# Equivalence of Deterministic Tree-to-String Transducers Is Decidable

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# Overview

Part 1: The General Setting

Part 2: Tree-to-Int Transducers

Part 3: Affine Spaces

Part 4: Polynomial Invariants

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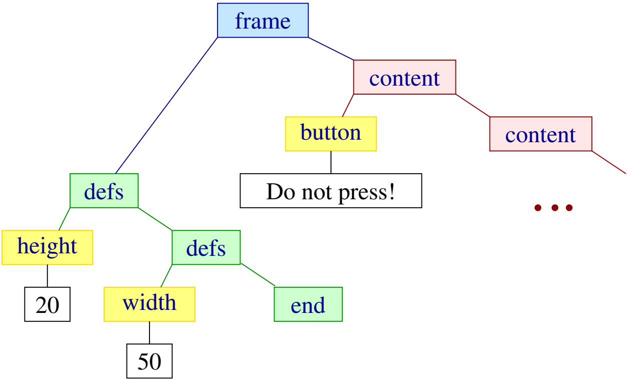
# Tree-to-String Translation

## Input



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## Output

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    ...
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## Realized by:

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q(\text{frame}(x_1, x_2)) & \rightarrow & \texttt{<frame } q_1(x_1)q(x_2)\texttt{</frame>} \\
q_1(\text{end}) & \rightarrow & \texttt{>} \\
q_1(\text{defs}(x_1, x_2)) & \rightarrow & q(x_1)q_1(x_2) \\
q(\text{height}(x_1)) & \rightarrow & \texttt{height = } q(x_1) \\
 & \ldots &
\end{array}
$$

# Tree-to-String Translation

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## Or realized by:

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q'_1(\text{end}) & \rightarrow & \epsilon \\
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versus

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q'(e) & \to & ab
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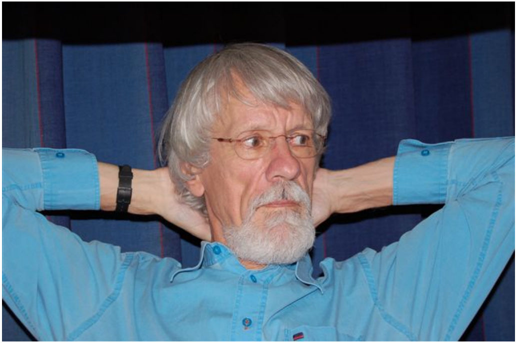
# Related Work

problem statement

Engelfriet, 1980



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| polynomial program invariants | Letichevsky, Lvov, 1996 |
| | Müller-Olm, S., 2004 |

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Obvious:

In-equivalence can be verified by counter example

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Required:

Complete proof system for equivalence

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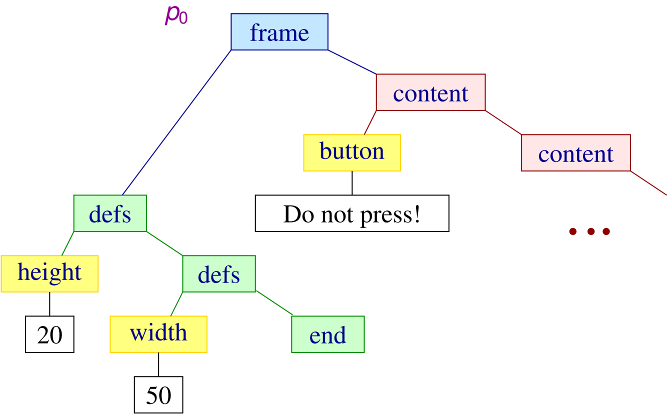
- There is a topdown-deterministic automaton $B$ with states $p \in P$

  // generalization of algebraic data type

  dom($p$) is the set of trees accepted at state $p$

  // trees of type $p$

# Topdown Automaton



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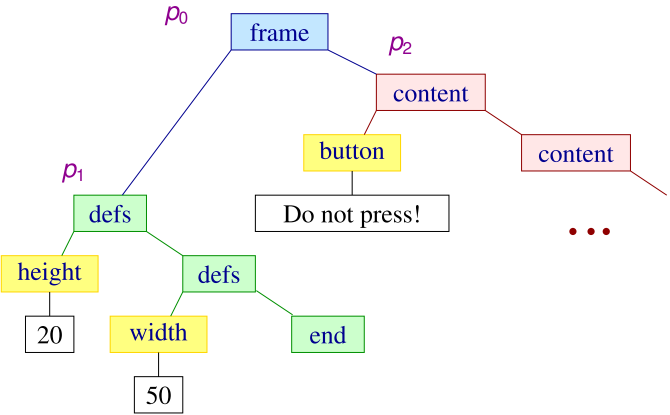

# Simplified Question

For states $q, q'$ of the transducer, $p_0 \in P$, does it hold that

$$[\![q]\!](t) = [\![q']\!](t) \qquad (t \in \mathrm{dom}(p_0))$$

# From Arbitrary Output to Ints

## Unary Output

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## Arbitary Output

| | | |
|---|---|---|
| letters $a, b, c, \ldots$ | $\hat{=}$ | digits $1, \ldots, h-1$ |
| string $aabc$ | $\hat{=}$ | $1 + h \cdot (1 + h \cdot (2 + h \cdot 3))$ |

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# Equivalence of Tree-to-Int Transducers

## Idea

- The semantics of a tree $t$ can be seen as

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- Consider $H(\mathbf{v}) = \mathbf{v}_q - \mathbf{v}_{q'}$.
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  1. $q, q'$ agree on inputs from $\mathcal{L}(B)$
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  3. $H(\mathbf{v}) = 0 \qquad (\mathbf{v} \in \mathrm{Aff}(V_{p_0}))$

     // affine closure

# Computing Affine Closures

Define

$$\begin{array}{rcl} [\![f]\!](\mathbf{x}_1,\ldots,\mathbf{x}_k) & = & ([\![T_1]\!](\mathbf{x}),\ldots,[\![T_n]\!](\mathbf{x})) \quad \text{where} \\ & & q(f(x_1,\ldots,x_k)) \rightarrow T_q \end{array}$$

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$$[\![3\cdot q(x_1)+q'(x_1)+2\cdot q'(x_2)+5]\!](\mathbf{x}_1,\mathbf{x}_2) \quad =$$

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$\Longrightarrow$ $\quad [\![f]\!] : \mathbb{Q}^n \times \ldots \times \mathbb{Q}^n \to \mathbb{Q}^n$ is affine.

# Computing Affine Closures (cont.)

## Consequence

$V'_p = \mathrm{Aff}(V_p)$ is the least solution of:

$$V'_p \supseteq [\![f]\!](V'_{p_1}, \ldots, V'_{p_k})$$

$((p, f) \mapsto p_1 \ldots p_k$ transition of $B)$ over the complete lattice of affine sub-spaces of $\mathbb{Q}^n$ !

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## Theorem

- Equivalence of total tree-to-int transducers relative to some $B$ is decidable in polynomial time.

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## Theorem

- Equivalence of total tree-to-int transducers relative to some $B$ is decidable in polynomial time.
- In-Equivalence of linear tree-to-string transducers is decidable in randomized polynomial time.

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# Tree-to-int Transducers with Parameters

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\begin{array}{lcl}
q(f(x_1, x_2), y) & \rightarrow & q_1(x_1, q_1(x_2, 1)) \\
q_1(a(x_1), y) & \rightarrow & y + q_1(x_1, y) \\
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The semantics of a tree $t$ is a vector

$$[\![t]\!] : (\mathbb{Q}^l \to \mathbb{Q})^n$$

of affine functions in the parameters

$\Longrightarrow$

can be represented by a matrix $([\![t]\!]_{qi}) \in \mathbb{Q}^{n\times(l+1)}$

# The Semantics of Constructors

$$[\![f]\!] : (\mathbb{Q}^{n\times(l+1)} \times \ldots \times \mathbb{Q}^{n\times(l+1)}) \to \mathbb{Q}^{n\times(l+1)}$$

thus is of the form:

$$([\![f]\!](\mathbf{x}_1, \ldots, \mathbf{x}_k))_{qj} = \text{polynomial in the } \mathbf{x}_{iq'j'}$$

# In the Example

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([\![f]\!](\mathbf{x}_1, \mathbf{x}_2))_{q0} & = & \mathbf{x}_{1q_10} + \mathbf{x}_{1q_11} \cdot (\mathbf{x}_{2q_10} + \mathbf{x}_{2q_11} \cdot 1) \\
([\![f]\!](\mathbf{x}_1, \mathbf{x}_2))_{q'0} & = & \mathbf{x}_{2q_10} + \mathbf{x}_{2q_11} \cdot (\mathbf{x}_{1q_10} + \mathbf{x}_{1q_11} \cdot 1)
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# Polynomial Invariant

polynomial equality:

$$\mathbf{z}_{q1} \cdot \mathbf{z}_{q'1} \cdot \mathbf{z}_{q'0} - 2 \cdot \mathbf{z}_{q0} + 3 \doteq 0$$

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$r_1 \doteq 0 \wedge \ldots \wedge r_m \doteq 0$ invariant at $p$ iff

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can be described by polynomial ideals ...

# Polynomial Ideals: A Primer

$R$ ring. $I \subseteq R$ ideal, if

- $a + b \in I$ whenever $a, b \in I$;
- $r \cdot a \in I$ whenever $a \in I$ and $r \in R$.

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$R = \mathbb{Q}[\mathbf{z}]$ polynomial ring

# Polynomial Ideals — Basis Theorem

David Hilbert (1890)

Every ideal of $\mathbb{Q}[\mathbf{z}]$ is finitely generated !

# Consequence

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- There are effective algorithms for
  - membership
  - inclusion
  - equality

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$p \mapsto I_p \subseteq \mathbb{Q}[\mathbf{z}]$ is inductive if for $p \to f(p_1, \ldots, p_k)$,

$$I_p \subseteq \{ r \in \mathbb{Q}[\mathbf{z}] \mid r[r^{(f)}/\mathbf{z}] \in I_{p_1} \oplus \ldots \oplus I_{p_k} \}$$

holds.

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holds.

# Main Theorem

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- For $p \in P$, let
$$\bar{I}_p = \{r \in \mathbb{Q}[\mathbf{z}] \mid r(\llbracket t \rrbracket) = 0 \quad (t \in \mathrm{dom}(p))\}$$
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## Corollary

Let $H(\mathbf{z}) = \mathbf{z}_{q0} - \mathbf{z}_{q'0}$. Then $q, q'$ are equivalent (relative to $p_0$) iff
$$H \in I_{p_0}$$
for some inductive invariant.

# Discussion

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- All potential counter examples can be enumerated ...

# Wrap-up

## Theorem

- Equivalence of deterministic tree-to-int transducers with parameters is decidable.

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- Equivalence of deterministic tree-to-int transducers with parameters is decidable.
- Equvalence of general deterministic tree-to-string transducers is decidable.

# Summary

Parameters allow to encode general output alphabets by means of unaries, i.e., numbers.

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Equivalence for unary transducers can be handled by means of techniques from precise program analysis, i.e., program proving.

Thank you!