Edge Partition of Toroidal Graphs into Forests in Linear Time

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Abstract

In this paper we give a linear algorithm to edge partition a toroidal graph, i.e., graph that can be embedded on the orientable surface of genus one without edge crossing, into three forests plus a set of at most three edges. For triangulated toroidal graphs, this algorithm gives a linear algorithm for finding three edge-disjoint spanning trees. This is in a certain way an extension of the well-known algorithm of Schnyder's decomposition for planar graph.

Keywords: Graph Partition, Graph on surfaces, Graph Algorithm, Spanning Tree, Schnyder Trees

1 Motivation and Background

The problem of finding the maximum number of edge-disjoint spanning trees arises in the context of constructing efficient multicast communication in

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wormhole-routed networks. In wormhole routing [6], each intermediate node forwards the unit of information transfer called worm to the desired output port as soon as the head of the worm is received. Although one spanning tree is enough to construct a deadlock-free multicast routing algorithm in wormhole-routed networks [5,9], it is clear that finding multiple edge-disjoint spanning trees in the network allows us to use more edges of the graph and so to decrease communication latency.

In 1973, Kundu has shown that triangulated (or maximal) toroidal graphs have 3 edge-disjoint spanning trees [4]. Unfortunately, his proof does not lead to an algorithm. Up to now, the most efficient algorithm is the $O(m^2)$ algorithm time of Roskind and Tarjan [7] designed for finding the maximum number of edge-disjoint spanning trees in a graph with m edges. So we have an $O(n^2)$ algorithm to find the 3 spanning trees (at most) into toroidal graphs due to the Euler's formula ($m \leq 3n$).

One of the most known algorithms for tree partition of graph is the Schnyder's one [8]. This linear algorithm partition the edges of a triangulated planar graph into 3 trees $\{T_1, T_2, T_3\}$ each rooted on a vertex of the outer face such that the neighborhood of each inner vertex v consists in 6 blocks U_1, D_3, U_2, D_1, U_3 and D_2 in counter-clockwise order where U_j (resp. D_j) consists in the parent (resp. the children) of v in T_j for $j \in \{1, 2, 3\}$.

The *orientable surface* of genus g is denoted $\mathbb{S}_g$. For instance $\mathbb{S}_0$ is the sphere, $\mathbb{S}_1$ the torus. The *genus* of a graph G, denoted $g(G)$, is the minimum g such that G can be embedded in $\mathbb{S}_g$ without edge crossing. For the rest of the paper, we will consider triangulations of torus, i.e., graphs of genus 1 that are embedded on the torus (see linear time algorithm of [3]) and such that all its faces are triangles. In the case of a non-triangulated embedding, we use a $O(n)$ time algorithm to triangulate the graph, without adding multiple edges, adding $O(n)$ vertices. An embedding is 2-*cell* if all its faces are 2-cells, i.e., a region homeomorphic to the open unit disk in $\mathbb{R}^2$.

Let us consider an embedding G of $\mathbb{S}_g$. Let C be a cycle of G. C is *contractible* if it divides the surface $\mathbb{S}_g$ into two disjoint regions so that one of them is a 2-cell. Otherwise C is *non-contractible*. If C is *non-contractible*, then it is either *separating* if it divides the surface $\mathbb{S}$ into two disjoint regions, or *non-separating* otherwise. The interesting property of non-contractible non-separating cycles is that their removal reduces the genus of a graph by one [1].

2 Sketch of the Algorithm

The main result of this paper is the following theorem.

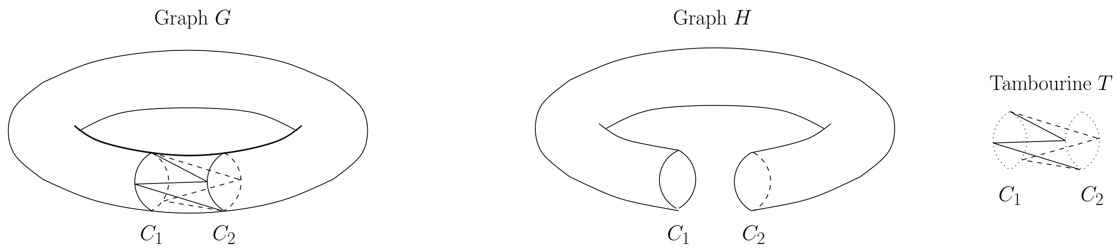

Fig. 1. Planarization of graph G.

Theorem 2.1 *In linear time, the edges of every toroidal graph can be partitioned into three forests plus at most three edges.*

We make a few remarks about this result. First, this partition is tight in the sense that we need to remove 3 edges to partition into 3 trees any triangulation of the torus. Indeed, such graphs have $3n$ edges due to Euler's formula and the total of edges of 3 trees in the graph is at most $3n-3$. The second remark we can make is that a simple planarization combined with the Schnyder's trees do not suffice because we may need to remove at least $\Omega(\sqrt{n})$ vertices or $\Omega(\sqrt{n\Delta})$ edges to obtain a planar graph [2] (Δ being the maximum degree of the graph). So, we cannot simply planarize the graph G and find a partition of G from H without extra properties on the partition of H and the planarizing edge set. Finally, we can remark that the 3 edges that are not in the 3 trees cannot be chosen arbitrarily. In the case of a triangulated graph, the graph resulting of the removal of the 3 edges must be 3 edge-connected and therefore the 3 edges cannot be contained in a separating triangle. The algorithm consists in 3 steps described briefly bellow.

- **Step 1:** Graph partition into a planar subgraph H and a tambourine T.

  The first step of the algorithm consists in a planarization of the graph. We need a special planarization to use partition of the resulting planar graph H. We decompose the graph into a planar graph H plus a set of edges T defined by a pair of non-contractible non-separating cycles (C_1, C_2) called *tambourine*. Edges of T have one end in C_1 and in C_2. Moreover, the edge-set T is the set of edges adjacent to C_1 (resp. C_2) that are on one side of C_1 (resp. C_2) (see Fig. 1 for a schema of planarization). With these properties, we obtain a graph H entirely triangulated except the 2 faces bounded by C_1 and C_2. To do this decomposition, we use the $O(n)$ algorithm of Djidjev and Venkatesan [2] to find a non-separating non-contractible cycle C. From this cycle, we compute the tambourine (C_1, C_2).

- **Step 2:** Edge partition of H into 3 forests A_1, A_2 and A_3 plus a set of at most 3 edges.

For this purpose, we find an edge partition of the graph H such that the edges of the tambourine can be partioned into the three forests and added to H without creating a cycle. The most important property used is that every vertex x of $C_1 \cup C_2$, except at most 3 vertices has a missing color in the graph H, i.e., there is a color $i \in \{1, 2, 3\}$ such that the connected component of the forest A_i which contains the vertex x does not contain other vertices of $C_1 \cup C_2$. To have this property, we contract each of the two faces C_1 and C_2 of H obtaining a new graph H'. We then compute a realizer of H' and from this realizer, we deduce the partition wanted for H.

- **Step 3:** Edge partition into 3 forests of G from the edge partition of H.

  We start with the partition of H computed in Step 2 and we insert each edge of the tambourine T into one of the forests F_i according to the missing color of one of its ends. The choice is done such that each vertex is used for only one edge and there is no path in the tambourine constituted of edges of one forest A_i between two vertices from which the missing color is different from i. That ensures that there is no cycle in A_i in G.

3 Possible Improvements and Open Problems

A possible improvement of the result would be to find a partition that have extra properties about the edge ordering around each vertex in spite of the three Schnyder's trees for plane triangulation. Note that in our partition all vertices at distance at least two from the tambourine have Schnyder ordering. Another example of desirable property would be to bound the diameter of any tree by $O(D)$, where D is a diameter of the graph.

A possible extension of our result is the generalization for graphs of genus $g > 1$. The first idea that comes to mind is to use recursion with the torus as a basis. We start from a graph G of genus g and we decompose it into a subgraph H of genus $g-1$ plus a tambourine T. Then, from a partition of H we obtain a partition of G. Unfortunately, our method for finding tambourine for triangulation of torus does not apply directly for higher genus.

Another possible improvement is to use the result of Thomassen [10] on graph with large edge-width. Such graphs have a collection of cycles that are far apart and whose removal results in a planar graph. So, for such graphs we can find a collection of tambourines that are far apart and whose removing results in a planar graph H. So we can directly use the algorithm on H and find a partition optimal in term of removed edges.

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