Dynamics on Games: Simulation-Based Techniques and Applications to Routing

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Slides partly borrowed from Thomas Brihaye and Marion Hallet
Work published at FSTTCS 2019
Two suspects are arrested by the police. The police, having separated both prisoners, visit each of them to offer the same deal.

- If one testifies (Defects) for the prosecution against the other and the other remains silent (Cooperate), the betrayer goes free and the silent accomplice receives the full 10-years sentence.
- If both remain silent, both are sentenced to only 3-years in jail.
- If each betrays the other, each receives a 5-years sentence.

How should the prisoners act?
The prisoner dilemma - the (matrix) game

The matrix associated with the prisoner dilemma:

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<tr>
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<tr>
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<td>(-3, -3)</td>
<td>(-10, 0)</td>
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<tr>
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The prisoner dilemma - the (matrix) game

The matrix associated with the prisoner dilemma:

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & (-3, -3) & (-10, 0) \\
D & (0, -10) & (-5, -5) \\
\end{array}
\]

Equivalently (since only the relative order of payoffs matters):

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & (3, 3) & (1, 4) \\
D & (4, 1) & (2, 2) \\
\end{array}
\]
The first point of view: strategic games

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Rules of the game

- The game is played only once by two players
- The players choose simultaneously their actions (no communication)
- Each player receives his payoff depending of all the chosen actions
- The goal of each player is to maximise his own payoff
The first point of view: strategic games

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(D, D) is the only rational choice!

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Hypotheses made in strategic games

- The players are **intelligent** (i.e. they reason perfectly and quickly)
- The players are **rational** (i.e. they want to maximise their payoff)
- The players are **selfish** (i.e. they only care for their own payoff)
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The strategy D is evolutionary stable, facing an invasion of the mutant strategy C.

Rules of the game

- We have a large population of individuals
- Individuals are repeatedly drawn at random to play the above game
- The payoffs are supposed to represent the gain in biological fitness or reproductive value

Hypotheses made in evolutionary games

Each individual is genetically programmed to play either C or D.

The individuals are no more intelligent, nor rational, nor selfish.
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Outline

1. A brief review of strategic games
   - Nash equilibrium et al
   - Symmetric two-player games

2. Evolutionary game theory
   - Evolutionary Stable Strategy
   - The Replicator Dynamics
   - Other Selections Dynamics

3. Games played on graphs
   - Two examples of dynamics
   - Relations that maintain termination
   - More realistic conditions
   - Application to interdomain routing
A strategic game $G$ is a triple $(N, (A_i)_{i \in N}, (P_i)_{i \in N})$ where:

- $N$ is the finite and non empty set of players,
- $A_i$ is the non empty set of actions of player $i$,
- $P_i : A_1 \times \cdots \times A_N \rightarrow \mathbb{R}$ is the payoff function of player $i$.

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Nash equilibrium

Nash Equilibrium - Definition
Let \((N, A_i, P_i)\) be a strategic game and \(a = (a_i)_{i \in N}\) be a strategy profile. We say that \(a = (a_i)_{i \in N}\) is a Nash equilibrium iff

\[
\forall i \in N \quad \forall b_i \in A_i \quad P_i(b_i, a_{-i}) \leq P_i(a_i, a_{-i})
\]

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & (3, 3) & (1, 4) \\
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\]

\((D, D)\) is the unique Nash equilibrium
Do all the finite matrix games have a Nash equilibrium?
Do all the finite matrix games have a Nash equilibrium?

No: matching pennies

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Mixed strategies

Notations

Given $E$, we denote $\Delta(E)$ the set of *probability distribution over* $E$.

Assuming $E = \{e_1, \ldots, e_n\}$, we have that:

$$\Delta(E) = \{(p_1, \ldots, p_n) \mid p_i \geq 0 \text{ and } p_1 + \ldots + p_n = 1\}.$$
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Mixed strategy

If $A_i$ are strategies of player $i$, $\Delta(A_i)$ is his set of mixed strategies.

Expected payoff

Given $(N, (A_i)_i, (P_i)_i)$. Let $(\sigma_1, \ldots, \sigma_n)$ be a mixed strategies profile. The expected payoff of player $i$ is

$$P_i(\sigma_1, \ldots, \sigma_n) = \sum_{(a_1, \ldots, a_N) \in A_1 \times \ldots \times A_N} \left( \prod_{i \in N} \sigma_i(a_i) \right) P_i(a_1, \ldots, a_N)$$

probability of $(a_1, \ldots, a_N)$
Nash equilibria in mixed strategies

The following profile is a *Nash equilibrium in mixed strategies*:

\[
\sigma_1 = \begin{cases} 
L & \text{with proba } \frac{1}{2} \\
R & \text{with proba } \frac{1}{2}
\end{cases} \quad \text{and} \quad \sigma_2 = \begin{cases} 
L & \text{with proba } \frac{1}{2} \\
R & \text{with proba } \frac{1}{2}
\end{cases}
\]

whose expected payoff is 0.
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**Nash Theorem [1950]**

*Every finite game admits mixed Nash equilibria.*
Symmetric games

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A symmetric game is a game \((N, (A_i)_{i \in N}, (P_i)_{i \in N})\) where:

- \(A_1 = A_2 = \cdots = A_N\)
- \(\forall (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N, \forall \pi \text{ permutations}, \forall k, \text{ we have that}\)
  \[ P_{\pi(k)}(a_1, \ldots, a_N) = P_k(a_{\pi(1)}, \ldots, a_{\pi(k)}) \]
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- Special case of 2-players: \(\forall (a_1, a_2) \in A_1 \times A_2, P_2(a_1, a_2) = P_1(a_2, a_1)\)
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Symmetric Nash Equilibrium

A Nash equilibrium \((\sigma_1, \ldots, \sigma_N)\) is said symmetric when \(\sigma_1 = \cdots = \sigma_N\).
Example 1: 2 × 2 games - The 4 categories

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- Cat 1: $\alpha < 0$ et $\beta > 0$. $\text{NE} = \{(Y, Y)\}$
- Cat 2: $\alpha, \beta > 0$. $\text{NE} = \{(X, X), (Y, Y), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 3: $\alpha, \beta < 0$. $\text{NE} = \{(X, Y), (Y, X), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 4: $\alpha > 0$ et $\beta < 0$. $\text{NE} = \{(X, X)\}$
Example 2: The generalised Rock-Scissors-Paper Games

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(The original RPS game is obtained when $a = 0$)
Example 2: The generalised Rock-Scissors-Paper Games

Uta stansburiana - The side-blotched lizard

The populations for these lizards cycle on a six year basis. When he read that lizards of the species Uta stansburia were essentially engaged in a game with rock-paper-scissors structure John Maynard Smith exclaimed:

They have read my book!

\[
\begin{array}{c|ccc}
 & R & S & P \\
\hline
R & (1, 1) & (2 + a, 0) & (0, 2 + a) \\
S & (0, 2 + a) & (1, 1) & (2 + a, 0) \\
P & (2 + a, 0) & (0, 2 + a) & (1, 1) \\
\end{array}
\]

(The original RPS game is obtained when \( a = 0 \))

A unique Nash equilibrium \((\sigma, \sigma, \sigma)\), where \(\sigma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\).
Some results on symmetric games

**Theorem [Cheng et al, 2004]**

Every 2-strategy symmetric game (i.e. $|A_i| = 2$) admits a (pure) Nash equilibrium. *But it might not be symmetric...*
Some results on symmetric games

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- no longer true if not “2-strategy”: RPS...
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- no longer true if not “2-strategy”: RPS...
- no longer true if not “symmetric”: Matching pennies

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\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & (1, -1) & (-1, 1) \\
R & (-1, 1) & (1, -1) \\
\end{array}
\]

- not necessarily symmetric: anti-coordination game

\[
\begin{array}{c|cc}
 & X & Y \\
\hline
X & (0, 0) & (1, 1) \\
Y & (1, 1) & (0, 0) \\
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Evolutionary game theory

We completely change the point of view!

**Rules of the game**

- We have a **large** population of individuals.
- Individuals are repeatedly drawn at random to play a symmetric game.
- The payoffs are supposed to represent the gain in biological fitness or reproductive value.

**Hypotheses made in evolutionary games**

- Each individual is **genetically programmed** to play a strategy.
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**.

Can an existing population resist to the invasion of a mutant?
Evolutionary Stable Strategy: robustness to mutations

Evolutionary Stable Strategy

We say that $\sigma$ is an **evolutionary stable strategy (ESS)** if

- $(\sigma, \sigma)$ is a Nash equilibrium
- $\forall \sigma' (\neq \sigma) \quad P(\sigma', \sigma) = P(\sigma, \sigma) \implies P(\sigma', \sigma') < P(\sigma, \sigma')$

Thus if $(\sigma, \sigma)$ is a **strict** Nash equilibrium, then $\sigma$ is an ESS.

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- $(A,A)$, $(B,B)$ and $(C,C)$ are Nash equilibria.
- $A$ is not an **ESS**.
- $B$ and $C$ are **ESS**.
Imagine a population composed of a unique species $\sigma$

A small proportion $\epsilon$ of the population mutes to a new species $\sigma'$

The new population is thus $\epsilon\sigma' + (1 - \epsilon)\sigma$

**Proposition**

A strategy $\sigma$ is an ESS iff $\forall \sigma' (\neq \sigma) \exists \epsilon_0 \in (0, 1) \forall \epsilon \in (0, \epsilon_0) \ P(\sigma, \epsilon\sigma' + (1 - \epsilon)\sigma) > P(\sigma', \epsilon\sigma' + (1 - \epsilon)\sigma)$
Evolutionary Stable Strategy - Alternative definition

- Imagine a population composed of a unique species $\sigma$
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$$P(\sigma, \epsilon \sigma' + (1 - \epsilon)\sigma) > P(\sigma', \epsilon \sigma' + (1 - \epsilon)\sigma)$$

Static concept: it suffices to study the one-shot game
## Evolutionary Stable Strategy - 2 × 2 games

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- **Cat 1**: NE = \{ (Y, Y) \}  
  ESS = \{ Y \}
- **Cat 2**: NE = \{ (X, X), (Y, Y), (σ, σ) \}  
  ESS = \{ X, Y \}
- **Cat 3**: NE = \{ (X, Y), (Y, X), (σ, σ) \}  
  ESS = \{ σ \}
- **Cat 4**: NE = \{ (X, X) \}  
  ESS = \{ X \}
The evolution of a population - intuitively

Population composed of several species

Variation of popu. the species = Popu. of the species \times \text{Advantage of the species}

Advantage of the species = \text{Fitness of the species} - \text{Average fitness of all species}
The evolution of a population - more formally (1)

- We consider a population where individuals are divided into $n$ species. Individuals of species $i$ are programmed to play the pure strategy $a_i$.
- We denote by $p_i(t)$ the number of individuals of species $i$ at time $t$.
- The **total population at time** $t$ is given by
  \[ p(t) = p_1(t) + \cdots + p_n(t) \]
- The **population state at time** $t$ is given by
  \[ \sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t)) \] where \[ \sigma_i(t) = \frac{p_i(t)}{p(t)} \]
The evolution of a population - more formally (2)

The evolution of the state of the population is given by:

**The replicator dynamics (RD)**

$$\frac{d}{dt}\sigma_i(t) = (P(a_i, \sigma(t)) - P(\sigma(t), \sigma(t))) \cdot \sigma_i(t)$$

**Theorem**

Given any initial condition $\sigma(0) \in \Delta(A)$, the above system of differential equations always admits a unique solution.
The replicator dynamics - $2 \times 2$ games

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Cat 1 $\rightarrow$ Cat 2

\[
\frac{d}{dt} \sigma_1(t) = (\alpha \sigma_1(t) - \beta \sigma_2(t)) \cdot \sigma_1(t) \sigma_2(t)
\]

\[
\frac{d}{dt} \sigma_2(t) = (\beta \sigma_2(t) - \alpha \sigma_1(t)) \cdot \sigma_1(t) \sigma_2(t)
\]

$\Delta(A) = \{(\sigma_1, \sigma_2) \in [0, 1]^2 \mid \sigma_1 + \sigma_2 = 1\} \simeq [0, 1]$, where $\sigma_1$ is the proportion of X.

The solutions $(\sigma_1(t), 1 - \sigma_1(t))$ of the (RD) behave as follows:

Cat 1 $\rightarrow$ Cat 2 $\rightarrow$ Cat 3 $\rightarrow$ Cat 4 $\rightarrow$ Cat 1

$\frac{\beta}{\alpha + \beta}$
Various concept of stability

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be smooth enough and consider:

$$\frac{d}{dt} x(t) = f(x(t)).$$

Let $\varphi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a maximal solution of the above equation.

Let $x_0 \in \mathbb{R}^n$, we say that

- $x_0$ is a **stationary point** iff $\forall t \in \mathbb{R} \quad \varphi(x_0, t) = x_0$
- $x_0$ is **Lyapunov stable** iff

  $$\forall U(x_0) \subseteq \mathbb{R}^n \quad \exists V(x_0) \subseteq \mathbb{R}^n \quad \forall x \in V(x_0) \quad \forall t \in \mathbb{R} \quad \varphi(x, t) \in U(x_0)$$

- $x_0$ is **asymptotically stable** iff $x_0$ is a Lyapunov stable point and

  $$\exists W(x_0) \quad \forall x \in W(x_0) \quad \lim_{t \rightarrow +\infty} \varphi(x, t) = x_0$$
**2 × 2 games - Stability**

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<tr>
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<th>X</th>
<th>Y</th>
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<tbody>
<tr>
<td>X</td>
<td>(α, α)</td>
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<tr>
<td>Y</td>
<td>(0, 0)</td>
<td>(β, β)</td>
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</table>

- **Cat 1**: Asymptotically stable
- **Cat 2**: Stationary
- **Cat 3**: Stationary
- **Cat 4**: Stationary

The diagram illustrates the stability analysis of a 2 × 2 game with payoffs (α, α) for Cat 1 and (β, β) for Cat 4, with (0, 0) being a common point for both players. Arrow directions and positions indicate the movement leading to stationary points.
Rock-Scissors-Paper

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is Lyapunov stable but not asymptotically stable.

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The picture is taken from *Evolutionary game theory* by J.W. Weibull.
### 2 × 2 games - RD Vs ESS

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</table>

- **Cat 1**: Asymptotically stable
- **Cat 2**: ESS = \{Y\}
- **Cat 3**: ESS = \{σ\}
- **Cat 4**: ESS = \{X\}

The diagrams illustrate the transitions between strategies under different parameter values of \(α\) and \(β\). The green dots represent asymptotically stable states, while the red dots indicate stationary states.
The generalised Rock-Scissors-Paper Games

\[ a = 0 \]

\( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) is not an ESS

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\[ a > 0 \]

\( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) is an ESS

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\[ a < 0 \]

\( \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \) is not an ESS

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The pictures are taken from *Evolutionnary game theory* by J.W. Weibull.
Results

There are several results relating various notions of “static” stability:
- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy...

with various notions of “dynamic” stability:
- stationary points,
- Lyapunov stable points,
- asymptotically stable point ...

Theorems

- If $\sigma \in \Delta$ is Lyapunov stable, then $\sigma$ is a NE.
- If $\sigma \in \Delta$ is an ESS, then $\sigma$ is asymptotically stable.
An alternative dynamics

**Replicator dynamics**

Variation of popu. the species = Popu. of the species × Advantage of the species

Advantage of the species = Fitness of the species − Average fitness of all species
An alternative dynamics

Replicator dynamics
Variation of popu. the species = Popu. of the species × Advantage of the species
Advantage of the species = Fitness of the species − Average fitness of all species

Alternative hypothesis: offspring react *smartly* to the mixture of past strategies played by the opponents, by playing a **best-reply strategy** to this mixture

Best-reply dynamics
Variation of Strategy Mixture = Best-Reply Strategy − Current Strategy Mixture
Replicator Vs Best-reply

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Replicator dynamics

Best-reply dynamics

Pictures taken from *Evolutionnary game theory* by W. H. Sandholm
Other dynamics

![Diagram of five basic deterministic dynamics in standard Rock-Paper-Scissors. Colors represent speeds: red, orange, yellow, green, blue, and cyan.](image)

Figure 1: Five basic deterministic dynamics in standard Rock-Paper-Scissors. Colors represent speeds: red, orange, yellow, green, blue, and cyan.
Static vs dynamic approach

Static approach                     Dynamic approach
Equilibria                          Stable Points

Picture taken from *Evolutionnary game theory* by W. H. Sandholm
Static vs dynamic approach

Static approach  Dynamic approach

Equilibria  Stable Points

If we discover a new game
- Find immediately a good strategy is concretely impossible
Static vs dynamic approach

**Static approach**

- Equilibria

**Dynamic approach**

- Stable Points

---

**If we discover a new game**

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
If we discover a new game

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
- With enough different plays, will we eventually stabilize?
Static vs dynamic approach

**Static approach**  **Dynamic approach**

Equilibria  Stable Points

If we discover a new game

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
- With enough different plays, will we eventually stabilize?
- If so, will this strategy be a good strategy?
Static vs dynamic approach

**Static approach**

* Equilibria

**Dynamic approach**

* Stable Points

---

**If we discover a new game**

- Find immediately a good strategy is concretely impossible
- If we play several times, we will improve our strategy
- With enough different plays, will we eventually stabilize?
- If so, will this strategy be a *good* strategy?

---

**Our Goal**

- Apply this idea of improvement/mutation on games played on graphs
- Prove stabilisation via reduction/minor of games
- Show some links with interdomain routing
Interdomain routing problem

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\perp$. 
Interdomain routing problem

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\perp$. 
Interdomain routing problem

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\bot$.

$v_1$ prefers the route $v_1 v_2 v_\bot$ to the route $v_1 v_\bot$ (preferred to $(v_1 v_2)^\omega$).

$v_2$ prefers the route $v_2 v_1 v_\bot$ to the route $v_2 v_\bot$ (preferred to $(v_2 v_1)^\omega$).
Interdomain routing problem as a game played on a graph

Two service providers: $v_1$ and $v_2$ want to route packets to $v_\bot$.

$v_1$ prefers the route $v_1 v_2 v_\bot$ to the route $v_1 v_\bot$ (preferred to $(v_1 v_2)^\omega$)

$v_2$ prefers the route $v_2 v_1 v_\bot$ to the route $v_2 v_\bot$ (preferred to $(v_2 v_1)^\omega$)

$v_1 v_\bot \prec_1 v_1 v_2 v_\bot$ and $v_2 v_\bot \prec_2 v_2 v_1 v_\bot$
Games played on a graph – The strategic game approach

2 Nash equilibria: \((c_1, s_2)\) and \((s_1, c_2)\)

Static vision of the game: players are perfectly informed and supposed to be intelligent, rational and selfish
Games played on a graph – The evolutionnary approach
Games played on a graph – The evolutionnary approach
Asynchronous nature of the network could block the packets in an undesirable cycle...
Interdomain routing problem - open problem

The game $G(c_1, c_2) = (s_1, c_2)$, $G(c_1, s_2) = (s_1, s_2)$

The graph of the dynamics: $G\langle\rightarrow\rangle$

Identify necessary and sufficient conditions on $G$ such that $G\langle\rightarrow\rangle$ has no cycle

Ideally, the conditions should be algorithmically simple, locally testable...

Numerous interesting partial solutions proposed in the literature

Daggitt, Gurney, Griffin. Asynchronous convergence of policy-rich distributed Bellman-Ford routing protocols. 2018
Games played on a graph – The evolutionnary approach

Different dynamics

\[
\begin{align*}
\text{\(D_1\) with no cycle} & \quad \text{\(D_2\) with a cycle} \\
(c_1, c_2) & \rightarrow (s_1, c_2) \\
(c_1, s_2) & \leftarrow (s_1, s_2) \\
(c_1, c_2) & \rightarrow (s_1, c_2) \\
(c_1, s_2) & \leftarrow (s_1, s_2)
\end{align*}
\]
Outline

1. A brief review of strategic games
   - Nash equilibrium et al
   - Symmetric two-player games

2. Evolutionary game theory
   - Evolutionary Stable Strategy
   - The Replicator Dynamics
   - Other Selections Dynamics

3. Games played on graphs
   - Two examples of dynamics
   - Relations that maintain termination
   - More realistic conditions
   - Application to interdomain routing
Positional 1-step dynamics $^{P_1}$

If:

- a single player changes at a single node
- this player improves his own outcome
Positional 1-step dynamics $\xrightarrow{P_1}$

profile$_1$ $\xrightarrow{P_1}$ profile$_2$

if:

- a single player changes at a single node
- this player improves his own outcome

$G^{P_1}$:

$$(c_1, c_2) \xrightarrow{P_1} (s_1, c_2)$$

$$(c_1, s_2) \xrightarrow{P_1} (s_1, s_2)$$
Positional Concurrent Dynamics

\[ \text{profile}_1 \xrightarrow{PC} \text{profile}_2 \]

if

- one or several players change at a single node
- all players that change intend to improve their outcome
- but synchronous changes may result in worst outcomes...
Positional Concurrent Dynamics $\xrightarrow{\text{PC}}$

$\text{profile}_1 \xrightarrow{\text{PC}} \text{profile}_2$

if

- one or several players change **at a single node**
- all players that change **intend** to improve their outcome
- but synchronous changes may result in worst outcomes...

$G^{\xrightarrow{\text{PC}}}$:

$(c_1, c_2) \xrightarrow{\text{PC}} (s_1, c_2)$

$(c_1, s_2) \xleftarrow{\text{PC}} (s_1, s_2)$
Positional Concurrent Dynamics $\xrightarrow{\text{PC}}$

\[
\text{profile}_1 \xrightarrow{\text{PC}} \text{profile}_2
\]

if

- one or several players change at a single node
- all players that change intend to improve their outcome
- but synchronous changes may result in worst outcomes...

both players intend to reach their best outcome ($\nu_1 \nu_\perp \prec_1 \nu_1 \nu_2 \nu_\perp$ and $\nu_2 \nu_\perp \prec_2 \nu_2 \nu_1 \nu_\perp$), even if they do not manage to do it (as the reached outcome is $(\nu_1 \nu_2)^\omega$ and $(\nu_2 \nu_1)^\omega$)
Questions

What condition $G$ should satisfy to ensure that $G\langle\rightarrow\rangle$ has no cycles, i.e. dynamics $\rightarrow$ terminates on $G$?
Questions

What condition $G$ should satisfy to ensure that

$$G\langleightarrow\rangle\text{ has no cycles, i.e. dynamics }\rightarrow\text{ terminates on } G?$$

What relations $\rightarrow_1$ and $\rightarrow_2$ should satisfy to ensure that

$$G\langleightarrow_1\rangle\text{ has no cycles if and only if } G\langleightarrow_2\rangle\text{ has no cycles?}$$
Questions

What condition $G$ should satisfy to ensure that $G\langle\rightarrow\rangle$ has no cycles, i.e. dynamics $\rightarrow$ terminates on $G$?

What relations $\rightarrow_1$ and $\rightarrow_2$ should satisfy to ensure that $G\langle\rightarrow_1\rangle$ has no cycles if and only if $G\langle\rightarrow_2\rangle$ has no cycles?

What should $G_1$ and $G_2$ have in common to ensure that $G_1\langle\rightarrow\rangle$ has no cycles if and only if $G_2\langle\rightarrow\rangle$ has no cycles?
Simulation relation on dynamics graphs

$G$ simulates $G'$ ($G' \sqsubseteq G$) if all that $G'$ can do, $G$ can do it too.

\[
\forall \text{profile'}_1 \rightarrow \forall \text{profile'}_2
\]

\[
\forall \text{profile}_1
\]
Simulation relation on dynamics graphs

$G$ simulates $G'$ ($G' \sqsubseteq G$) if all that $G'$ can do, $G$ can do it too.

$$
\forall \text{profile}'_1 \quad \forall \text{profile}'_2

\text{profile}'_1 \rightarrow \text{profile}'_2

\exists \text{profile}_1 \quad \forall \text{profile}_2

\text{profile}_1 \rightarrow \text{profile}_2
$$
Simulation relation on dynamics graphs

\( G \) simulates \( G' \) (\( G' \subseteq G \)) if all that \( G' \) can do, \( G \) can do it too.

\[
\forall \text{ profile}'_1 \rightarrow \text{ profile}'_2
\]

\[
\forall \exists \text{ profile}_1 \rightarrow \text{ profile}_2
\]

Folklore

If \( G_1\langle \rightarrow_1 \rangle \) simulates \( G_2\langle \rightarrow_2 \rangle \) and the dynamics \( \rightarrow_1 \) terminates on \( G_1 \), then the dynamics \( \rightarrow_2 \) terminates on \( G_2 \).
Relation between games

$G'$ is a minor of $G$ if it is obtained by a succession of operations:

- deletion of an edge (and all the corresponding outcomes);
- deletion of an isolated node;
- deletion of a node $v$ with a single edge $v \rightarrow v'$ and no predecessor $u \rightarrow v$ such that $u \rightarrow v'$.
Relation between games

$G'$ is a minor of $G$ if it is obtained by a succession of operations:

- deletion of an edge (and all the corresponding outcomes);
- deletion of an isolated node;
- deletion of a node $v$ with a single edge $v \to v'$ and no predecessor $u \to v$ such that $u \to v'$.

\[ V_1 \rightarrow V_2 \]
\[ V_3 \rightarrow V_4 \]
\[ V_5 \]

\[ V_1 \rightarrow V_2 \]
\[ V_3 \rightarrow V_4 \]
\[ V_{\perp} \rightarrow V_5 \]

\[ V_1 \rightarrow V_2 \]
\[ V_3 \rightarrow V_5 \]
\[ V_{\perp} \rightarrow V_5 \]

\[ V_1 \rightarrow V_2 \]
\[ V_3 \rightarrow V_5 \]
\[ V_{\perp} \rightarrow V_5 \]
Relation between simulation and minor

Theorem

If $G'$ is a minor of $G$, then $G\langle P_1 \rangle$ simulates $G'\langle P_1 \rangle$. In particular, if $P_1$ terminates for $G$, it terminates for $G'$ too.
Relation between simulation and minor

**Theorem**

If $G'$ is a minor of $G$, then $G^{\langle \text{P1} \rangle}$ simulates $G'^{\langle \text{P1} \rangle}$. In particular, if $\rightarrow^{\text{P1}}$ terminates for $G$, it terminates for $G'$ too.

**Theorem**

If $G'$ is a minor of $G$, then $G^{\langle \text{PC} \rangle}$ simulates $G'^{\langle \text{PC} \rangle}$. In particular, if $\rightarrow^{\text{PC}}$ terminates for $G$, it terminates for $G'$ too.

Remark: $G^{\langle \text{P1} \rangle} \sqsubseteq G^{\langle \text{PC} \rangle}$
More realistic conditions

Adding fairness

- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of *fair termination*
More realistic conditions

Adding fairness

- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of *fair termination*

More realistic dynamics

Consider *best reply* variants \( bP_1 \) and \( bPC \) of the two dynamics, where each player that modifies its strategy changes in the best possible way...
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G\langle \rightarrow^{PC} \rangle$ simulates $G'\langle \rightarrow^{PC} \rangle$. In particular, if $\rightarrow^{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow^{bP1}$ and $\rightarrow^{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$.
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G^{\text{PC}}$ simulates $G'^{\text{PC}}$. In particular, if $\rightarrow$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow^{bP1}$ and $\rightarrow^{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$. 

![Diagram](image-url)
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G\langle PC \rangle$ simulates $G'\langle PC \rangle$. In particular, if $\rightarrow$ terminates for $G$, it terminates for $G'$ too.

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What results?

Previous theorem

If $G'$ is a minor of $G$, then $G\langle \rightarrow^{\text{PC}} \rangle$ simulates $G'\langle \rightarrow^{\text{PC}} \rangle$. In particular, if $\rightarrow^{\text{PC}}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow^{\text{bP1}}$ and $\rightarrow^{\text{bPC}}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$
- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not *terminate*) but not for $G'$
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G^{\langle \xrightarrow{PC} \rangle}$ simulates $G'^{\langle \xrightarrow{PC} \rangle}$. In particular, if $\xrightarrow{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\xrightarrow{bP1}$ and $\xrightarrow{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$
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What results?

**Previous theorem**

If $G'$ is a minor of $G$, then $G\langle \rightarrow^\text{PC} \rangle$ simulates $G'\langle \rightarrow^\text{PC} \rangle$. In particular, if $\rightarrow^\text{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow^\text{bP1}$ and $\rightarrow^\text{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$
- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not *terminate*) but not for $G'$
What results?

**Previous theorem**

If $G'$ is a minor of $G$, then $G^{\langle \rightarrow \rangle}_{PC}$ simulates $G'^{\langle \rightarrow \rangle}_{PC}$. In particular, if $\rightarrow_{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow_{bP1}$ and $\rightarrow_{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$.
- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not *terminate*) but not for $G'$.
- The reciprocal does not hold...
What results?

**Previous theorem**

If $G'$ is a minor of $G$, then $G'\langle \text{PC}\rangle$ simulates $G'\langle \text{PC}\rangle$. In particular, if $\text{PC}$ terminates for $G$, it terminates for $G'$ too.

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- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not terminate) but not for $G'$
- The reciprocal does not hold...

**Theorem**

If $G'$ is a dominant minor of $G$, then $\text{bPC}$/ $\text{bP1}$ fairly terminates for $G$ if and only if it fairly terminates for $G'$. 
What results?

Previous theorem

If $G'$ is a minor of $G$, then $G\langle \rightarrow^\text{PC}\rangle$ simulates $G'\langle \rightarrow^\text{PC}\rangle$. In particular, if $\rightarrow^\text{PC}$ terminates for $G$, it terminates for $G'$ too.

- Becomes false for best reply dynamics $\rightarrow^\text{bP1}$ and $\rightarrow^\text{bPC}$: the best reply dynamics could terminate in $G$ but not in the minor $G'$
- Does not apply to fair termination: the dynamics could fairly terminate for $G$ (and not terminate) but not for $G'$
- The reciprocal does not hold...

Theorem

If $G'$ is a dominant minor of $G$, then $\rightarrow^\text{bPC} / \rightarrow^\text{bP1}$ fairly terminates for $G$ if and only if it fairly terminates for $G'$.

- Use of simulations that are partially invertible...
Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

**Theorem [Sami, Shapira, Zohar, 2009]**

If $G$ is a one-target game for which $\text{bPC}$ fairly terminates, then it has exactly one *equilibrium*.

**Theorem [Griffin, Shepherd, Wilfong, 2002]**

There exists a pattern, called *dispute wheel* such that if $G$ is a one-target game that has no dispute wheels, then $\text{bPC}$ fairly terminates.
Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

**Theorem [Griffin, Shepherd, Wilfong, 2002]**

There exists a pattern, called *dispute wheel* such that if $G$ is a one-target game that has no dispute wheels, then $\text{bPC}$ fairly terminates.

\[ \forall 1 \leq i \leq k \quad \pi_i \prec_{u_i} h_i \pi_{i+1} \]

![Diagram showing dispute wheel pattern with nodes and edges labeled with $h_i$, $\pi_i$, and $u_i$.]
Reciprocal?

**Theorem**

There exists a stronger pattern, called *strong dispute wheel*, such that if $\text{PC} \rightarrow$ terminates for $G$, then $G$ has no strong dispute wheel.
Theorem
There exists a stronger pattern, called *strong dispute wheel*, such that if \( PC \) terminates for \( G \), then \( G \) has no strong dispute wheel.

Theorem
If \( G \) satisfies a locality condition on the preferences, then \( PC \) fairly terminates for \( G \) if and only if \( G \) has no strong dispute wheel.

\[ \text{Griffin et al} \]

\[ \begin{align*}
\text{bPC} & \quad \text{does not fairly terminate for } G \\
& \quad \text{Griffin et al} \\
\text{G has a dispute wheel} & \quad \Downarrow \\
\rightarrow & \quad \text{PC} \quad \text{does not fairly terminate for } G \\
& \quad \text{if neighbour game} \\
\text{PC} & \quad \text{does not terminate for } G \\
& \quad \text{G has a strong dispute wheel}
\end{align*} \]
Theorem
There exists a stronger pattern, called *strong dispute wheel*, such that if \( \text{PC} \rightarrow \) terminates for \( G \), then \( G \) has no strong dispute wheel.

Theorem
If \( G \) satisfies a locality condition on the preferences, then \( \text{PC} \rightarrow \) fairly terminates for \( G \) if and only if \( G \) has no strong dispute wheel.

Theorem
Finding a strong dispute wheel in \( G \) can be tested by searching whether \( G \) contains the following game as a minor:
Summary

- Looking for equilibria in dynamics of $n$-player games
- Different possible dynamics
- Conditions for (fair) termination
- Use of game minors and graph simulations
- In the article, non-positional strategies are also considered

Perspectives

Still open to find a forbidden pattern/minor for fair termination of $bPC$ in one-target games
Consider games with imperfect information: model of malicious router
A better model of asynchronicity?
Model fairness using probabilities?

Thank you! Questions?
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