The Complexity of Rational Synthesis

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Motivation

- Classical reactive system synthesis:
  - One system and one antagonist environment
  - Synthesize a system to ensure the specification

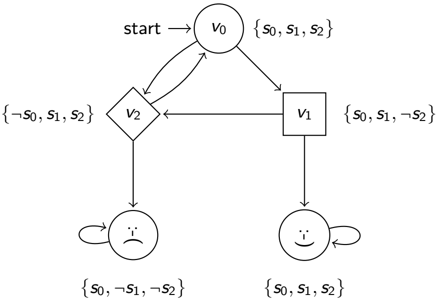
Environment Model $\|$? $\models$ Specification

  - Synthesis $\approx$ two-player zero-sum game

- Rational synthesis:
  - Multi-component environment
  - Non-antagonist objectives
  - Rational synthesis $\approx$ multiplayer turn-based game

Multiplayer Games

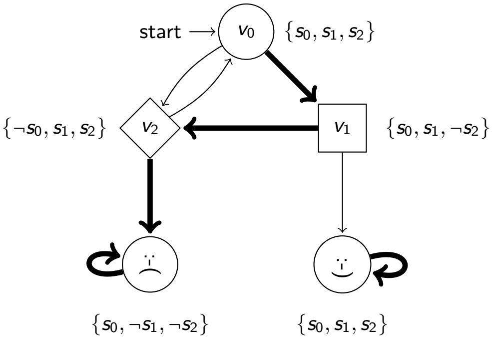
- $\mathcal{G} = \langle \Omega, V, (V_i)_{i\in\Omega}, E, v_0, (\mathcal{O}_i)_{i\in\Omega} \rangle$ where $\Omega = \{0, 1, \ldots, k\}$ and $\mathcal{O}_i \subseteq V^\omega$



- $S_0 = \{v_0, v_1, ☺, ☹\}$ $S_1 = \{v_0, v_1, v_2, ☺\}$ $S_2 = \{v_0, v_2, ☺\}$
- $\mathcal{O}_i = (S_i)^\omega$, $0 \leq i \leq 2$

Strategies and Nash Equilibria

- Strategy of Player i : $\sigma_i : V^* V_i \to V$
- Strategy profile $\bar{\sigma} = (\sigma_i)_{i \in \Omega}$,
- $pay(\bar{\sigma}) \in \{0,1\}^n$ s.t. $pay(\bar{\sigma})[i] = 1$ iff $out(\bar{\sigma}) \in \mathcal{O}_i$



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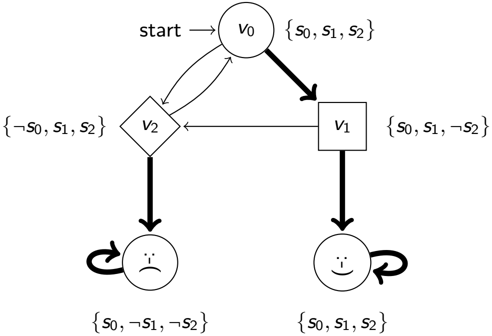

Nash Equilibrium

Definition (Nash Equilibrium (Nash51))

$\bar{\sigma}$ is Nash Equilibrium iff no incentive to deviate

$$pay(\bar{\sigma}_{-i}, \tau_i)[i] \leq pay(\bar{\sigma})[i] \ \forall i \in \Omega \text{ and } \tau_i \text{ strategy of Player } i$$

- $\bar{\sigma}$ is 0-fixed Nash Equilibrium iff

$$pay(\bar{\sigma}_{-i}, \tau_i)[i] \leq pay(\bar{\sigma})[i] \ \forall i \in \Omega \setminus \{0\} \text{ and } \tau_i \text{ strategy of Player } i$$

Rational Synthesis

- Rational synthesis = find winning strategy for the system (Player 0) against an multi-component environment (Players $1, ..., k$) with rational behavior.

Definition (Rational Synthesis Problems)

Given as input a game $\mathcal{G}$ with winning objectives $(\mathcal{O}_i)_{i \in \Omega}$, the two settings:

cooperative:[1] Is there a 0-fixed Nash equilibrium $\bar{\sigma}$ such that $pay(\bar{\sigma})[0] = 1$?

non-cooperative:[2] Is there a strategy σ_0 for Player 0 such that for any 0-fixed Nash equilibrium $\bar{\sigma} = \langle \sigma_0, \dots, \sigma_k \rangle$, we have $pay(\bar{\sigma})[0] = 1$?

[1] D. Fisman, O. Kupferman, and Y. Lustig. Rational synthesis. CoRR, abs/0907.3019, 2009.

[2] O. Kupferman, G. Perelli, and M. Y. Vardi. Synthesis with rational environments. In Multi-Agent Systems - 12th European Conference, EUMAS 2014

Rational Synthesis with LTL Objectives[Kupferman et al.]

$\bar{\sigma}$ *is* 0*-fixed NE iff* $\psi_{0Nash}(\bar{\sigma}) := \bigwedge_{i=1}^{k} [\![\tau_i]\!] \big(\flat(\bar{\sigma}_{-i}, \tau_i)\varphi_i \to \flat(\bar{\sigma})\varphi_i \big)$ *holds*

- Reduce to Model Checking of SL[NG] formulas with depth 1:

$$\text{Cooperative: } \psi_{cRS} := \langle\!\langle \sigma_0 \rangle\!\rangle \langle\!\langle \sigma_1 \rangle\!\rangle \ldots \langle\!\langle \sigma_k \rangle\!\rangle \big(\psi_{0Nash}(\bar{\sigma}) \wedge \varphi_0 \big)$$

$$\text{non-Cooperative: } \psi_{noncRS} := \langle\!\langle \sigma_0 \rangle\!\rangle [\![\sigma_1]\!] \ldots [\![\sigma_k]\!] \big(\psi_{0Nash}(\bar{\sigma}) \to \varphi_0 \big)$$

Theorem (Cooperative and non-cooperative rational-synthesis complexity)

The cooperative and non-cooperative rational-synthesis problems are 2EXPTIME-complete.

Rational Synthesis with Particular Objectives

	Cooperative		Non-Cooperative	
	Unfixed k	Fixed k	Unfixed k	Fixed k
Safety	NP-c	PTIME-c	PSPACE-c	PTIME-c
Reachability	NP-c	PTIME-c	PSPACE-c	PTIME-c
Büchi	PTIME-c[3]	PTIME-c[3]	PSPACE-c	PTIME-c
co-Büchi	NP-c[3]	PTIME-c	PSPACE-c	PTIME-c
Parity	NP-c[3]	$UP \cap co-UP$, parity-h	EXPTIME, PSPACE-h	PSPACE, NP-h, coNP-h
Streett	NP-c[3]	NP[3], NP-hard	EXPTIME,PSPACE-h	PSPACE-c
Rabin	P^{NP}, NP-h, CONP-h	P^{NP}, CONP-h	EXPTIME, PSPACE-h	PSPACE-c
Muller	PSPACE-c	PSPACE-c	EXPTIME, PSPACE-h	PSPACE-c
LTL	2EXPTIME-C[2]	2EXPTIME-C[2]	2EXPTIME-C[2]	2EXPTIME-C[2]

Table: Complexity of rational synthesis for k players.

$\text{Safe}(S) = \{\pi \in V^\omega \mid \forall n \geq 0 : \pi(n) \in S\}$ Büchi $(F) = \{\pi \in V^\omega \mid inf(\pi) \cap F \neq \varnothing\}$

$\text{Reach}(T) = \{\pi \in V^\omega \mid \exists n \geq 0 : \pi(n) \in T\}$ $\text{Muller}(\mu) = \{\pi \in V^\omega \mid inf(\pi) \vDash \mu\}$

If $p : V \to \mathbb{N}$, $\text{Parity}(p) = \{\pi \in V^\omega \mid min\{p(\pi(n)) \mid n \geq 0 \text{ and } \pi(n) \in inf(\pi)\} \text{ is even }\}$

[2] O. Kupferman, G. Perelli, and M. Y. Vardi. Synthesis with rational environments. In Multi-Agent Systems - 12th European Conference, EUMAS 2014

[3] M. Ummels. The complexity of Nash Equilibria in infinite multiplayer games. In Foundations of Software Science and Computational Structures, 11th International Conference, FOSSACS 2008

LTL Characterization of 0-fixed Nash Equilibria

For Safety, Reachability and tail objectives

- Compute W_i : the set of states from which Player i has a winning strategy
- If $\mathcal{O}_i$ are either all reachability or all tail objectives definable by LTL formula φ_i:

$$\phi^{\mathcal{G}}_{\text{0Nash}} = \bigwedge_{i=1}^{k} (\neg\varphi_i \rightarrow \Box\neg W_i^{\mathcal{G}})$$

- If $\mathcal{O}_i = \text{Safe}(S_i)$ for some $S_i \subseteq V$:

$$\phi^{\mathcal{G}}_{\text{0Nash}} = \bigwedge_{i=1}^{k} ((\neg W_i^{\mathcal{G}} \ \mathcal{U} \ \neg S_i) \vee \Box S_i)$$

Lemma (Characterization of 0-fixed Nash Equilibria)

Let $\mathcal{G}$ be a multiplayer game with either all safety, all reachability, or all tail objectives, definable in LTL$[\mathcal{G}]$. Then, the following hold:

1. *For all $\pi \in Plays(\mathcal{G})$, if $\pi \models \phi^{\mathcal{G}}_{0Nash}$, then $\exists\bar{\sigma}$ a 0-fixed Nash equilibrium in $\mathcal{G}$ s.t. $out(\bar{\sigma}) = \pi$,*
2. *For all 0-fixed Nash equilibrium $\bar{\sigma}$ in $\mathcal{G}$, $out(\bar{\sigma}) \models \phi^{\mathcal{G}}_{0Nash}$.*

Cooperative Rational Synthesis Problem

Lemma

There is a solution to the cooperative synthesis problem iff there exists a path $\pi \in Plays(\mathcal{G})$ such that $\pi \models \phi_{0Nash}^{\mathcal{G}} \wedge \varphi_0$.

- If it exists such π, it exists $\pi = x(y)^\omega$ with $|xy|$ polynomial in $\mathcal{G}$

Theorem

The CRSP is

- PTIME *for Büchi objectives (by Ummels).*
- NP-COMPLETE *for Safety, Reachability, co-Büchi, Parity and Streett objectives*
- PSPACE, *NP-h and co-NP-hard for Rabin objectives*
- PSPACE-COMPLETE *for Muller objectives*

Non-Cooperative Rational Synthesis Problem

non-cooperative: *Is there a strategy* σ_0 *for Player* 0 *such that for any* 0*-fixed Nash equilibrium* $\bar{\sigma} = \langle \sigma_0, \ldots, \sigma_k \rangle$*, we have* $pay(\bar{\sigma})[0] = 1$ *?*

- First attempt: two player zero-sum game with objective

$$\mathcal{O} = \{\pi \mid \pi \models \phi_{0\mathsf{Nash}}^{\mathcal{G}} \rightarrow \varphi_0\}$$

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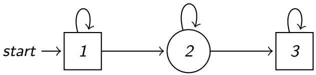
(Counterexample!)



Reachability objectives: $R_0 = \{2\}$, $R_1 = \{3\}$
$W_1 = \{3\}$
Zero-sum game objective: $(\Box\neg 3 \rightarrow \Box\neg 3) \rightarrow \Diamond 2 \equiv \Diamond 2$

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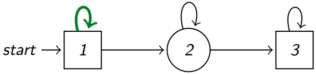

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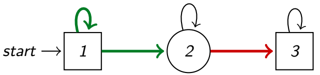

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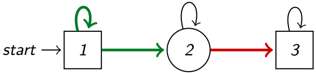

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Fix σ_0. Only 0-fixed NE w.r.t. σ_0 should be considered !!!

NCRSP Solution

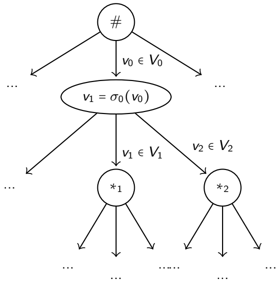
- Desired objective: find σ_0 s.t. $\forall\pi$ in $\mathcal{G}[\sigma_0]$,

$$\pi \models \phi_{0\text{Nash}}^{\mathcal{G}[\sigma_0]} \rightarrow \varphi_0$$

- May be difficult to compute $W_i^{\mathcal{G}[\sigma_0]}$!

Solution:

- Encode $\sigma_0 : V^* V_0 \rightarrow V$ as a $(V \cup \{*_i \mid 1 \leq i \leq k\})$-labelled V-tree t_{σ_0}



Define a nondeterministic tree automaton $\mathcal{T}$ s.t.

$$\mathcal{L}(\mathcal{T}) = \{t_{\sigma_0} \mid \sigma_0 \text{ is solution to NCRSP}\}$$

- $\mathcal{T}$ guesses sufficient states in $W_i^{\mathcal{G}[\sigma_0]}$

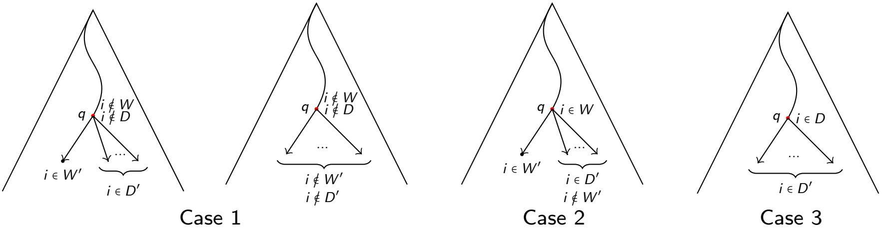
Nondeterministic Tree automaton $\mathcal{T}$

$$\mathcal{T} = \mathcal{C} \times \mathcal{U}$$

- Deterministic Safety tree automaton $\mathcal{C}$:
  - accepts only proper encodings of strategies σ_0 of Player 0
  - polynomial size in $\mathcal{G}$
- Nondeterministic tree automaton $\mathcal{U}$
  - for each branch π of t_{σ_0} compatible to σ_0, check that:
    - $\pi \in \mathcal{O}_0$ or
    - guess at least one player that wants to deviate from π and check he has a winning strategy under σ_0
  - exponential size in $\mathcal{G}$

Nondeterministic Tree automaton $\mathcal{U}$

- States: $q = (W, D, v) \in 2^{\Omega} \times 2^{\Omega} \times V$
- W : the set of players that have winning strategy from v
- D : the set of players that have a winning deviation from the current prefix
- $\delta(q, v') = ((W, D, v'), v')$ for $q = (W, D, v)$
- $\delta(q, *_i)$ for $q = (W, D, v)$:
  1. $i \in \bar{W} \cap \bar{D}$: either do not guess anything or guess that Player i has a winning strategy
  2. $i \in W \cap \bar{D}$: guess the next move according to the winning strategy
  3. $i \in \bar{W} \cap D$: just propagate the sets D and W
  4. $i \in D \cap W$: never reachable by construction



Nondeterministic Tree automaton $\mathcal{U}$

- On each branch η of a run in $\mathcal{U}$, (W, D) is monotone w.r.t.

$$(W, D) \sqsubseteq (W', D') \text{ iff } D \subseteq D' \text{ and } W \cup D \subseteq W' \cup D'$$

  - D and W stabilize on $lim_D(\eta)$ and $lim_W(\eta)$
- Accepting condition: branches η s.t.

$$\Big(\eta|_V \in \mathcal{O}_0 \vee \bigvee_{i=1}^{k} \big(\eta|_V \notin \mathcal{O}_i \wedge \varphi_{\exists dev}(i, \eta)\big)\Big) \wedge \bigwedge_{i \in \lim_W(\eta)} \eta|_V \in \mathcal{O}_i$$

  - $\forall i \in lim_W(\eta)$, Player i wins
  - and
    - either Player 0 wins
    - or $\exists i \in \Omega$ s.t. Player i loses but has a winning deviation ($i \in lim_D(\eta)$ for tail objectives)

$\mathcal{T}$ as a two-player game $\mathcal{G}_\mathcal{T}$

- Two-player zero-sum game $\mathcal{G}_\mathcal{T}$:
  - Eve: constructs a tree and a run in $\mathcal{T}$ on this tree
    - guesses σ_0, W_i and constructs winning strategy for Player i from states in W_i
  - Adam: prove the run is not accepting by choosing directions in the tree
    - plays for environment components (players $1 \dots k$)
- Eve's objective: the accepting condition of $\mathcal{T}$

Solve $\mathcal{G}_{\mathcal{T}}$ for particular objectives

- Safety, Reachability, Büchi, co-Büchi: reduce to finite-duration game $\mathcal{G}_{\mathcal{T}}^{f}$
  - The plays of the game $\mathcal{G}_{\mathcal{T}}^{f}$ are of polynomial length in the size of the initial game $\mathcal{G}$.
  - Solve $\mathcal{G}_{\mathcal{T}}^{f}$ on-the-fly by a PTIME alternating algorithm
- Muller: reduce to a two-player zero-sum parity game with an exponential number of states but a polynomial number of priorities
  - use Last Appearance Record (LAR) construction
  - solve in EXPTIME in number of priorities

Solve $\mathcal{G}_\mathcal{T}$ for particular objectives

Theorem

For each $\mathcal{X} \in \{\text{Reach}, \text{Safe}, \text{Buchi}, \text{coBuchi}, \text{Street}, \text{Rabin}, \text{Parity}, \text{Muller}\}$*, the non-cooperative rational synthesis problem in multiplayer* $\mathcal{X}$*-games is* PSPACE-HARD.

Proof by reduction from QBF. □

	Cooperative		Non-Cooperative	
	Unfixed k	Fixed k	Unfixed k	Fixed k
Safety	NP-c	PTIME-c	PSPACE-c	PTIME-c
Reachability	NP-c	PTIME-c	PSPACE-c	PTIME-c
Büchi	PTIME-c[3]	PTIME-c[3]	PSPACE-c	PTIME-c
co-Büchi	NP-c[3]	PTIME-c	PSPACE-c	PTIME-c
Parity	NP-c[3]	$UP \cap co - UP$, parity-h	EXPTIME, PSPACE-h	PSPACE, NP-h, coNP-h
Streett	NP-c[3]	NP[3], NP-hard	EXPTIME,PSPACE-h	PSPACE-c
Rabin	P^{NP}, NP-h, coNP-h	P^{NP}, coNP-h	EXPTIME, PSPACE-h	PSPACE-c
Muller	PSPACE-c	PSPACE-c	EXPTIME, PSPACE-h	PSPACE-c
LTL	2EXPTIME-c[2]	2EXPTIME-c[2]	2EXPTIME-c[2]	2EXPTIME-c[2]

Table: Complexity of rational synthesis for k players.

	Cooperative		Non-Cooperative	
	Unfixed k	Fixed k	Unfixed k	Fixed k
Safety	NP-c	PTIME-c	PSPACE-c	PTIME-c
Reachability	NP-c	PTIME-c	PSPACE-c	PTIME-c
Büchi	PTIME-c[3]	PTIME-c[3]	PSPACE-c	PTIME-c
co-Büchi	NP-c[3]	PTIME-c	PSPACE-c	PTIME-c
Parity	NP-c[3]	$UP \cap co-UP$, parity-h	EXPTIME, PSPACE-h	PSPACE, NP-h, coNP-h
Streett	NP-c[3]	NP[3], NP-hard	EXPTIME,PSPACE-h	PSPACE-c
Rabin	P^{NP}, NP-h, CONP-h	P^{NP}, CONP-h	EXPTIME, PSPACE-h	PSPACE-c
Muller	PSPACE-c	PSPACE-c	EXPTIME, PSPACE-h	PSPACE-c
LTL	2EXPTIME-C[2]	2EXPTIME-C[2]	2EXPTIME-C[2]	2EXPTIME-C[2]

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- Future work: other notions of rationality, e.g. secure equilibria, doomsday equilibria or subgame perfect equilibria

Thank you!