# On Universal Graphs of Minor Closed Families

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**Abstract**

For a family $\mathcal{F}$ of graphs, a graph $U$ is said to be $\mathcal{F}$-universal if every graph of $\mathcal{F}$ is a subgraph of $U$. Similarly, a graph is said to be $\mathcal{F}$-induced-universal if every graph of $\mathcal{F}$ is an induced subgraph of $U$. In this paper, we give constructive proofs of new upper bounds for size and order of such minimal graphs for the family of graphs with no $K$-minor and more particularly on graphs of bounded treewidth.

*Keywords:* Combinatorial problems, Universal graphs, Minor, treewidth.

## 1 Introduction

For a finite family $\mathcal{F}$ of graphs, a graph $U$ is said to be $\mathcal{F}$-universal if every graph in $\mathcal{F}$ is a subgraph of $U$. For instance, if we denote by $\mathcal{F}_n$ the family of all graphs with at most $n$ vertices, then the complete graph $K_n$ is $\mathcal{F}_n$-universal. The problem of finding small universal graphs was originally motivated by circuit design for computer chips [5]. Nowadays, this application is no longer of interest but the problem was still studied as a beautiful combinatorial problem. Formally, the problem consists in finding for a family of graph of at most $n$ vertices a $n$-vertex universal graph with minimal number of edges. Numerous

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families of graphs were examined for this problem such as forests [9], bounded-degree forests [3,4], and bounded-degree graphs [1].

The notion of induced-universal graph can be similarly defined. For a family $\mathcal{F}$ of graphs, a graph $U$ is *$\mathcal{F}$-induced-universal* if every graph in $\mathcal{F}$ is an induced subgraph of $U$. The family of all graphs on $n$ vertices was considered by Moon [13], while Chung considered trees, planar graphs, and graphs with bounded arboricity [10].

The problem of finding a small induced-universal graph is strongly related to a notion of distributed data structure known as *adjacency labeling scheme*. An adjacency labeling scheme for a family $\mathcal{F}$ of graphs consists in a labeling function that assigns labels to the vertices of any graph of $\mathcal{F}$ such that the adjacency can be decided between any two vertices by only looking at their labels. The problem of finding an adjacency labeling scheme with small labels was introduced by Breuer [7,8]. Kannan, Naor and Rudich [12] established that there is an adjacency labeling scheme with labels of size $k(n)$ bits for the family $\mathcal{F}_n$ if and only if there exists an $\mathcal{F}_n$-induced-universal graphs with $2^{k(n)}$ vertices. This strong link between the two notions implies that any result on one notion has direct consequences on the other. For instance, the best known induced-universal graph for the class of forests is deduced from a labeling scheme [2].

Our main contribution is universal and induced-universal graphs for bounded treewidth and no $K$-minor graphs. Our universal graph for $n$-vertex graphs of treewidth $k$ has $n$ vertices and $O(k^2 n \log^2 n)$ edges and our induced-universal graph has $O((k \log n)^{O(k)} n)$ vertices (previous best known bound of $O(n^k)$ [12]). For $n$-vertex graphs with no $K$-minor, our universal graphs has $O(n^{3/2}|V(K)|^{3/2})$ edges and our induced-universal graph has $O((r \log n)^{O(r)} n^2)$ vertices (previous best known bound of $O(n^r)$ [12]) with $r$ a constant depending only on $K$.

## 2 Universal graphs using bisector

In this section, we show how to use special separator to construct universal graph. A *separator* of a $n$-vertex graph $G$ is a set $S \subseteq V(G)$ such that every connected components of the graph $G[V(G) \setminus S]$ obtained by removing $S$ from $G$ has at most $n/2$ vertices. A graph is said to be $s(n)$-separable if every of its $k$-vertex subgraph enjoys a separator of size $s(k)$. A biseparator of a graph $G$ consists in a set $B \subset V(G)$ such that there is a vertex partition of $V(G) \setminus B$ into two parts $V_1$ and $V_2$ of equal size and no edge linking a vertex of $V_1$ to a vertex of $V_2$. A graph is said to be $b(n)$-bisectable if every of its $k$-vertex subgraph enjoys a biseparator of size $b(k)$. There is a strong link between these two notions as stated by the following lemma.

**Lemma 2.1** *Let $G$ be a $n$-vertex graph. If $G$ is $s(n)$-separable then it is $2\left(\sum_{i=0}^{\log n} s(n/2^i)\right)$-bisectable.*

**Proof.** A bisector $B$ of a subgraph of $G$ is constructed by finding a $s(n)$-separator, putting it into $B$, then grouping all the resulting connected components except the greatest into two sets $V_1, V_2$ each with $\leqslant n/2$ nodes and iteratively repeating this process on the only remaining component if not empty. By this way, there is no edges of $G$ between $V_1$ and $V_2$ since we put each connected component entirely into one of the sets $V_1, V_2$. There is almost $\log n + 1$ iteration of this process since the size the connected component is halved at each step. So, the size of $B$ is $\sum_{i=0}^{\log n} s(n/2^i)$. Actually, we need that $|V_1| = |V_2|$ and not $|V_1|, |V_2| \leqslant n/2$. We can achieve this property by putting vertices of the greatest of the two parts $V_1$ and $V_2$ into $B$ until parts are equals, almost doubling size of $B$.

**Lemma 2.2** *Graphs of treewidth $k$ are $(2(k+1)(\log n + 1))$-bisectable.*

**Proof.** Graphs of treewidth $k$ are $(k+1)$-separable [6]. So, from lemma 2.1 they are $f(n)$-bisectable where $f(n) = 2\sum_{i=0}^{\log n} k + 1 = 2(k+1)(\log n + 1)$.

**Lemma 2.3** *The $n$-vertex graphs with no $K$-minor are $\left(2(2+\sqrt{2})|V(K)|^{3/2} n^{1/2}\right)$-bisectable.*

**Proof.** Graphs with no $K$-minor are $\left((2+\sqrt{2})|V(K)|^{3/2} n^{1/2}\right)$-separable [6]. So, from lemma 2.1 they are $f(n)$-bisectable where:

$$f(n) = \sum_{i=0}^{\log n} \left((2+\sqrt{2})h^{3/2}\frac{n}{2^i}\right)^{1/2} \leqslant (2+\sqrt{2})h^{3/2}n^{1/2}$$

Let $\mathcal{H}_{n,k}$ denote the family of $n$-vertex graphs of treewidth $k$.

**Theorem 2.4** *There is an $\mathcal{H}_{n,k}$-universal graph with $n$ vertices and $O(k^2 n \log^2 n)$ edges.*

**Proof.** We use a construction of [4]. Let $H_{n,k}$ be the $\mathcal{H}_{n,k}$-universal graph we construct. The idea is that we can construct recursively $H_{n,k}$ by linking each vertex of a clique $R$ of size $2(k+1)(\log n + 1)$ to all vertices of two copies $S_1, S_2$ of $H_{n/2,k}$. The base graph of the recursion is $H_{1,k} = K_1$. It is easy to prove by induction that we can embed any graph $G$ of $\mathcal{H}_{n,k}$ into $H_{n,k}$ in an induced way. Indeed, it suffices to compute a bisector $B$ of $G$, then to embed $B$ into $R$ and each part $V_i$ for $i = 1, 2$ into $G_i$. We have the following set of equations:

$$|V(H_{n,k})| = n$$
$$|E(H_{n,k})| = 2|E(H_{n/2,k})| + 2(k+1)n(\log n + 1)$$

After some computation, we obtain the desired asymptotic bounds.

Let $\mathcal{F}_{n,K}$ denote the family of $n$-vertex graphs with no $K$-minor.

**Theorem 2.5** *There is an $\mathcal{F}_{n,K}$-universal graph with $n$ vertices and $O(n^{3/2}|V(K)|^{3/2})$ edges.*

**Proof.** Same construction as proof of Theorem 2.4 except that the size of the clique $R$ is equal to $2(2+\sqrt{2})|V(K)|^{3/2}n^{1/2}$.

# 3 From universal graphs to induced-universal graphs

In this section, we show how to construct induced-universal graph from universal graph using treewidth.

**Lemma 3.1** *Let $H_{n,k}$ an $\mathcal{H}_{n,k}$-universal graph. For $i \in \mathbb{N}$, let $c_i$ denotes the number of distinct cliques of size $i$ in $H_{n,k}$. There is an $\mathcal{H}_{n,k}$-induced-universal graph with $((k+1)c_{k+1} + kc_k)2^k$ vertices.*

**Theorem 3.2** *There is an $\mathcal{H}_{n,k}$-induced-universal graph with $O((k \log n)^{O(k)} n)$ vertices.*

**Proof.** The idea is to recursively count the number of cliques of universal graph $H_{n,k}$ obtained in Theorem 2.4 and then apply lemma 3.1 to construct the induced-universal graph.

**Corollary 3.3** *There is an $\mathcal{F}_{n,K}$-induced-universal graph of $O((r \log n)^{O(r)} n^2)$ vertices where $r = r(K)$ is a constant only depending on $K$.*

**Proof.** Graphs of $\mathcal{F}_{n,K}$ can be edge partitioned into two graphs of $\mathcal{H}_{n,r(K)}$ [11]. Using Corollary 3.2, we can construct a $\mathcal{H}_{n,r}$-universal graph with $O((k \log n)^{O(r)} n)$ vertices. Using a result of [10], we can deduce the desired $\mathcal{F}_{n,K}$-induced-universal graph.

# 4 Concluding remarks and open problems

There remains a gap between the lower bound of $O(n)$ and the upper bound of $\tilde{O}(n)$ for the minimal number of vertices for the family of bounded treewidth graphs. One may ask if it possible to reduce this gap.

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