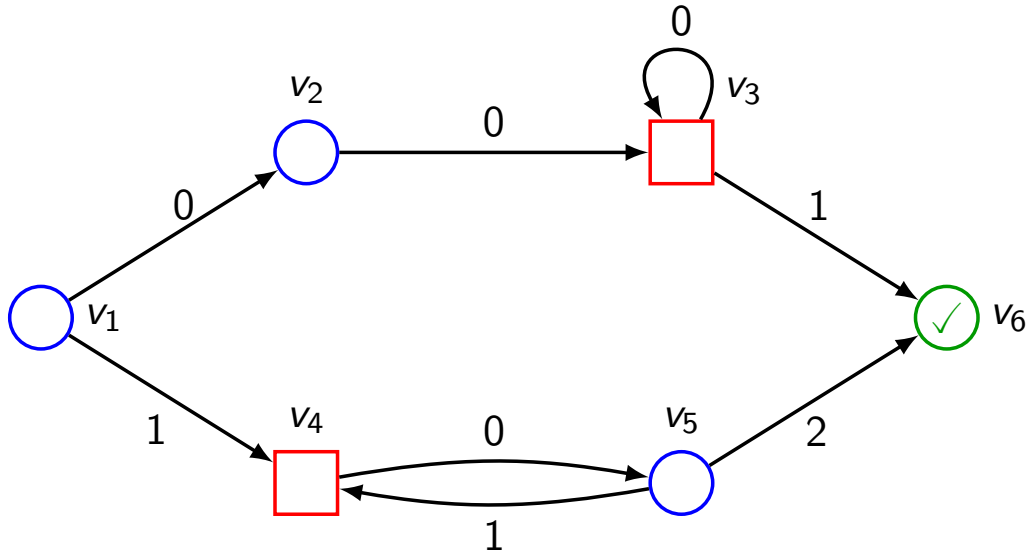

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$$v_1 \searrow v_4 \rightarrow v_5 \leftarrow v_4 \rightarrow v_5 \nearrow \checkmark$$
$$1 \qquad +1 \qquad +2 \qquad = 4$$

$$v_1 \nearrow v_2 \rightarrow v_3 \circlearrowright v_3 \circlearrowright v_3 \quad \cdots$$
$$\cdots \qquad = +\infty \ (\checkmark \text{ not reached})$$

Weight of a path: $\begin{cases} +\infty & \text{if } \checkmark \text{ not reached} \\ \text{total weight until } \checkmark & \text{otherwise} \end{cases}$

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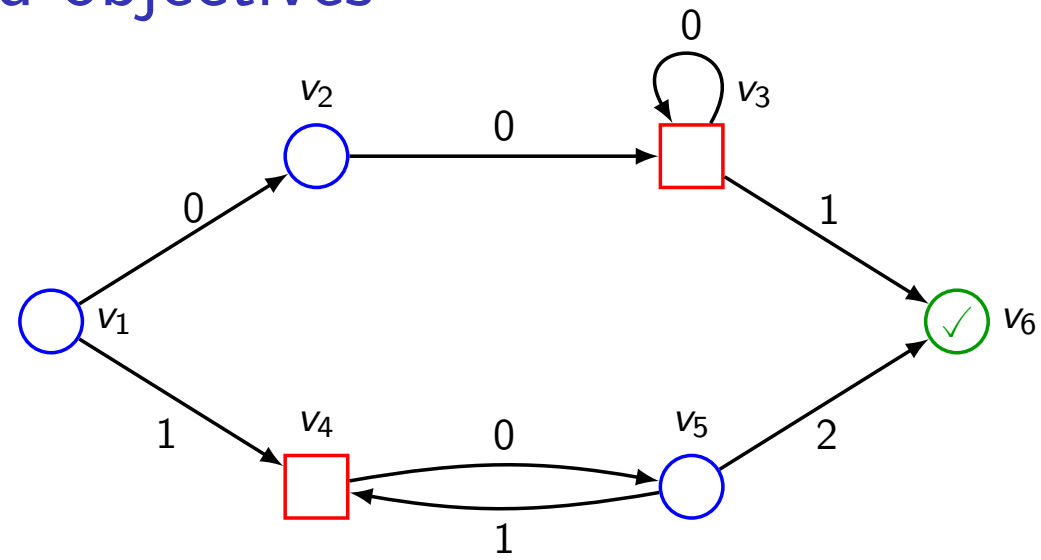

Player ○'s objective: reach ✓ **and** minimise the cost
Player □'s objective: avoid ✓ **or, if not possible,** maximise the weight

Strategy for a player: map finite paths to edges

Main object of interest:
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  "value iteration algorithm"

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- Value $-\infty$: detection is as hard as solving parity games ($\mathbf{NP} \cap \mathbf{co\text{-}NP}$)

Contribution 1: divergent weighted games

Property of the underlying graph:

Divergence

Every cycle have total weight either $\leqslant -1$ or $\geqslant 1$, i.e., no cycles of weight $= 0$.

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Theorem:

We can compute the value of a divergent weighted game in poly. time.

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Divergence

Every cycle have total weight either $\leqslant -1$ or $\geqslant 1$, i.e., no cycles of weight $= 0$.

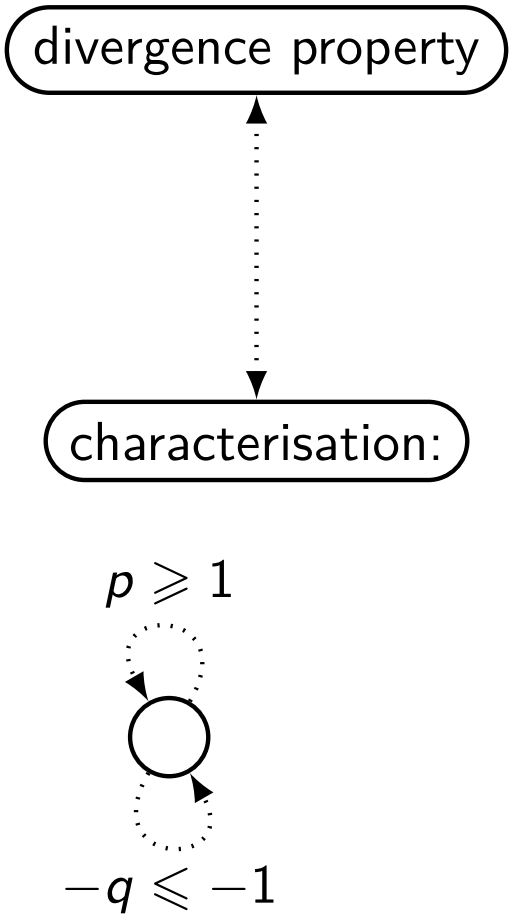
Theorem:

We can compute the value of a divergent weighted game in poly. time.

Theorem:

We can decide in PTIME if a weighted game is divergent.

Divergent weighted games analysis



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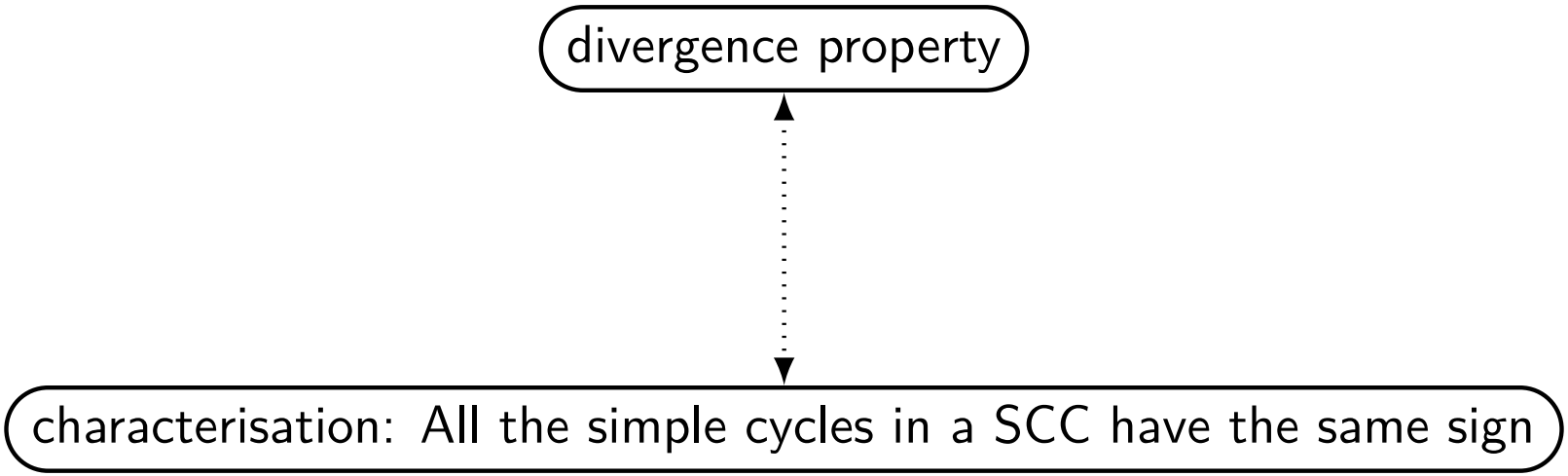

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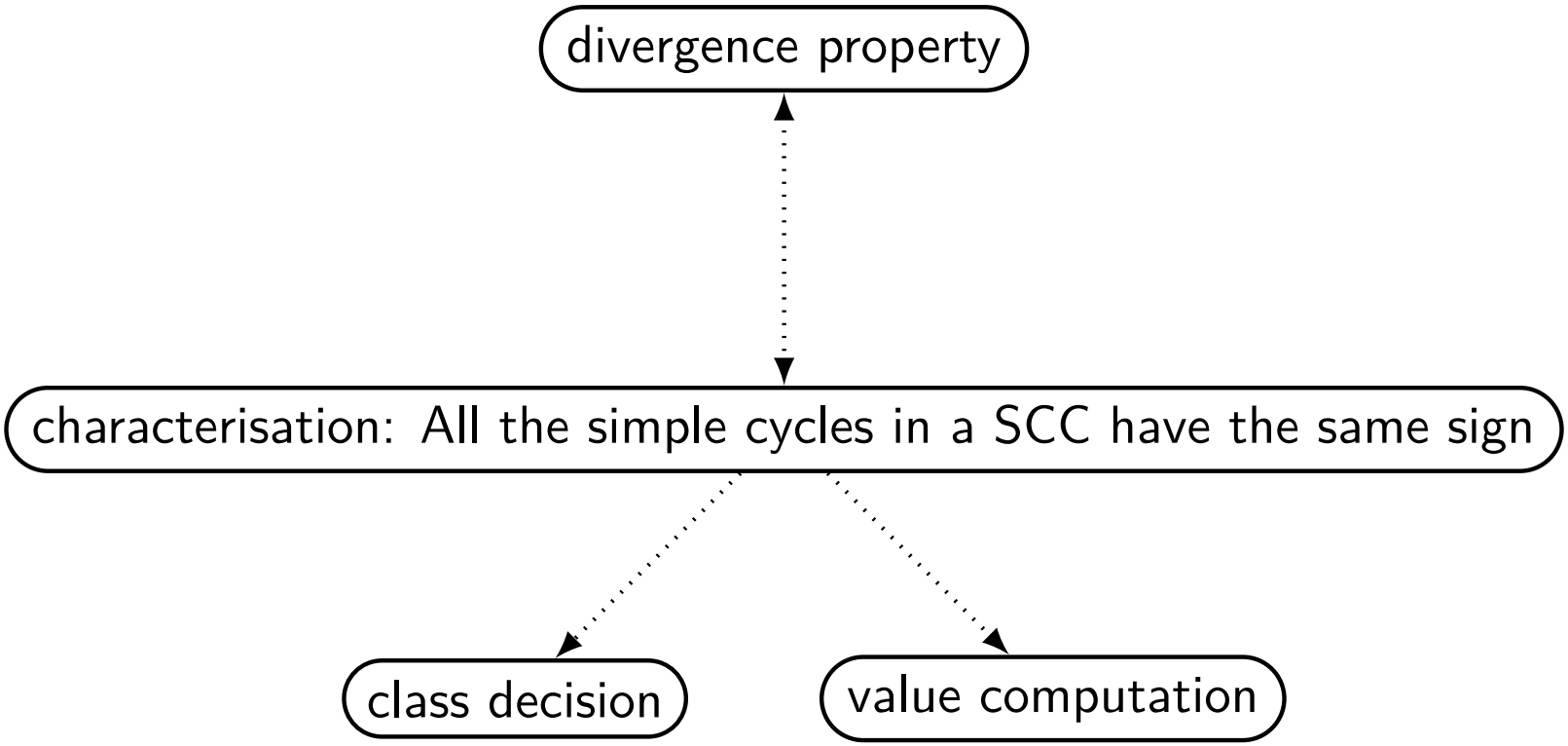

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- Therefore, stabilisation after n steps only

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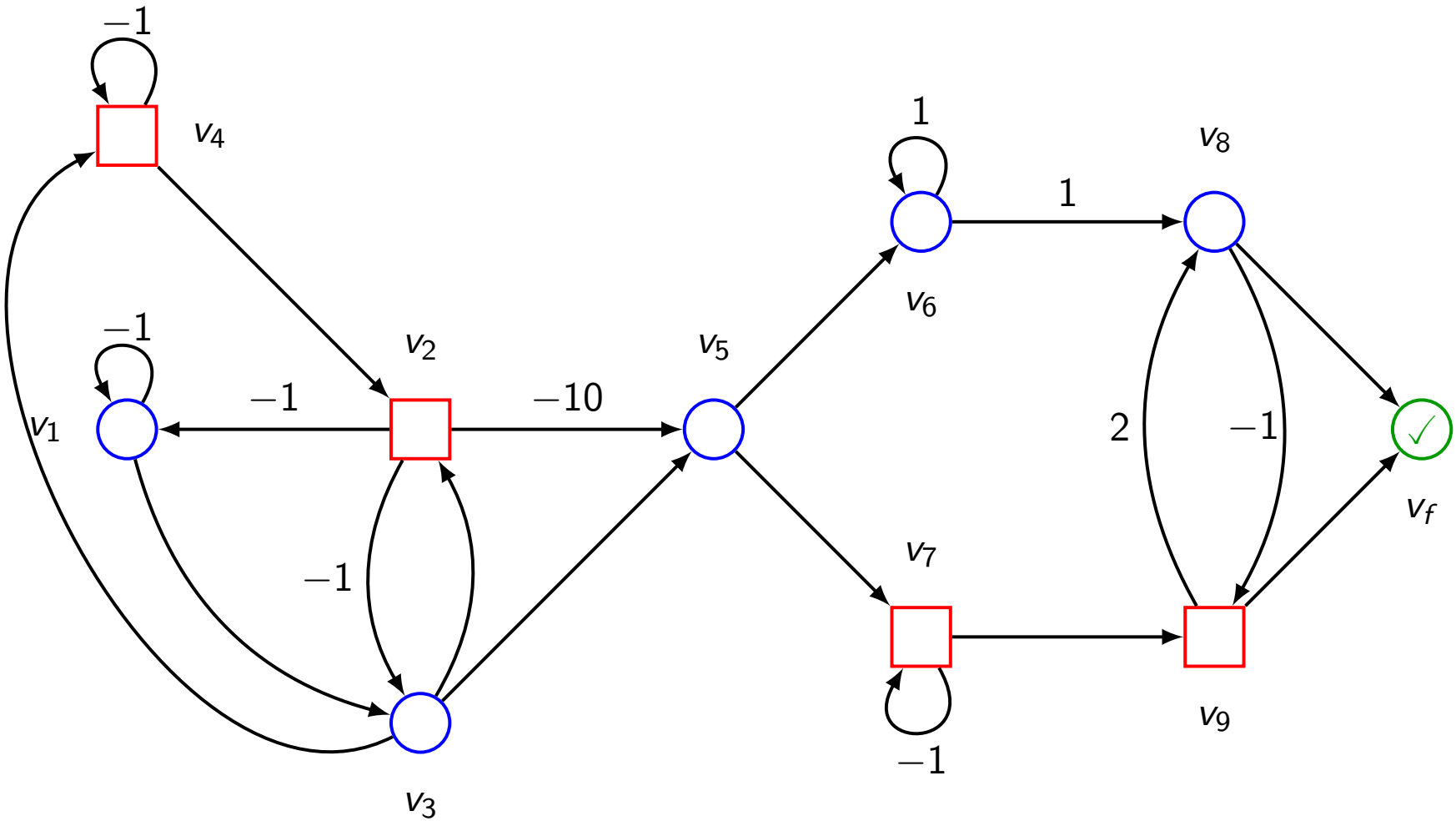
negative SCC

- Outside of the attractor of player □ toward ✓ $\Rightarrow -\infty$
- The "value iteration" algorithm converges in linear time with initialisation at $-\infty$

Example



Min = ○, Max = □

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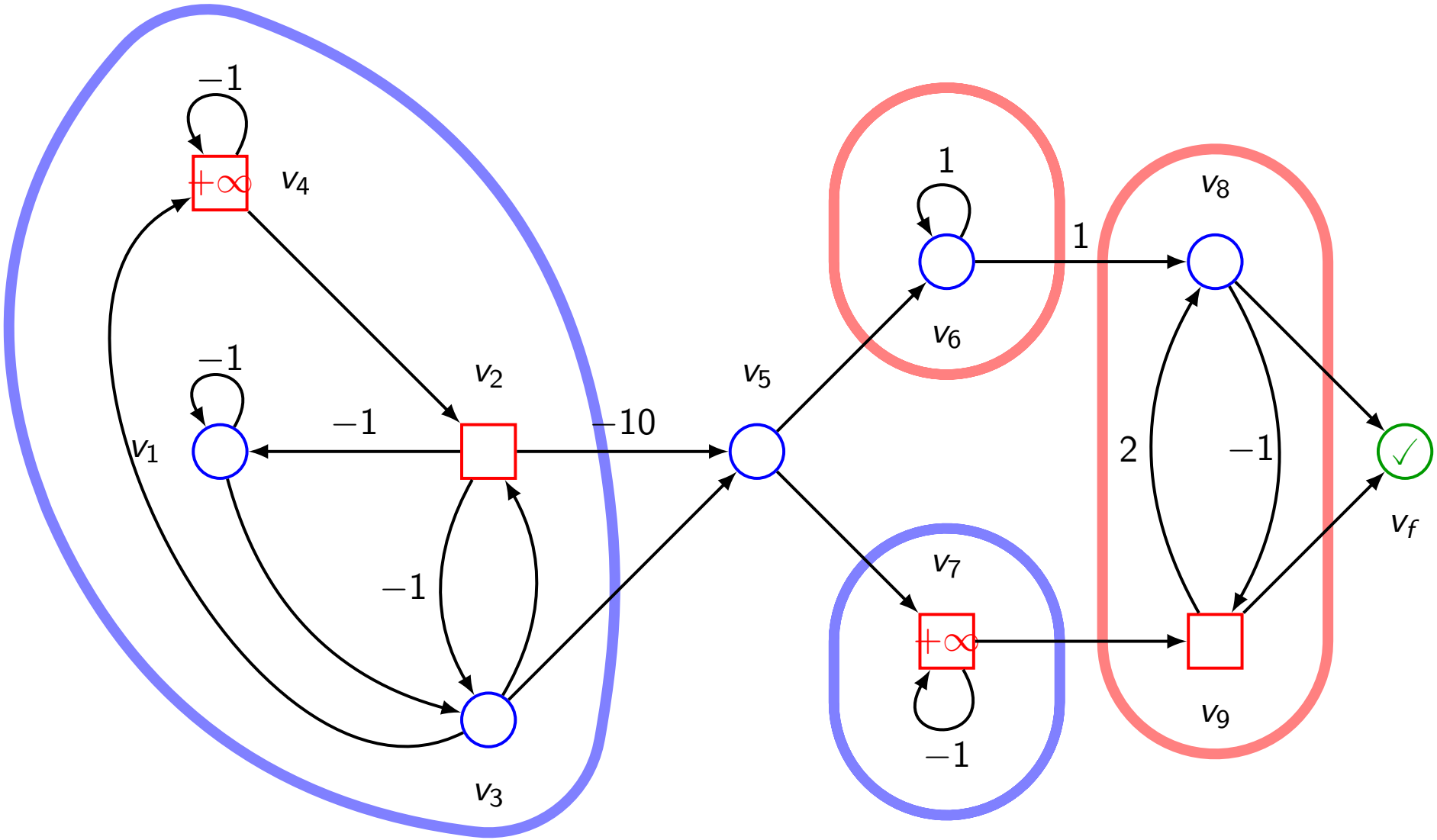

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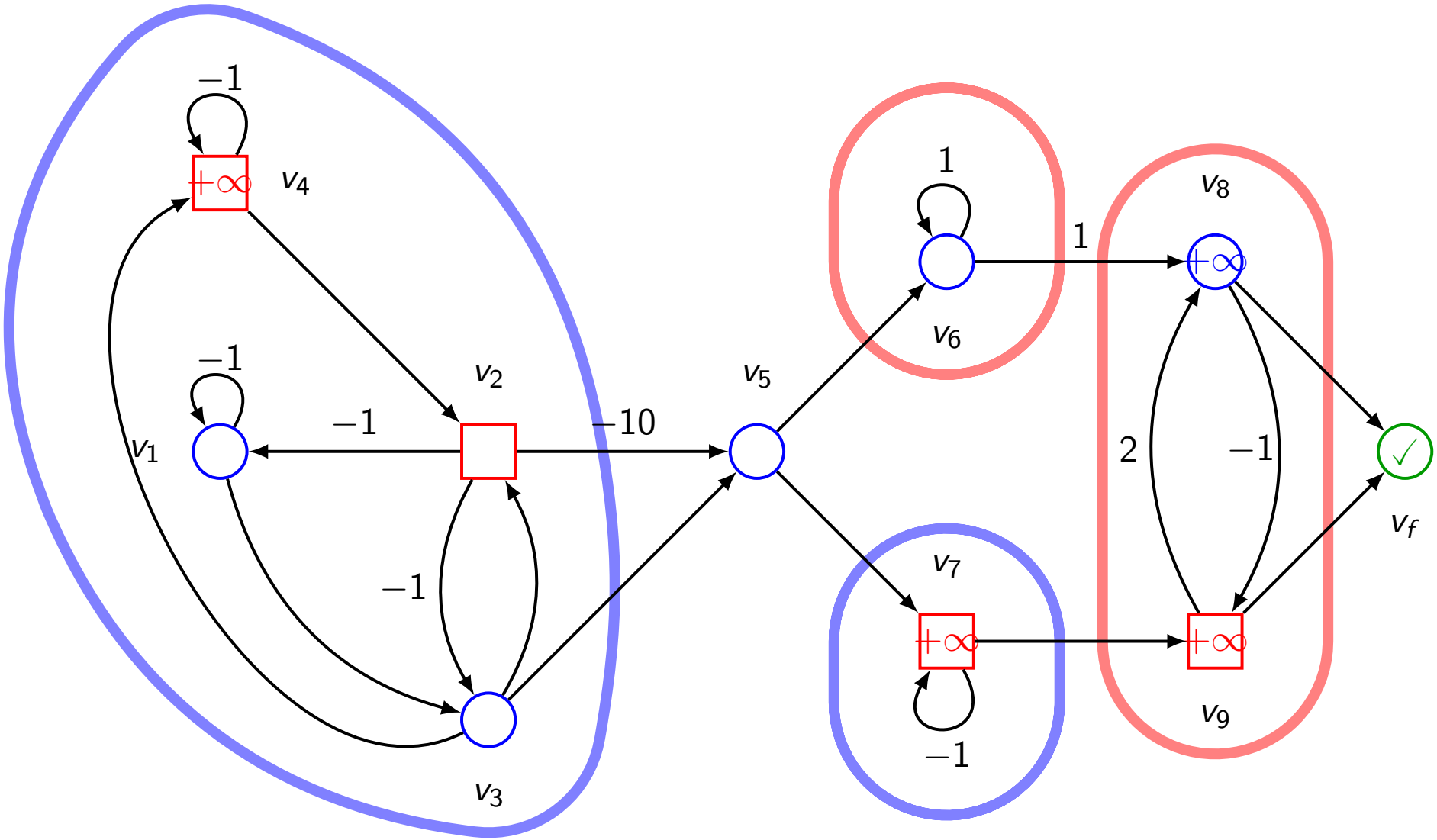

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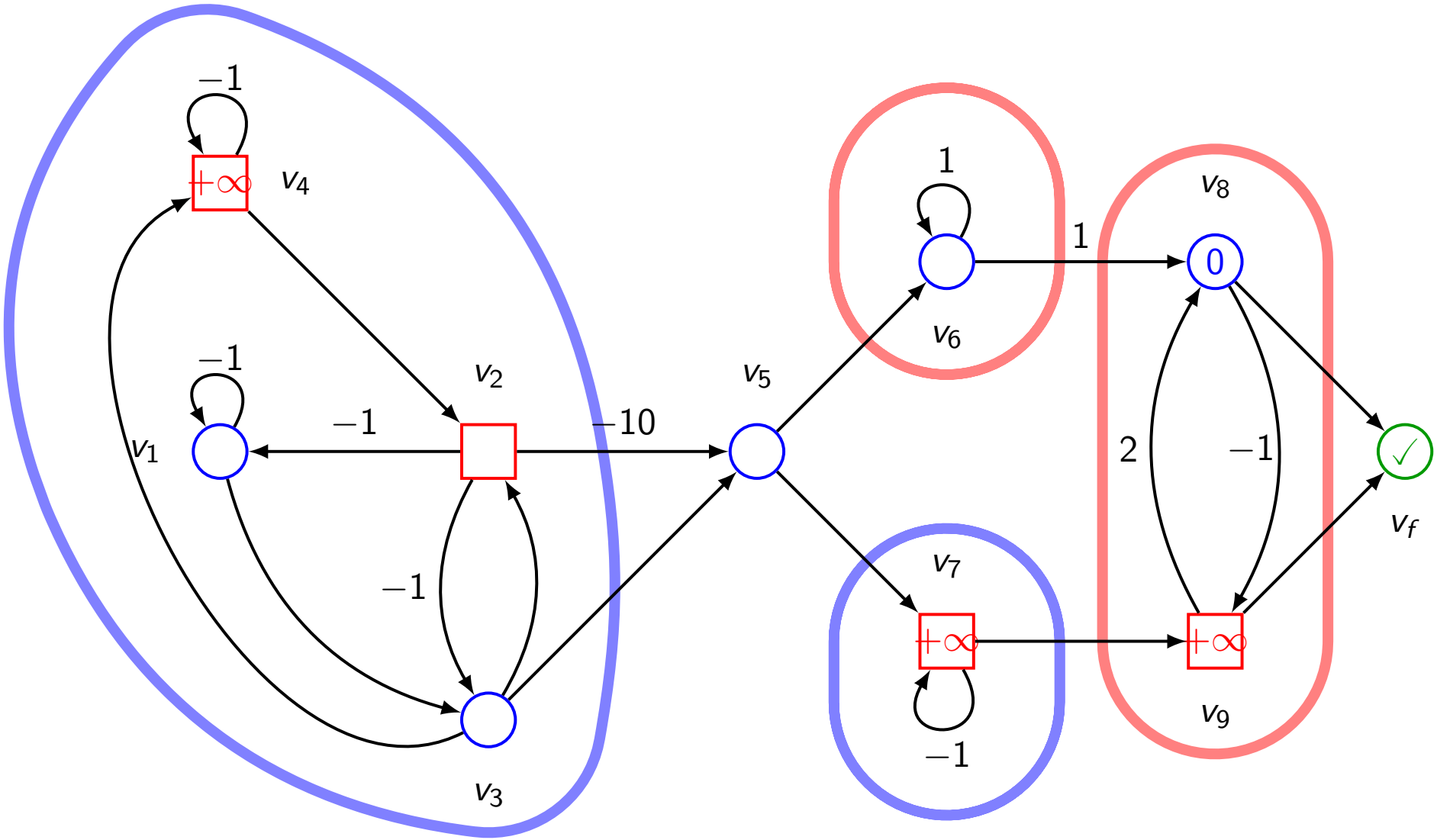

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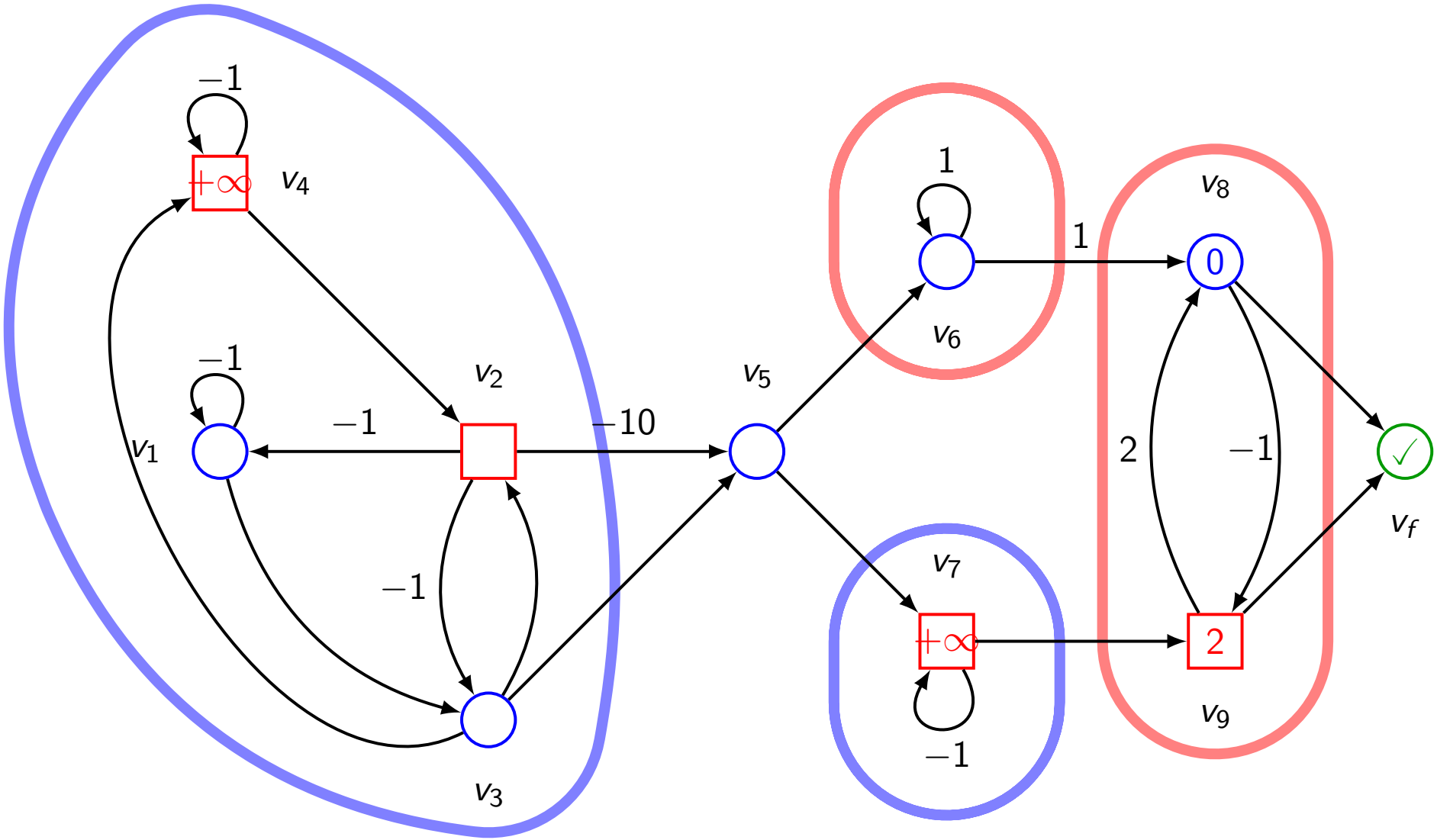

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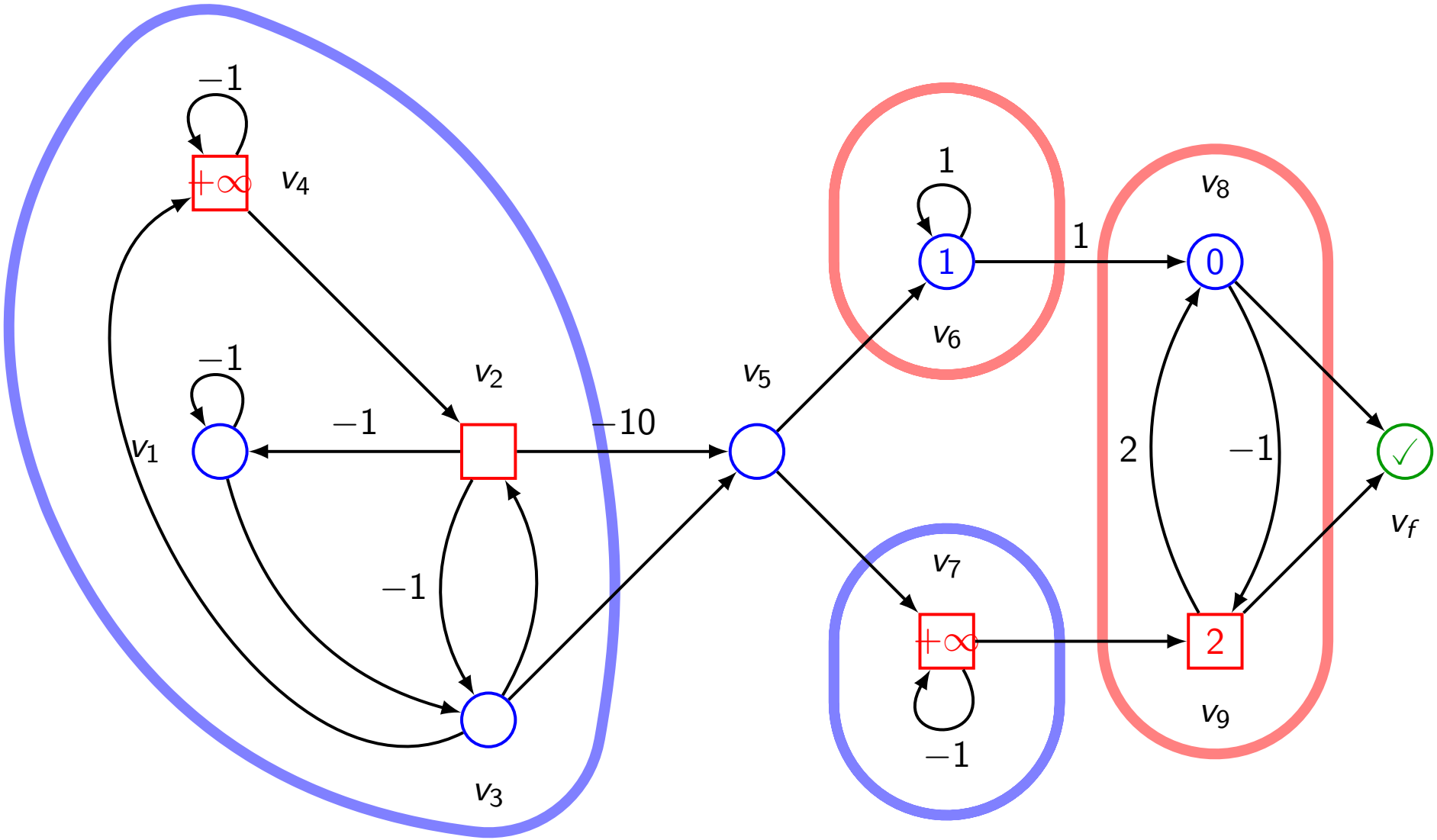

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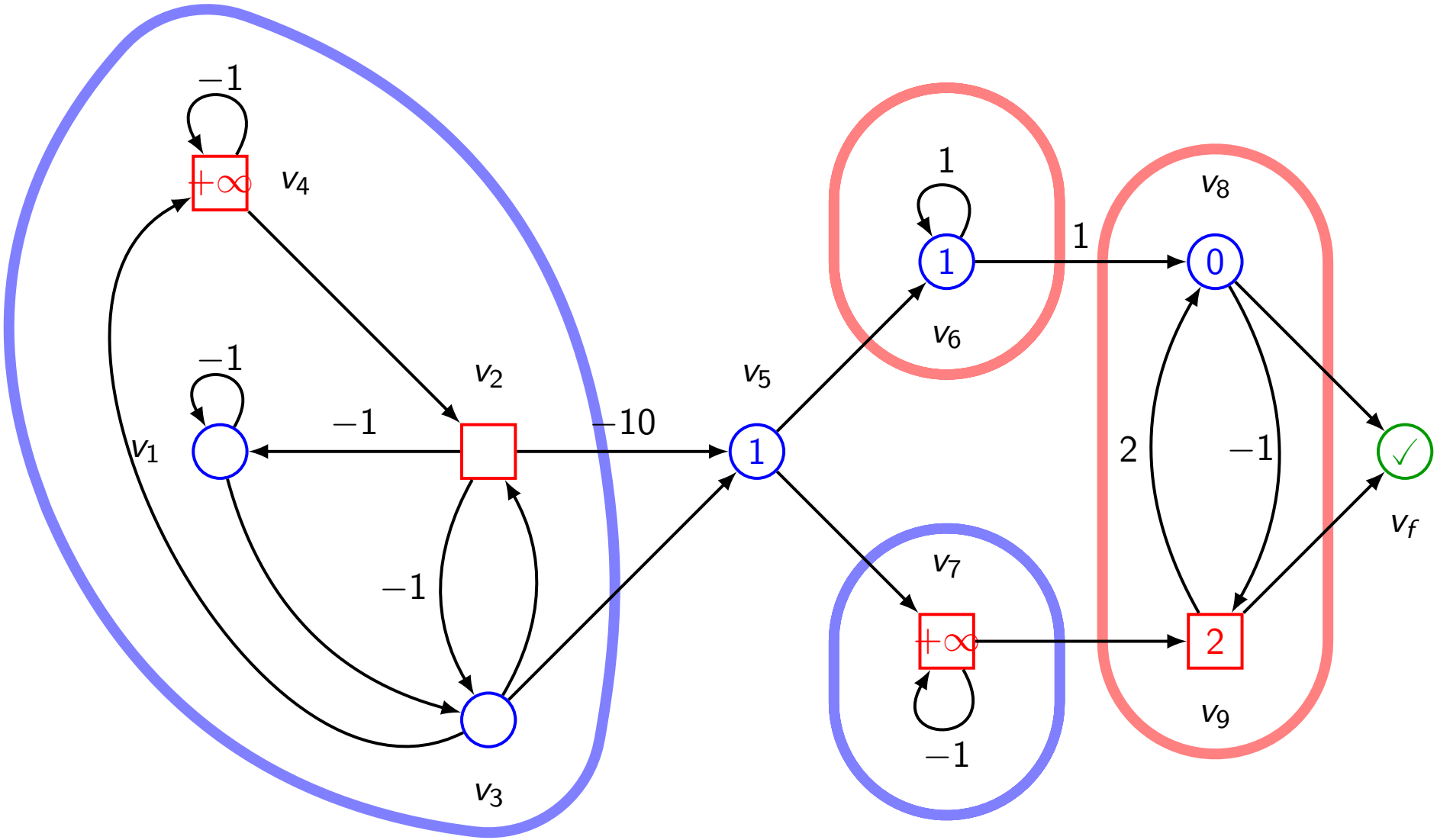

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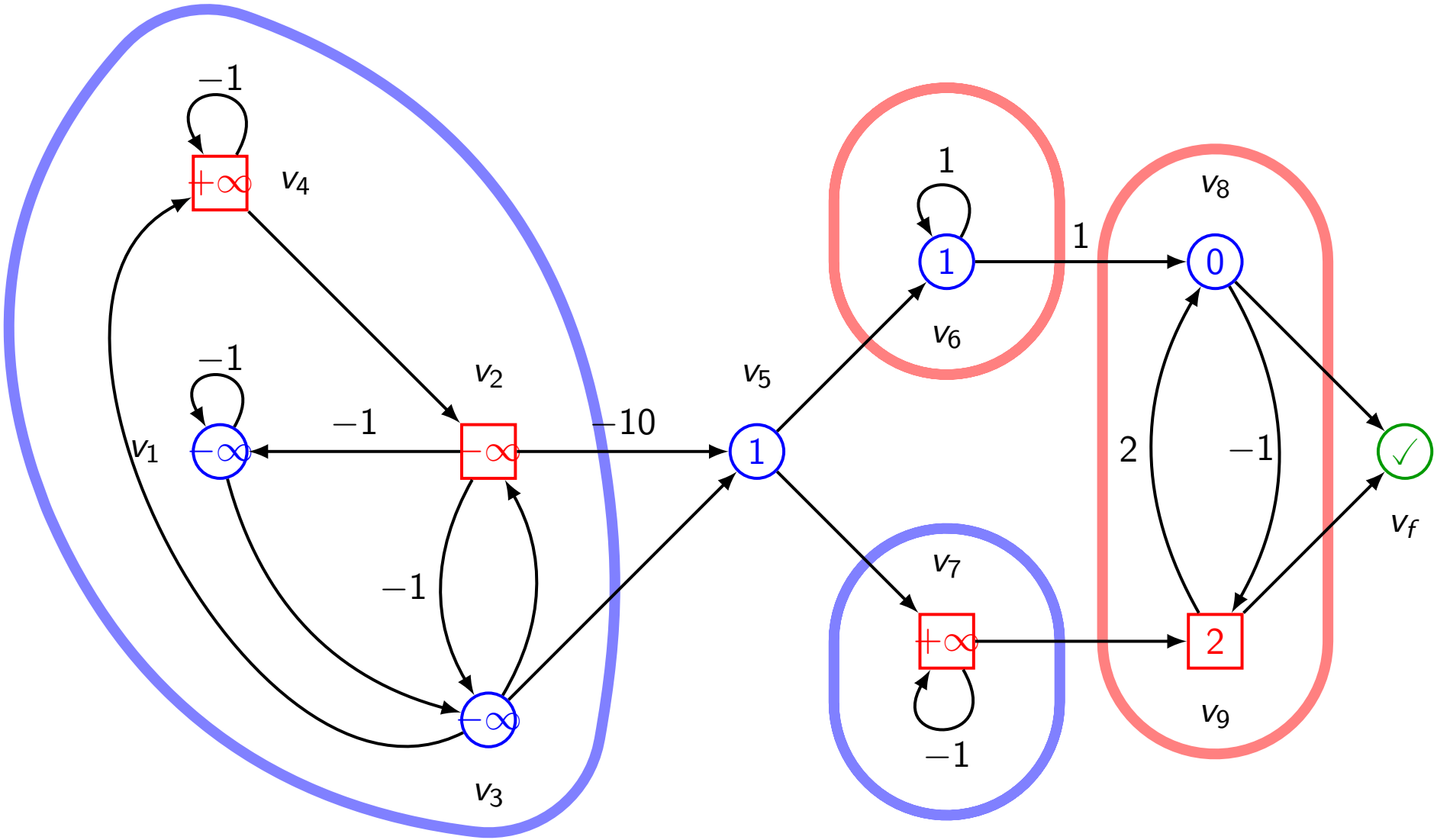

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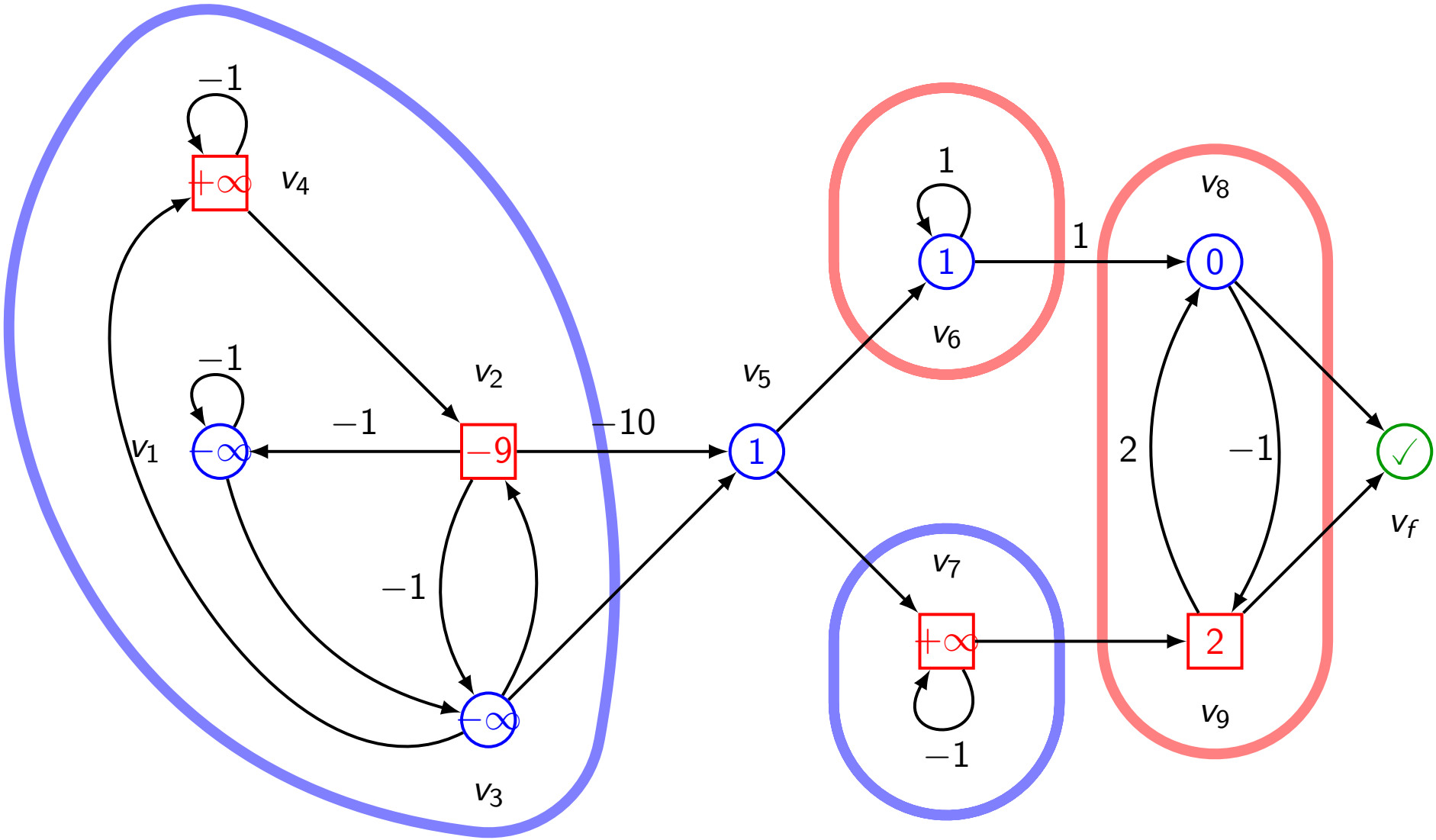

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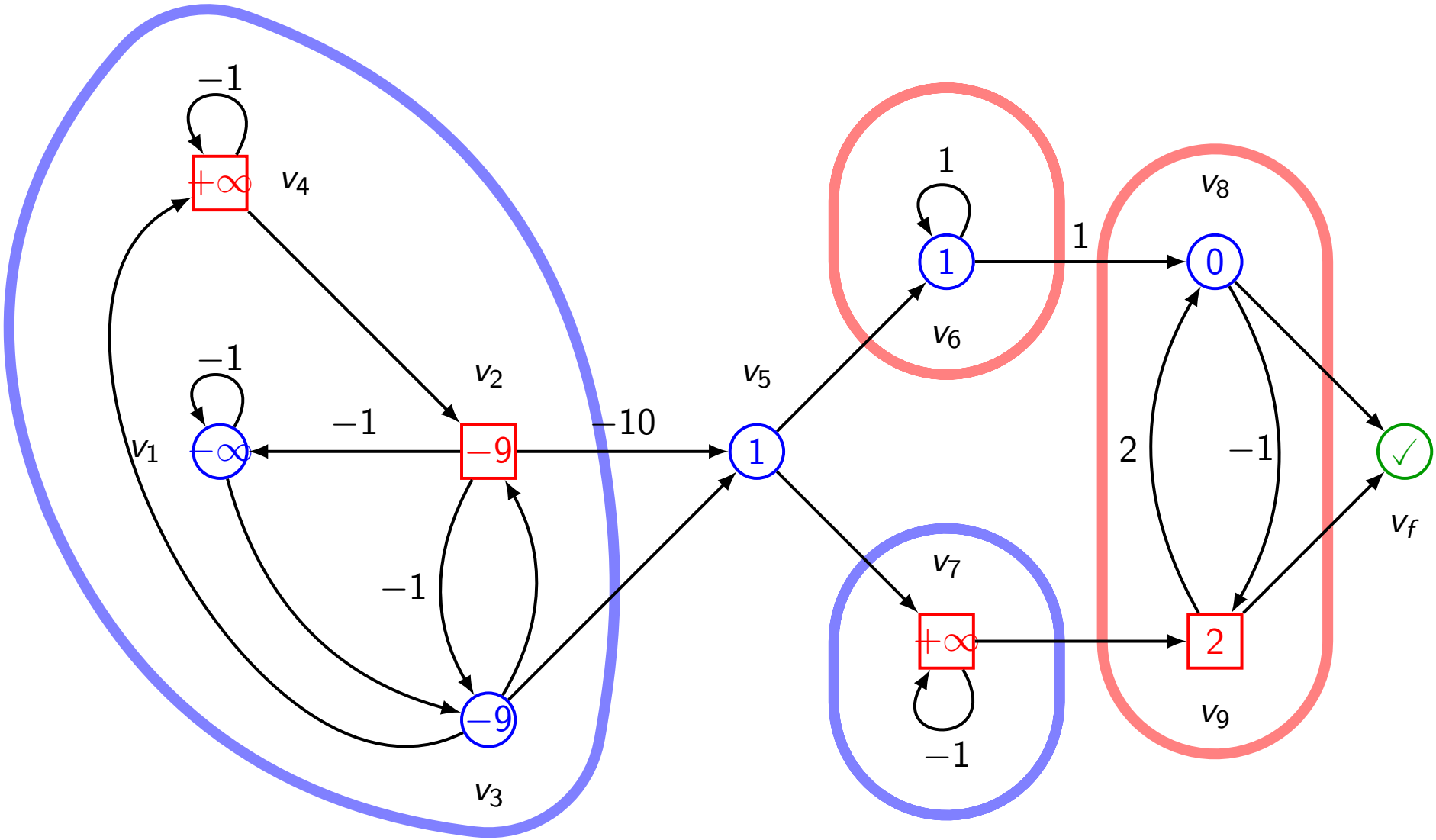

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Contribution 2: Divergent weighted timed games

Property of the underlying timed automaton:

Divergence

Every execution following a cycle of the region automaton has a total weight either $\leqslant -1$ or $\geqslant 1$, i.e., no execution following a cycle has weight in $(-1, 1)$

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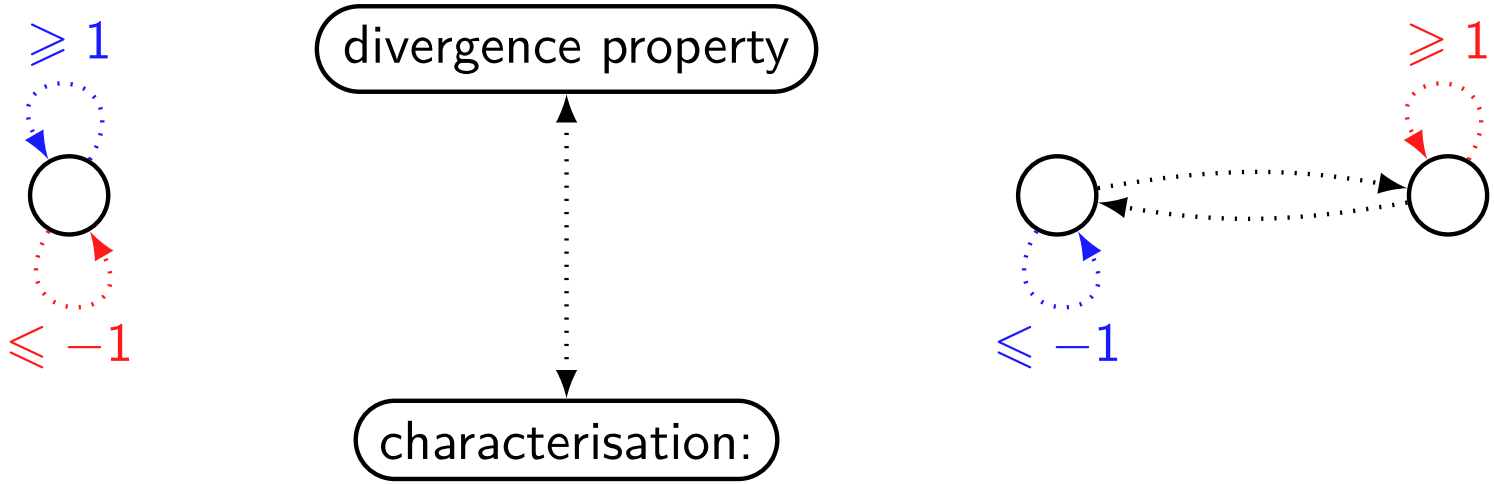
Observation

The set of weights of executions following a fixed region automaton cycle is an interval

Divergent WTG analysis: use of corner-point abstraction



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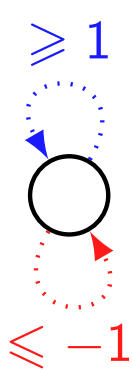
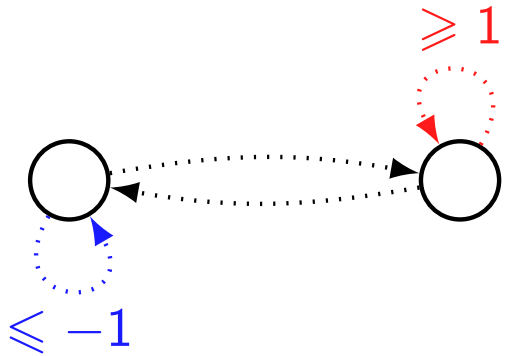
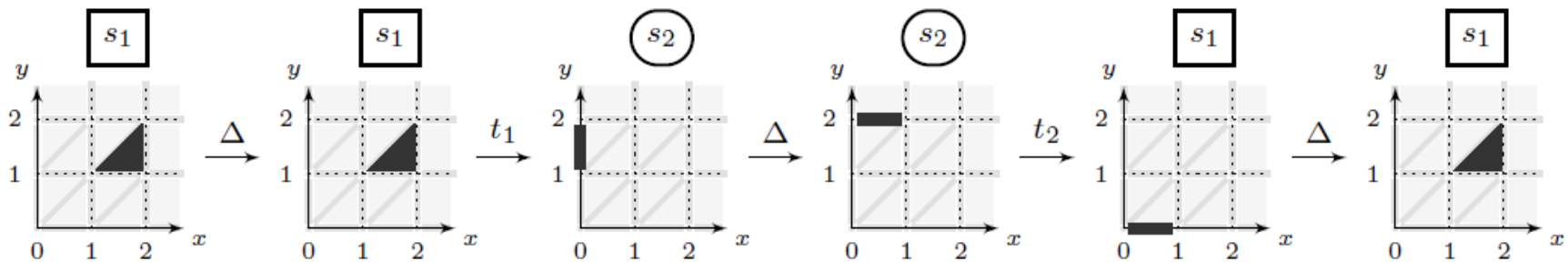
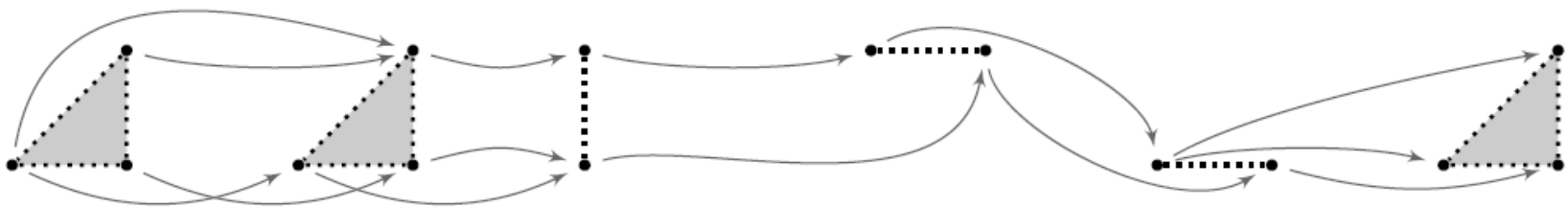


divergence property

characterisation:







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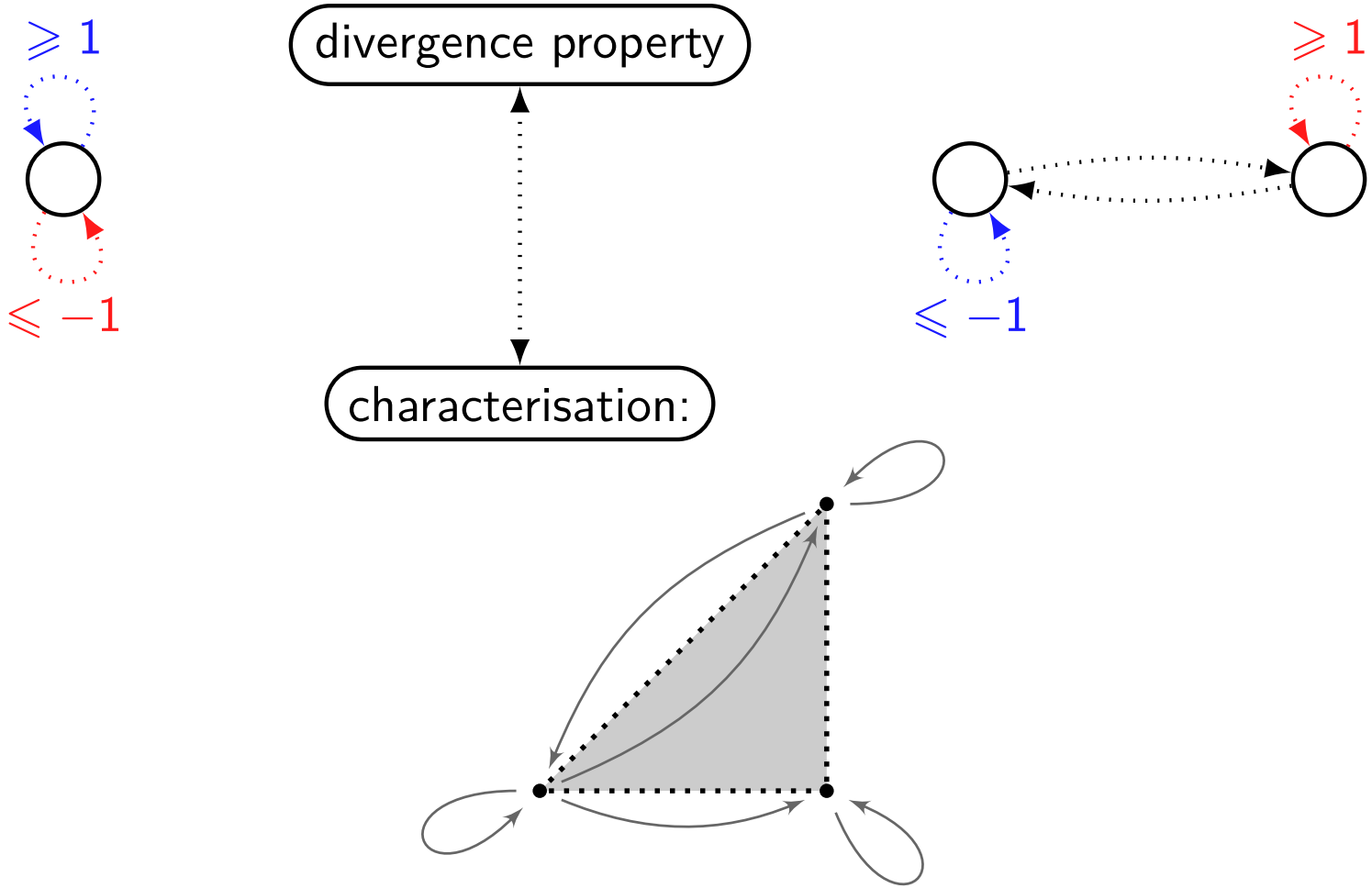

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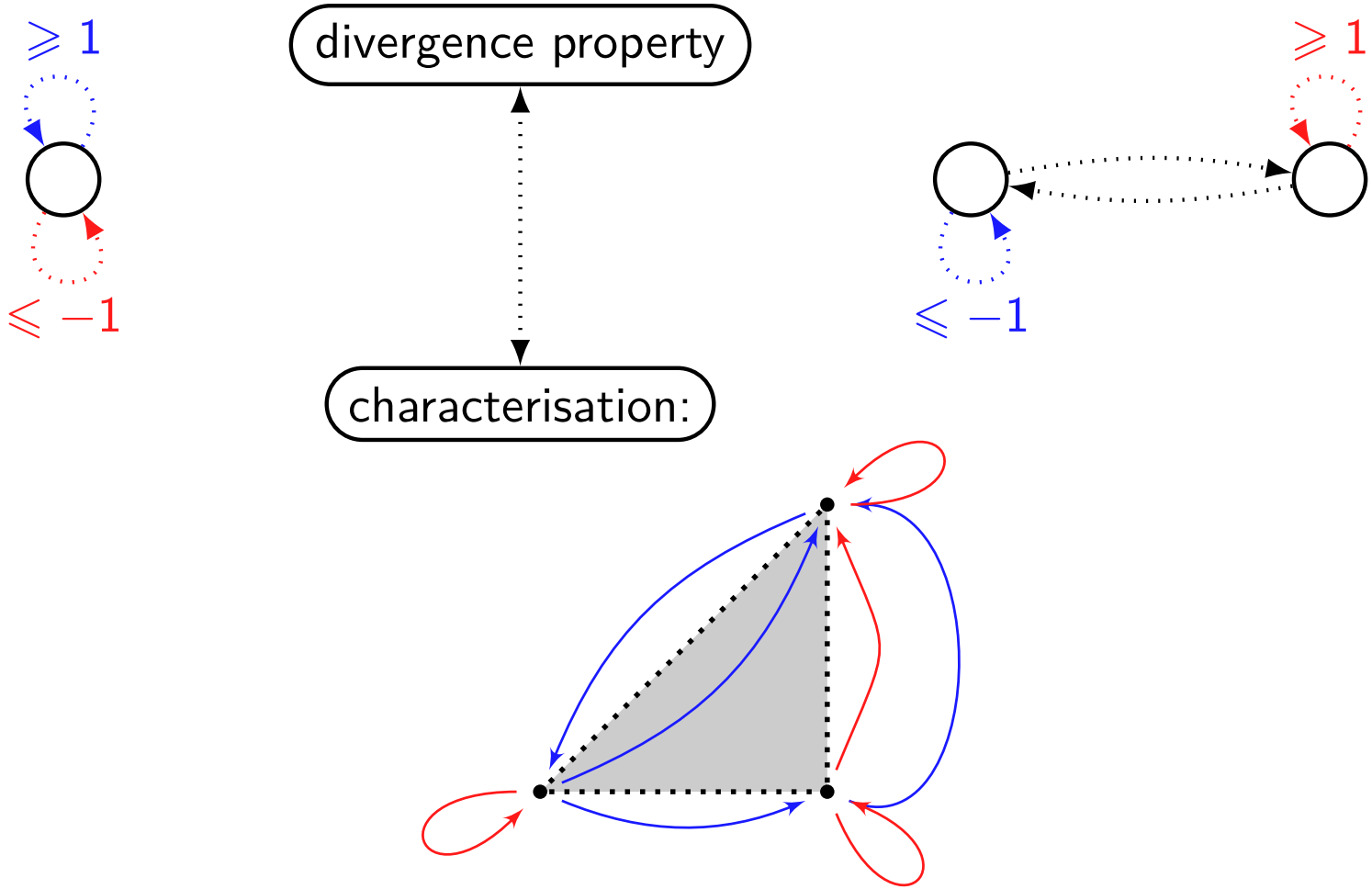

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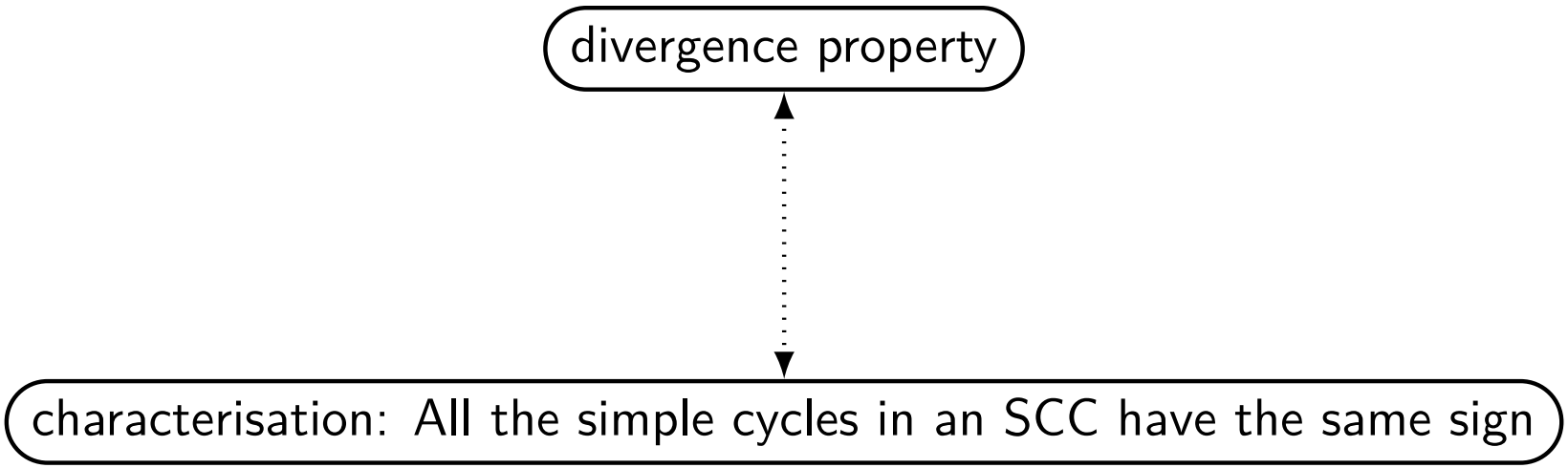

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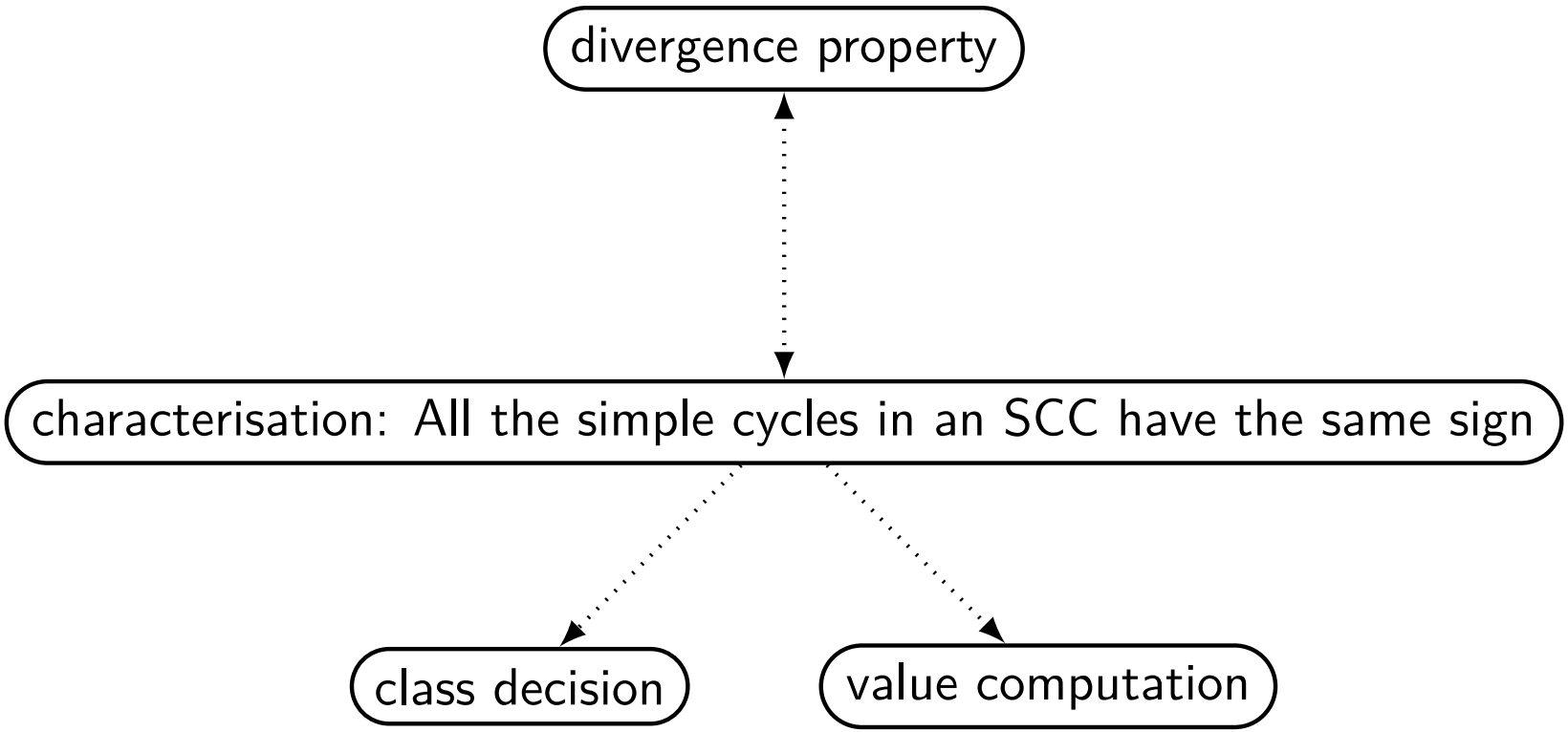

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- Outside of the attractor of player $\square$ toward $\checkmark$ $\Rightarrow -\infty$
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Conclusion

Value computation in weighted games:

- general case: **pseudo-polynomial** algorithm
- divergent weighted games: **P**-complete

Deciding if a weighted game is divergent: in **P**

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Value computation in weighted timed games:

- general case: value problem **undecidable**
- divergent weighted timed games: **2-exponential** algorithm

Deciding if a weighted timed game is divergent: **PSPACE**-complete

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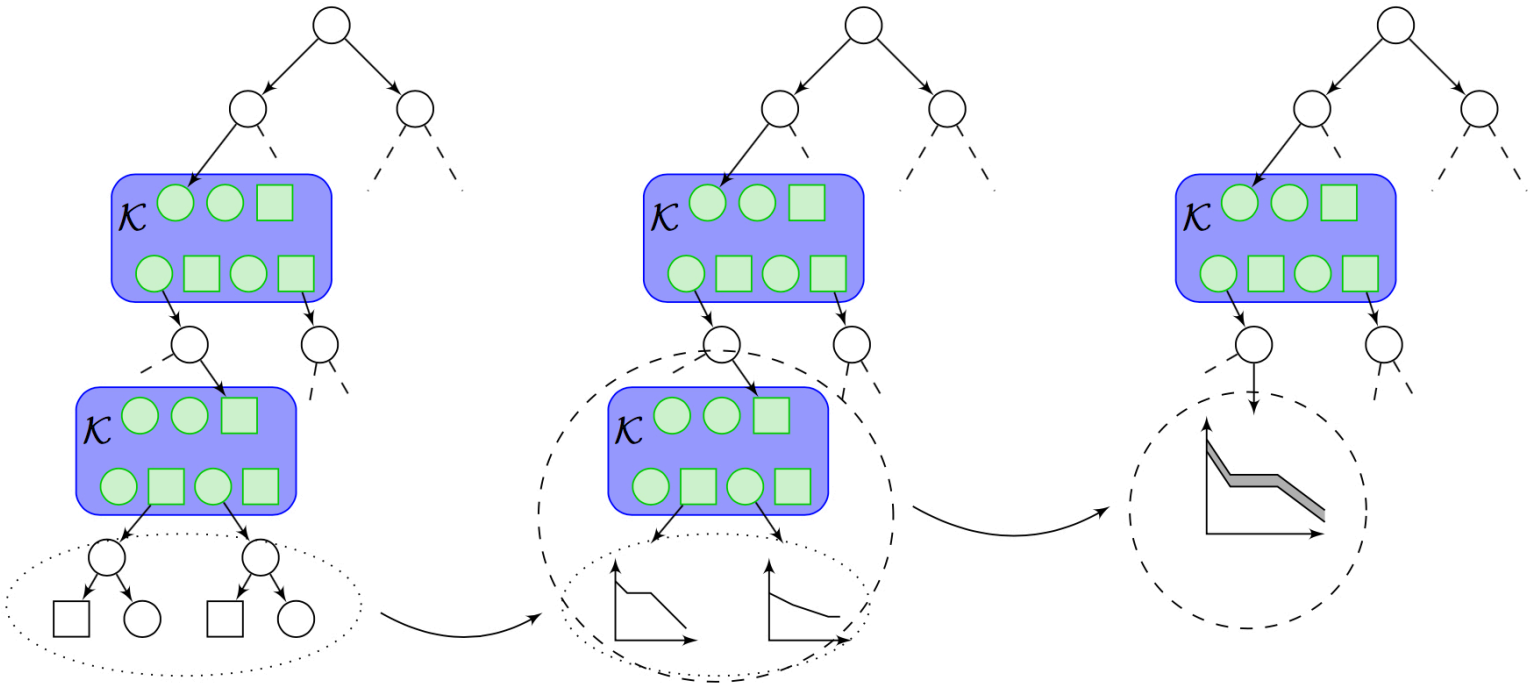

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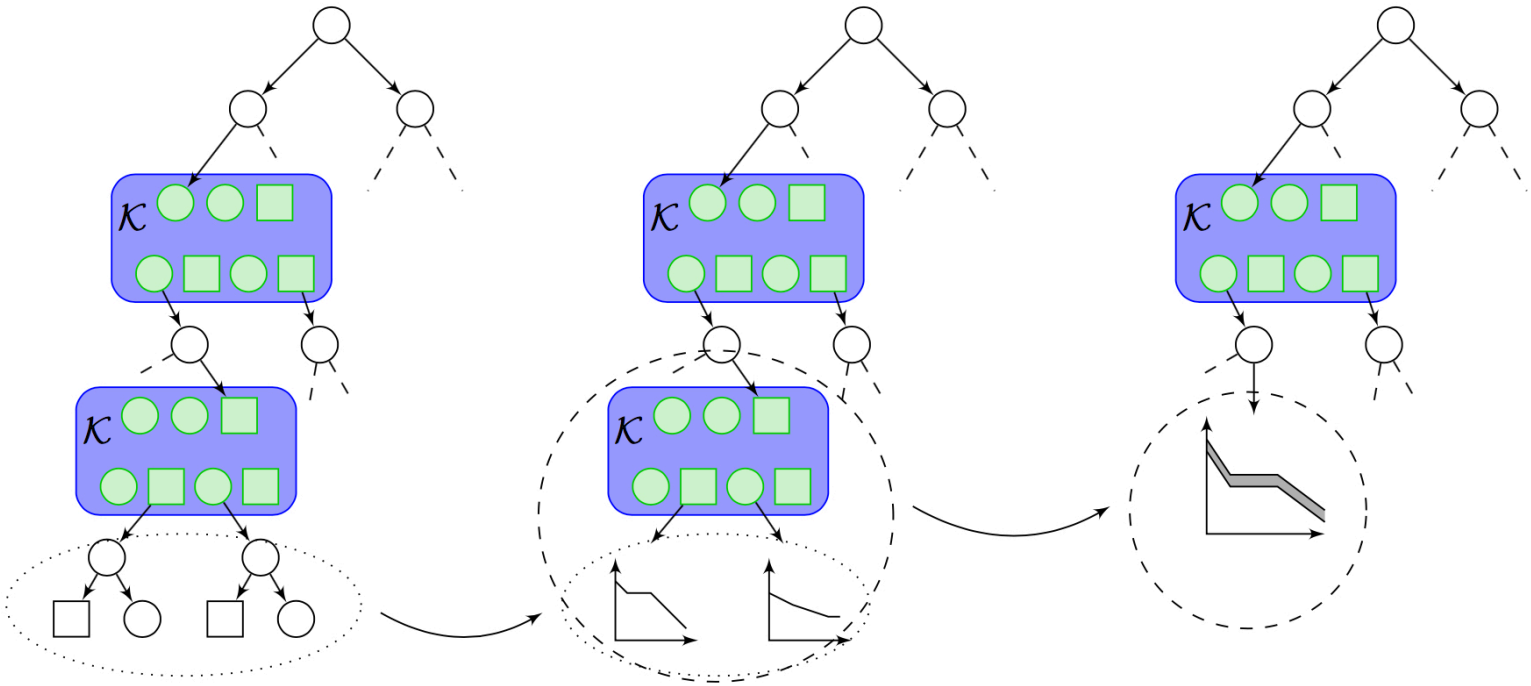

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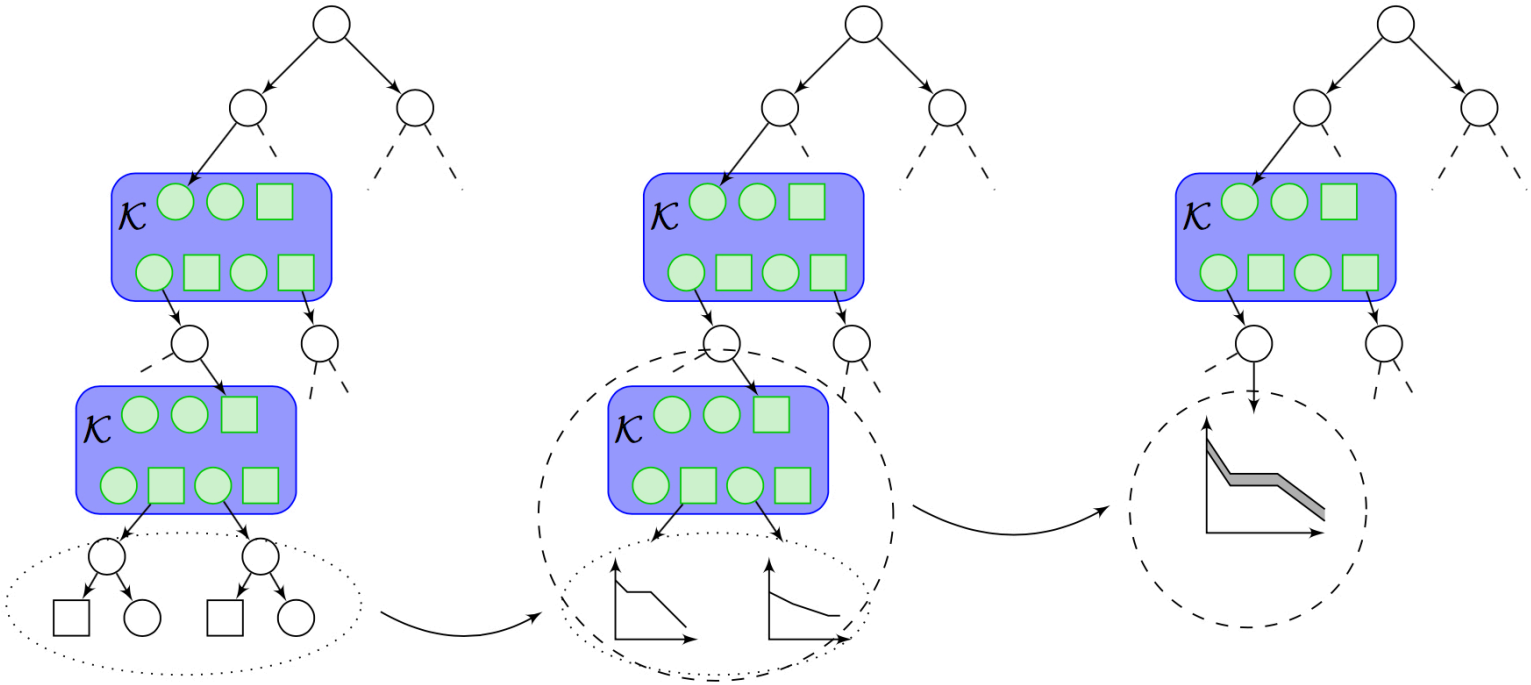

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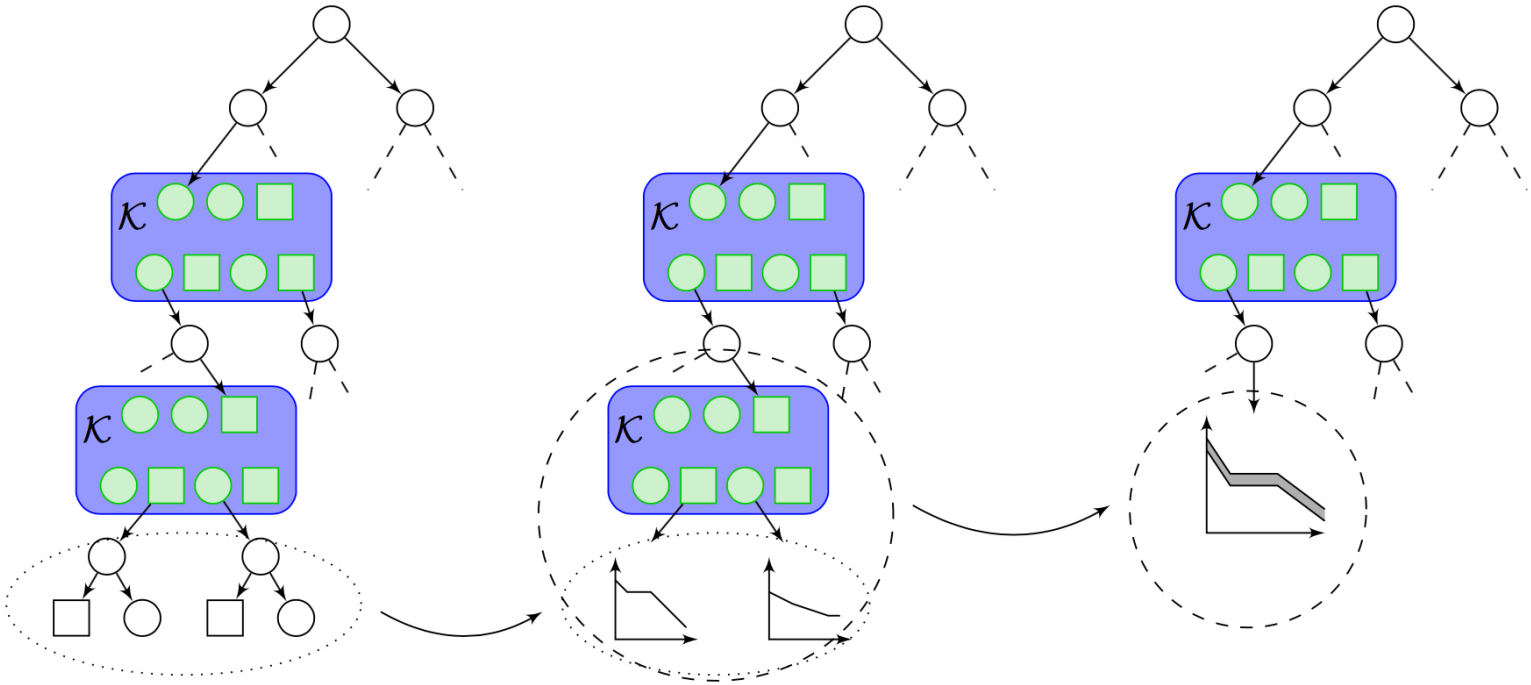

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New result

Approximation is decidable for almost-divergent WTG.

Thank you!

References I


Alur, Rajeev, Mikhail Bernadsky, and P. Madhusudan (2004). "Optimal Reachability for Weighted Timed Games". In: *Proceedings of the 31st International Colloquium on Automata, Languages and Programming (ICALP'04)*. Vol. 3142. LNCS. Springer, pp. 122–133.


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