15/02/2024

Séminaire MoVe

Minimization of Cost Register Automata over a Field

Yahia Idriss BENALIOUA

Nathan LHOTE & Pierre-Alain REYNIER

Aix*Marseille université
Socialement engagée

LIS
LABORATOIRE
D'INFORMATIQUE
& DES SYSTÈMES

Register minimization problem

In: f rational series given as a WA , $k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant k$ registers realizing f

???

Register minimization problem ???

In: f rational series given as a WA, $k \in \mathbb{N}$

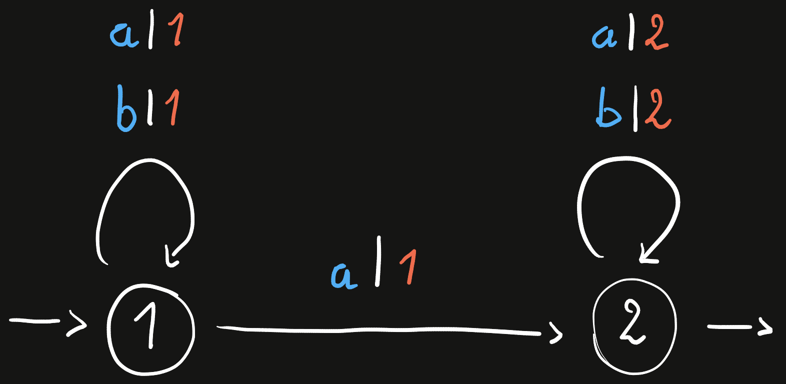
Q?: $\exists ?$ CRA with $\leqslant k$ registers realizing f

??? ???

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:

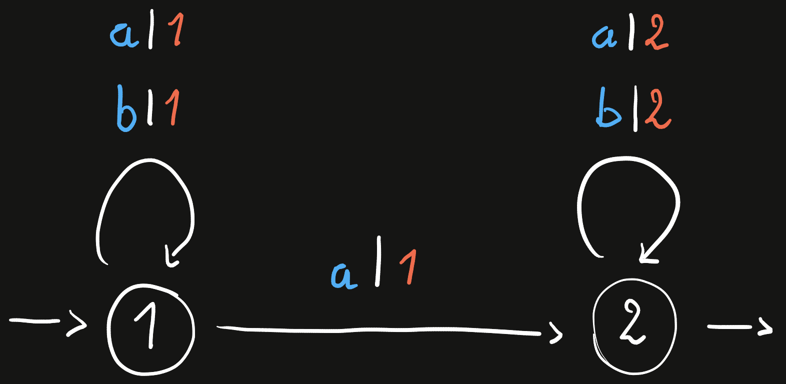


realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

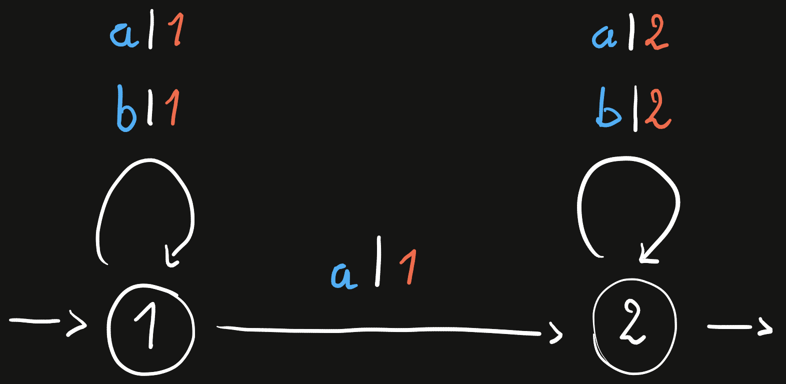
$a\,a\,b$:

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 1 \times 2 = 2$$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

aab :

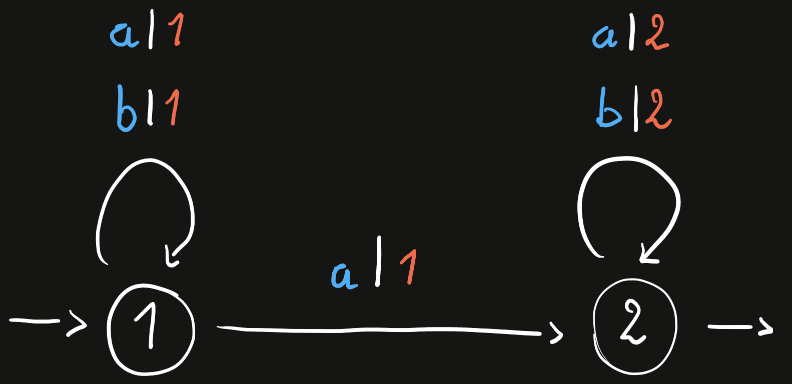
$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 1 \times 2 = 2$$

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 2 \times 2 = 4$$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

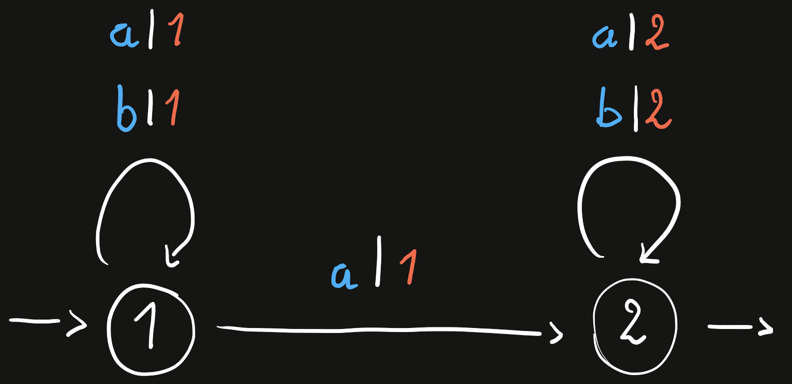
aab :

$$\begin{array}{l} w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 1 \times 2 = 2 \\ \hfill + \\ w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = \underline{1 \times 2 \times 2 = 4} \\ \hfill aab \mapsto 6 \end{array}$$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab :

$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 1 \times 2 = 2$

$+$

$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 2 \times 2 = 4$

$aab \mapsto 6$

Linear representation

(u, μ, ν)

initial vector

$u = (\qquad)$

terminal vector

$\nu = \left(\quad\right)$

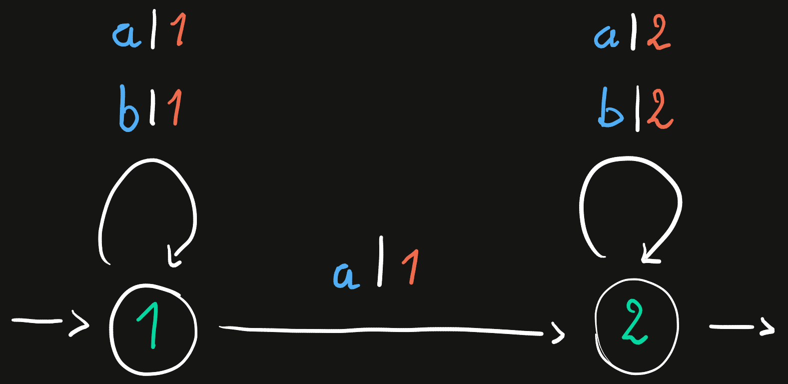
transition matrices

$\mu(a) = \left(\qquad\right)$ $\mu(b) = \left(\qquad\right)$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab :

$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 1 \times 2 = 2$

$+$

$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 2 \times 2 = 4$

$aab \longmapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ \left(\quad \right. & \left. \quad \right) \end{array}$$

terminal vector

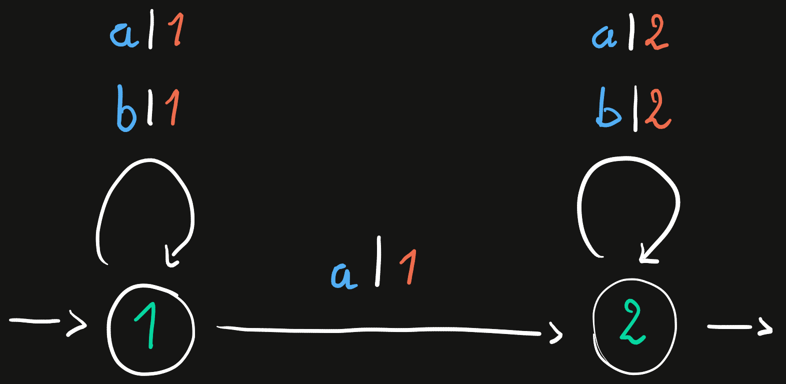
$$v = \begin{array}{cc} \begin{pmatrix} \; \\ \; \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} \end{array}$$

Weighted Automata
(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab:

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 1 \times 2 = 2$$

$+$

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 2 \times 2 = 4$$

$\text{aab} \longmapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

terminal vector

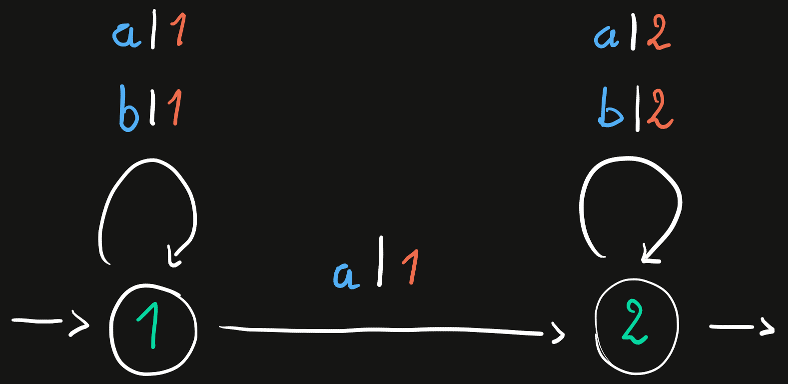
$$v = \begin{array}{cc} \begin{pmatrix} \ \\ \ \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \ & \ \\ \ & \ \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \ & \ \\ \ & \ \end{pmatrix} \end{array}$$

Weighted Automata
(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab :

$$w\left(① \xrightarrow{a|1} ① \xrightarrow{a|1} ② \xrightarrow{b|2} ②\right) = 1 \times 1 \times 2 = 2$$

$+$

$$w\left(① \xrightarrow{a|1} ② \xrightarrow{a|2} ② \xrightarrow{b|2} ②\right) = 1 \times 2 \times 2 = 4$$

$aab \longmapsto 6$

Linear representation
(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

terminal vector

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

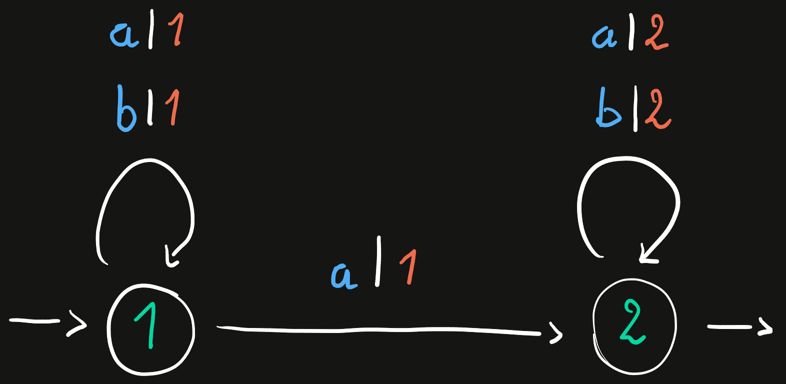
transition matrices

$$\mu(a) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \end{array}$$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab :

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 1 \times 2 = 2$$

$$+$$

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 2 \times 2 = 4$$

$$aab \mapsto 6$$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

terminal vector

$$v = \begin{array}{cc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

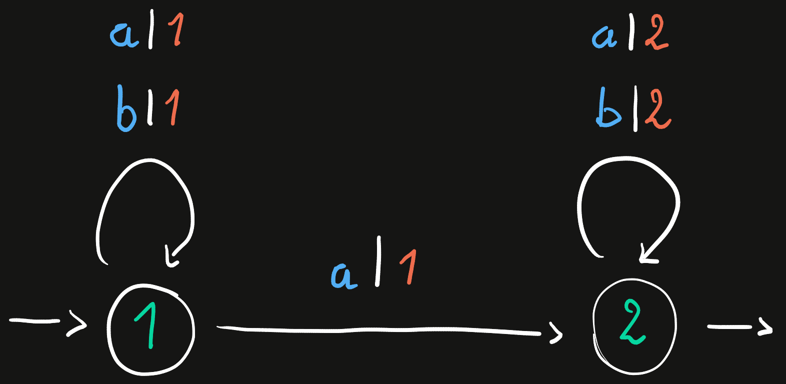
transition matrices

$$\mu(a) = \begin{array}{cc} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab :

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 1 \times 2 = 2$$

$$+$$

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 2 \times 2 = 4$$

$aab \mapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ \hline 1 & 0 \end{array}$$

terminal vector

$$v = \begin{array}{c|c} 0 & 1 \\ 1 & 2 \end{array}$$

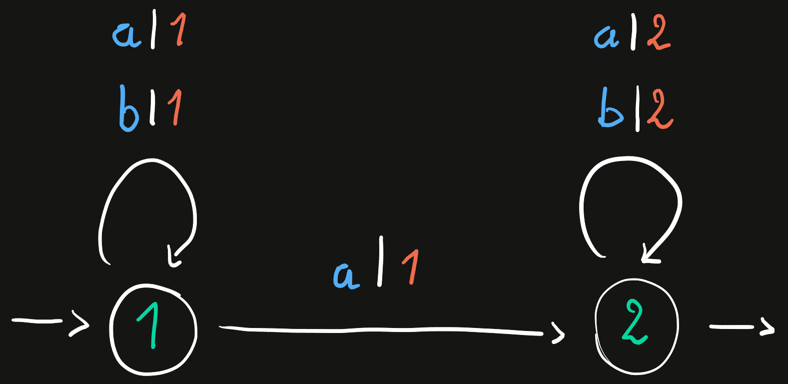
transition matrices

$$\mu(a) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 0 & 2 \end{array} \qquad \mu(b) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 0 \\ 2 & 0 & 2 \end{array}$$

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

aab :

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 1 \times 2 = 2$$

$+$

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 2 \times 2 = 4$$

$$\text{aab} \mapsto 6$$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array}$$

terminal vector

$$v = \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 0 & 2 \end{array} \qquad \mu(b) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 0 \\ 2 & 0 & 2 \end{array}$$

$$\text{aab} \mapsto u \cdot \mu(aab) \cdot v$$

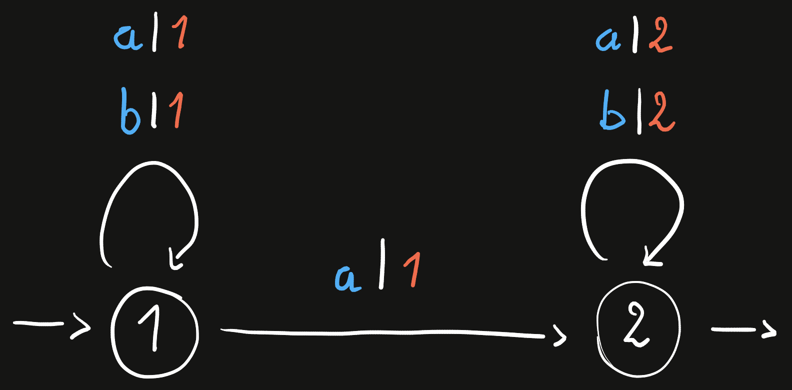
$$= u \cdot \mu(a) \cdot \mu(a) \cdot \mu(b) \cdot v$$

$$= (1 \quad 0) \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 6$$

Weighted Automata

(WA)



⚠ Not always equivalent to a sequential WA

(input deterministic)

Linear representation

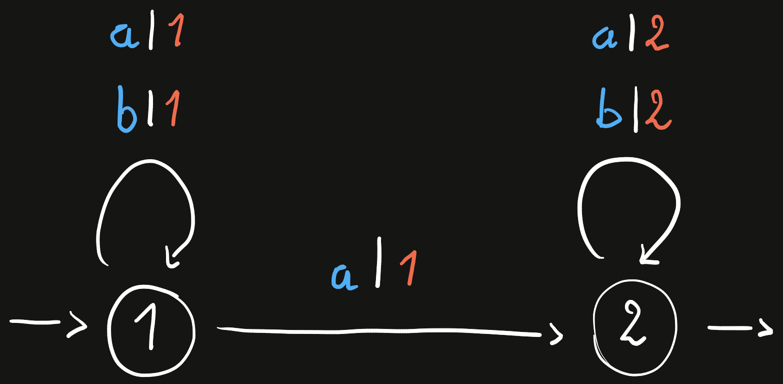
(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Not always equivalent to a sequential WA

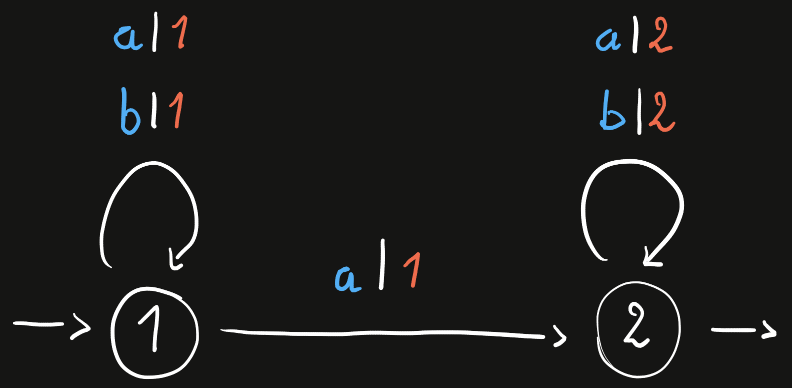
(input deterministic)

Some prop. of WA over a Field :

- Zeroness/equivalence is decidable

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

⚠ Not always equivalent to a sequential WA

(input deterministic)

Some prop. of WA over a Field :

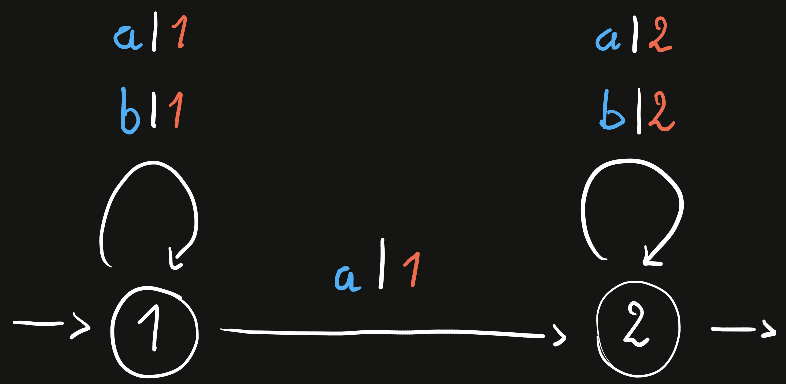
- Zeroness/equivalence is decidable

in poly. time

- ∃ computable minimal WA
  (unique up to change of basis)

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

⚠ Not always equivalent to a sequential WA

(input deterministic)

Some prop. of WA over a Field:

- Zeroness/equivalence is decidable

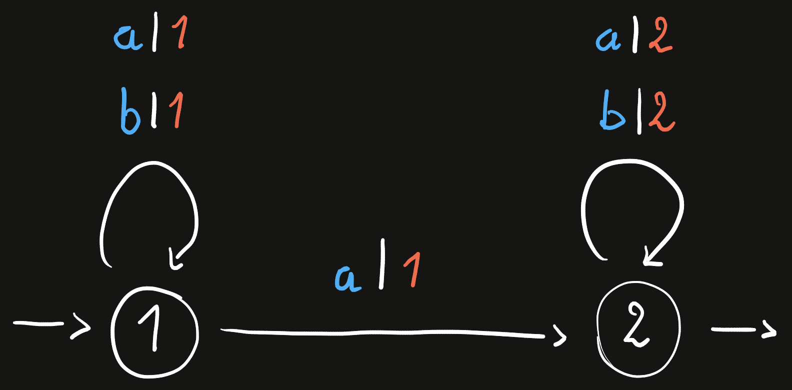
in poly. time

- ∃ computable minimal WA
  (unique up to change of basis)
- Unambiguity / Sequentiality is decidable

[Bell & Smertnig 2023]

Weighted Automata

(WA)



Linear representation

(u, μ, v)

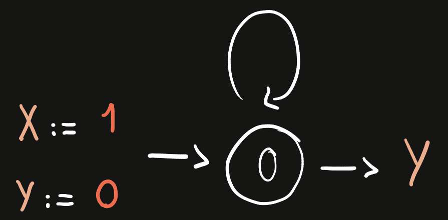
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

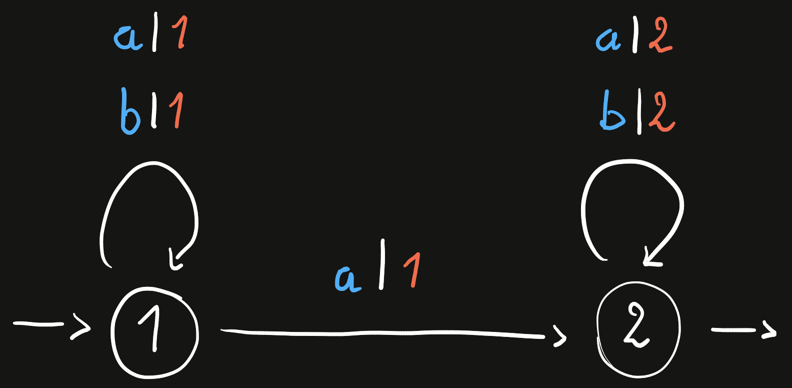
(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



Weighted Automata

(WA)



Linear representation

(u, μ, v)

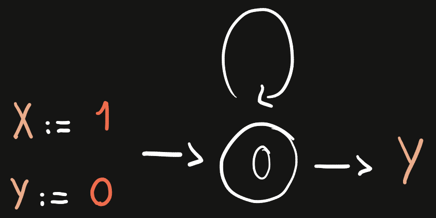
$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) [Alur et al. 2013]

$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right.$, $b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$



aab :

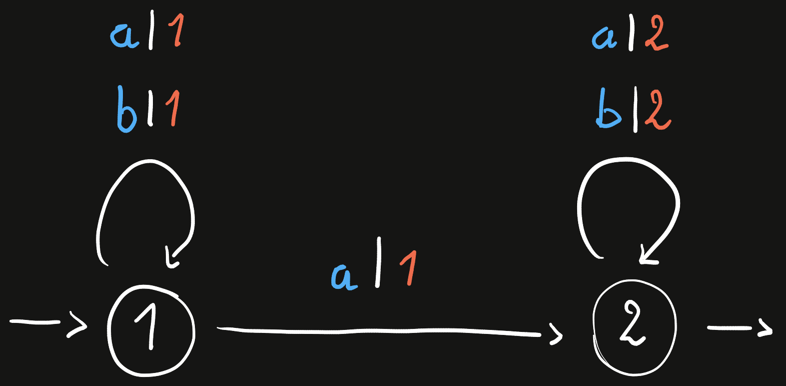
→ ⓪

$X = 1$

$Y = 0$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

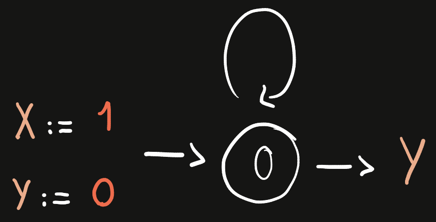
$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) [Alur et al. 2013]

$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$



aab :

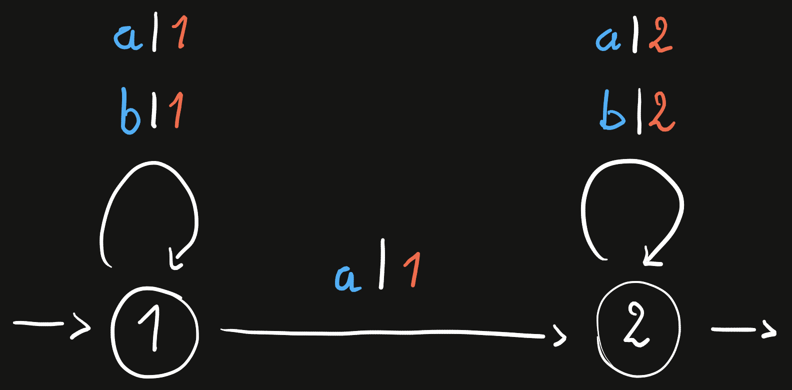
$\rightarrow \boxed{0} \xrightarrow{a} \boxed{0}$

$X = 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

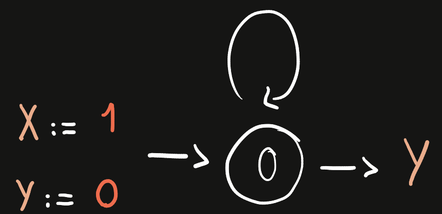
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \qquad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



aab :

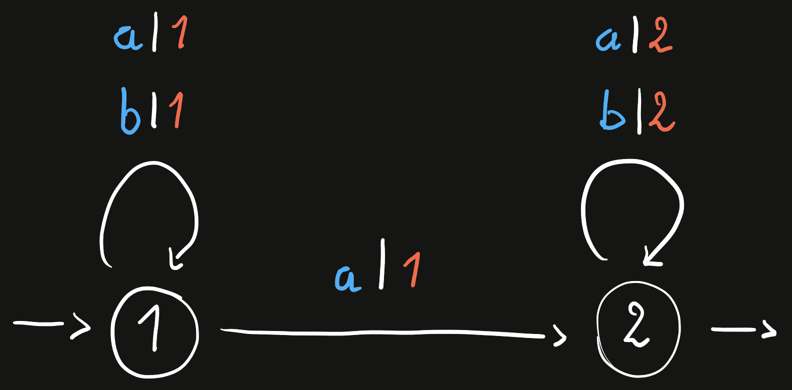
$$\to 0 \xrightarrow{a} 0 \xrightarrow{a} 0$$

$$X = 1 \longrightarrow 1 \longrightarrow 1$$

$$Y = 0 \longrightarrow 1 \longrightarrow 3$$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

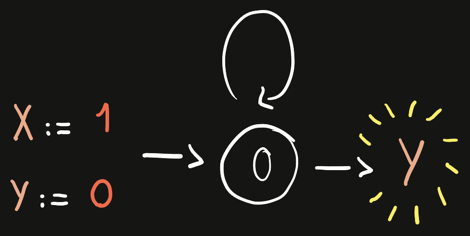
$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) [Alur et al. 2013]

$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right.$, $b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$



aab :

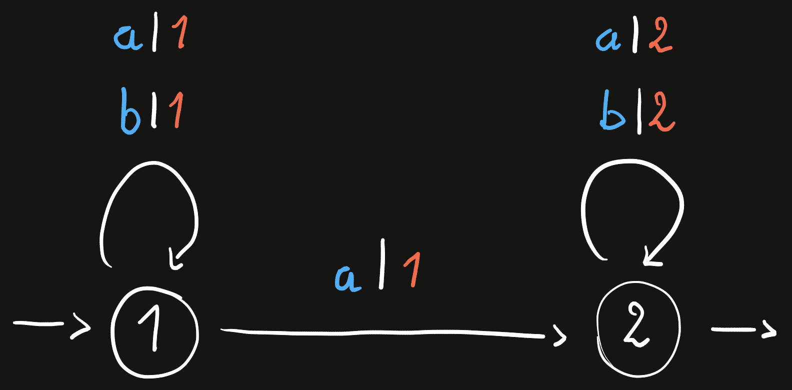
$X = 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1 \longrightarrow 3 \longrightarrow 6$

$aab \mapsto 6$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

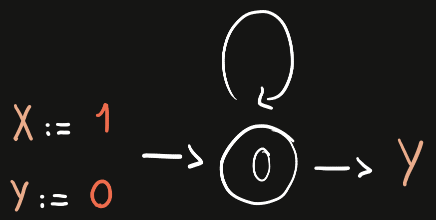
$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



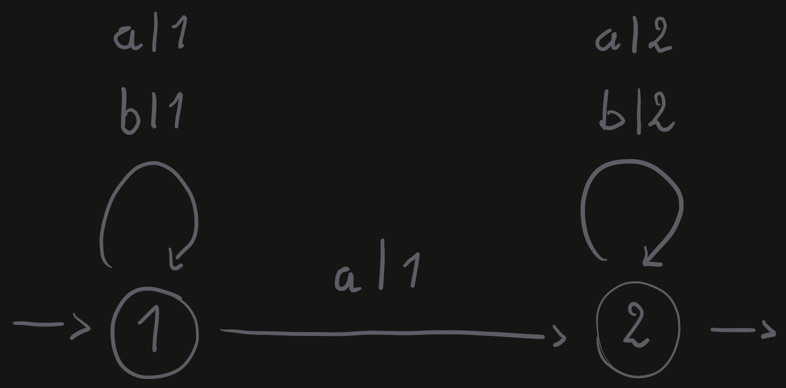
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

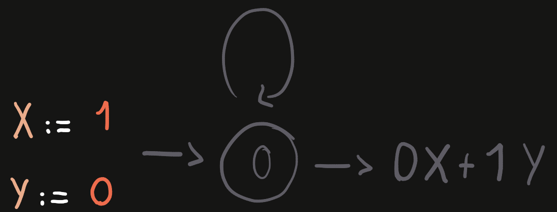
$$u = \begin{array}{cc} X & Y \\ (1 & 0) \end{array} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



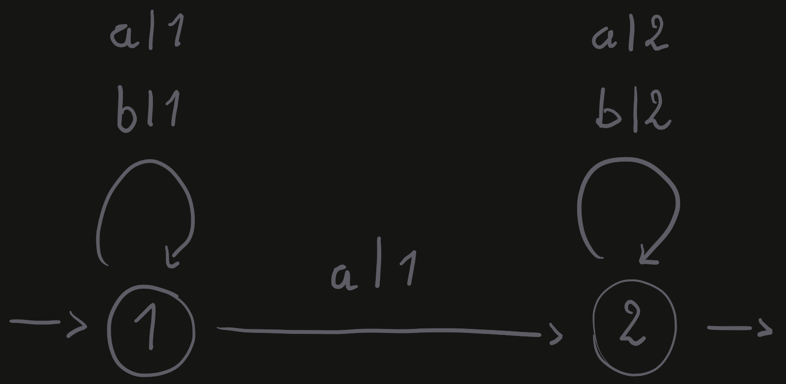
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

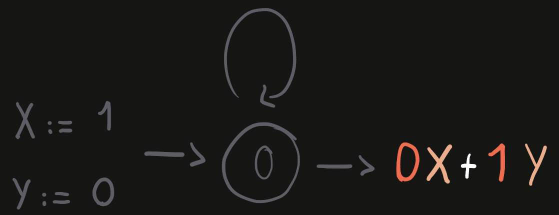
$$u = \begin{array}{cc} X & Y \\ \left(1\right. & \left.0\right) \end{array} \qquad v = \begin{array}{cc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{array}{c} X \\ Y \end{array} \end{array}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



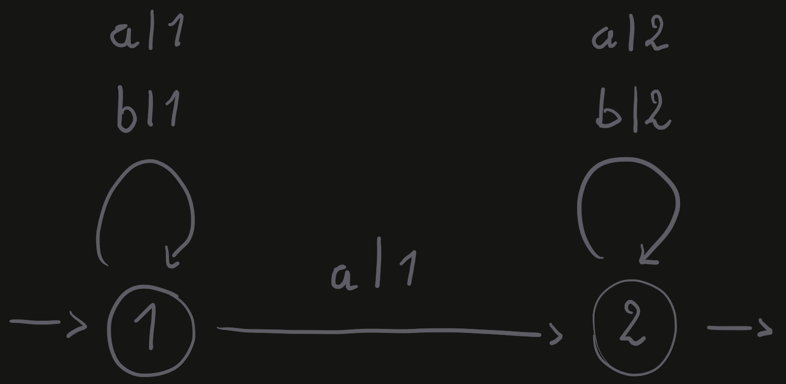
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

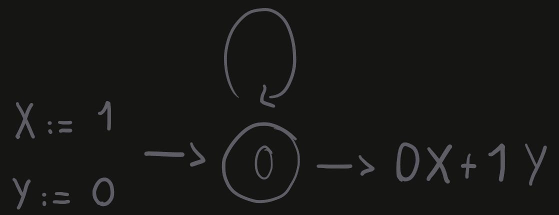
$$u = \begin{array}{cc} X & Y \\ \left(1\right. & \left.0\right) \end{array} \qquad v = \begin{array}{cc} \left(\begin{array}{c} 0 \\ 1 \end{array}\right) & \begin{array}{c} X \\ Y \end{array} \end{array}$$

$$\mu(a) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 1 \\ Y & 0 & 2 \end{array} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



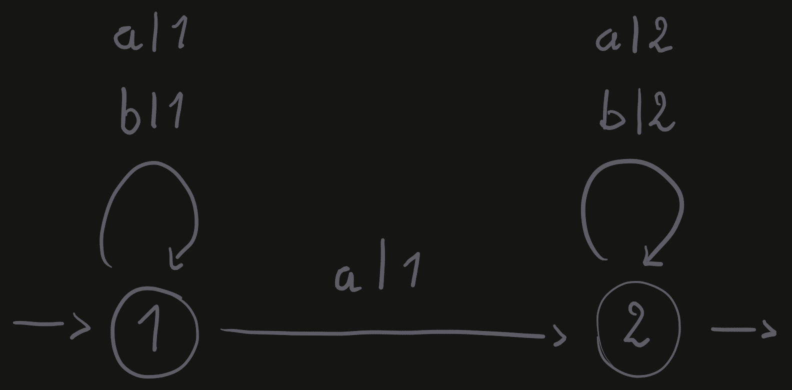
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

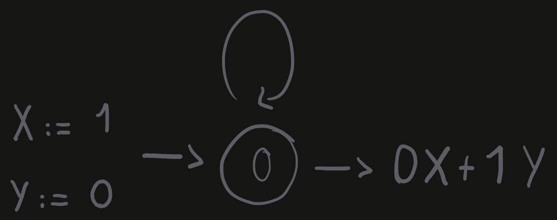
$$u = \begin{array}{c} \\ \end{array}\begin{array}{cc} X & Y \\ \end{array} \quad u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{array}{c} X \\ Y \end{array}$$

$$\mu(a) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 1 \\ Y & 0 & 2 \end{array} \qquad \mu(b) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 0 \\ Y & 0 & 2 \end{array}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



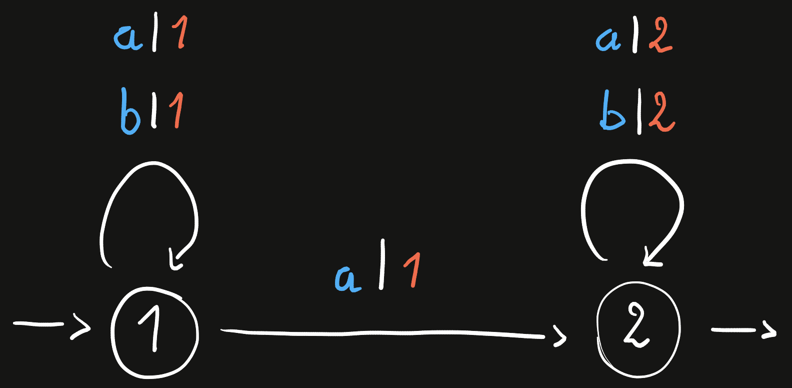
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

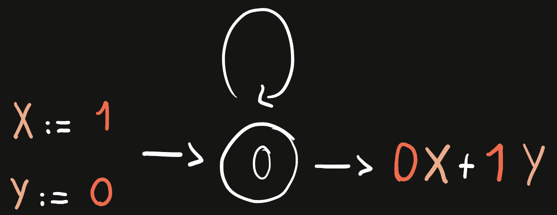
$$u = \begin{array}{cc} X & Y \\ \hline 1 & 0 \end{array} \qquad v = \begin{array}{c|c} 0 & X \\ 1 & Y \end{array}$$

$$\mu(a) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 1 \\ Y & 0 & 2 \end{array} \qquad \mu(b) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 0 \\ Y & 0 & 2 \end{array}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



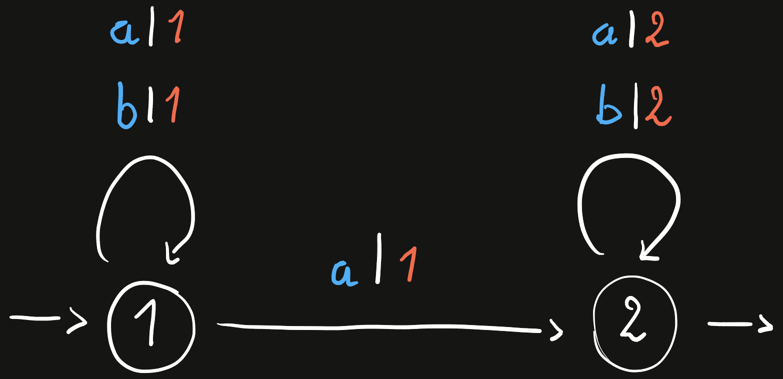
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

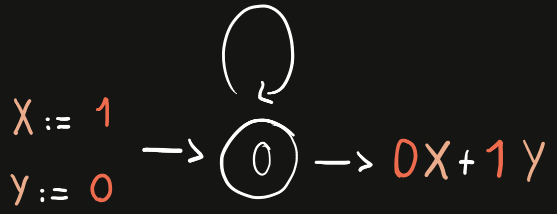
$$u = \begin{array}{cc} X & Y \\ \hline 1 & 0 \end{array} \qquad v = \begin{array}{c|c} 0 & X \\ 1 & Y \end{array}$$

$$\mu(a) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 1 \\ Y & 0 & 2 \end{array} \qquad \mu(b) = \begin{array}{c|cc} & X & Y \\ \hline X & 1 & 0 \\ Y & 0 & 2 \end{array}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := 1X + 0Y \\ Y := 1X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := 1X + 0Y \\ Y := 0X + 2Y \end{array} \right.$$



Prop.

- Linear CRA <=> WA
  $(X := \alpha X + \beta Y + \gamma Z)$

- 1 Register CRA <=> Sequential WA
  $(X := \alpha X)$

Def: Register complexity of a rational series f

= min nb. of registers needed by a CRA to realize f

Register minimization problem

In: f rational series given as a WA, $k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant k$ registers realizing f

Def: Register complexity of a rational series f

= min nb. of registers needed by a CRA to realize f

Register minimization problem

In: f rational series given as a WA, $k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant k$ registers realizing f

Def: State-Register complexity of a rational series f

= set of (n,k) s.t. $\exists$ CRA for f with n states & k registers
& $\forall$ CRA for f nb. states $> n$ or nb. registers $> k$

State-Register minimization problem

In: f rational series given as a WA, $n, k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant n$ states & $\leqslant k$ registers realizing f

Let Σ finite alphabet $\mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

$\mathbb{K}$ field

Def: Invariant of $\mathcal{R}$

set $\mathcal{I} \subseteq \mathbb{K}^d$ s.t.

- $u \in \mathcal{I}$
- $\forall a \in \Sigma, \forall x \in \mathcal{I}, \quad x \cdot \mu(a) \in \mathcal{I}$

Let Σ finite alphabet
$\mathbb{K}$ field

$\mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

Def: Invariant of $\mathcal{R}$

set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma, \forall x \in I, x \cdot \mu(a) \in I$

E.g.: $u \cdot \mu(\Sigma^*)$: Reachability set

$\mathbb{K}^d$

Let Σ finite alphabet $\quad \mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

$\mathbb{K}$ field

Def: Invariant of $\mathcal{R}$

set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma, \ \forall x \in I, \ x \cdot \mu(a) \in I$

strongest

E.g.: $u \cdot \mu(\Sigma^*)$: Reachability set

$\mathbb{K}^d$

weakest

Def: I is stronger than J

if $I \subseteq J$

Let Σ finite alphabet $\quad \mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

$\mathbb{K}$ field

Def: Invariant of $\mathcal{R}$

set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma, \forall x \in I, \; x \cdot \mu(a) \in I$

strongest

E.g.: $u \cdot \mu(\Sigma^*)$: Reachability set

$\mathbb{K}^d$

weakest

Def: I is stronger than J

if $I \subseteq J$

Def: Linear Zariski topology

[Bell & Smertnig 2021]

closed sets : finite unions of vector subspaces of $\mathbb{K}^d$

semilinear sets

Let Σ finite alphabet
$\mathbb{K}$ field

$\mathcal{R}=(u,\mu,v)$ d-dimensional WA on Σ over $\mathbb{K}$

Def: Invariant of $\mathcal{R}$

set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma$, $\forall x \in I$, $x \cdot \mu(a) \in I$

strongest

E.g.: $u \cdot \mu(\Sigma^*)$: Reachability set

$\mathbb{K}^d$

weakest

Def: I is stronger than J

if $I \subseteq J$

Def: Linear Zariski topology

[Bell & Smertnig 2021]

closed sets : finite unions of vector subspaces of $\mathbb{K}^d$

semilinear sets

irreducible components

$$S = V_1 \cup V_2 \cup \cdots \cup V_n$$

Length n = nb. of components

Dimension $k = \max_{1 \leq i \leq n} (\dim(V_i))$

Let Σ finite alphabet
$\mathbb{K}$ field

$\mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

Def: Semilinear Invariant of $\mathcal{R}$

semilinear set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma$, $\forall x \in I$, $x \cdot \mu(a) \in I$

strongest

E.g.: $\overline{u \cdot \mu(\Sigma^*)}^{\ell}$: (Linear Hull)

$\mathbb{K}^d$

weakest

Def: I is stronger than J

if $I \subseteq J$

Def: Linear Zariski topology

[Bell & Smertnig 2021]

closed sets: finite unions of vector subspaces of $\mathbb{K}^d$

semilinear sets

irreducible components

$$S = V_1 \cup V_2 \cup \cdots \cup V_n$$

Length n = nb. of components

Dimension $k = \max_{1 \leq i \leq n} (\dim(V_i))$

E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

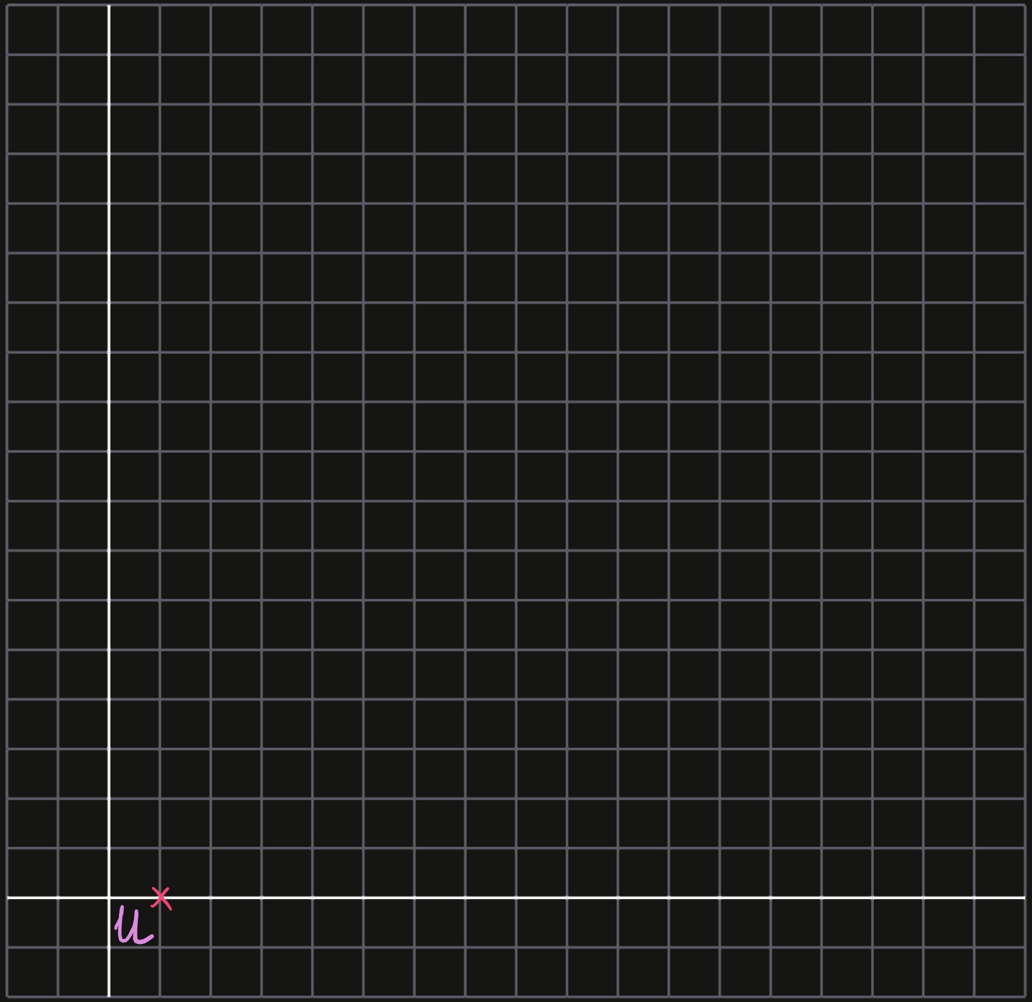
E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



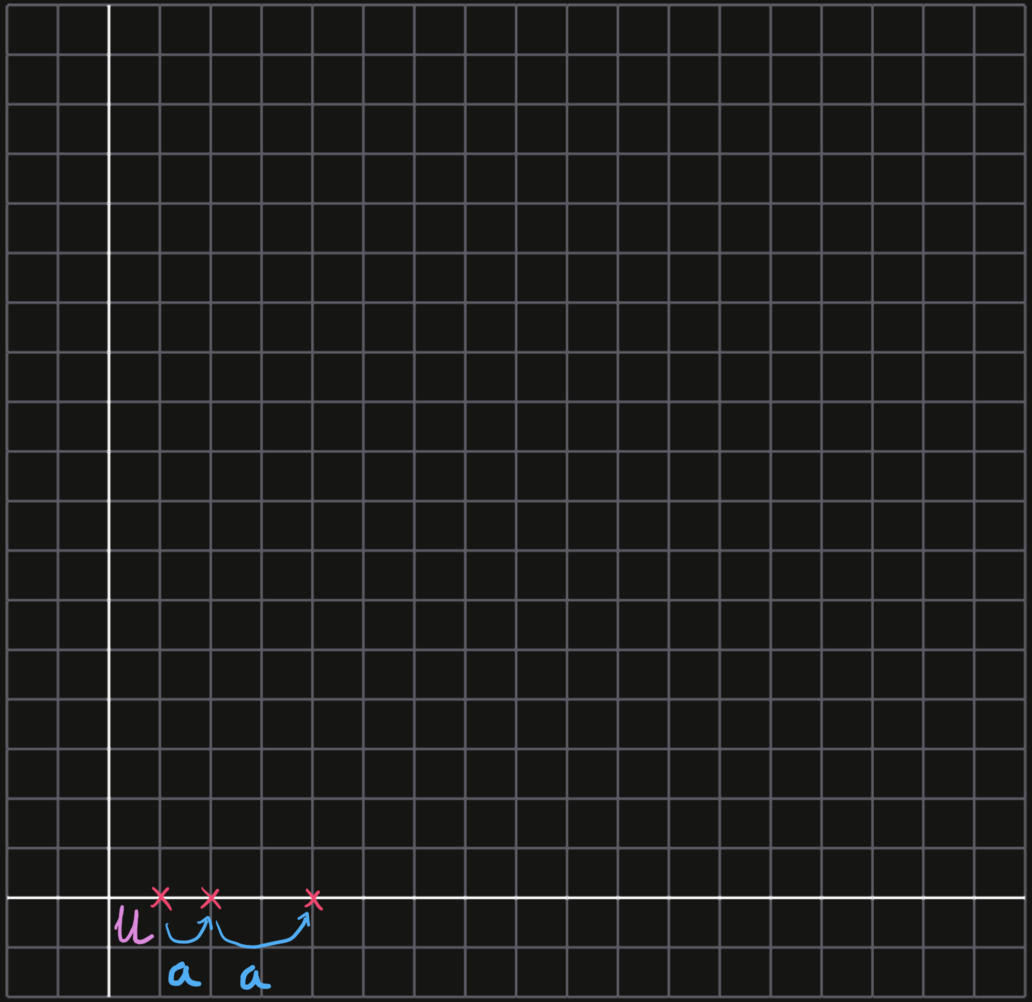
E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



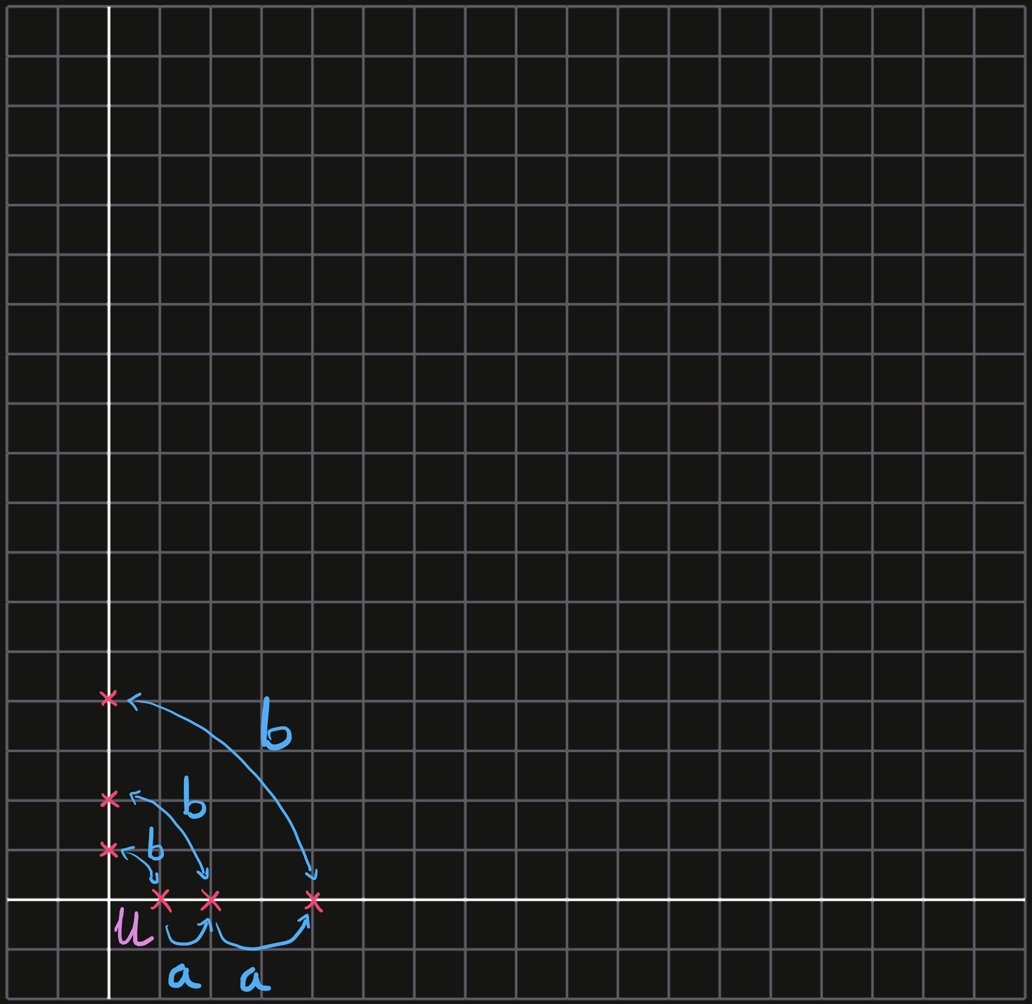
E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



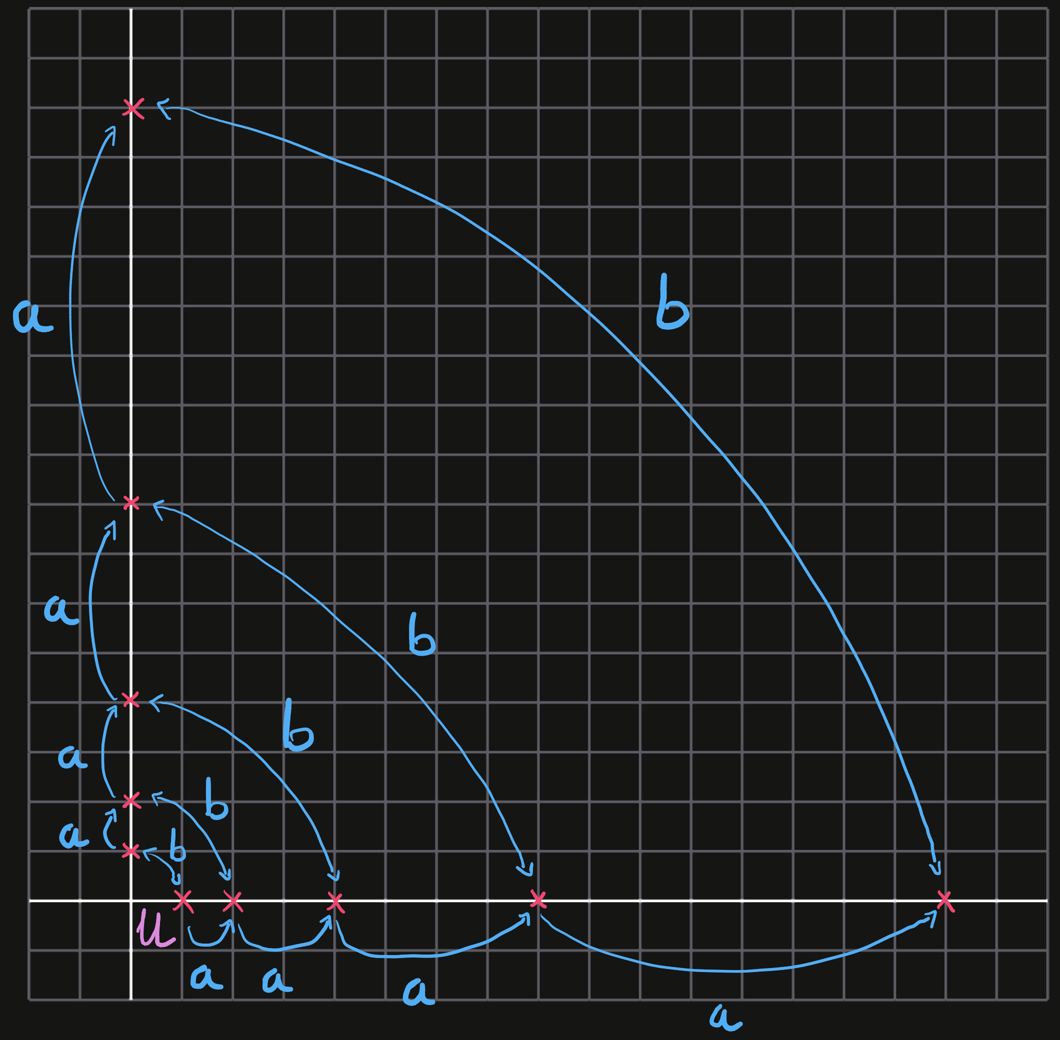
E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$w \longmapsto \begin{cases} 2^{|w|_a} & \text{if } |w|_b \text{ is even} \\ 0 & \text{else} \end{cases}$$

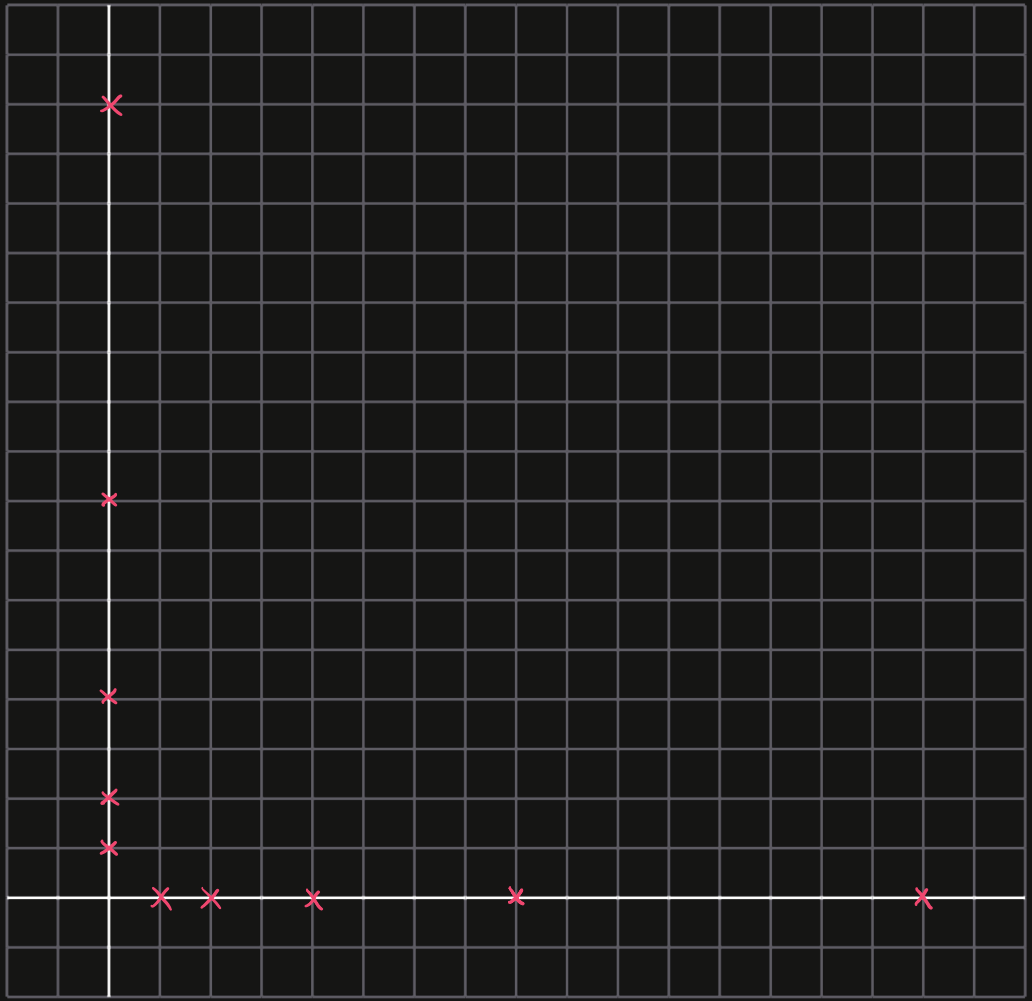
E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = (1 \ \ 0)$ $\quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$$w \mapsto \begin{cases} 2^{|w|_a} & \text{if } |w|_b \text{ is even} \\ 0 & \text{else} \end{cases}$$

$$u\mu(\Sigma^*) = \{(2^n \ \ 0),\ n \in \mathbb{N}\} \cup \{(0 \ \ 2^n),\ n \in \mathbb{N}\}$$

E.g. $\Sigma = \{a, b\}$ $\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$w \mapsto \begin{cases} 2^{|w|_a} & \text{if } |w|_b \text{ is even} \\ 0 & \text{else} \end{cases}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$

Length : 2

Dimension : 1

Let f be a rational series

Thm : [Bell & Smertnig 2021]

$\exists$ sequential WA for f

iff

$\forall$ minimal WA (u, μ, v) for f

$$\dim\left(\overline{u\mu(\Sigma^*)}^{!} \right) \leqslant 1$$

Let f be a rational series

Our results

Thm: [Bell & Smertnig 2021]

$\exists$ sequential WA for f

iff

$\forall$ minimal WA (u,μ,v) for f

$$\dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \leq 1$$

Thm: (Characterization)

$\exists$ CRA for f with n states & k registers

iff

$\forall$ minimal WA $\mathfrak{R}$ for f

$\exists$ semilinear invariant I of $\mathfrak{R}$ s.t.

$$\mathrm{length}(I) \leq n \quad \& \quad \dim(I) \leq k$$

Cor: Register complexity of f

$$= \dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)$$

where (u,μ,v): minimal WA for f

Let f be a rational series

Our results

Thm : [Bell & Smertnig 2021]

$\exists$ sequential WA for f

iff

$\forall$ minimal WA (u,μ,v) for f

$$\dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \leqslant 1$$

↳ [Bell & Smertnig 2023]

$\overline{u\mu(\Sigma^*)}^{\ell}$ is computable

=> Sequential? is decidable

(Unambiguous? is decidable)

Thm : (Characterization)

$\exists$ CRA for f with n states & k registers

iff

$\forall$ minimal WA $\mathcal{R}$ for f

$\exists$ semilinear invariant I of $\mathcal{R}$ s.t.

$$\mathrm{length}(I) \leqslant n \quad \& \quad \dim(I) \leqslant k$$

Cor : Register complexity of f

$$\shortparallel$$

$$\dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)$$

where (u,μ,v) : minimal WA for f

Let f be a rational series

Thm : [Bell & Smertnig 2021]

$\exists$ sequential WA for f

iff

$\forall$ minimal WA (u, μ, v) for f

$\dim(\overline{u\mu(\Sigma^*)}^{\ell}) \leqslant 1$

↳ [Bell & Smertnig 2023] $\overline{u\mu(\Sigma^*)}^{\ell}$ is computable

=> Sequential? is decidable

(Unambiguous? is decidable)

in 2-EXPTIME

Our results

Thm : (Characterization)

$\exists$ CRA for f with n states & k registers

iff

$\forall$ minimal WA $\mathcal{R}$ for f

$\exists$ semilinear invariant I of $\mathcal{R}$ s.t.

$\mathrm{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

↳ I is computable in NEXPTIME

=> Stt-Reg min pb. is decidable in NEXPTIME

Cor : Register complexity of f

$=$

$\dim(\overline{u\mu(\Sigma^*)}^{\ell})$

where (u, μ, v) : minimal WA for f

↳ $\overline{u\mu(\Sigma^*)}^{\ell}$ is computable in 2-EXPTIME

=> Reg min pb is decidable in 2-EXPTIME

Let f be a rational series

Proof sketch

Thm: (Characterization)

$\exists$ CRA for f with n states & k registers

iff

$\forall$ minimal WA $\mathfrak{R}$ for f

$\exists$ semilinear invariant I of $\mathfrak{R}$ s.t.

$$\mathrm{length}(I) \leqslant n \quad \& \quad \dim(I) \leqslant k$$

Proof sketch

Let f be a rational series

Prop. Let $\mathcal{R}$ be a WA for f

$\exists$ Semilinear invariant I of $\mathcal{R}$ s.t. $\mathrm{length}(I) = n$ & $\dim(I) = k$

$$\Downarrow$$

$\forall\, \mathcal{R}_m$ minimal WA for f

$\exists$ Semilinear invariant I_m of $\mathcal{R}_m$ s.t. $\mathrm{length}(I_m) \leqslant n$ & $\dim(I_m) \leqslant k$

Thm: (Characterization)

$\exists$ CRA for f with n states & k registers

iff

$\forall$ minimal WA $\mathcal{R}$ for f

$\exists$ semilinear invariant I of $\mathcal{R}$ s.t.

$\mathrm{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Proof sketch

Let f be a rational series

Prop. Let $\mathcal{R}$ be a WA for f

$\exists$ Semilinear invariant I of $\mathcal{R}$ s.t. $\mathrm{length}(I) = n$
& $\dim(I) = k$

$\Downarrow$

$\forall \mathcal{R}_m$ minimal WA for f

$\exists$ Semilinear invariant I_m of $\mathcal{R}_m$ s.t. $\mathrm{length}(I_m) \leqslant n$
& $\dim(I_m) \leqslant k$

Thm: (Characterization)

$\exists$ CRA for f with n states
& k registers

iff

$\forall$ minimal WA $\mathcal{R}$ for f

$\exists$ semilinear invariant I of $\mathcal{R}$ s.t.

$\mathrm{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Prop. (WA $\rightarrow$ CRA)

$\exists$ WA for f with a
semilinear invariant I s.t. $\mathrm{length}(I) = n$
& $\dim(I) = k$

$\Downarrow$

$\exists$ CRA for f with n states
& k registers

Prop. (CRA $\rightarrow$ WA)

$\exists$ CRA for f with n states
& k registers

$\Downarrow$

$\exists$ WA for f with a
semilinear invariant I s.t. $\mathrm{length}(I) = n$
& $\dim(I) \leqslant k$

WA $\to$ CRA

$\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathfrak{A} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$

WA → CRA

$\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$

WA → CRA

$\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$

WA → CRA

$\Sigma = \{a, b\}$

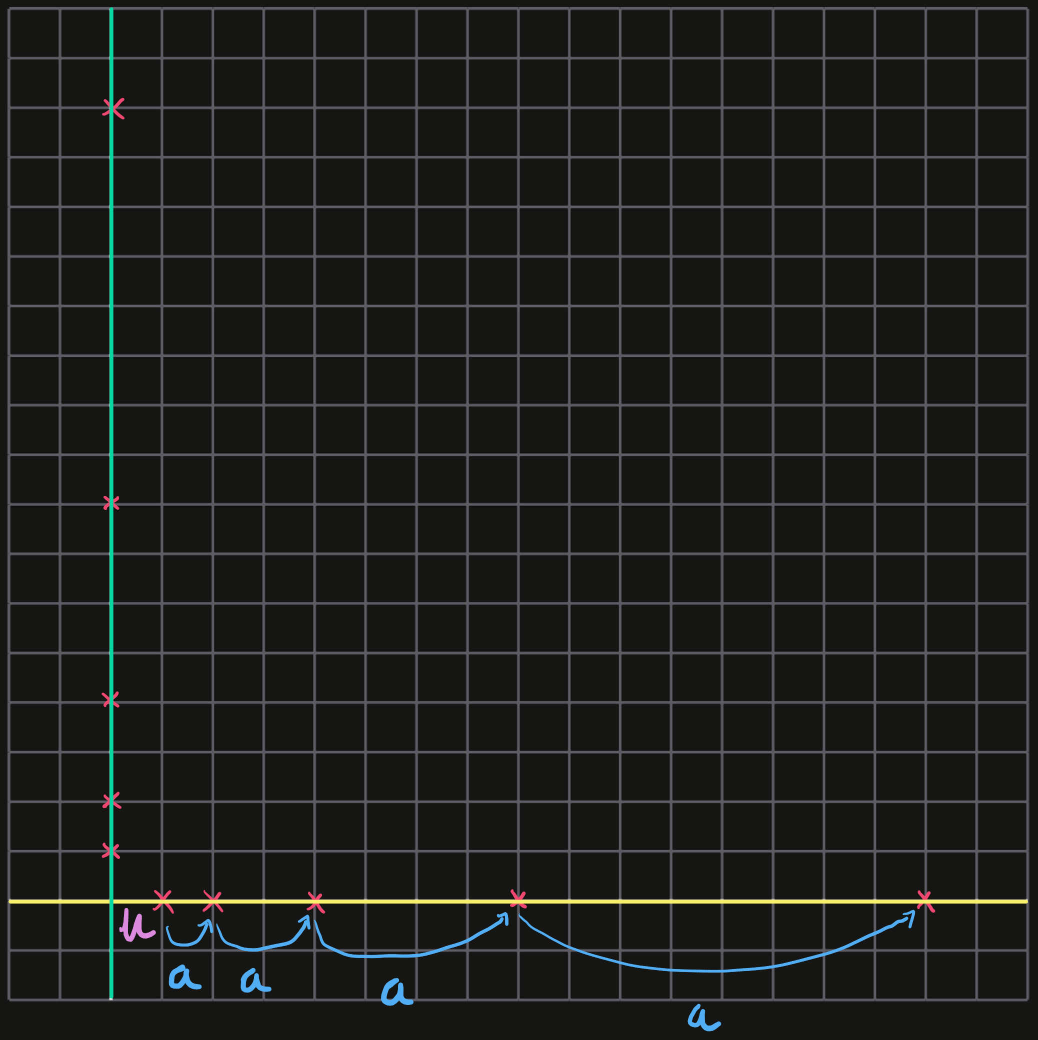
$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$



WA → CRA

$\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$

WA → CRA

$\Sigma = \{a, b\}$

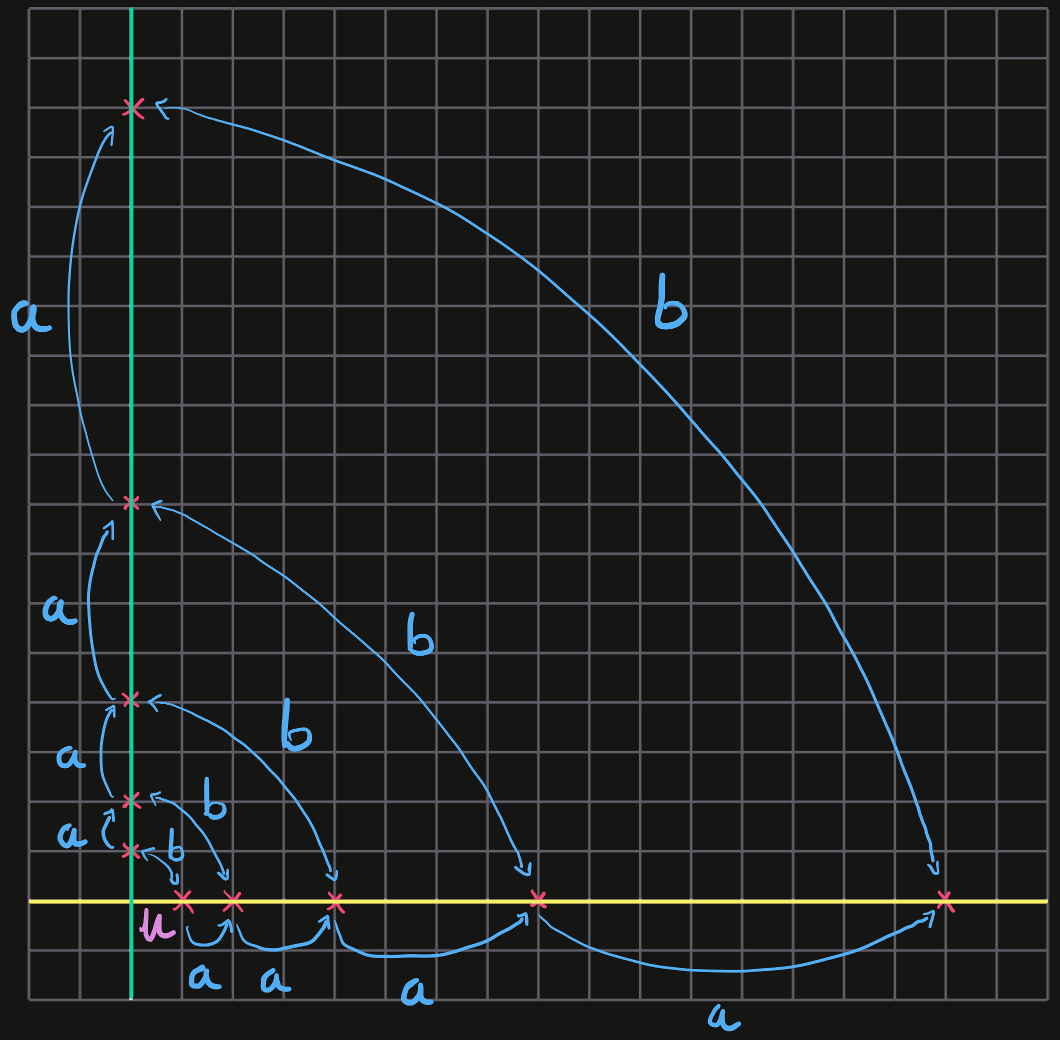
$\mathbb{K} = (\mathbb{R}, +, \cdot)$

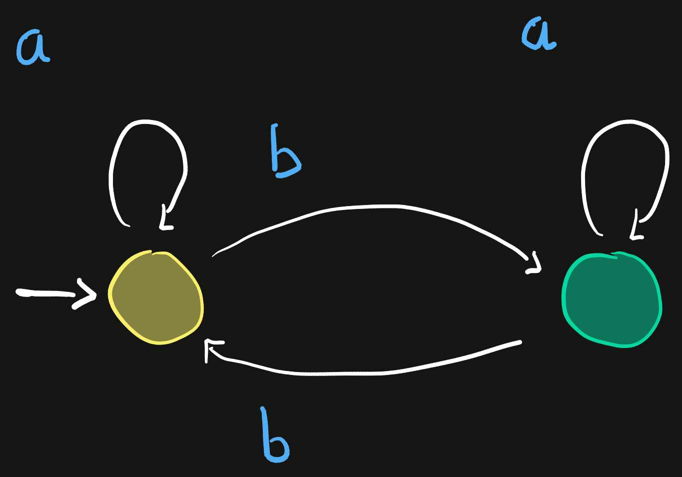
$\mathfrak{R} = (u, \mu, v)$

$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$





WA → CRA

$\Sigma = \{a, b\}$

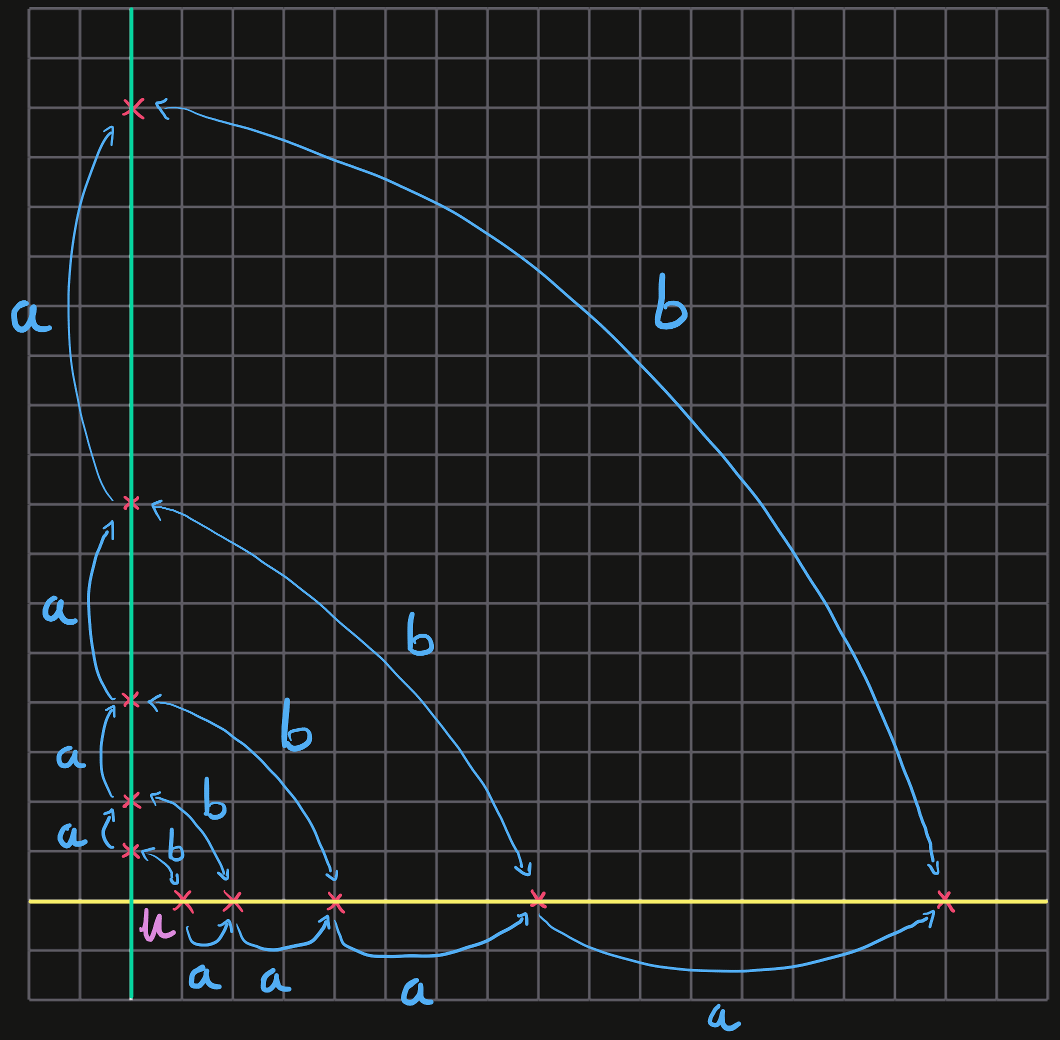
$\mathbb{K} = (\mathbb{R}, +, \cdot)$

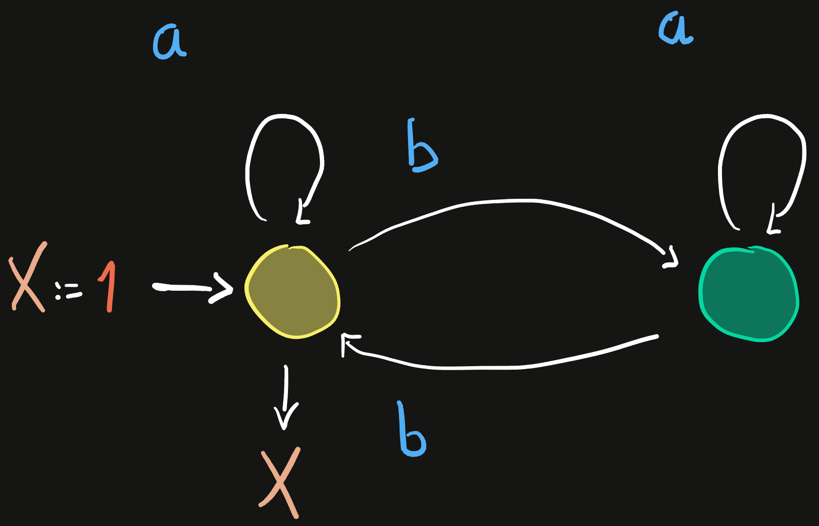
$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$





WA → CRA

$\Sigma = \{a, b\}$

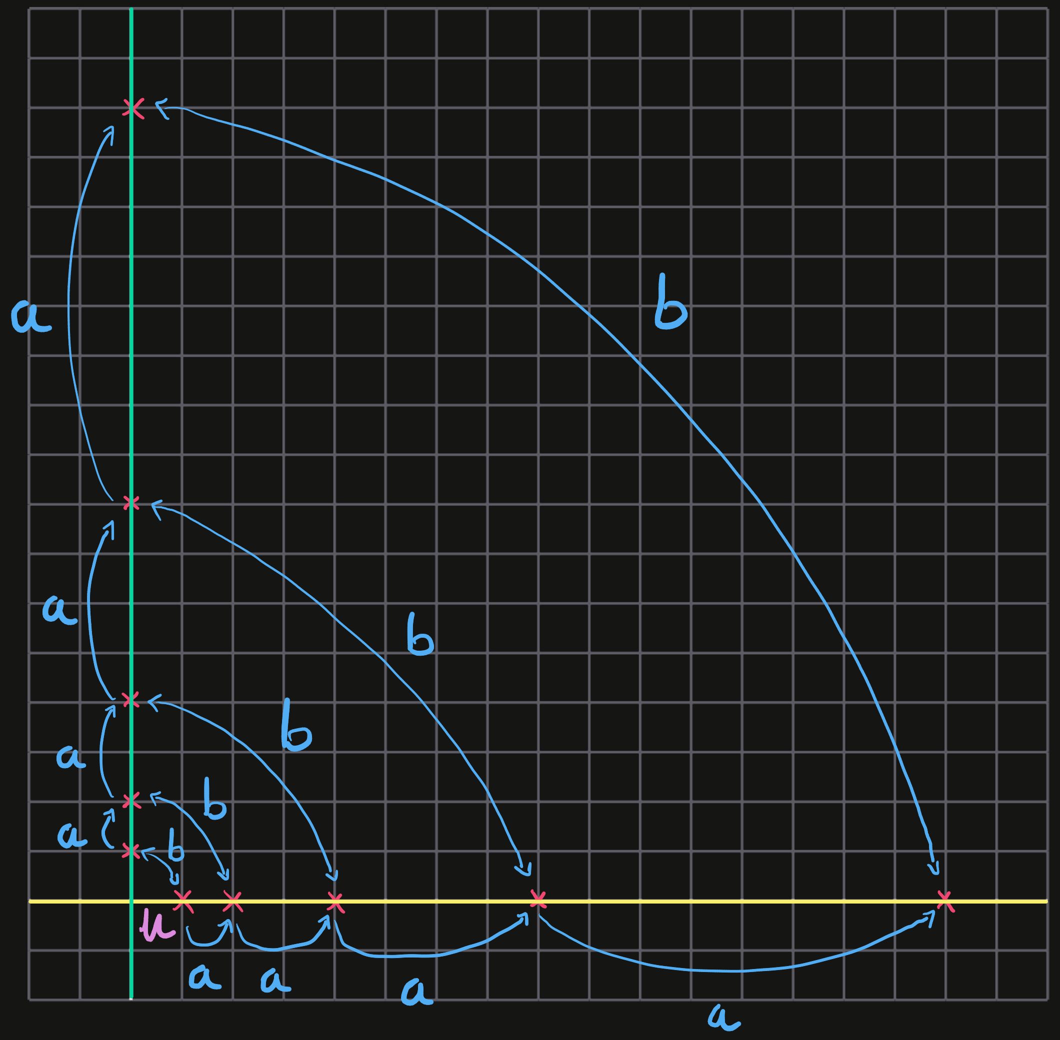
$\mathbb{K} = (\mathbb{R}, +, \cdot)$

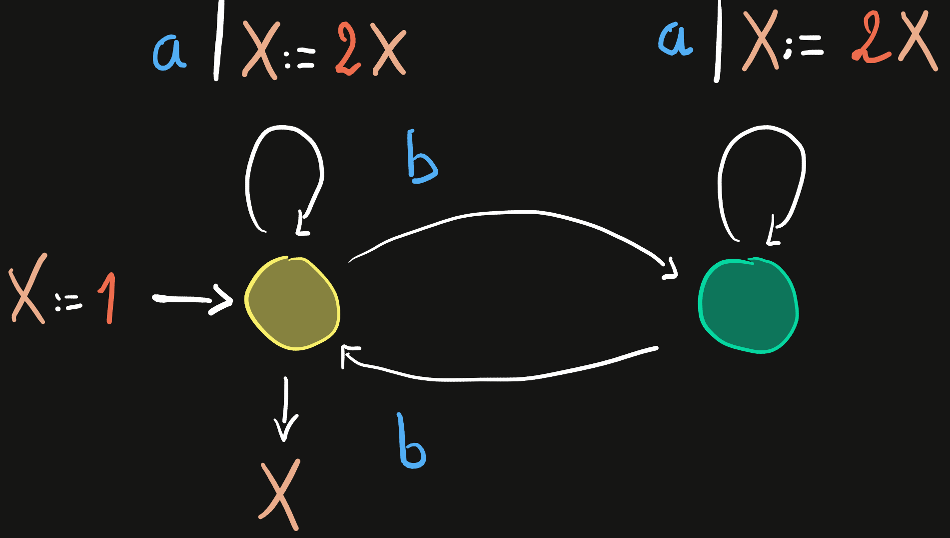
$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$





WA → CRA

$\Sigma = \{a, b\}$

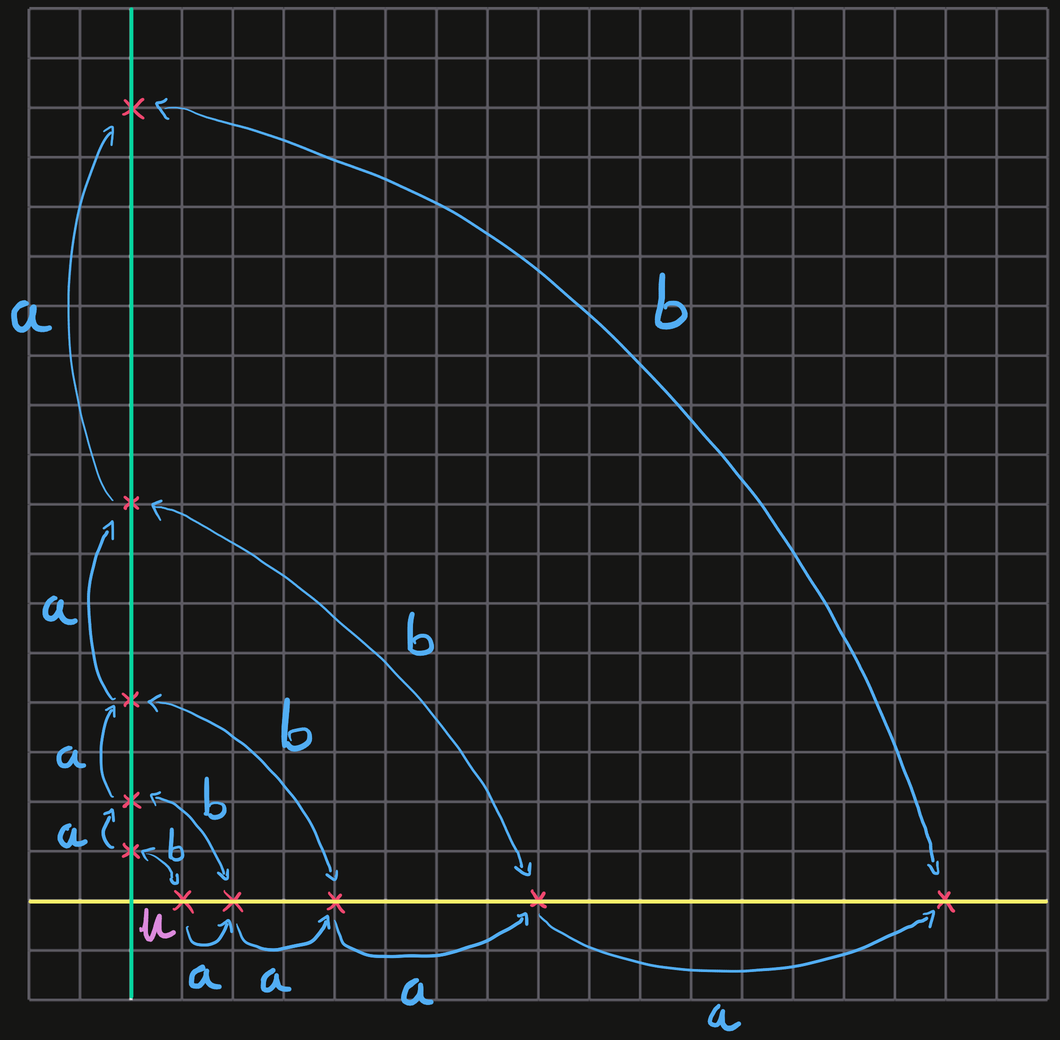
$\mathbb{K} = (\mathbb{R}, +, \cdot)$

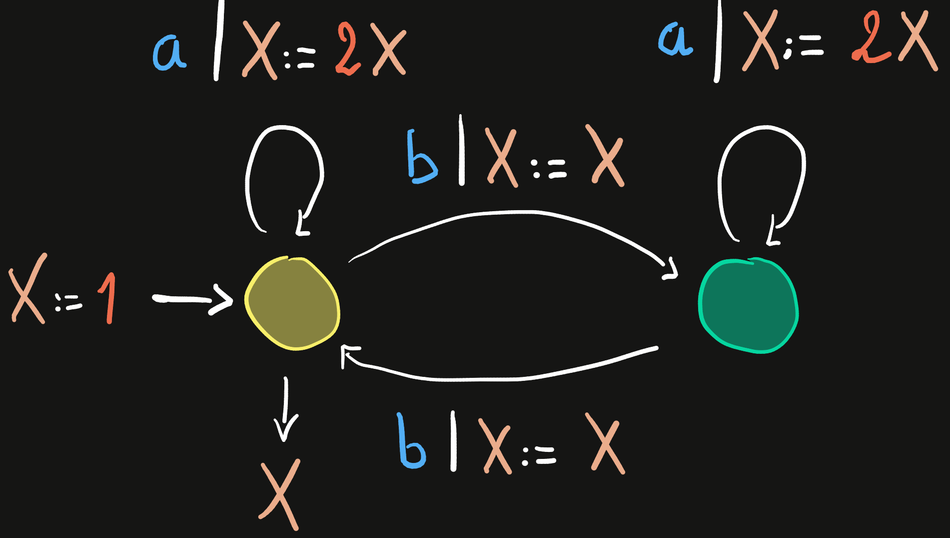
$\mathcal{R} = (u, \mu, \nu)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad \nu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$





WA → CRA

$\Sigma = \{a, b\}$

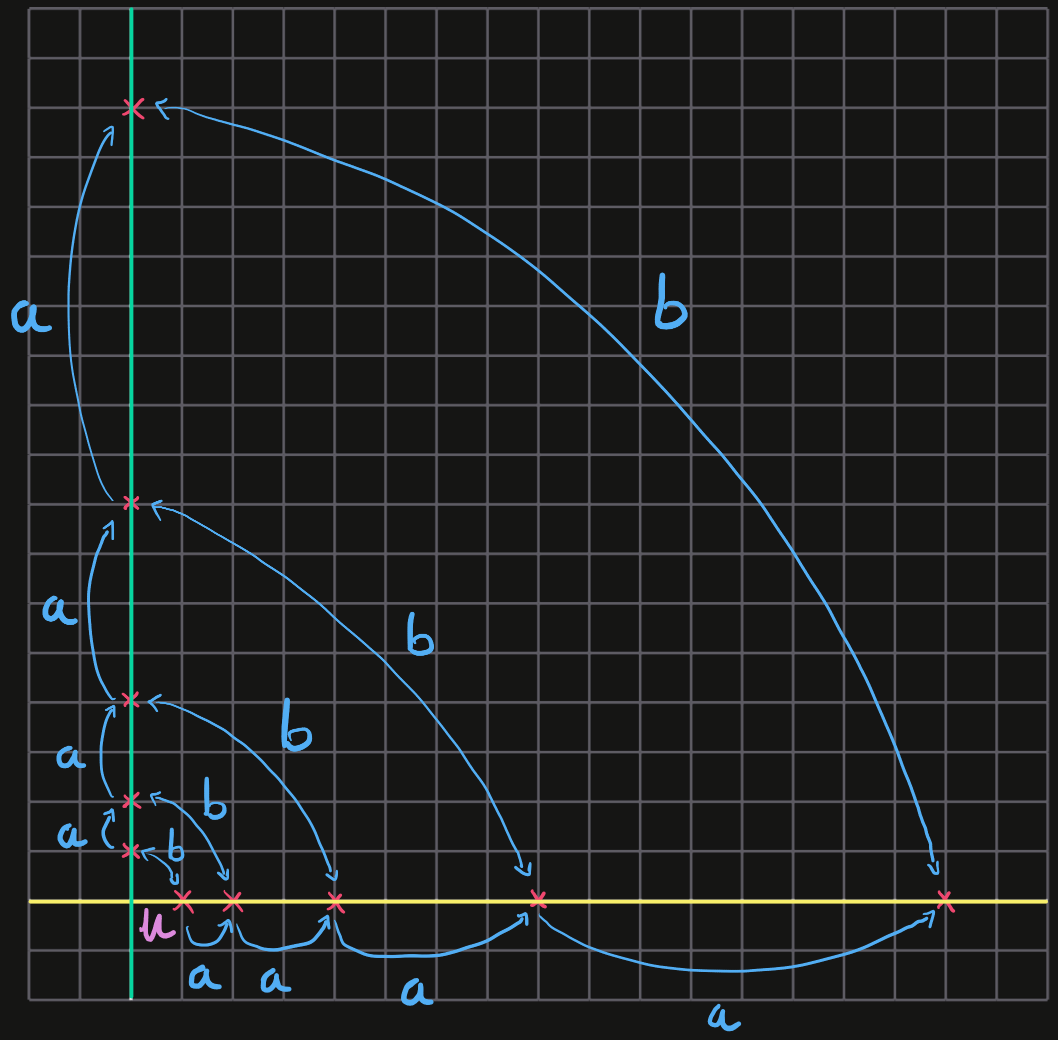
$\mathbb{K} = (\mathbb{R}, +, \cdot)$

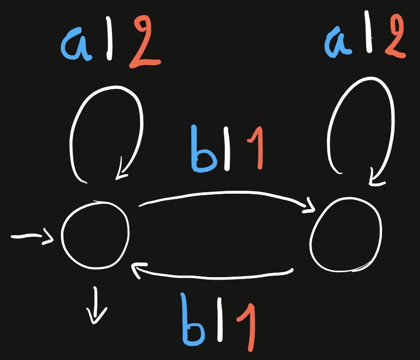
$\mathcal{R} = (u, \mu, v)$

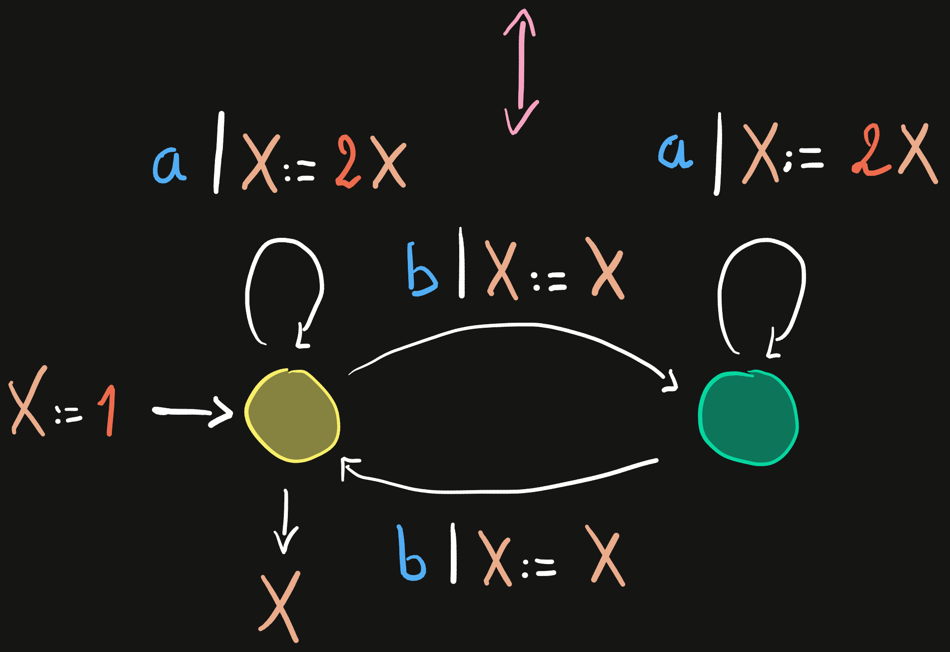
$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

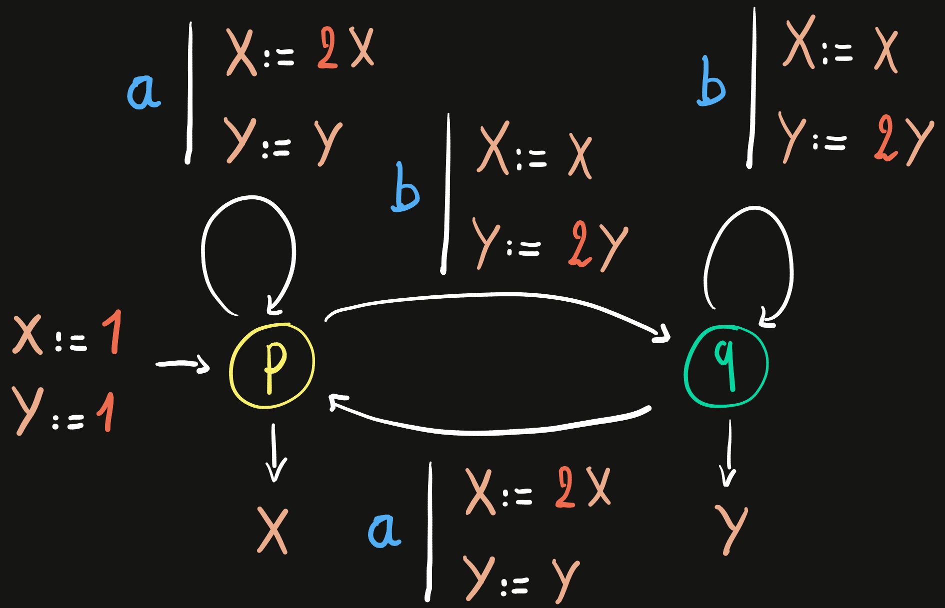
$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$





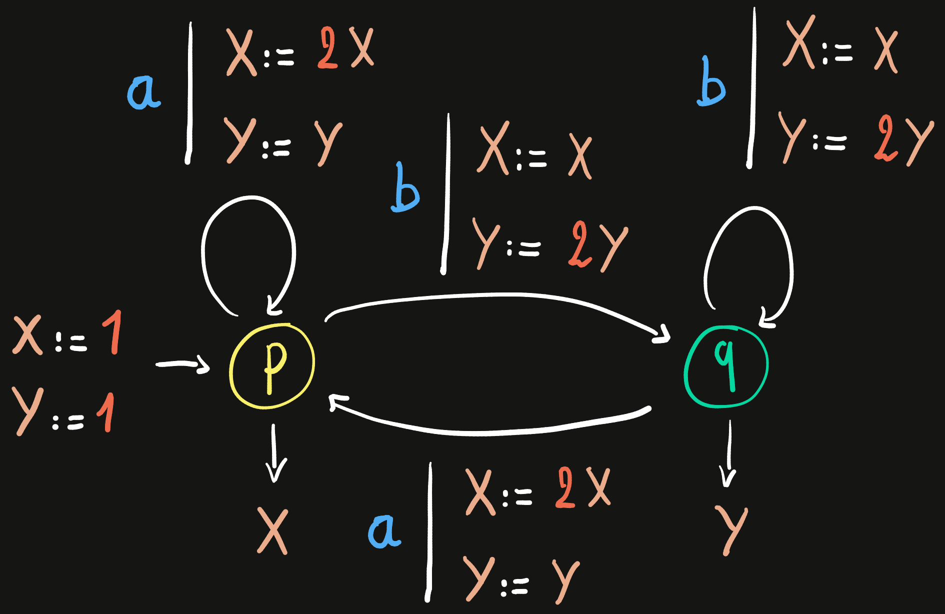


CRA → WA



$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA

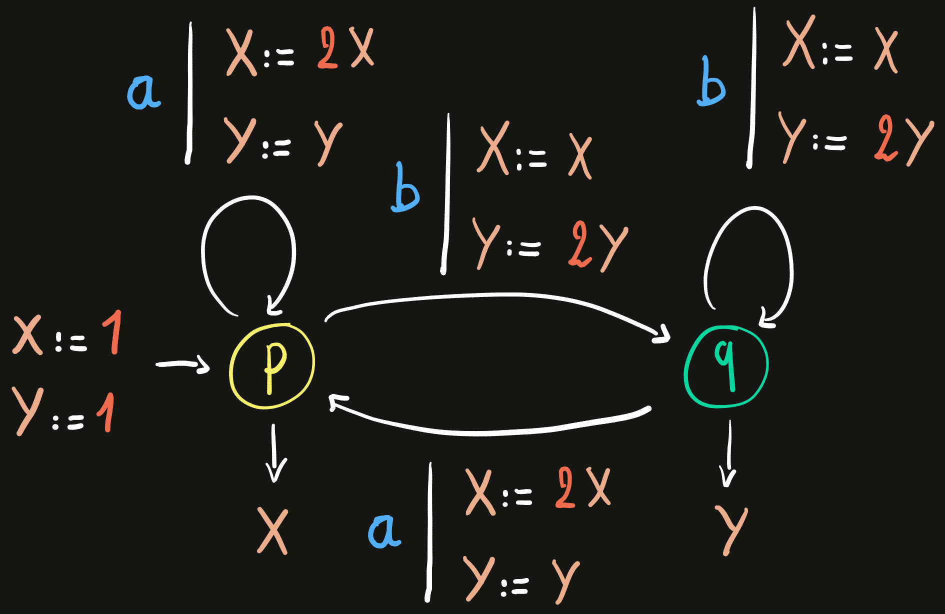


$$u = \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left(\begin{array}{c|c} \quad & \quad \end{array}\right) \end{array} \qquad v = \left(\begin{array}{c} \quad \\ \hline \quad \end{array}\right)\begin{array}{c} p \\ q \end{array}$$

$$\mu(a) = \begin{array}{c} \\ p \\ q \end{array} \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left(\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right) \end{array} \qquad \mu(b) = \begin{array}{c} \\ p \\ q \end{array} \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left(\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right) \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA

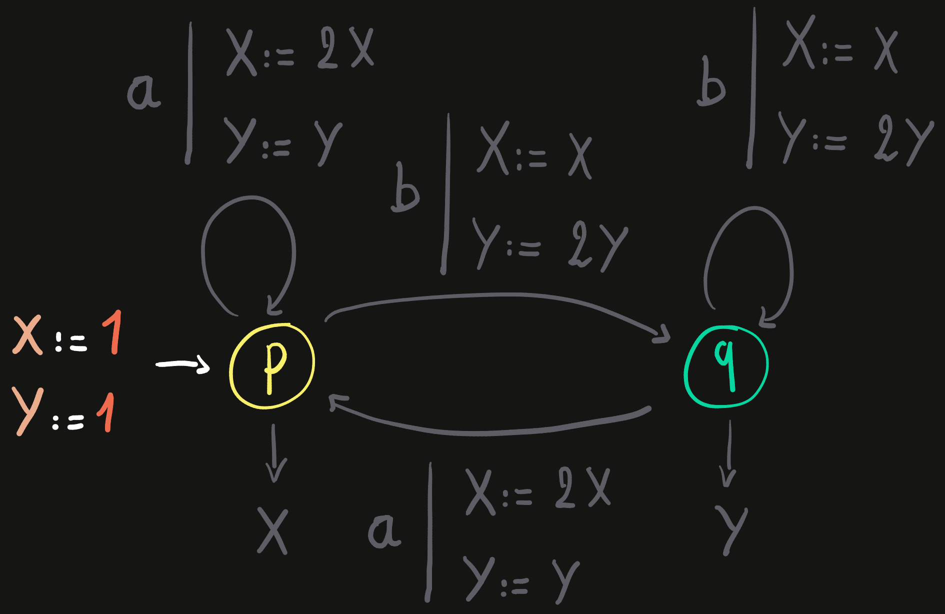


$$u = \left(\begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ \quad & \quad & \quad & \quad \\ x & y & x & y \end{array}\right) \qquad v = \begin{array}{c} x \\ y \\ x \\ y \end{array}\left(\begin{array}{c} \quad \\ \quad \\ \hline \quad \\ \quad \end{array}\right)\begin{array}{c} p \\ \\ q \\ \\ \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array} \qquad \mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} p & & q & \\ \left(1 \right. & 1 & \big| \; 0 & \left. 0 \right) \\ x & y & x & y \end{array}$$

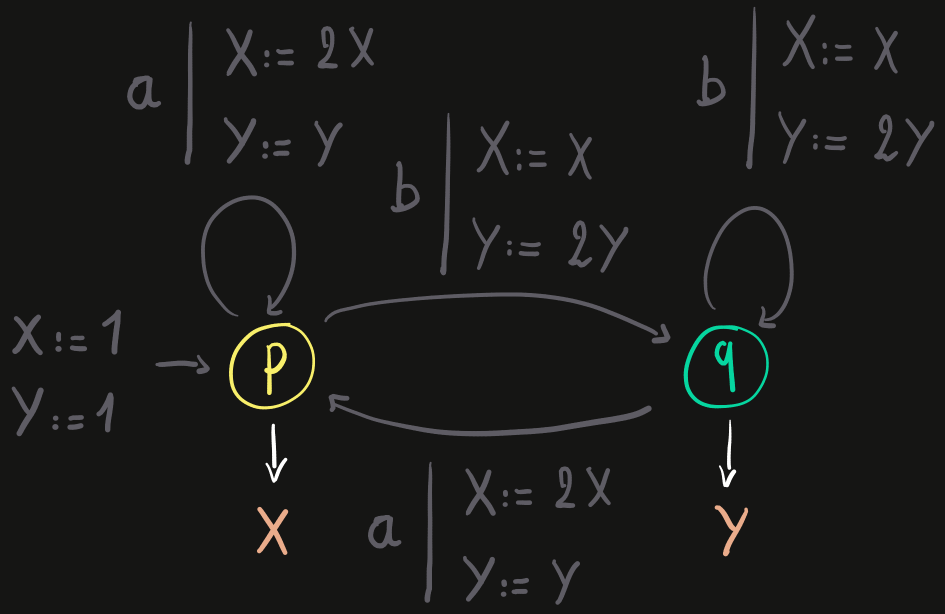
$$v = \begin{array}{cc} p & x \\ & y \\ q & x \\ & y \end{array}\left(\begin{array}{c} \\ \\ \hline \\ \\ \end{array}\right)\begin{array}{c} p \\ \\ q \\ \\ \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|c} & \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ & x & y & x & y \\ p \; x & & & & \\ y & & & & \\ q \; x & & & & \\ y & & & & \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|c} & \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ & x & y & x & y \\ p \; x & & & & \\ y & & & & \\ q \; x & & & & \\ y & & & & \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{cccc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ \hline 1 & 1 & 0 & 0 \\ \hline x & y & x & y \end{array}$$

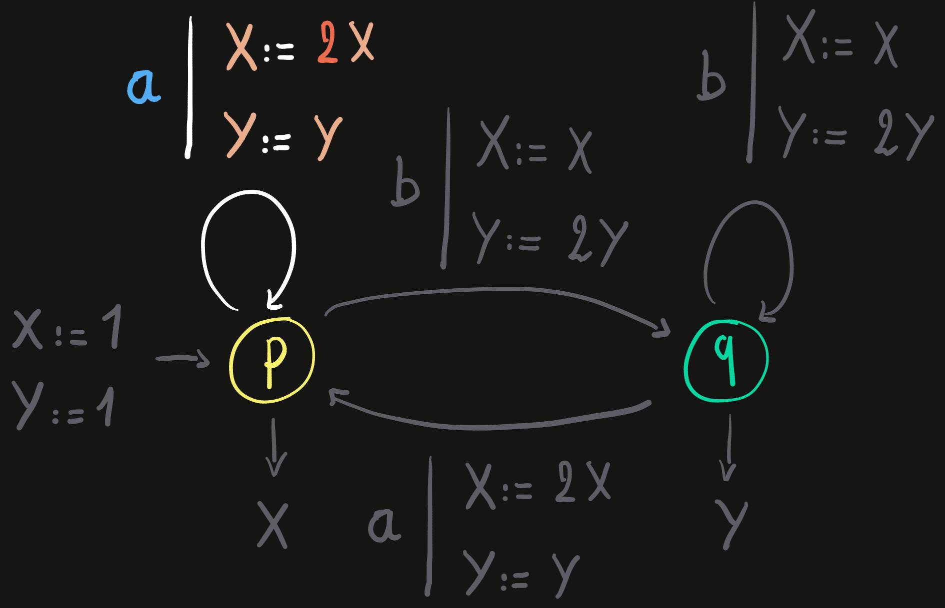
$$v = \begin{array}{cc|c} & & \\ x & 1 & p \\ y & 0 & \\ \hline x & 0 & q \\ y & 1 & \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \left(\begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ 1 & 1 & 0 & 0 \\ x & y & x & y \end{array}\right)$$

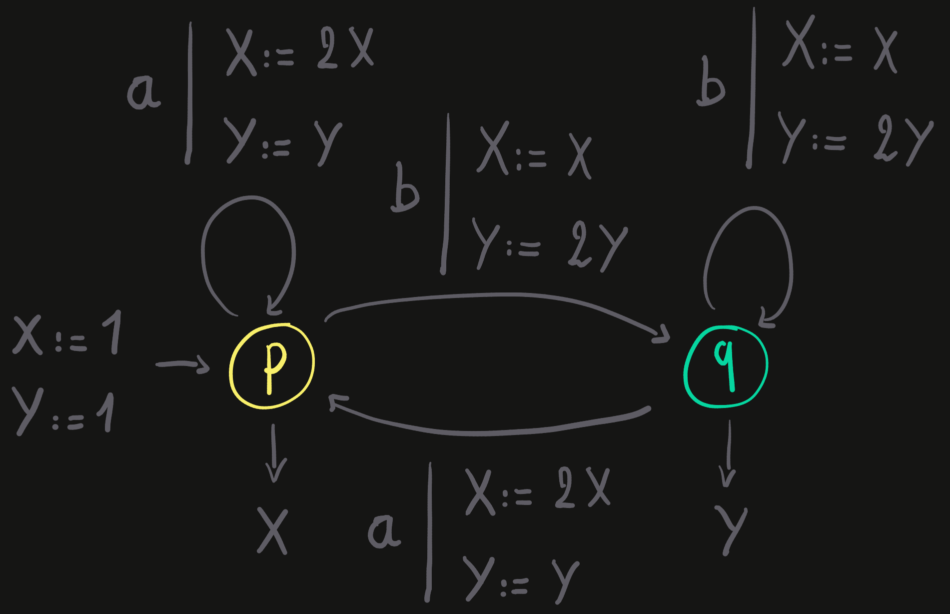
$$v = \begin{array}{c|c|c} x & 1 & \\ y & 0 & p \\ \hline x & 0 & \\ y & 1 & q \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & & & x \\ & y & 0 & 1 & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ (1 & 1 & 0 & 0) \\ x & y & x & y \end{array}$$

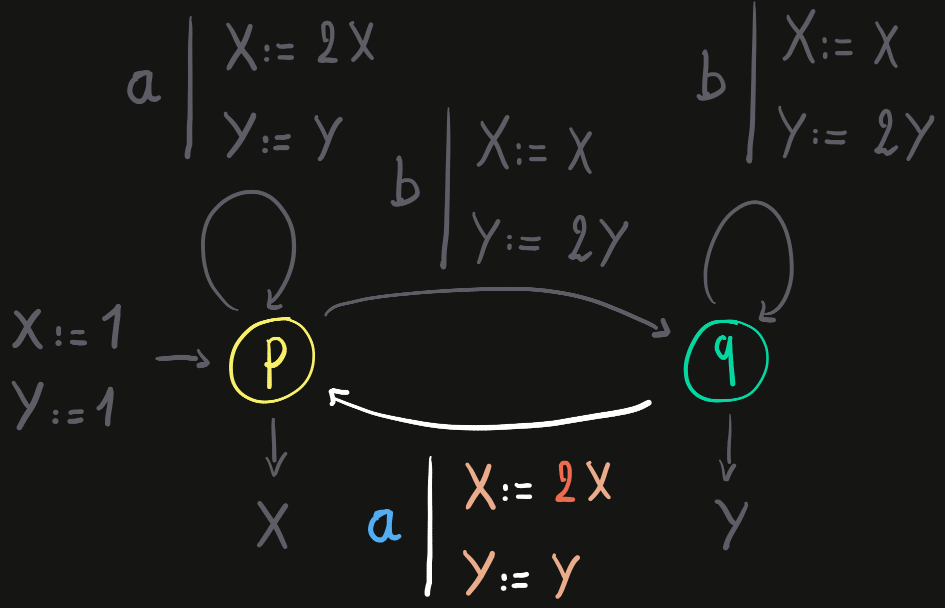
$$v = \begin{array}{c|c|c} x & 1 & p \\ y & 0 & \\ \hline x & 0 & q \\ y & 1 & \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



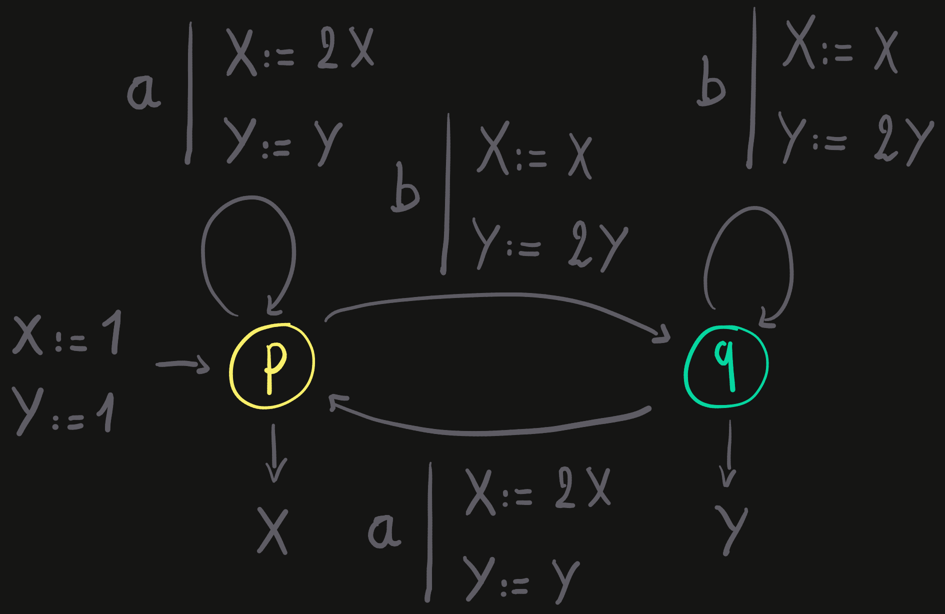
$$u = \left(\begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ 1 & 1 & 0 & 0 \\ x & y & x & y \end{array}\right) \qquad v = \left(\begin{array}{c|c|c} x & 1 & p \\ y & 0 & \\ \hline x & 0 & q \\ y & 1 & \end{array}\right)$$

$$\mu(a) = \left(\begin{array}{cc|cc|cc|c} & & p & & q & & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & 2 & 0 & & & x \\ & y & 0 & 1 & & & y \end{array}\right)$$

$$\mu(b) = \left(\begin{array}{cc|cc|cc|c} & & p & & q & & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}\right)$$

$$w\sigma \mapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ (1 & 1 & 0 & 0) \\ x & y & x & y \end{array}$$

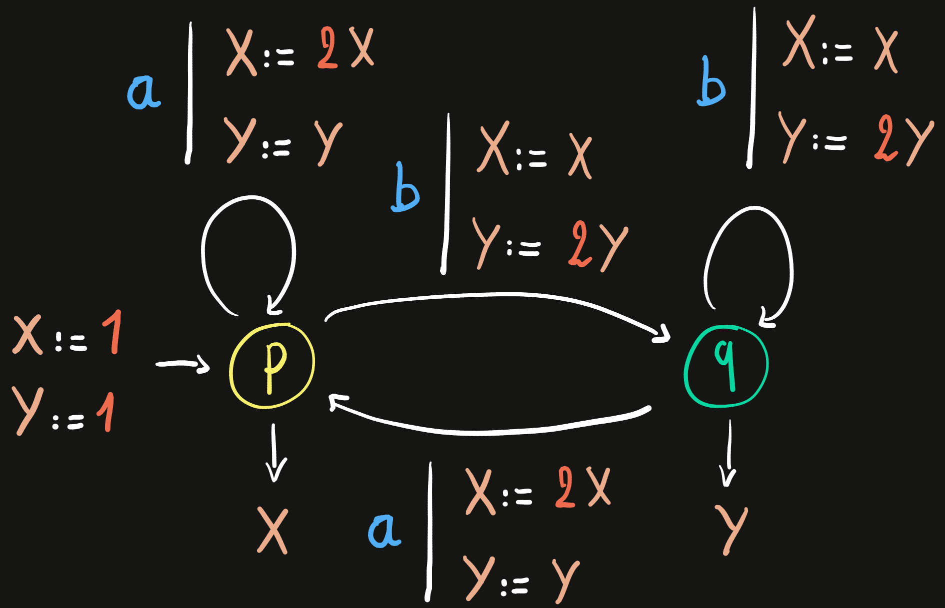
$$v = \begin{array}{c} x \\ y \\ x \\ y \end{array} \left(\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 1 \end{array}\right) \begin{array}{c} p \\ \\ q \\ \\ \end{array}$$

$$\mu(a) = \begin{array}{cc} & \begin{array}{cccc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ x & y & x & y \end{array} \\ \begin{array}{cc} p & x \\ & y \\ q & x \\ & y \end{array} & \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{array}{c} x \\ y \\ x \\ y \end{array} \end{array}$$

$$\mu(b) = \begin{array}{cc} & \begin{array}{cccc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ x & y & x & y \end{array} \\ \begin{array}{cc} p & x \\ & y \\ q & x \\ & y \end{array} & \left(\begin{array}{cc|cc} \; & \; & \; & \; \\ & & & \\ \hline & & & \\ & & & \end{array}\right) \begin{array}{c} x \\ y \\ x \\ y \end{array} \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{cccc} p & & q & \\ \left(1\right. & 1 \mid 0 & \left.0\right) \\ x & y & x & y \end{array}$$

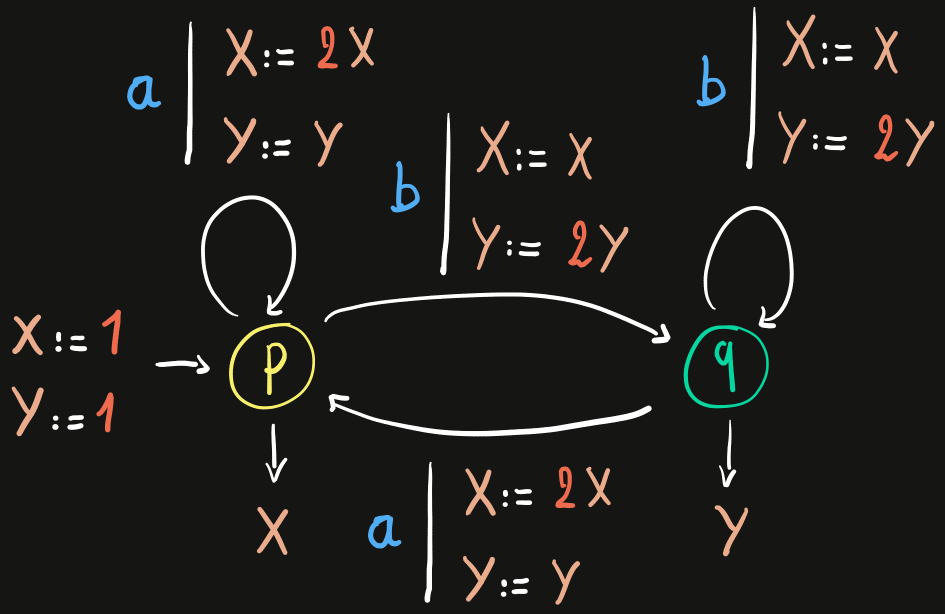
$$v = \begin{array}{c|c|c} x & 1 & \\ y & 0 & p \\ \hline x & 0 & \\ y & 1 & q \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & p & & q & & \\ & & x\ y & & x\ y & & \\ p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & p & & q & & \\ & & x\ y & & x\ y & & \\ p & x & 0 & 0 & 1 & 0 & x \\ & y & 0 & 0 & 0 & 2 & y \\ \hline q & x & 0 & 0 & 1 & 0 & x \\ & y & 0 & 0 & 0 & 2 & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{c} \begin{array}{cc} p & q \end{array} \\ \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x & y & x & y \end{array} \end{array} \qquad v = \begin{array}{c} x \\ y \\ x \\ y \end{array} \left(\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 1 \end{array}\right) \begin{array}{c} p \\ \\ q \\ \\ \end{array}$$

$$\mu(a) = \begin{array}{cc} & \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ x & y & x & y \end{array} \\ \begin{array}{cc} p & x \\ & y \\ q & x \\ & y \end{array} & \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{array}{c} x \\ y \\ x \\ y \end{array} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ x & y & x & y \end{array} \\ \begin{array}{cc} p & x \\ & y \\ q & x \\ & y \end{array} & \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right) \begin{array}{c} x \\ y \\ x \\ y \end{array} \end{array}$$

Configurations : $(* \; * \mid 0 \; 0)$ or $(0 \; 0 \mid * \; *)$

Semilinear invariant : $\mathbb{R}^2 \times \{0\}^2 \cup \{0\}^2 \times \mathbb{R}^2$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

Algorithms

Thm: (Characterization)

$\exists$ CRA for $\mathfrak{f}$ with n states & k registers

iff

$\forall$ minimal WA $\mathfrak{R}$ for $\mathfrak{f}$

$\exists$ semilinear invariant I of $\mathfrak{R}$ s.t.

$$\mathrm{length}(I) \leqslant n \quad \& \quad \dim(I) \leqslant k$$

↳ I is computable in NEXPTIME

⇒ Stt-Reg min pb. is decidable in NEXPTIME

Cor: Register complexity of $\mathfrak{f}$

$$\| \quad \dim\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)$$

where (u,μ,v): minimal WA for $\mathfrak{f}$

↳ $\overline{u\mu(\Sigma^*)}^{\ell}$ is computable in 2-EXPTIME

⇒ Reg min pb is decidable in 2-EXPTIME

Algorithm for the State-Register Minimization Problem

In: $\mathfrak{R} = (u, \mu, v)$ d-dimensional WA , $n, k \in \mathbb{N}$

Out: Semilinear invariant of $\mathfrak{R}$ (if $\exists$)

of length $\leqslant n$ & dimension $\leqslant k$

Algorithm for the State-Register Minimization Problem

In: $\mathcal{R} = (u, \mu, v)$ d-dimensional WA , $n, k \in \mathbb{N}$

Out: Semilinear invariant of $\mathcal{R}$ of length $\leqslant n$ & dimension $\leqslant k$ (if $\exists$)

Lem. Semilinear invariants of length $\leqslant n$ & dimension $\leqslant k$ can be represented in size $O(n^2 k^2)$

1. Guess a representation of a semilinear set S
2. if S is an invariant of $\mathcal{R}$: return S
   else : reject

Complexity: NEXPTIME

Algorithm for the Register Minimization Problem

In: $\mathfrak{R} = (u, \mu, v)$ d-dimensional WA , $c \in \mathbb{N}$

Out: Semilinear invariant of $\mathfrak{R}$

stronger than all the semilinear invariants of $\mathfrak{R}$ of length $\leqslant c$

Algorithm for the Register Minimization Problem

In: $\mathfrak{R} = (u, \mu, v)$ d-dimensional WA , $c \in \mathbb{N}$

Out: Semilinear invariant of $\mathfrak{R}$

stronger than all the semilinear invariants of $\mathfrak{R}$ of length $\leqslant c$

=> returns $\overline{u\mu(\Sigma^*)}^{\ell}$ if c is large enough

Complexity: $O(c^{P(d)})$ (for a polynomial P)

Algorithm for the Register Minimization Problem

In: $\mathcal{R} = (u, \mu, v)$ d-dimensional WA , $c \in \mathbb{N}$

Out: Semilinear invariant of $\mathcal{R}$

stronger than all the semilinear invariants of $\mathcal{R}$ of length $\leqslant c$

$\Rightarrow$ returns $\overline{u\mu(\Sigma^*)}^{\ell}$ if c is large enough

Complexity: $O(c^{P(d)})$ (for a polynomial p)

[Bell & Smertnig 2023]

length$\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \leqslant$ 2-EXP in d

↑

could be improved ???

Algorithm for the Register Minimization Problem

In: $\mathfrak{R} = (u, \mu, v)$ d-dimensional WA , $c \in \mathbb{N}$

Out: Semilinear invariant of $\mathfrak{R}$

stronger than all the semilinear invariants of $\mathfrak{R}$ of length $\leqslant c$

=> returns $\overline{u\mu(\Sigma^*)}^{\ell}$ if c is large enough

Complexity: $O(c^{P(d)})$ (for a polynomial P)

[Bell & Smertnig 2023]

$\text{length}\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \leqslant$ 2-EXP in d

↑

could be improved ???

$\forall \sigma, \tau \in \Sigma, \; \mu(\sigma\tau) = \mu(\tau\sigma)$

$\Downarrow$

$\text{length}\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \leqslant$ EXP in d

↑

tight bound

(example with $|\Sigma| = 1$)

Tradeoff States / Registers

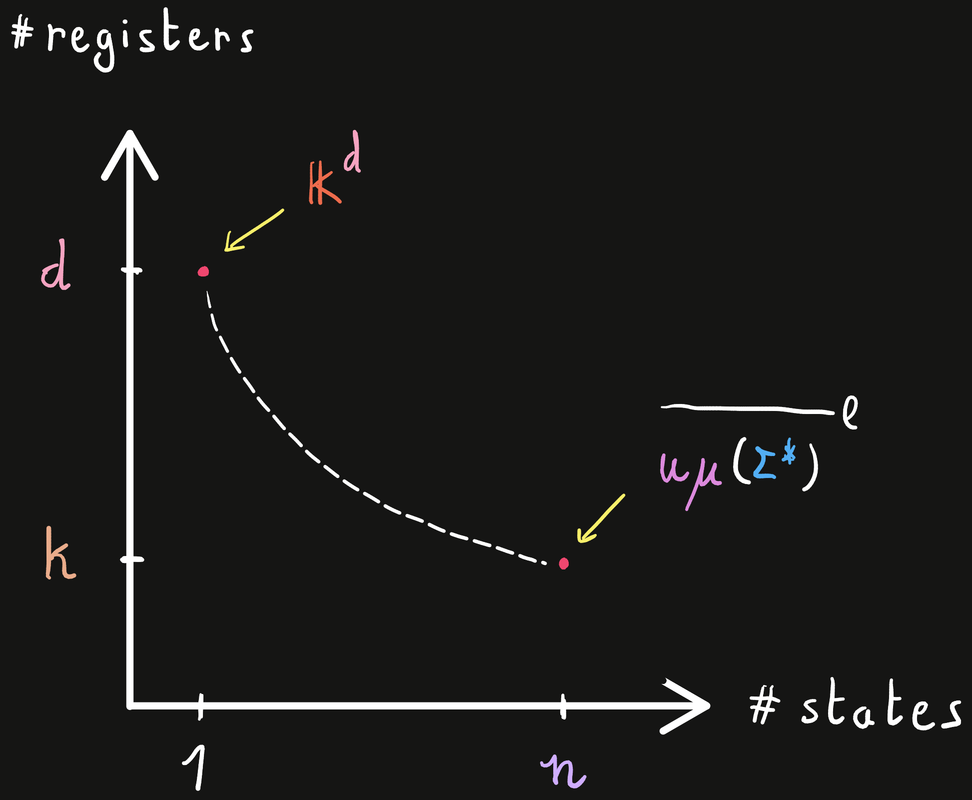
Let

$\mathfrak{R} = (u, \mu, \nu)$ be a

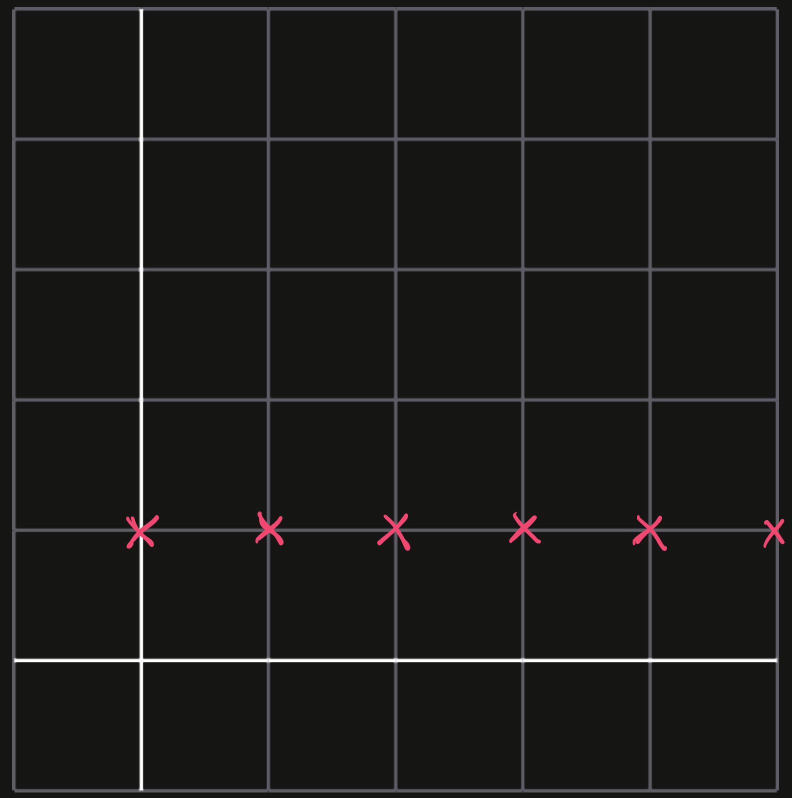
d-dimensional minimal WA

$n = \text{length}\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \lesssim$ 2-EXP in d

$k = \text{dimension}\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)$



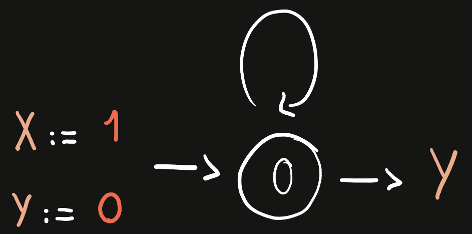
Affine CRA



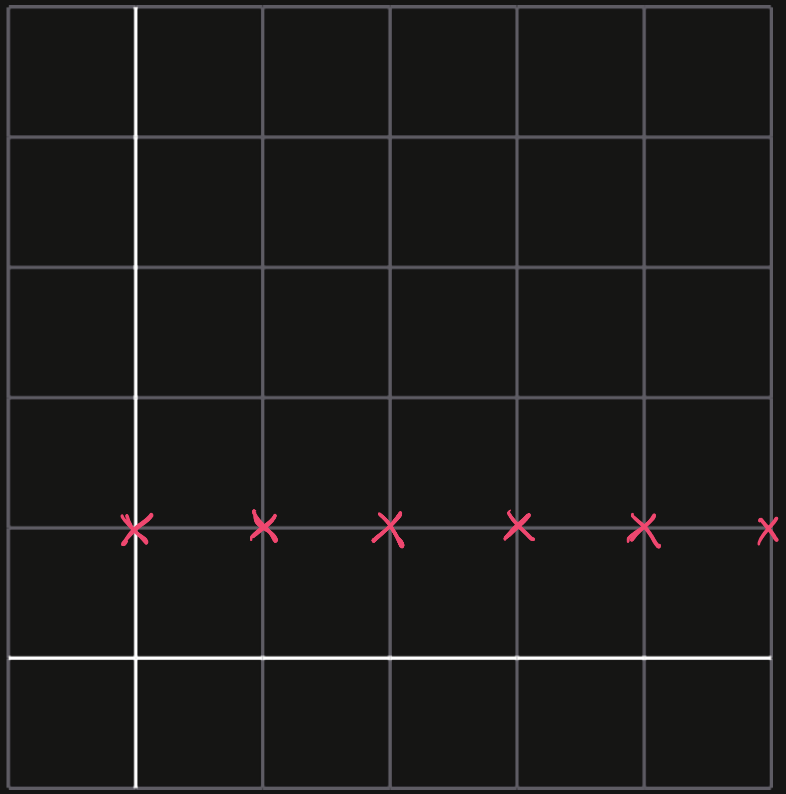
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



Affine CRA

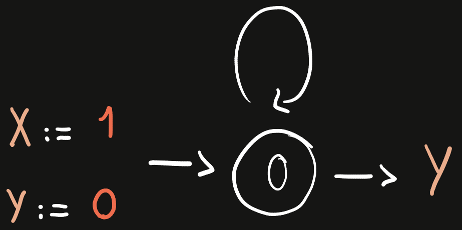


$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

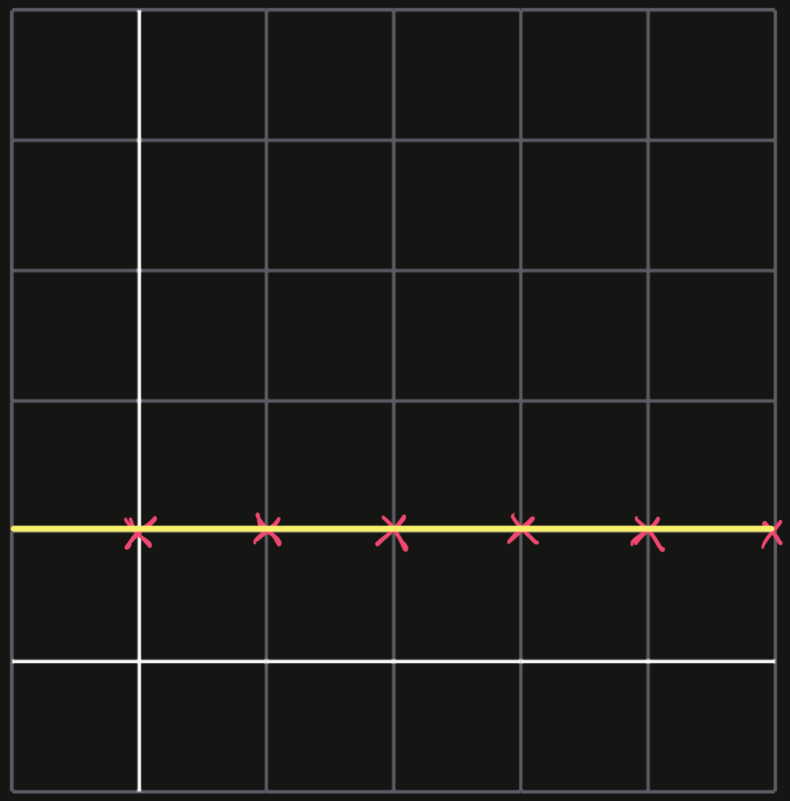
$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R}^2$$

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



Affine CRA



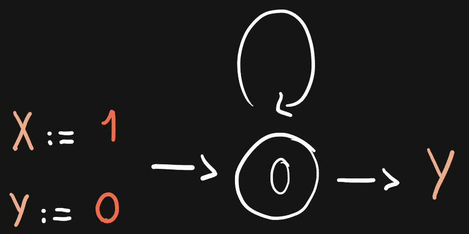
$$\overline{u\mu(\Sigma^*)}^{a} = (1\ 0) + \mathbb{R} \times \{0\}$$

$$u = (1 \quad 0) \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

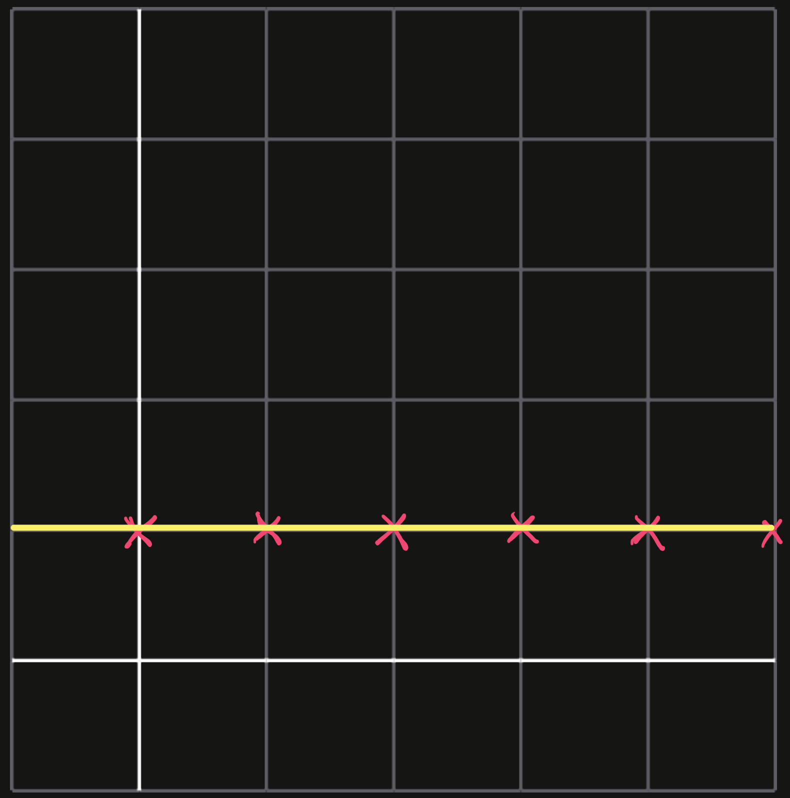
$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R}^2$$

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$

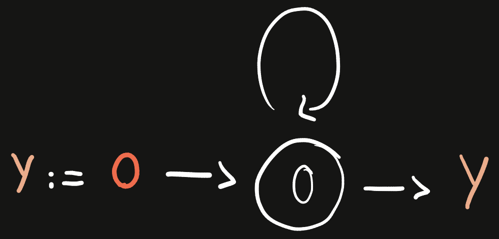


Affine CRA



$$\overline{u\mu(\Sigma^*)}^{a} = (1\ 0) + \mathbb{R}\times\{0\}$$

$a \mid Y := 2Y + 1$, $b \mid Y := 2Y$

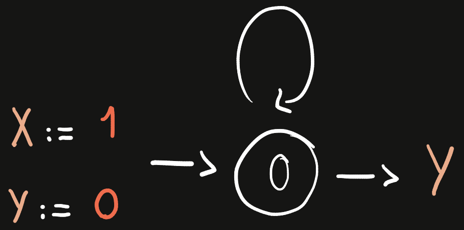


$$u = (1\quad 0) \qquad v = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\mu(a) = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix} \qquad \mu(b) = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$$

$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R}^2$$

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



Conclusion

Semilinear / semiaffine invariants can be used to solve:

State-Register minimization problem in NEXPTIME

Register minimization problem in 2-EXPTIME

for linear / affine CRA

Sequential? & Unambiguous? too

Conclusion

Semilinear / semiaffine invariants can be used to solve :

State-Register minimization problem in NEXPTIME

Register minimization problem in 2-EXPTIME

for linear / affine CRA

Sequential ? & Unambiguous ? too

Open questions

- better complexity ?
- other classes of CRA ?
- other semirings ?

Thank you

for your attention