Coloring a set of touching strings*

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Abstract

For a family of geometric objects in the plane $\mathcal{F} = \{S_1, \ldots, S_n\}$, define $\chi(\mathcal{F})$ as the least integer ℓ such that the elements of $\mathcal{F}$ can be colored with ℓ colors, in such a way that any two intersecting objects have distinct colors. When $\mathcal{F}$ is a set of Jordan regions that may only intersect on their boundaries, and such that any point of the plane is contained in at most k regions, it can be proven that $\chi(\mathcal{F}) \leq 3k/2 + o(k)$ since the problem is equivalent to cyclic coloring of plane graphs. In this paper, we study the same problem when Jordan regions are replaced by a family $\mathcal{F}$ of Jordan curves that do not cross. In other words, any two curves of $\mathcal{F}$ are only allowed to "touch" each other. We conjecture that also in this case, $\chi(\mathcal{F})$ only depends on the maximum number of curves containing a given point of the plane. To support this conjecture, we prove it when the curves are x-monotone (any vertical line intersect each curve in at most one point), and we give a bound in the general case that also depends on how many times two curves intersect.

Keywords: Intersection graphs, graph coloring, strings.

1 Introduction

For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of subsets of a set Ω, the *intersection graph* $G(\mathcal{F})$ of $\mathcal{F}$ is defined as the graph with vertex set $\mathcal{F}$, in which two vertices are adjacent if and only if the corresponding sets have non-empty intersection.

For a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the least number of colors needed in a proper coloring of G (a coloring such that any two adjacent vertices have distinct colors). When talking about a proper coloring of a family $\mathcal{F}$ of subsets of a given set, we implicitly refer to a proper coloring of the intersection graph of $\mathcal{F}$, thus the chromatic number $\chi(\mathcal{F})$ is defined in a natural way.

The chromatic number of families of geometric objects in the plane have been extensively studied since the sixties [2, 7, 10, 11, 12]. Since it is possible to construct sets of pairwise intersecting segments of any size, the chromatic number of sets of segments is unbounded in general. However, a famous conjecture of Erdős states that triangle-free intersection graphs of straight-line segments in the plane have a bounded chromatic number (see [6]). This initiated the study of the chromatic number of families of geometric objects in the plane as a function

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of their *clique number*, the maximum size of subsets of the family that pairwise intersect [4]. In this paper, we consider families of geometric objects in the plane for which the chromatic number only depends of local properties of the families, such as the maximum number of objets containing a given point of the plane.

Consider a set $\mathcal{F} = \{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ of Jordan regions (subsets of the plane which are homeomorphic to a closed disk) such that the intersection of the interiors of any two regions is empty. Let $\mathcal{H}_\mathcal{F}$ be the planar hypergraph with vertex set $\mathcal{F}$, in which the hyperedges are the maximal sets of regions whose intersection is non-empty. A proper coloring of $\mathcal{F}$ is equivalent to a coloring of $\mathcal{H}_\mathcal{F}$ in which all the vertices of every hyperedge have distinct colors. If every point is contained in at mosy k regions, Borodin conjectured that there exists such a coloring of $\mathcal{H}_\mathcal{F}$ with at most $\frac{3}{2}k$ colors [3]. It was recently proven that this conjecture holds asymptotically [1]. As a consequence, $\mathcal{F}$ can be properly colored with $\frac{3}{2}k + o(k)$ colors.

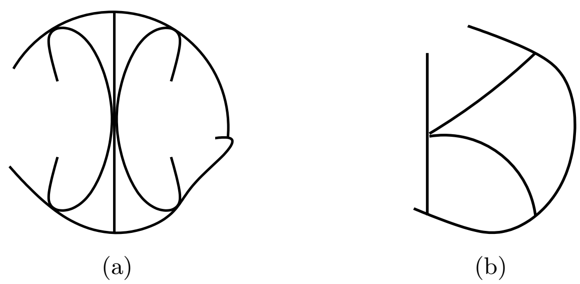


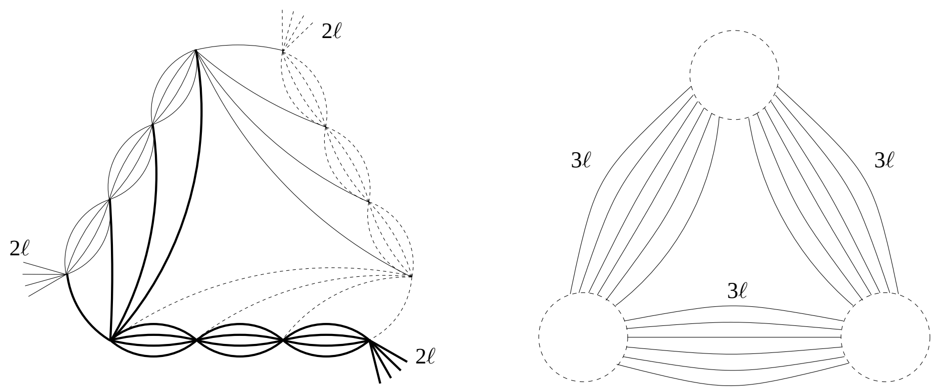
Figure 1: (a) A 3-touching set of strings $\mathcal{S}_1$ with $G(\mathcal{S}_1) \cong K_5$. (b) A one-sided 3-contact representation of curves $\mathcal{S}_2$ with $G(\mathcal{S}_2) \cong K_4$.

It seems natural to investigate the same problem when Jordan regions are replaced by Jordan curves. Consider a set $\mathcal{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of simple Jordan curves in the plane (the range of continuous injective functions from $[0,1]$ to $\mathbb{R}^2$). These curves are usually referred to as *strings*. We say that $\mathcal{S}$ is *k-touching* if no pair of strings of $\mathcal{S}$ cross, and at most k strings can "touch" in any point of the plane, i.e., any point of the plane is contained in at most k strings (see Figure 1(a) for an example).

Note that the family of all touching sets of strings contains all *curve contact representations*, defined as sets of strings such that the interior of any two strings have empty intersection. In other words, if c is a contact point in the interior of a string s, all the curves containing c distinct from s end at c. In [8], the author studied curve contact representations such that all the curves ending at c leave from the same side of s. Such a representation is said to be *one-sided* (see Figure 1(b) for an example). It was proved in [8] that any one-sided k-contact representation of curves (that is, such that any point is contained by at most k curves) can be colored with at most $2k$ colors.

In this paper, our aim is to study k-touching sets of strings in their full generality. Observe that if $\mathcal{S}$ is a k-touching, k might be much smaller than the maximum degree of $\mathcal{S}$. However, based on the cases of jordan regions and contact representation of curves, we conjecture the following:

Conjecture 1 *For some constant $c > 0$, any k-touching set $\mathcal{S}$ of strings can be colored with ck colors.*



(a) (b)

Figure 2: (a) The construction of a 2ℓ-sun. (b) A set $\mathcal{S}$ of k-touching strings requiring $\lceil \frac{9k}{2} \rceil - 5$ distinct colors.

We first show that the constant c in Conjecture 1 is at least $\frac{9}{2}$.

Claim 1 *For every odd $k \geq 1$, $k = 2l+1$, there exists a set of k-touching strings $\mathcal{S}_k$ such that the strings of $\mathcal{S}_k$ pairwise touch and such that $|\mathcal{S}_k| = 9l = 9(k-1)/2$. Thus $\chi(\mathcal{S}_k) \geq 9l = \lceil \frac{9k}{2} \rceil - 5$.*

Consider n touching strings $s_1, \ldots, s_n$ that all intersect n points $c_1, \ldots, c_n$ in the same order (see the set of bold strings in Figure 2(a) for an example when $n = 4$), and call this set of strings an *n-braid*. For some $\ell > 0$, take three 2ℓ-braids $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and for $i = 1, 2, 3$, connect each of the strings of $\mathcal{S}_i$ to a different intersection point of $\mathcal{S}_{f(i)}$, where $f(i) = (1+i)$ mod 3, while keeping the family of strings touching (see Figure 2(a)). We call this set of touching strings a *2ℓ-sun*. Observe that a 2ℓ-sun contains 6ℓ strings (the *rays*) that pairwise intersect, and that each intersection point contains at most $2\ell + 1$ strings.

We now take three 2ℓ-suns and paste their respective rays as depicted in Figure 2(b). We obtain a $(2\ell+1)$-touching set containing 9ℓ pairwise intersecting strings. Hence we need at least $\lceil \frac{9k}{2} \rceil - 5$ colors in any proper coloring of this set of strings.

Before proving any result on the structure of sets of k-touching strings in general, we make the following important observation:

Observation 2 *The family of intersection graphs of 2-touching strings is exactly the class of planar graphs.*

The class of planar graphs being exactly the class of intersection graphs of 2-touching Jordan regions (see [9]) it is clear that planar graphs are intersection graphs of 2-touching strings (by taking the boundaries of the Jordan regions). Furthermore, every intersection graph of 2-touching strings is contained in the intersection graph of 2-touching Jordan regions, and is thus planar. Indeed, it is easy given a set of 2-touching strings $\mathcal{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ to draw a set of 2-touching Jordan regions $\mathcal{F} = \{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ such that $\mathcal{C}_i \subset \mathcal{R}_i$ for every $i \in [1, n]$.

2 μ-intersecting strings

Let $\mathcal{S}$ be a k-touching set of strings. The set $\mathcal{S}$ is said to be *μ-intersecting* if any two strings intersect in at most μ points. We denote by $H(\mathcal{S})$ the multigraph associated to $\mathcal{S}$: the vertices of $H(\mathcal{S})$ are the strings of $\mathcal{S}$, and two strings with t common points correspond to two vertices connected by t edges in $H(\mathcal{S})$.

Our first bound on the chromatic number of k-touching sets of strings depends on how many times two strings intersect, although Conjecture 1 states that the bound should only depend on k. This is not completely surprising: for instance, the best known upper bound on the chromatic number of K_k-free intersection graphs of curves in the plane also depends on how many times two curves intersect, see [4]. We prove the following theorem.

Theorem 3 *Any k-touching set $\mathcal{S}$ of μ-intersecting strings can be properly colored with $3\mu k$ colors.*

The proof is based on an upper bound on the number of edges of such graphs.

Lemma 4 *If $\mathcal{S}$ is a k-touching set of n μ-intersecting strings, then $H(\mathcal{S})$ has less than $\frac{3}{2}\mu kn$ edges.*

Proof. For a string s, let $d(s)$ denote the number of strings intersecting s, and let n denote the number of strings of $\mathcal{S}$. Let us denote by $D(c)$ the number of strings containing an intersection point c (for any c, $D(c) \geq 2$ by definition), and let N denote the number of intersection points of $\mathcal{S}$. Note that k is the maximum of $D(c)$ over all intersection points c.

Let us slightly modify $\mathcal{S}$ in order to obtain a set $\mathcal{S}'$ of 2-touching and μ-intersecting strings. For that, repeat the following operation while there exists an intersection point c with $D(c) > 2$. Consider a string $s \ni c$ such that all the other strings at c are on the same side of s (possible since the strings do not cross each other). Move s along another string $s' \ni c$ to a new intersction point c' in such way that the strings remain touching (see Figure 3).

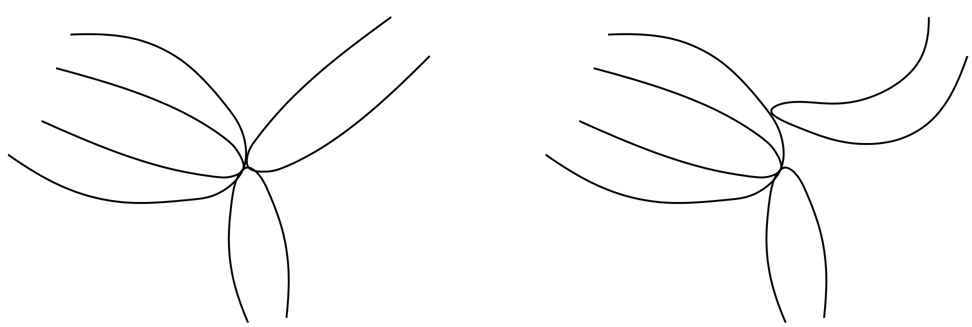


Figure 3: Operation reducing the number of strings at an intersection point c.

Each intersection point c in $\mathcal{S}$ corresponds to a set X_c of intersection points in $\mathcal{S}'$. Let N' be the number of intersection points in $\mathcal{S}'$. By construction, each X_c has size exactly $D(c) - 1$, hence $N' = \sum_c |X_c| = \sum_c (D(c) - 1)$. Observe that $H(\mathcal{S}')$ is a planar subgraph of $H(\mathcal{S})$, so $N' \leq (3n - 6)\mu$. As for any intersection point c in $\mathcal{S}$, $D(c) \leq k$, we have

$$\sum_c D(c)(D(c) - 1) \;\leq\; kN' \;\leq\; (3n - 6)\mu k \;<\; 3\mu kn.$$

Finally, since the number of edges of $H(\mathcal{S})$ is at most $\frac{1}{2}\sum_{s\in\mathcal{S}}\sum_{c\in s}(D(c)-1)$ which equals $\frac{1}{2}\sum_{c} D(c)(D(c)-1)$, we have that $H(\mathcal{S})$ has less than $\frac{3}{2}\mu kn$ edges. $\square$

Proof of Theorem 3. If $H(\mathcal{S})$ has less than $\frac{3}{2}\mu kn$ edges, its average degree is less than $3\mu k$. Furthermore, since the class of graphs defined by k-touching and μ-intersecting segment sets is closed under taking induced subgraphs, $H(\mathcal{S})$ is $(3\mu k-1)$-degenerate and thus $3\mu k$-colorable. $\square$

In particular, if a k-touching set $\mathcal{S}$ of strings is such that any two strings intersect in at most one point, Theorem 3 yields a bound of $3k$ for the chromatic number of $\mathcal{S}$. We suspect that it is far from tight:

Conjecture 2 *For some constant $c > 0$, every k-touching set $\mathcal{S}$ of 1-intersecting strings can be properly colored with $k + c$ colors.*

This is interesting to notice that even though the bound for k-touching μ-intersecting graphs in Conjecture 1 does not depend on μ, the chromatic number of these graphs has some connection with μ: sets of strings with $\mu = 1$ have chromatic number at most $3k$, whereas there exists sets of strings with large μ and chromatic number at least $\frac{9}{2}k - 5$.

Figure 4: (a) A 3-touching set of 1-intersecting strings requiring 7 colors (b) A k-touching set of 1-intersecting strings requiring $k+2$ colors (here $k=4$).

Note that the constant c in Conjecture 2 is at least 4. Figure 4(a) depicts a 3-touching set of seven 1-intersecting strings, in which any two strings intersect. Hence, this set requires seven colors. However this construction does not extends to k-touching sets with $k \geq 4$, it might be that the constant is smaller for higher k. In Figure 4(b), the k-touching set $\mathcal{S}_k$ contains $k+2$ non-crossing strings, and is such that any two strings intersect. Hence, $k+2$ colors are required in any proper coloring.

3 Contact representation of segments

An interesting example of k-touching sets of 1-intersecting strings is the family of non-crossing (straight-line) segments in the plane. This family is also known as *contact representation of segments*, and has been studied in [5], where the authors prove that any bipartite planar graph has a contact representation with horizontal and vertical segments. In [14], it is proven that the intersection graph of any one-sided 2- or 3-contact representation of segments is planar.

Moreover, the author proved that it is NP-complete to determine whether a one-sided 2-contact representation of segments is 3-colorable.

By Theorem 3, the chromatic number of any k-touching set $\mathcal{S}$ of segments is at most $3k$. The next theorem shows that this bound can be further improved:

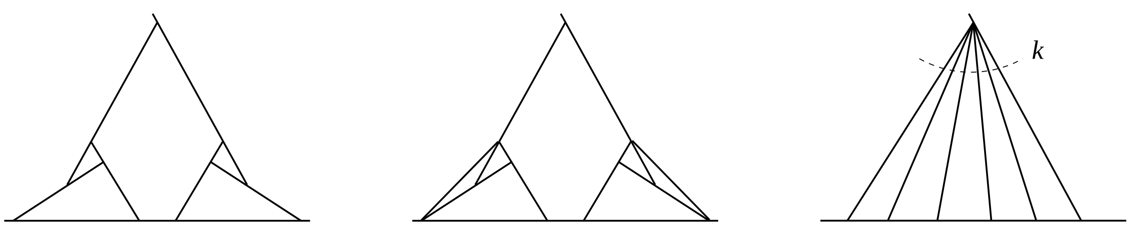
Theorem 5 *Any k-touching set $\mathcal{S}$ of segments can be properly colored with $2k$ colors.*

The proof relies on the same reasoning as the proof of Theorem 3.

Lemma 6 *If $\mathcal{S}$ is a k-touching set of n segments, then $G(\mathcal{S})$ has less than kn edges.*

Proof. The proof follows the same lines as the proof of Lemma 4. By slightly modifying $\mathcal{S}$ around each contact point, we obtain a 2-touching set of segments $\mathcal{S}'$. Note that by slightly lengthening some of the segments, we can assume without loss of generality that no two segments of $\mathcal{S}'$ end at the same point. This means that every contact point of $\mathcal{S}'$ is the end of exactly one segment. Moreover, if s is a segment of $\mathcal{S}'$ with the bottom-left-most end, then this end is not a contact point. Hence, the number of contact points of $\mathcal{S}'$ is less than twice the number of segments of $\mathcal{S}'$.

The remaining of the proof proceeds exactly as in the proof of Lemma 4. □



(a) (b) (c)

Figure 5: (a) A 2-touching set of segments requiring 4 colors (b) A 3-touching set of segments requiring 5 colors (c) A k-touching set of segments requiring $k+1$ colors.

Figure 5 depicts k-touching sets of segments requiring $k+2$ colors, for $k = 2, 3$. However it does not appear to be trivial to extend this construction for any $k \geq 4$. Note that Figure 5(b) also shows that there are intersection graphs of 3-contact representations of segments (with two-sided contact points) that are not planar.

4 x-monotone strings

An *x-monotone string* is a string such that every vertical line intersects it in at most one point. Alternatively, it can be defined as the curve of a continuous function from an interval of $\mathbb{R}$ to $\mathbb{R}$. Our aim in this section is to find a non-trivial bound for the chromatic number of k-touching x-monotone strings which does not depend on how many times two curves intersect.

Similarly as in Observation 2, it is easy to see that the family of intersection graphs of x-monotone 2-touching sets of strings in the plane is exactly the family of planar graphs. To see this, take a representation of a planar graph G as a contact of disks, and rotate the representation so that all the contact points have distinct x-coordinates. In each disk, order the contact points $p_1, \ldots, p_\ell$ on this disk by increasing x-coordinate, and for any $1 \le i \le \ell - 1$, connect the points p_i and p_{i+1} by a straight-line segment. We obtain a 2-touching representation of G, in which all the strings are x-monotone.

We can also observe that the strings in Figure 2(a) can be made x-monotone. This k-touching set contains $3k-3$ strings $s_1, \ldots, s_{3k-3}$. Next, take $k-1$ new x-monotone strings intersecting $3k-3$ times, at the points $c_1, \ldots, c_{3k-3}$. It is now possible to connect s_i to c_i, for $1 \le i \le 3k-3$, so that the all the strings remain x-monotone. This k-touching set $\mathcal{S}_k$ contains $4k-4$ x-monotone strings. Hence, $\chi(\mathcal{S}_k) \ge 4k-4$.

We now prove that this bound is tight up to a factor of $\frac{3}{2}$.

Theorem 7 *If $\mathcal{S}$ is a k-touching set of x-monotone strings, then $\chi(\mathcal{S}) \le 6k-6$.*

As in the previous sections, the proof relies on a upper bound of the number of edges.

Lemma 8 *If $\mathcal{S}$ is a k-touching set of n x-monotone strings, then $G(\mathcal{S})$ has less than $3n(k-1)$ edges.*

Proof. We consider each string of $\mathcal{S}$ as a continuous function $f_i : I_i = [h_i, t_i] \to \mathbb{R}$. Without loss of generality, we can assume that for any $i \neq j$, the function $f_i - f_j$ has a finite number of zeros, and that all the elements of $A = \{h_1, \ldots, h_n, t_1, \ldots, t_n\}$ are distinct.

We say that $f_i \preceq f_j$ if and only if there exists $x \in I_i \cap I_j$ such that $f_i(x) \le f_j(x)$. Observe that this definition is consistent, as both functions are continuous and the set is k-touching. For any i, j with $f_i \prec f_j$ and any $x \in I_i \cap I_j$, define $\theta_x(i,j)$ as the number of integers k such that $x \in I_k$ and $f_i \prec f_k \prec f_j$.

Consider the graph G' with vertex set $\{f_1, \ldots, f_n\}$, and with an edge between f_i and f_j if and only if $f_i \prec f_j$ and $\theta_x(i,j) \le k-2$ for some x. Observe that $G(\mathcal{S})$ is a subgraph of G' since the strings are k-touching. Hence, any upper bound on the number of edges of G' will give an upper bound on the number of edges of $G(\mathcal{S})$.

Consider an edge $f_i f_j$ in G' with $f_i \prec f_j$ and take $h = \max(h_i, h_j)$. If $\theta_h(i,j) \le k-2$, we say that the edge $f_i f_j$ is *h-realized*. Otherwise take the smallest $t = t_k$, for some $1 \le k \le n$, such that $\theta_t(i,j) = k-1$ and $\theta_{t+\epsilon}(i,j) = k-2$ with $\epsilon = \frac{1}{2} \min_{(x,y) \in A^2} |x-y|$. Such a value t must exist, since otherwise f_i and f_j would not be adjacent in G' (moreover we have that $f_i \prec f_k \prec f_j$). In this case we say that the edge $f_i f_j$ is *t-realized*. Observe that every edge of G' is either h-realized or t-realized for some h or t in A. We now count the number of h_i-realized edges and the number of t_i-realized edges, for any $1 \le i \le n$.

The h_i-realized edges correspond to functions f_j with $h_j < h_i$ and such that either $f_i \prec f_j$ and $\theta_{h_i}(i,j) \le k-2$, or $f_j \prec f_i$ and $\theta_{h_i}(j,i) \le k-2$. It is easy to see that there are at most $2k-2$ such functions f_j. The t_i-realized edges correspond to pairs of function (f_j, f_k) with $f_j \prec f_i \prec f_k$, and $\theta_t(i,j) = k-1$ and $\theta_{t_i+\epsilon}(i,j) = k-2$. There are at most $k-1$ such pairs (f_j, f_k).

Since there is no h_j-realized edges and t_k-realized edges for $h_j = \min_{1 \le i \le n} h_i$ and $t_k = \max_{1 \le i \le n} t_i$, G' (and so $G(\mathcal{S})$) contains at most $(n-1)(3k-3) < 3n(k-1)$ edges. $\square$

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