On induced-universal graphs for the class of bounded-degree graphs

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April 17, 2008

Abstract

For a family $\mathcal{F}$ of graphs, a graph U is said to be $\mathcal{F}$-induced-universal if every graph of $\mathcal{F}$ is an induced subgraph of U. We give a construction for an induced-universal graph for the family of graphs on n vertices with degree at most k. For k even, our induced-universal graph has $O(n^{k/2})$ vertices and for k odd it has $O(n^{\lceil k/2 \rceil - 1/k} \log^{2+2/k} n)$ vertices. This construction improves a result of Butler by a multiplicative constant factor for the even case and by almost a multiplicative $n^{1/k}$ factor for the odd case. We also construct induced-universal graphs for the class of oriented graphs with bounded incoming and outgoing degree, slightly improving another result of Butler.

1 Introduction

All graphs are assumed to be without loops or multiples edges. For a graph G we denote by $V(G)$ its vertex set and by $E(G)$ its edge or arc set. Our terminology is standard and any undefined term can be found in standard theory books [11].

For a finite family $\mathcal{F}$ of graphs, a graph U is said to be $\mathcal{F}$-*universal* if every graph in $\mathcal{F}$ is a subgraph of U. For instance, if we denote by $\mathcal{F}_n$ the family of all graphs with at most n vertices, then the complete graph K_n is $\mathcal{F}_n$-universal. The universal graph problem consists in finding a n-vertex universal graph with the minimal number of edges for specific subfamilies of $\mathcal{F}_n$. This problem was originally motivated by circuit design for computer chips [4]. Several families of graphs have been studied for this problem, including forests [10], bounded-degree forests [2, 3], and bounded-degree graphs [1].

The notion of induced-universal graph can be similarly defined. For a family $\mathcal{F}$ of graphs, a graph U is $\mathcal{F}$-*induced-universal* if every graph in $\mathcal{F}$ is an induced subgraph of U. The induced-universal graph problem consists in finding an induced-universal graph of the minimal number of vertices for specific subfamilies of $\mathcal{F}_n$. The family $\mathcal{F}_n$ itself was considered by Moon [13], while Chung considered trees, planar graphs, and graphs with bounded arboricity on n vertices [9].

The induced-universal problem is strongly related to a notion of distributed data structure known as *adjacency labeling scheme* or *implicit representation*. An implicit representation for

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a family $\mathcal{F}$ of graphs consists of two functions: a labeling function that assigns labels to the vertices of any graph of $\mathcal{F}$ and an adjacency function that determines the adjacency between two vertices only by looking at their labels. The problem of finding an implicit representation with small labels for specific families of graphs was first introduced by Breuer [6, 7]. Kannan, Naor and Rudich [12] established the strong relation between the two problems by proving that the existence of an implicit representation using $k(n)$ bits per vertex for a family $\mathcal{F}_n$ is equivalent to the existence of an $\mathcal{F}_n$-induced-universal graph with $2^{k(n)}$ vertices.

In this paper, we focus on induced-universal graphs for bounded-degree graphs. We construct an induced-universal graph for the family $\mathcal{F}_{k,n}$ of n-vertex graphs of degree at most k. For k even, our induced-universal graph has $O(n^{k/2})$ vertices and for k odd our induced-universal graph has $O(n^{\lceil k/2 \rceil - 1/k} \log^{2+2/k} n)$ vertices. Our result for graphs with maximum degree $k \equiv 0 \pmod 2$ is deduced from a construction similar to that of [8] but with an improvement of the base graph of the construction (Section 3). Actually, our $\mathcal{F}_{2,n}$-induced-universal graph forming the basis of the construction has $5n/2 + O(1)$ vertices while the best lower bound known for the order of such graphs is $11n/6 + \Omega(1)$. Our result for graphs with maximum degree $k \equiv 1 \pmod 2$ is deduced from a recent result of Alon and Capalbo [1] on universal graphs for bounded-degree graphs, combined with a construction of [9] that gives an interesting connection between induced-universal graphs and universal graphs (Section 4). Given that the best known lower bound for the number of vertices of an $\mathcal{F}_{k,n}$-induced-universal graph is $\Omega(n^{k/2})$ [8], our result for k even is tight up to a multiplicative constant and our result for k odd is equal to $O(n^{1/2-1/k} \log^{2+2/k} n)$ times the lower bound. We also give a generalization of our result for oriented graphs of bounded degree (Section 5). In Section 6, we show how to construct an induced-universal graph for all orientations of the graphs of a family $\mathcal{F}$, only using a specific $\mathcal{F}$-induced-universal graph. We conclude the paper with some open problems (Section 7).

2 A small induced-universal graph for graphs with degree at most two

Our main concern here is to find an $\mathcal{F}_{k,n}$-induced-universal graphs for every k. We first investigate the case $k = 2$.

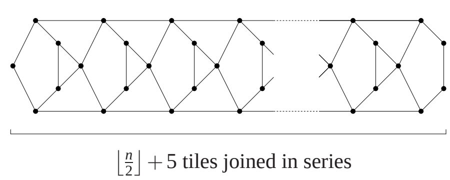


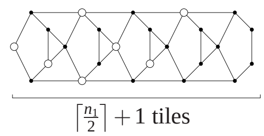
Figure 1: The $\mathcal{F}_{2,n}$-induced-universal graph U_n.

Lemma 1 *The graph U_n depicted in Figure 1 is an $\mathcal{F}_{2,n}$-induced-universal graph.*

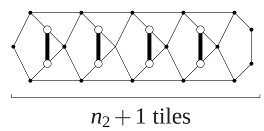
Proof. It is sufficient to prove that any graph $G \in \mathcal{F}_{2,n}$ is an induced subgraph of the graph U_n depicted in Figure 1. For $1 \leq i \leq n$, let n_i be the number of connected components of G

with i vertices. The degree of G is bounded by 2 so G contains n_1 isolated vertices, n_2 disjoint K_2's, and for $i \geq 3$, n_i cycles or paths of i vertices. We embed the connected components of G into U_n from left to right after having sort them by increasing size. The graph U_n is made of cycles of size 5 called *tiles* that are joined in series by 4 edges. Let us prove that we can embed all the connected components of G in an induced way using at most $\lfloor \frac{n}{2} \rfloor + 5$ tiles.

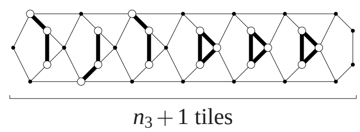
- The embedding of the stable set of size n_1, using $\lceil \frac{n_1}{2} \rceil + 1$ tiles.



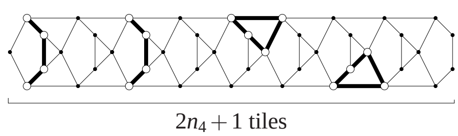
- The embedding of n_2 K_2's, using $n_2 + 1$ tiles.



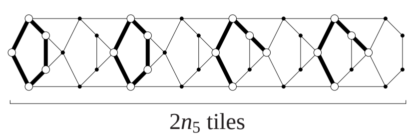
- The embedding of n_3 connected components of size 3, using $n_3 + 1$ tiles.



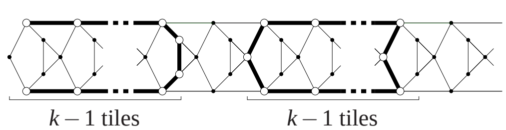
- The embedding of n_4 connected components of size 4, using $2n_4 + 1$ tiles.



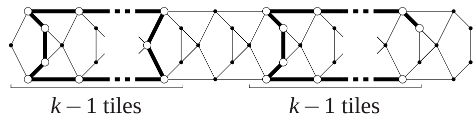
- The embedding of n_5 connected components of size 5, using $2n_5$ tiles.



- For $k \geq 3$, the embedding of n_{2k} connected components of size $2k$, using kn_{2k} tiles.



- For $k \geq 3$, the embedding of n_{2k+1} connected components of size $2k+1$, using kn_{2k+1} tiles.



Observe that for each i the embedding of connected components of size i is induced. Moreover, at the end of the embedding of all connected components of size i, there is a tile in which no vertex of G is embedded. So, there are no edges of U_n between the embeddings of two connected components of different sizes. Hence, the embedding of G into U_n is induced. It remains to upper bound the number l of tiles used by such an embedding.

$$\begin{aligned} l &= \frac{n_1}{2} + 2 + n_2 + 1 + n_3 + 1 + 2n_4 + 1 + 2n_5 + \sum_{k=3}^{\lfloor n/2 \rfloor} 2kn_{2k} + \sum_{k=3}^{\lfloor n/2 \rfloor} 2kn_{2k+1} \\ &\leq 5 + \sum_{i=1}^{n} i\frac{n_i}{2} \\ &\leq 5 + \left\lfloor \frac{n}{2} \right\rfloor \text{, since } \sum_{i=1}^{n} in_i = n \text{ and the number of tiles is an integer.} \end{aligned}$$

□

A natural question is to investigate whether this construction is optimal. We now prove that it is optimal up to a constant multiplicative factor of approximately $\frac{3}{2}$.

Claim 1 *Every $\mathcal{F}_{2,n}$-induced-universal graph has at least $11\left\lfloor \frac{n}{6} \right\rfloor$ vertices.*

Proof. Let $n \in \mathbb{N}$ be a multiple of 6. Let $\mathcal{H}_n$ be the family containing the following three graphs:

- the stable set of n vertices,
- the disjoint union of $n/2$ K_2,
- the disjoint union of $n/3$ K_3.

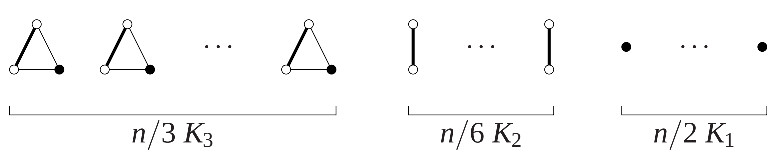


Figure 2: An induced subgraph of U_n.

Note that these three graphs have n vertices and degree at most two. Let U_n be an $\mathcal{H}_n$-induced-universal graph. Then U_n must contain $n/3$ triangles as induced subgraphs.

Since each of the triangles intersects at most one induced K_2, the graph U_n must contain an induced matching of size at least $n/2 - n/3 = n/6$ disjoint from the triangles. Since each K_2 and each triangle contains at most one isolated vertex as an induced subgraph, U_n must contain a stable set of size $n - n/3 - n/6 = n/2$ disjoint from the triangles and the induced matching (see Figure 2). Eventually, U_n has at least $3n/3 + 2n/6 + n/2 = 11n/6$ vertices and so any $\mathcal{F}_{2,n}$-induced-universal graph needs $11 \lfloor n/6 \rfloor$ vertices because $\mathcal{H}_{6\lfloor n/6 \rfloor} \subseteq \mathcal{F}_{2,n}$. □

We believe that the results in this section are not sharp. Indeed, we conjecture that there exists an $\mathcal{F}_{2,n}$-induced-universal graph with $2n + o(n)$ vertices, and that this is optimal.

3 Induced-universal graphs for graphs with even maximum degree

We now use our construction of an $\mathcal{F}_{2,n}$-induced-universal graph to construct an $\mathcal{F}_{k,n}$-induced-universal graph for k even (the same method was already used in [8]).

Theorem 1 *Let $k \geq 2$ be an even integer. There is an $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ such that*

$$|V(U_{k,n})| = (1 + o(1)) \left(\frac{5n}{2}\right)^{k/2} \text{ and } |E(U_{k,n})| = \left(\frac{9k}{10} + o(1)\right)\left(\frac{5n}{2}\right)^{k-1}.$$

Proof. To prove this theorem, we first reduce the problem to the construction of an $\mathcal{F}_{2,n}$-induced-universal graph. Petersen [14] proved that any k-regular graph with k even can be decomposed into $k/2$ edge-disjoint graphs of degree at most 2. In [9], Chung proved that for two families of graphs $\mathcal{F}$ and $\mathcal{H}$ such that any graph of $\mathcal{F}$ can be decomposed into k graphs of $\mathcal{H}$, if we have an $\mathcal{H}$-induced-universal graph W, we can construct an $\mathcal{F}$-induced-universal graph U such that:

$$|V(U)| = |V(W)|^k \text{ and } |E(U)| = k|V(W)|^{2k-2}|E(W)|.$$

Using Lemma 1, we construct an $\mathcal{F}_{2,n}$-induced-universal graph U_n with $|V(U_n)| = \frac{5}{2}n + O(1)$ and $|E(U_n)| = \frac{9}{2}n + O(1)$. Therefore, using the fact that any graph of $\mathcal{F}_{k,n}$ can be decomposed into $k/2$ graphs of $\mathcal{F}_{2,n}$, we obtain an $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ such that:

$$\begin{aligned}
|V(U_{k,n})| &= |V(U)|^{k/2} = \left(\frac{5}{2}\right)^{k/2} n^{k/2} + o(n^{k/2}) \\
|E(U_{k,n})| &= \frac{k}{2}|V(U)|^{k-2}|E(U)| = \frac{k}{2} \cdot \frac{9}{2}\left(\frac{5}{2}\right)^{k-2} n^{k-1} + o(n^{k-1}).
\end{aligned}$$

□

4 Induced-universal graphs for graphs with odd maximum degree

To the best of your knowledge, there is no good result on edge decomposition for graphs belonging to $\mathcal{F}_{k,n}$ with k odd. Nevertheless, we can use $U_{k+1,n}$ as an $\mathcal{F}_{k,n}$-induced-universal graph since $\mathcal{F}_{k,n} \subset \mathcal{F}_{k+1,n}$. The graph obtained is from a multiplicative factor of $O(n^{1/2})$ of the best known lower bound for the number of vertices of $\mathcal{F}_{k,n}$-induced-universal graphs. We now show how to reduce the gap between lower and upper bounds with a construction deduced from universal graphs.

Theorem 2 *Let $k \geq 3$ be an odd integer. There is an $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ such that*

$$|V(U_{k,n})| = c_1(k)n^{\lceil k/2 \rceil - 1/k} \log^{2+2/k} n \text{ and } |E(U_{k,n})| = c_2(k)n^{k-2/k} \log^{4+4/k} n$$

Proof. The induced-universal graph is deduced from the $\mathcal{F}_{k,n}$-universal graph obtained by Alon and Capalbo [1], using a result of Chung [9] that gives a general construction of an induced-universal graph from an universal graph.

The construction of Chung [9] depends on the degree of the universal graph and the arboricity of graphs of the family. Indeed, if we consider a family A_r of graphs with arboricity at most r and an A_r-universal graph G, then the construction produces an A_r-induced-universal graph H such that :

$$|V(H)| = \sum_{v \in V(G)} (d_G(v)+1)^r \text{ and } |E(H)| = \sum_{uv \in E(G)} (d_G(u)+1)^r d_G(v)^{r-1}.$$

The arboricity of graphs of the family $\mathcal{F}_{k,n}$ is at most $\lceil k/2 \rceil$. Moreover, the $\mathcal{F}_{k,n}$-universal graph described in [1] has degree at most $c(k)n^{2-2/k} \log^{4/k} n$. Hence, there is an induced-universal graph $U_{k,n}$ for the family $\mathcal{F}_{k,n} = \mathcal{A}_{\lceil k/2 \rceil}$ such that:

$$\begin{aligned}
|V(U_{k,n})| &= \sum_{v \in V(H_{k,n})} (d_{H_{k,n}}(v)+1)^{\lceil k/2 \rceil} \\
&\leq |V(H_{k,n})|(2d_{H_{k,n}})^{\lceil k/2 \rceil} \\
&\leq n(2c(k)n^{1-2/k} \log^{4/k} n)^{\lceil k/2 \rceil} \\
&\leq c_1(k)n^{\lceil k/2 \rceil - 1/k} \log^{2+2/k} n \text{ , where } c_1(k) = (2c(k))^{\lceil k/2 \rceil}
\end{aligned}$$

$$\begin{aligned}
|E(U_{k,n})| &= \sum_{uv \in E(H_{k,n})} (d_{H_{k,n}}(u)+1)^{\lceil k/2 \rceil} d_{H_{k,n}}(v)^{\lceil k/2 \rceil - 1} \\
&\leq |E(H_{k,n})|(2d_{H_{k,n}})^{\lceil k/2 \rceil}(d_{H_{k,n}})^{\lceil k/2 \rceil - 1} \\
&\leq c(k)n^{2-2/k} \log^{4/k} n(2c(k)n^{1-2/k} \log^{4/k} n)^{\lceil k/2 \rceil}(c(k)n^{1-2/k} \log^{4/k} n)^{\lceil k/2 \rceil - 1} \\
&\leq c_2(k)n^{k-2/k} \log^{4+4/k} \text{ , where } c_2(k) = (2c(k))^{k+1}.
\end{aligned}$$

□

5 Induced-universal graphs for bounded-degree oriented graphs

An *orientation* $\overrightarrow{G}$ of a graph G consists in assigning to every edge of G one of its two possible orientations. $\overrightarrow{G}$ is called an oriented graph and by definition, it cannot have loops nor opposite arcs. The construction of Section 3 can be easily generalized to the family $\mathcal{O}_{k,n}$ of all the orientations of the graphs from $\mathcal{F}_{2k,n}$ having incoming and outgoing degree at most k. Indeed, any graph of $\mathcal{O}_{k,n}$ can be decomposed into k graphs of $\mathcal{O}_{1,n}$ [14] and the construction of induced universal graph using decomposition works in the oriented case.

Theorem 3 *There is an $\mathcal{O}_{k,n}$-induced-universal oriented graph $\overrightarrow{O_{k,n}}$ such that*

$$|V(\overrightarrow{O_{k,n}})| = (1+o(1))\,(3n)^k \text{ and } |E(\overrightarrow{O_{k,n}})| = (2k+o(1))\,(3n)^{2k-1}.$$

Proof. The construction of an induced-universal graph for $\mathcal{O}_{k,n}$ is almost the same as the construction for $\mathcal{F}_{2k,n}$ presented in Section 3. Any graphs with outgoing and incoming degree at most k can be decomposed into k edge-disjoint graphs having outgoing and incoming degree at most 1 [14]. Let $\overrightarrow{O_n}$ be the graph depicted in Figure 3. If $\overrightarrow{O_n}$ is $\mathcal{O}_{1,n}$-induced-universal then, using the construction of Chung [9], we can construct an $\mathcal{O}_{k,n}$-induced-universal graph $\overrightarrow{O_{k,n}}$ having $|V(\overrightarrow{O_{k,n}})| = (1+o(1))\,(3n)^k$ vertices and $|E(\overrightarrow{O_{k,n}})| = (2k+o(1))\,(3n)^{2k-1}$ edges. So, the only thing we need to prove is that $\overrightarrow{O_n}$ is $\mathcal{O}_{1,n}$-induced-universal.

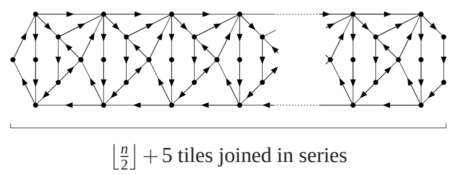
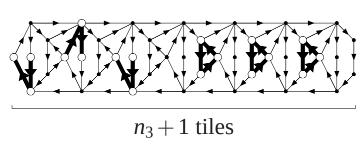


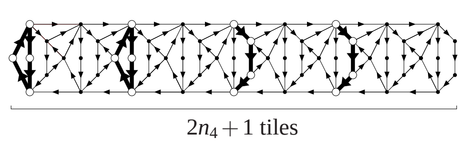
Figure 3: The $\mathcal{O}_{1,n}$-induced-universal graph $\overrightarrow{O_n}$.

Let $\overrightarrow{G}$ be any graph of $\mathcal{O}_{1,n}$. The connected components of $\overrightarrow{G}$ are either directed paths (oriented paths with exactly one sink and one source) or directed cycles (oriented cycles with no source). We embed $\overrightarrow{G}$ in $\overrightarrow{O_n}$ almost the same way we embedded graphs of $\mathcal{F}_{2,n}$ in U_n in Section 2. The only differences are for the embeddings of connected components of size 3 or more that slightly differ from the non-oriented case.

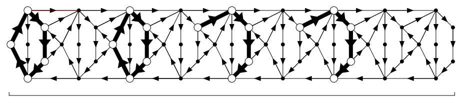
- The embedding of n_3 connected components of size 3, using n_3+1 tiles.



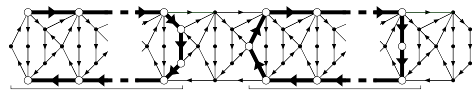
- The embedding of n_4 connected components of size 4, using $2n_4+1$ tiles.



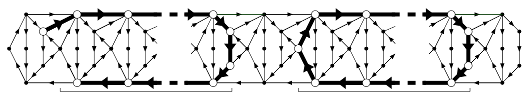
- The embedding of n_5 connected components of size 5, using $2n_5 + 1$ tiles.



- For $k \geq 3$, the embedding of n_{2k} connected components of size $2k$, using kn_{2k} tiles.



- For $k \geq 3$, the embedding of n_{2k+1} connected components of size $2k+1$, using kn_{2k+1} tiles.



We use for embeddings exactly the same number of tiles as for the non-oriented case, so the graph $\overrightarrow{O_n}$ has also $\lfloor \frac{n}{2} \rfloor + 5$ tiles. □

6 From induced-universal graphs to oriented induced-universal graphs

In Section 5, we constructed an induced-universal graph for a family of orientations of graphs in $\mathcal{F}_{2,n}$ by orienting the edges and adding some vertices to the non-oriented induced-universal graph. Let $\mathcal{F}$ be a family of graphs and $\overrightarrow{\mathcal{F}}$ be a family of orientations of graphs from $\mathcal{F}$. One may ask if, taking an $\mathcal{F}$-induced-universal graph U, it is always possible to construct an $\overrightarrow{\mathcal{F}}$-induced-universal graph $\overrightarrow{U}$.

Given two graphs G and H, a *homomorphism* from G to H is a mapping $f : V(G) \to V(H)$ satisfying $[x, y] \in E(G) \Rightarrow [f(x), f(y)] \in E(H)$. In fact, the construction is possible if there is a graph $\overrightarrow{H}$ into which each graph of $\overrightarrow{\mathcal{F}}$ has a homomorphism. In this case, the graph $\overrightarrow{H}$ is said to be an *$\overrightarrow{\mathcal{F}}$-universal graph for homomorphism*. For instance, the directed cycle of length three is an universal graph for homomorphism for the family of orientation of trees. The graph $\overrightarrow{U}$ can be obtained by making a special product of the two graphs $\overrightarrow{H}$ and U. The *oriented tensor product* $G \times \overrightarrow{H}$ of a non-oriented graph G and an oriented graph $\overrightarrow{H}$ is defined to have vertex set $V(G \times \overrightarrow{H}) = V(G) \times V(\overrightarrow{H})$ and arc set $E(G \times \overrightarrow{H}) = \left\{[(x, u), (y, v)] \mid xy \in E(G) \text{ and } uv \in E(\overrightarrow{H})\right\}$.

Theorem 4 *Let U and $\overrightarrow{H}$ be two graphs. If U is $\mathcal{F}$-induced-universal and $\overrightarrow{H}$ is $\overrightarrow{\mathcal{F}}$-universal for homomorphism then $U \times \overrightarrow{H}$ is $\overrightarrow{\mathcal{F}}$-induced-universal.*

Proof. It suffices to show that we can embed an arbitrary graph $\overrightarrow{G} \in \overrightarrow{\mathcal{F}}$ as an induced subgraph of $U \times \overrightarrow{H}$. Let $v \in \overrightarrow{G}$. There is a homomorphism of $\overrightarrow{G}$ to $\overrightarrow{H}$ since $\overrightarrow{H}$ is $\overrightarrow{\mathcal{F}}$-universal for homomorphism. We denote by $h(v) \in V(\overrightarrow{H})$ the vertex into which v is mapped. If we forget about the orientation, we can embed $\overrightarrow{G}$ into U since U is $\mathcal{F}$-induced-universal. Let $u(v) \in V(U)$ denote the vertex into which v is embedded. The embedding of $\overrightarrow{G}$ into $U \times \overrightarrow{H}$ consists in embedding each vertex v of G into the vertex $(u(v), h(v))$ of $U \times \overrightarrow{H}$. The embedding is correct in the sense that if there is an arc $[x, y]$ in $\overrightarrow{G}$ then there is an arc $[(u(x), h(x)), (u(y), h(y))]$ in $U \times \overrightarrow{H}$. Indeed, there is an edge $[u(x), u(y)]$ in U due to the non-oriented embedding of $\overrightarrow{G}$ into U and an arc $[h(x), h(y)]$ in $\overrightarrow{H}$ due to the mapping of $\overrightarrow{G}$ into $\overrightarrow{H}$. Moreover, the embedding is induced. Indeed, if two vertices x and y of G are not adjacent then $u(x)$ and $u(y)$ are not adjacent in U because the non-oriented embedding of $\overrightarrow{G}$ into U is induced. So, by construction, $(u(x), h(x))$ and $(u(y), h(y))$ are not adjacent in $U \times \overrightarrow{H}$. □

Families such as trees, planar graphs, partial 2-trees, outerplanar graphs, and subcubic graphs are known to have universal graphs for homomorphism with constant number of vertices [5, 15]. So for these families, induced-universal graphs and induced-universal oriented graphs have asymptotically the same order.

7 Concluding remarks and open problems

In Section 2, we proved that a minimal $\mathcal{F}_{2,n}$-induced-universal graph has at least $11n/6+\Omega(1)$ vertices, and at most $5n/2 + O(1)$ vertices. The natural question that arises is whether it is possible to reduce the gap between 5/2 and 11/6 for the multiplicative constant. This question seems to be quite difficult, even though graphs of $\mathcal{F}_{2,n}$ have a very simple structure. For k odd, if we drop the polylogarithmic factor, there remains a multiplicative factor of $n^{1/2-1/k}$ between the lower and the upper bound for the number of vertices in a minimal $\mathcal{F}_{k,n}$-induced-universal graph. An interesting problem would be to lower this factor, especially for large values of k. In our construction, for k even, our $\mathcal{F}_{k,n}$-induced-universal graph have maximum degree $4^{k/2}$ depending only on k whereas for k odd, it has maximum degree $c_2(k)n^{k-1-2/k}\log^{4+4/k} n$. Considering that for k even our construction is almost tight whereas for k odd it is not, we conjecture that $\mathcal{F}_{k,n}$-induced-universal graphs with minimal number of vertices and edges have degree only depending on k. In other words, we conjecture that there is a function $f(k)$ such that the existence of a $\mathcal{F}_{k,n}$-induced-universal graph $U_{k,n}$ implies that there exists another one with at most the same number of vertices, but with degree at most $f(k)$.

A more general problem concerning induced-universal graphs should be to solve the induced-universal version of the implicit graph conjecture of Kannan, Naor and Rudich [12]:

Conjecture 1 (Implicit Graph Conjecture (induced-universal version)) *Every hereditary class of graphs which contains $2^{O(n \log n)}$ graphs on n vertices admits an induced-universal graph with $n^{O(1)}$ vertices.*

Solving this conjecture seems rather difficult even if it is known that families of graphs closed by taking minor fulfill the conjecture since they admit induced-universal graph of $n^{O(1)}$ vertices.

References

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