26/07/2023

Highlights '23

Kassel

Refinement problems for recognizable relations

(ongoing work)

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LIS
LABORATOIRE
D'INFORMATIQUE
& SYSTÈMES

Motivation

Min. Extension Problem:

I: $f : \Sigma^* \to \Sigma^*$ partial fct. def. by a seqential transducer, $K \in \mathbb{N}$

Q: $\exists ?\, g : \Sigma^* \to \Sigma^*$ total fct. def. by a seqential transducer, with $\leqslant K$ states, s.t. $g_{|dom(f)} = f$

Linked to
register min. problem for
Streaming String Transducers
(SST)

Let $(M, *)$ be a monoid

Def: (right) Congruence on M

binary relation $\sim$ that is:

Equivalence relation $\begin{cases} \text{- Reflexive } (x \sim x) \\ \text{- Symmetric } (x \sim y \Leftrightarrow y \sim x) \\ \text{- Transitive } (x \sim y \wedge y \sim z \Rightarrow x \sim z) \end{cases}$

- Compatible with $*$ $(x \sim y \Rightarrow x * z \sim y * z)$
  (on the right)

$\rightarrow$ $|M/_\sim|$ is called the index of $\sim$

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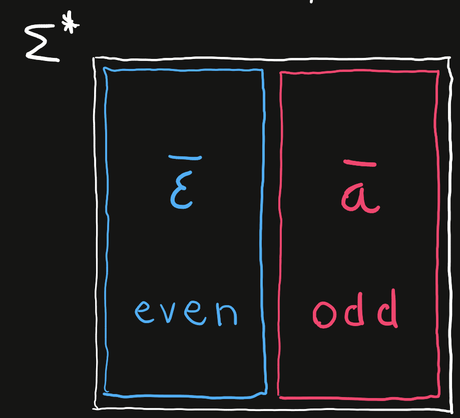
E.g.: $\Sigma = \{a, b\}$

$\forall u, v \in \Sigma^*,\ u \sim v$ iff $|u| \equiv |v| \bmod 2$



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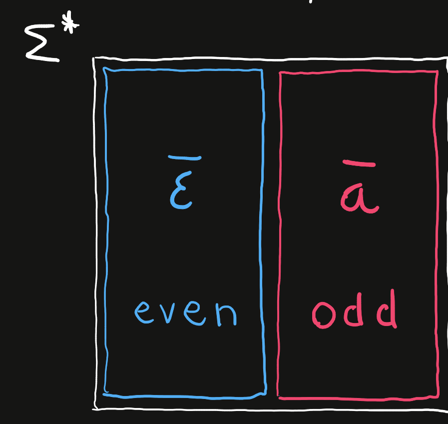
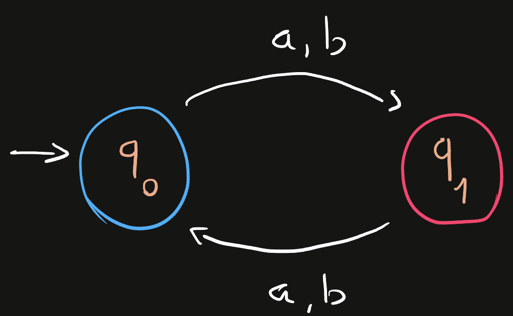


$\Leftrightarrow$ $q_0 \xrightarrow{u} q$, $q_0 \xrightarrow{v} q$



Deterministic Finite Automaton
(DFA)

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Def.: Refinement

R_1 is finer than R_2

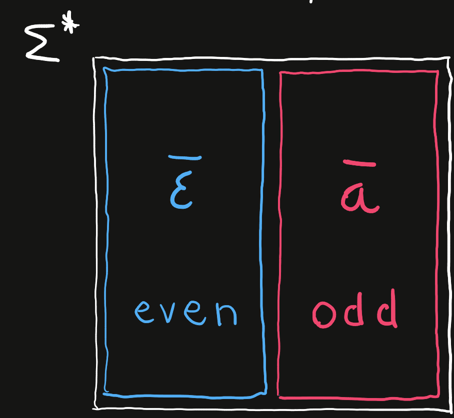
R_2 is coarser than R_1 if:

$\forall x, y$ if $x R_1 y$ then $x R_2 y$

Notation: $R_1 \sqsubseteq R_2$

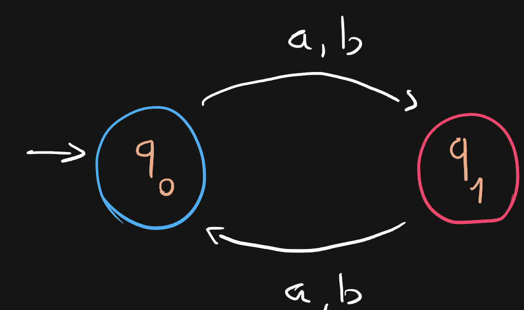
E.g.: $\Sigma = \{a, b\}$

$\forall u, v \in \Sigma^*$, $u \sim v$ iff $|u| \equiv |v| \mod 2$



$\Leftrightarrow q_0 \xrightarrow{u} q$, $q_0 \xrightarrow{v} q$

$\Updownarrow$



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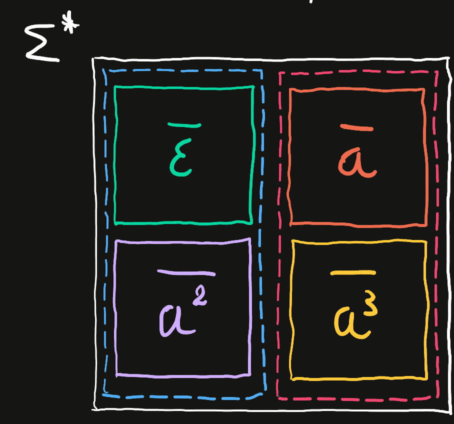
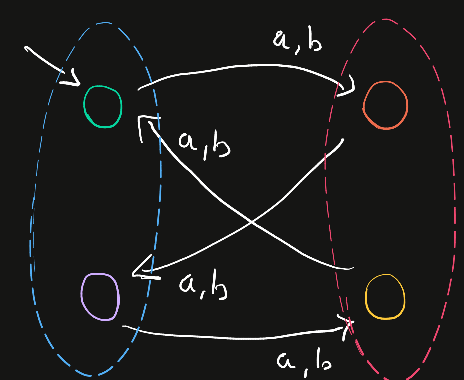


$\iff q_0 \overset{u}{\rightsquigarrow} q$, $q_0 \overset{v}{\rightsquigarrow} q$



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Def.: Precongruence on M

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- Reflexive ($x R x$)
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Def: Precongruence on M

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Min. Refinement Problem:

I: $\sim$ congruence on Σ^*,

$\bar{R}$ precongruence on $\Sigma^*/{\sim}$, $K \in \mathbb{N}$

Q: $\exists ?$ $\approx$ congruence on Σ^*

of index $\leq K$ s.t. $\approx \sqsubseteq R$

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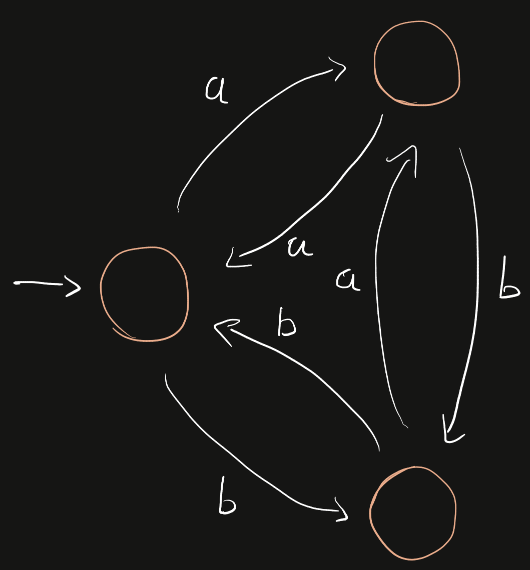

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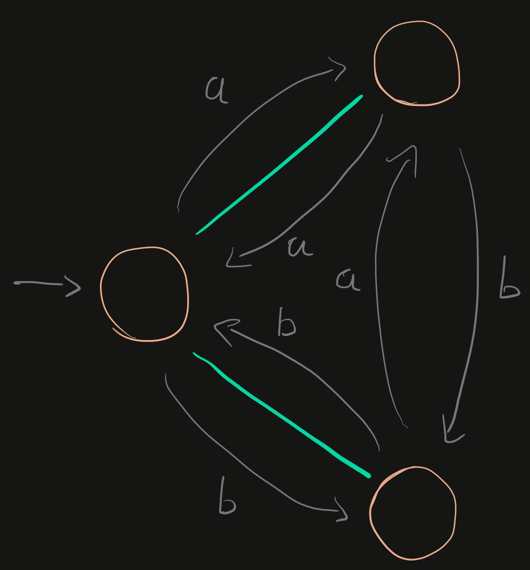

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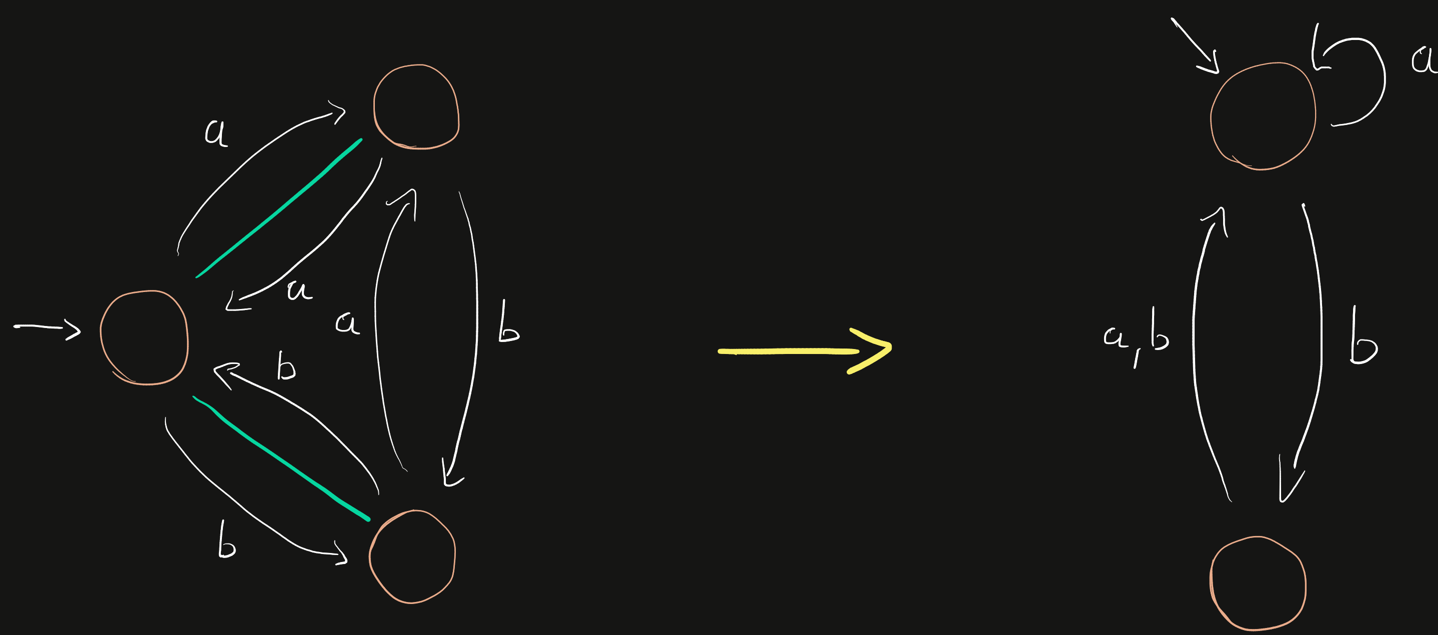

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Result 1:

$\Updownarrow$ Many-one reducible

Min. Extension Problem:

I: $f: \Sigma^* \longrightarrow \Sigma^*$ partial fct. def. by a seqential letter-to-letter transducer, $K \in \mathbb{N}$

Q: $\exists ? g: \Sigma^* \longrightarrow \Sigma^*$ total fct. def. by a seqential transducer, with $\leqslant K$ states, s.t. $g_{|dom(f)} = f$

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NP-Complete

[Pfleeger 1973]

Let V be a variety of languages

V-Refinement Problem:

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Let V be a variety of languages

E.g.: FO-definable $\iff$ Star-free

$\iff$ Recognizable by an aperiodic congruence

$(\exists n \in \mathbb{N}, \forall w \in \Sigma^*, \; w^n \sim w^{n+1})$

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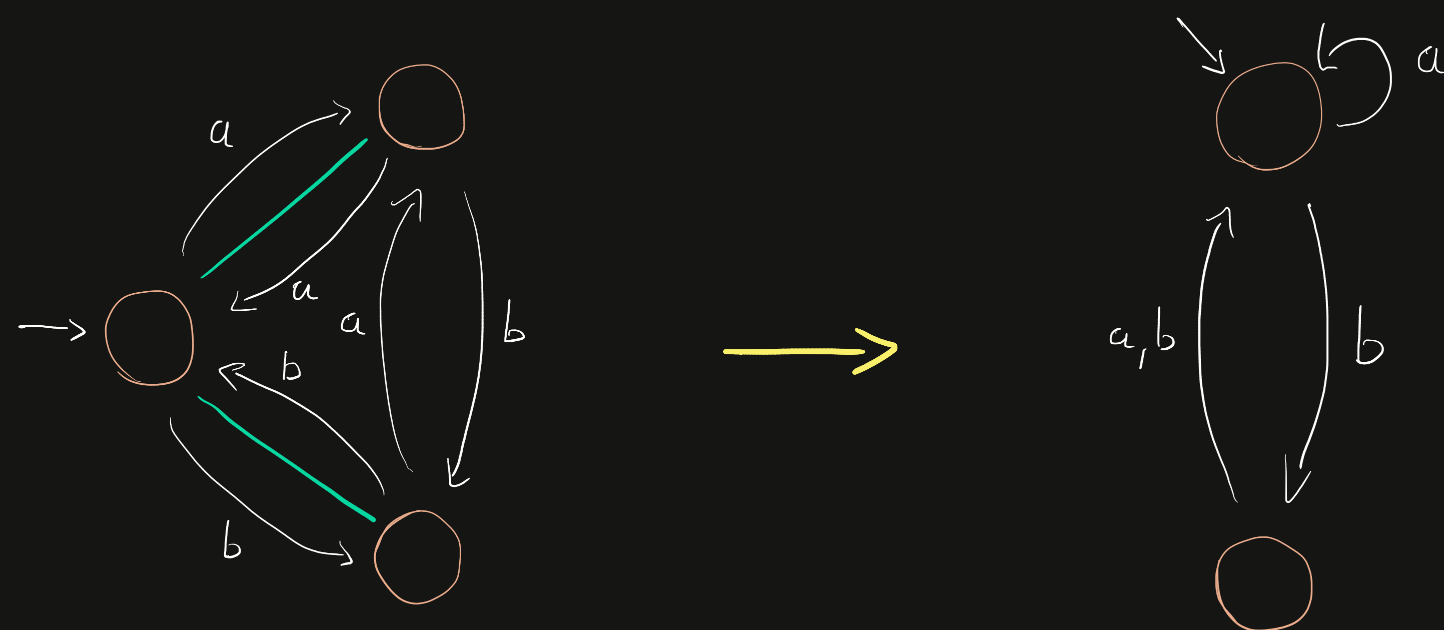

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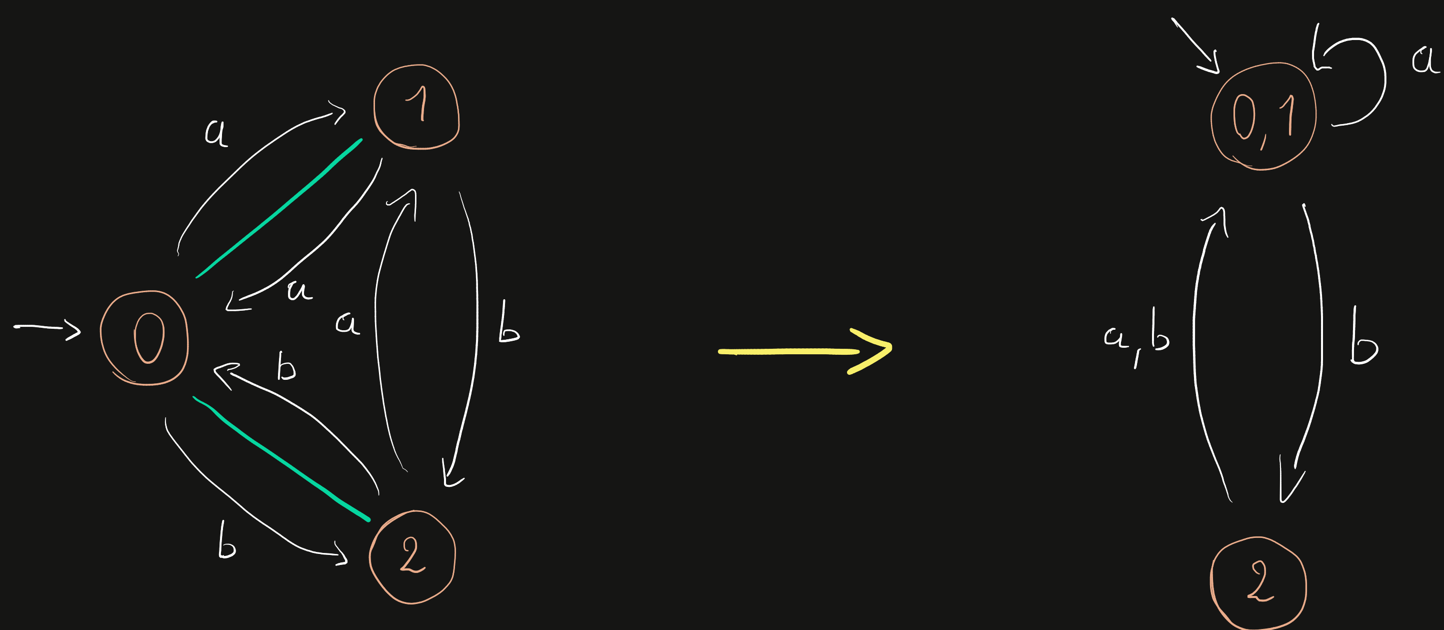

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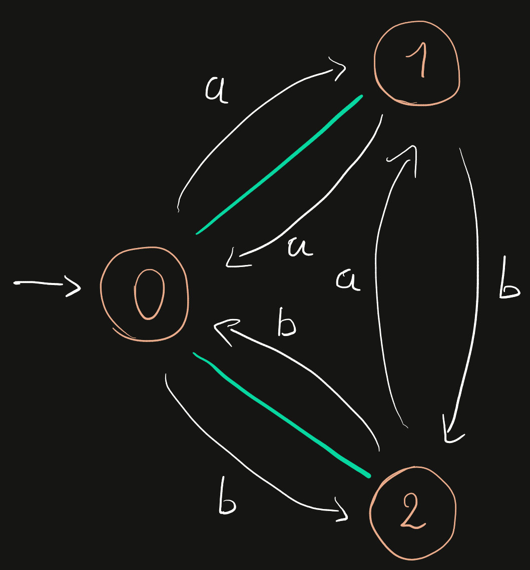

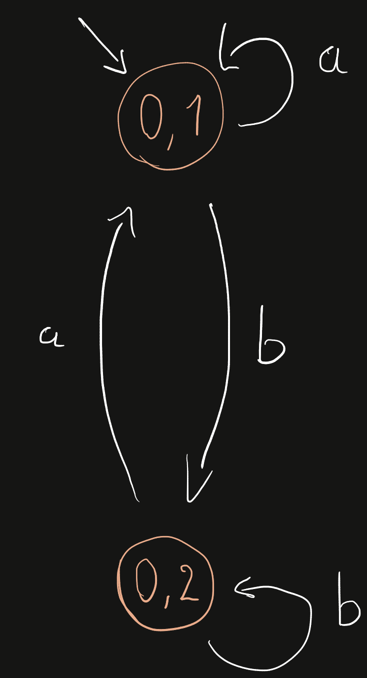

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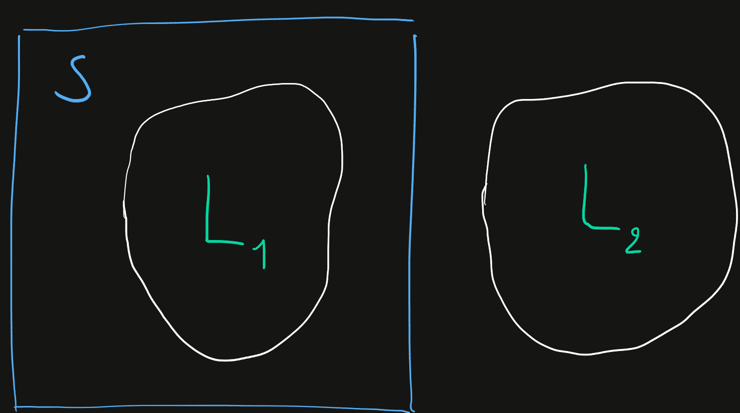
Result 2:



S in V

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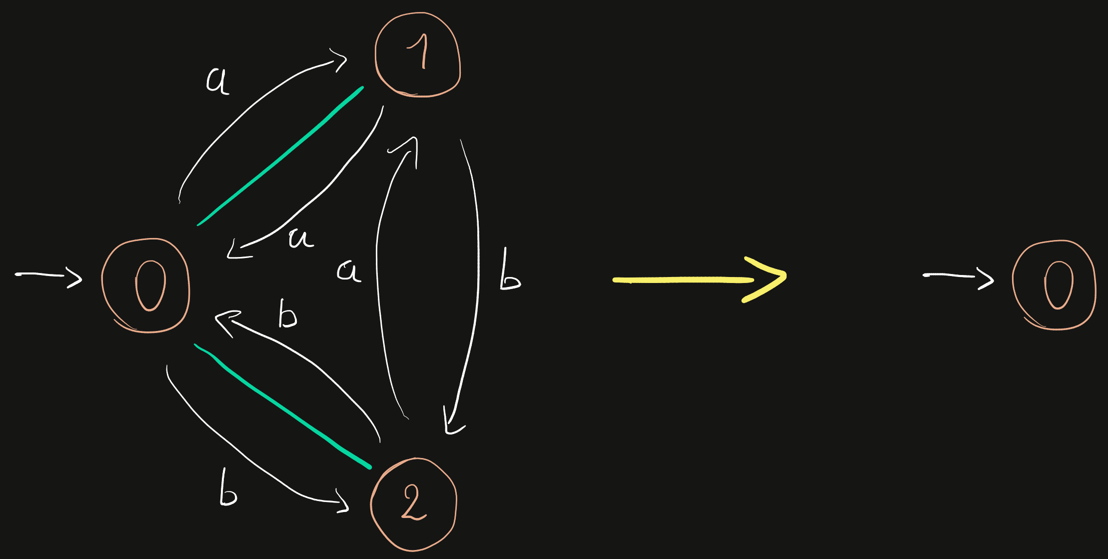
Many-one reducible $\uparrow$ $\downarrow$ Turing reducible

V-Separation Problem:

I: L_1, L_2 regular languages

Q: $\exists ?$ V-separator of L_1 and L_2

Subset Expansion



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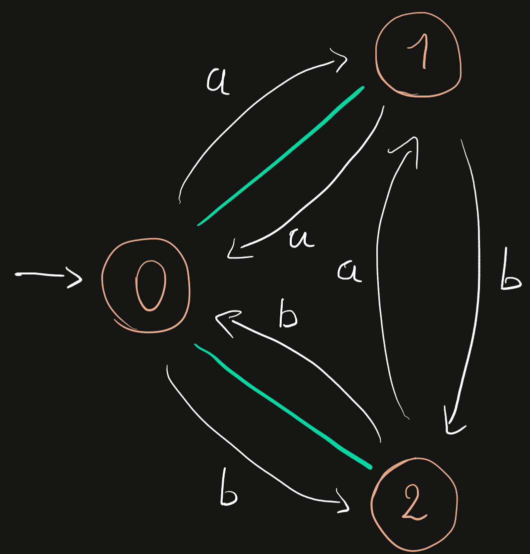

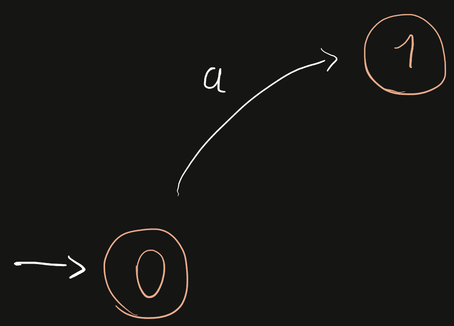

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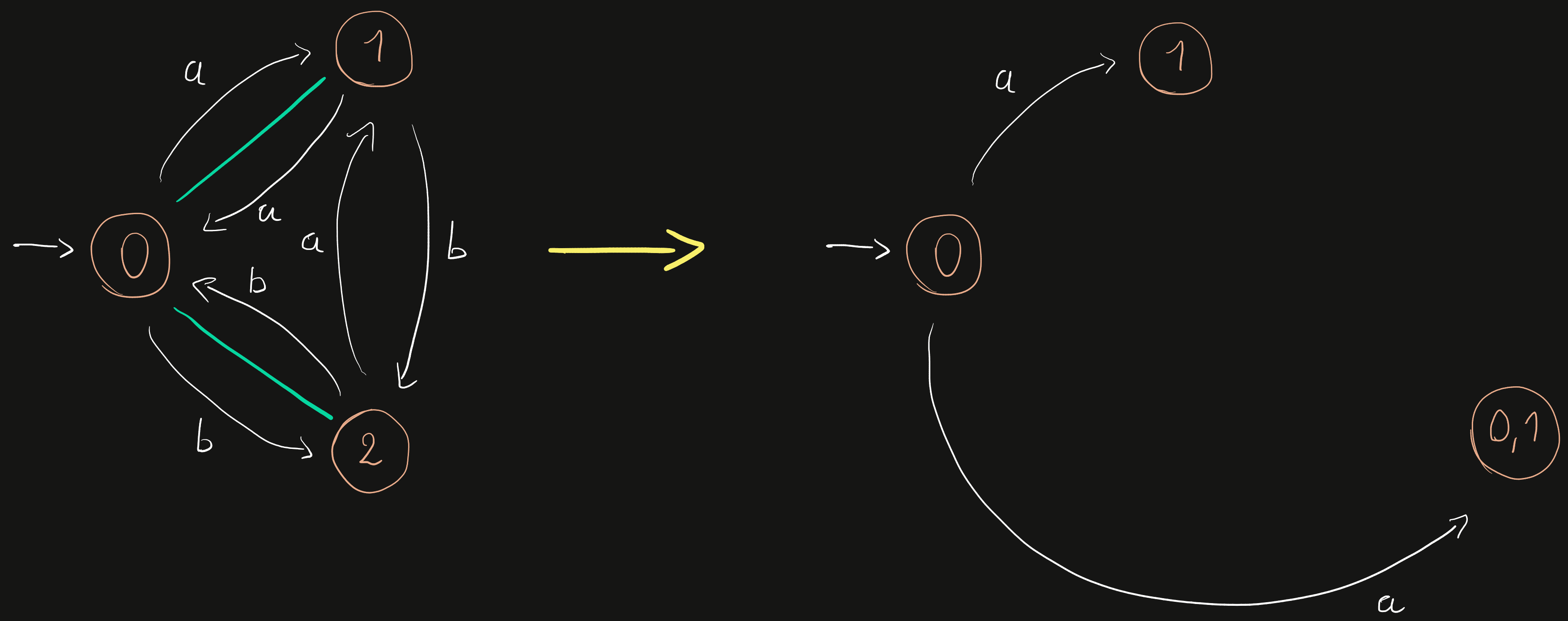

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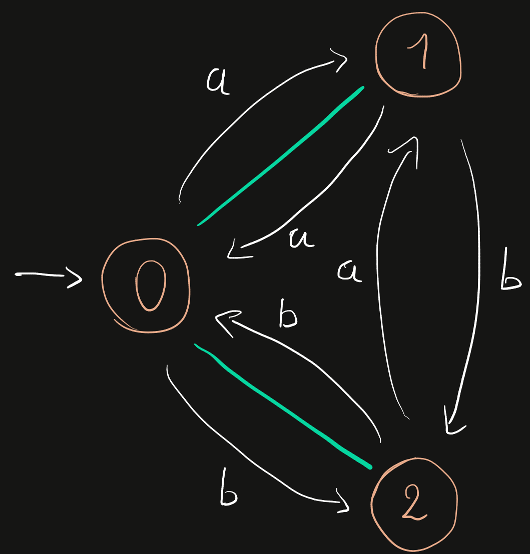

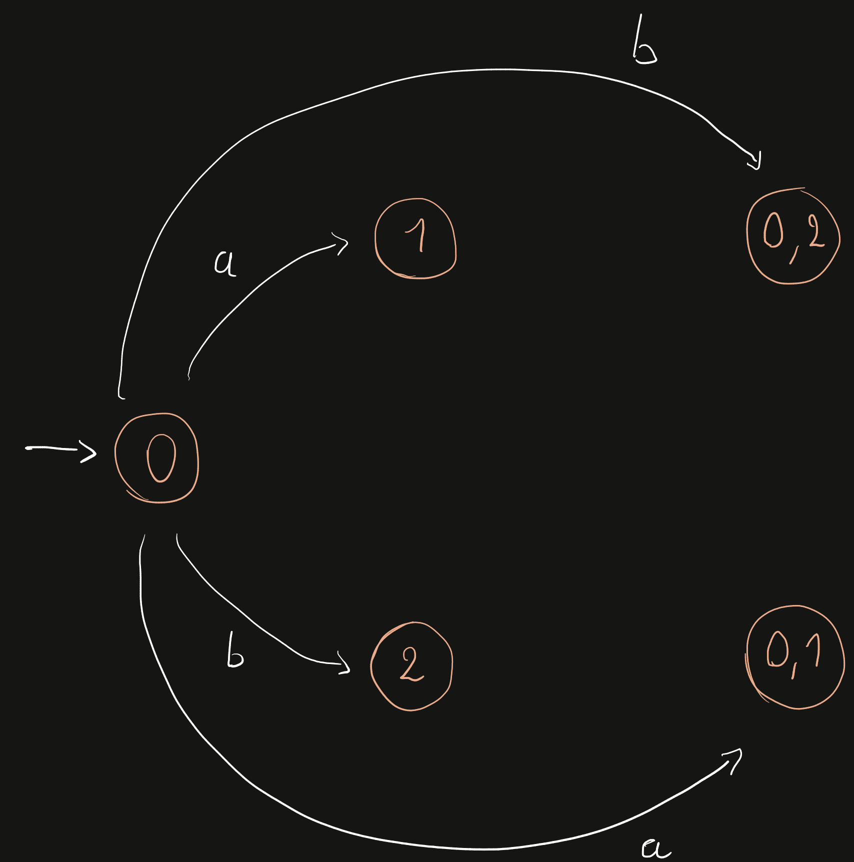

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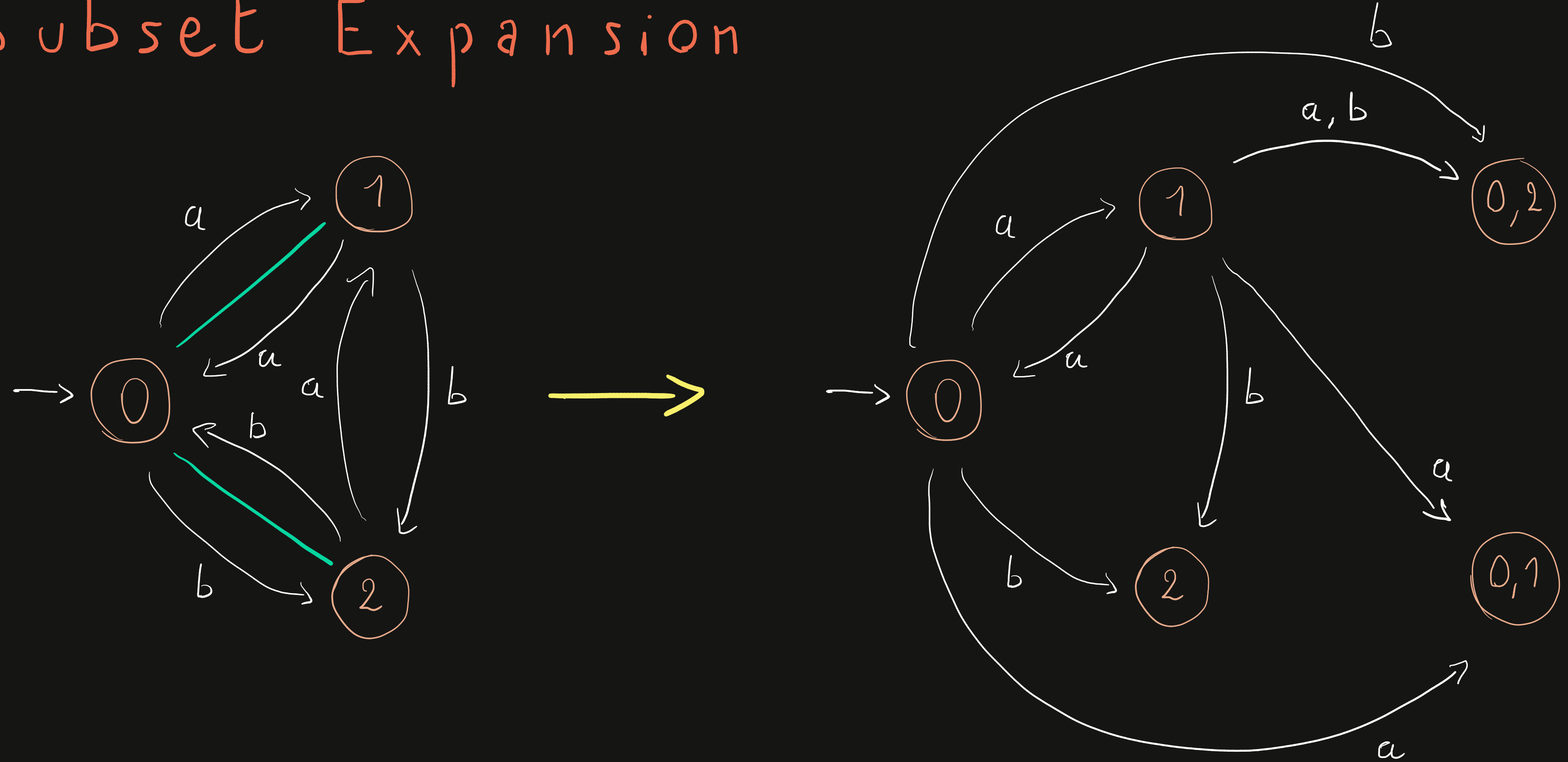

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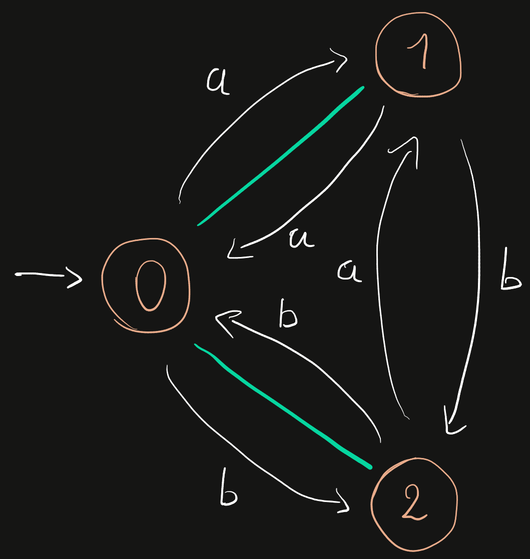

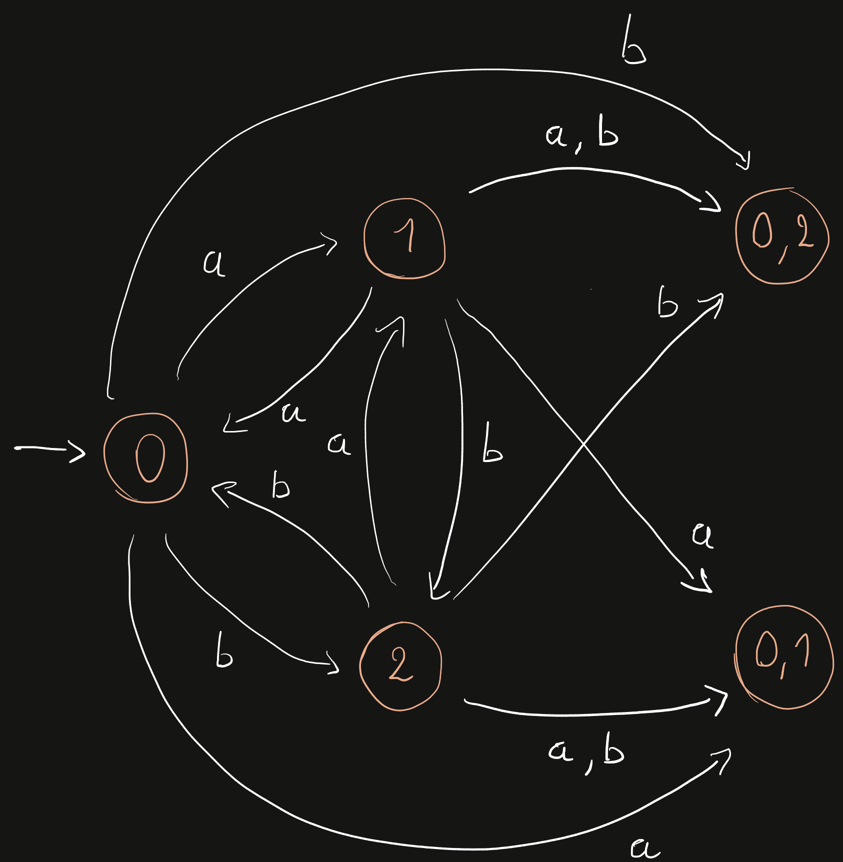

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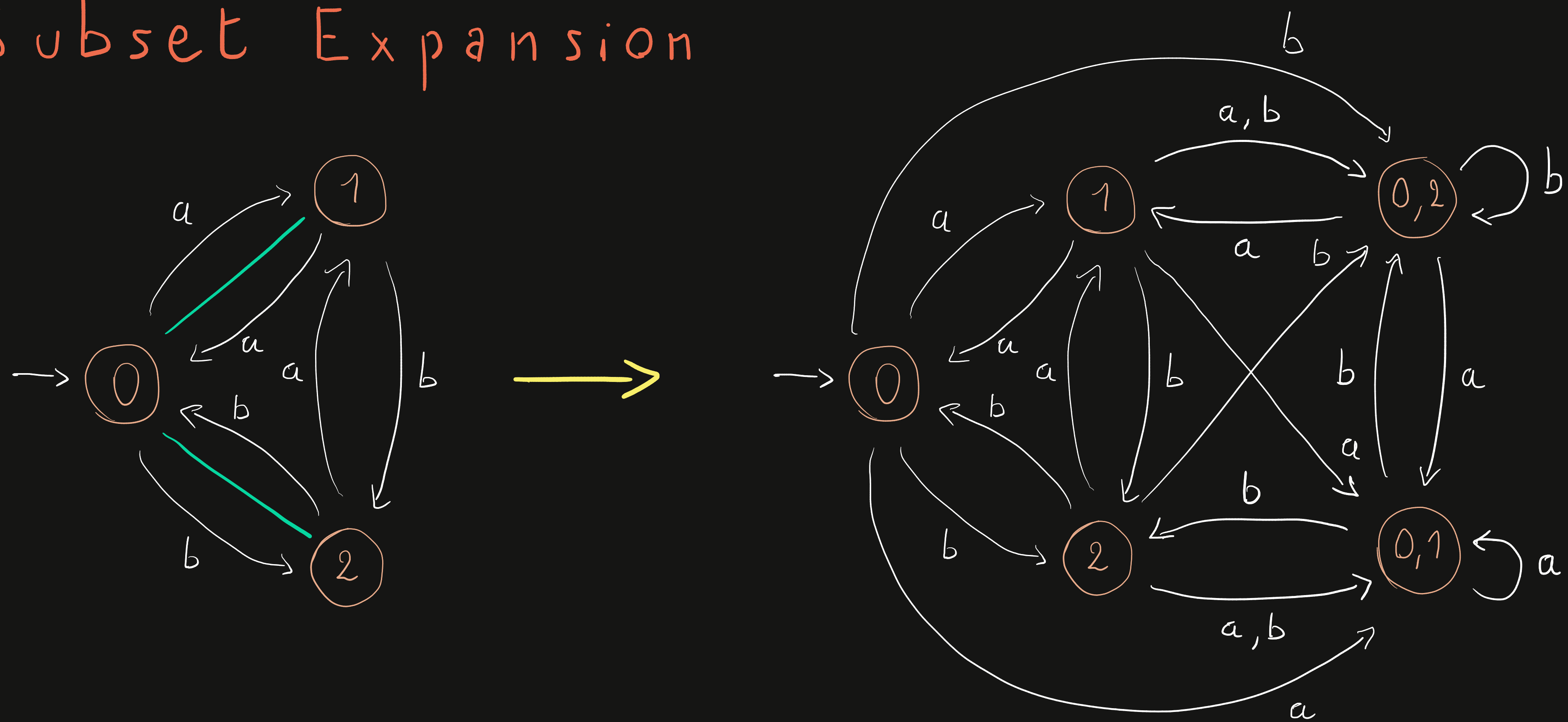

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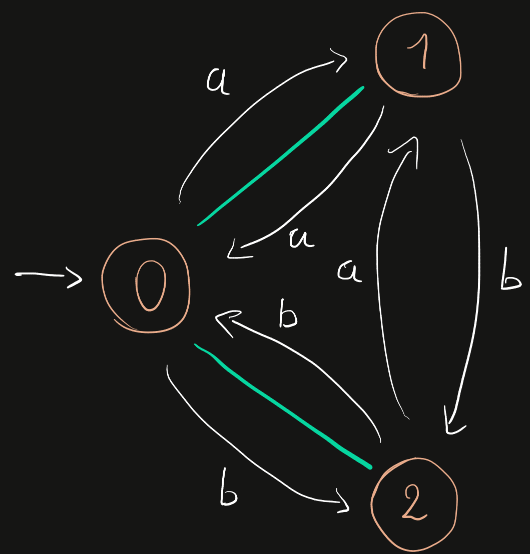

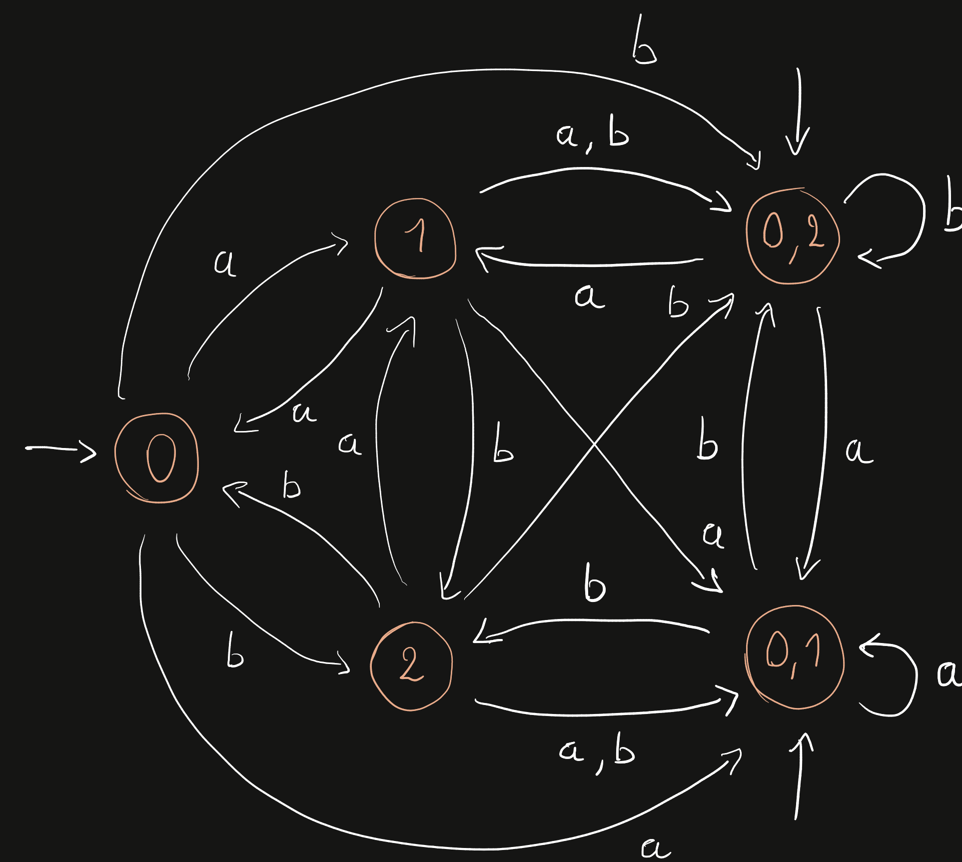

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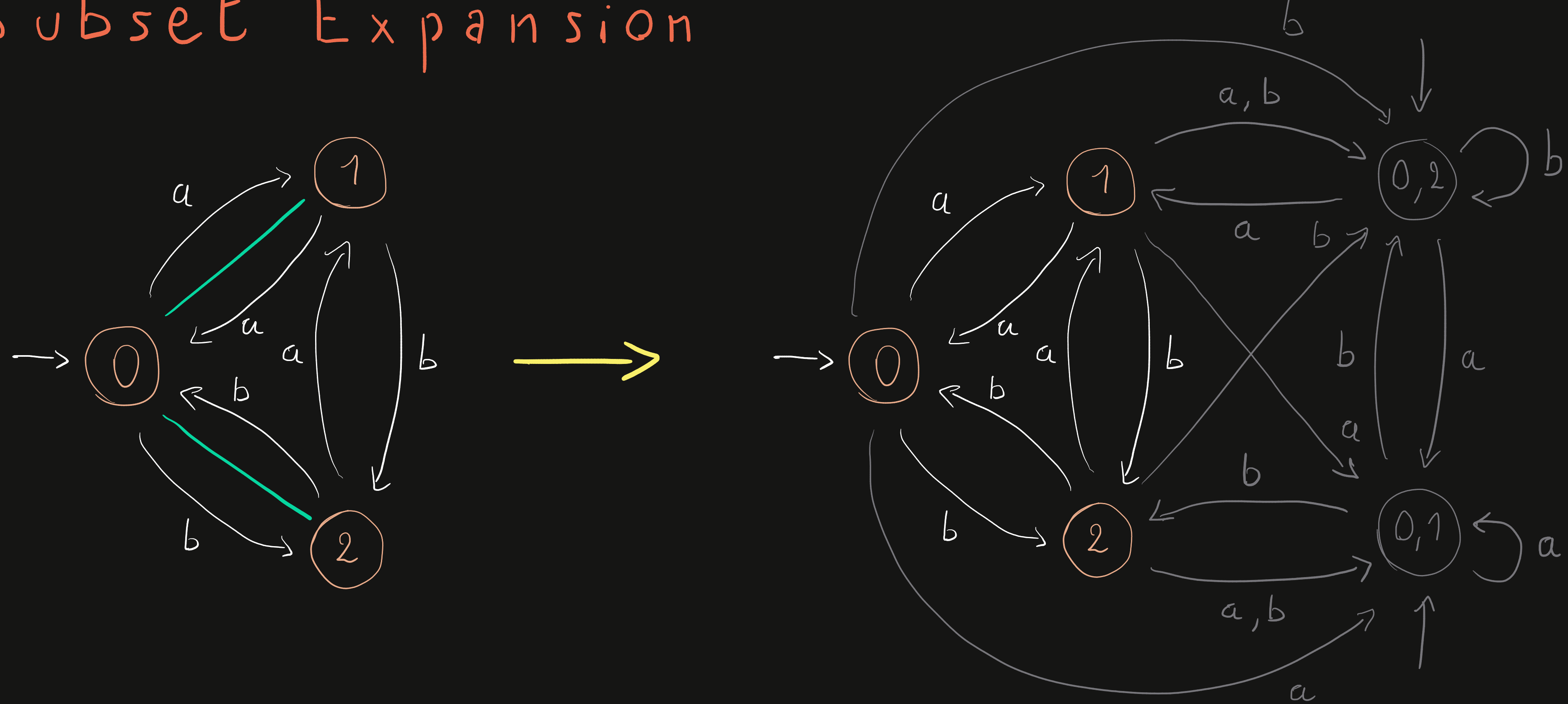

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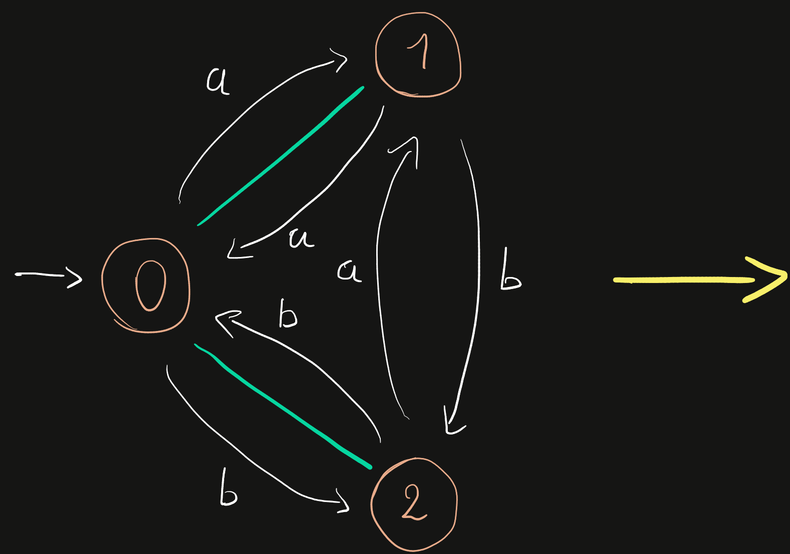

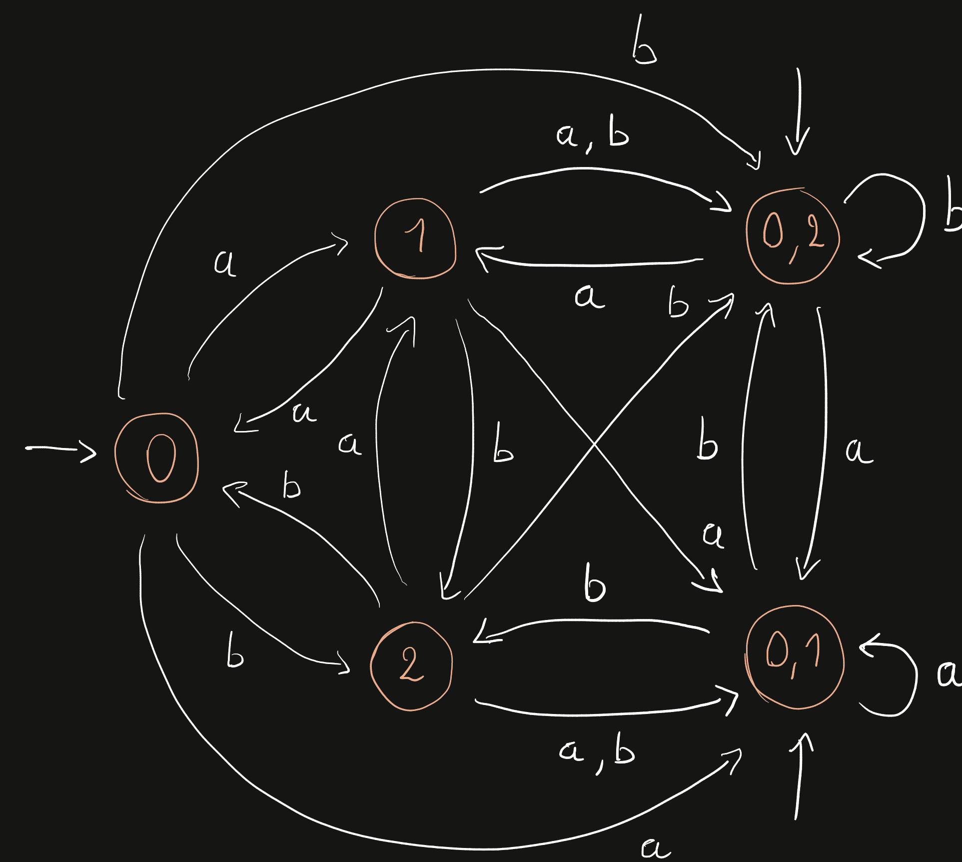

Result 3:

every minimal refinement of R is $\simeq$ to sub-automaton of the subset expansion

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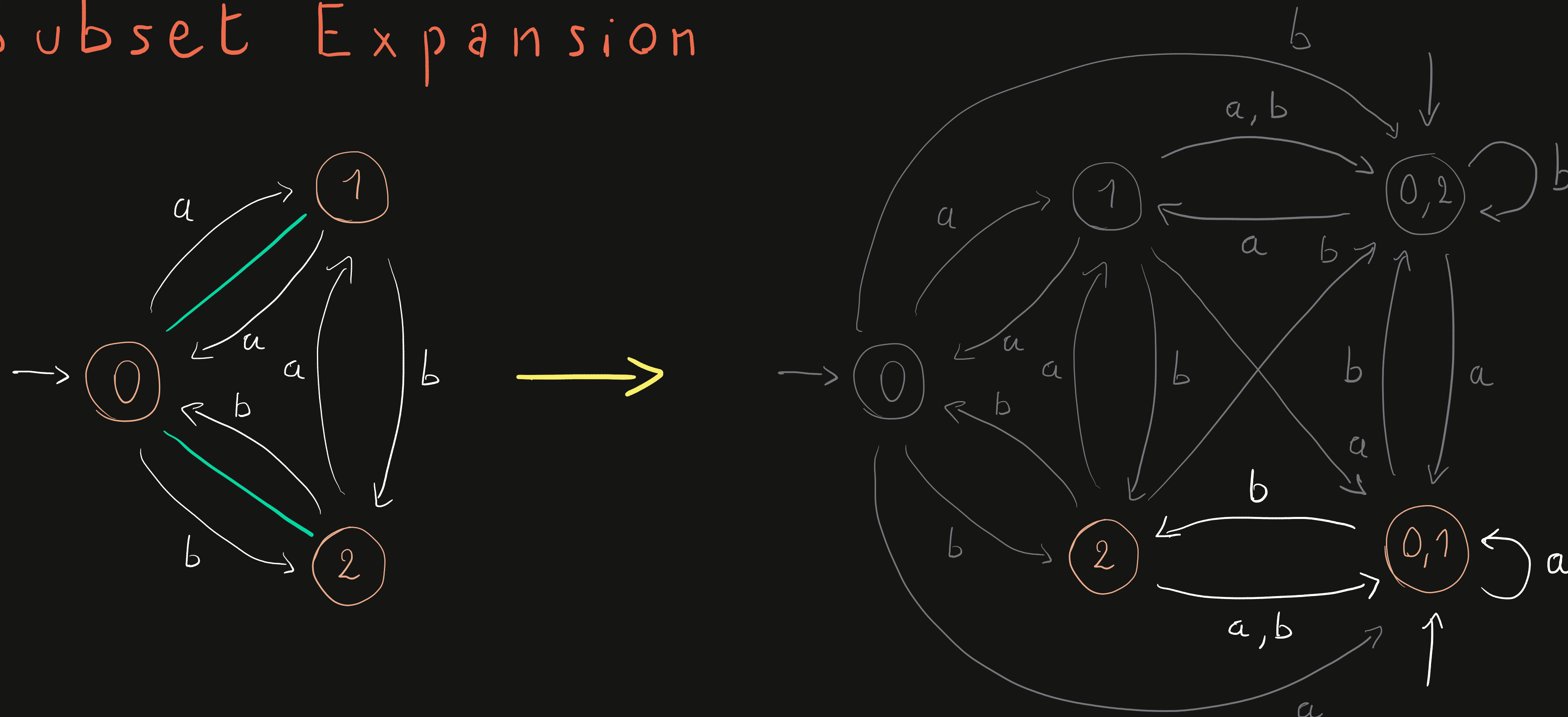

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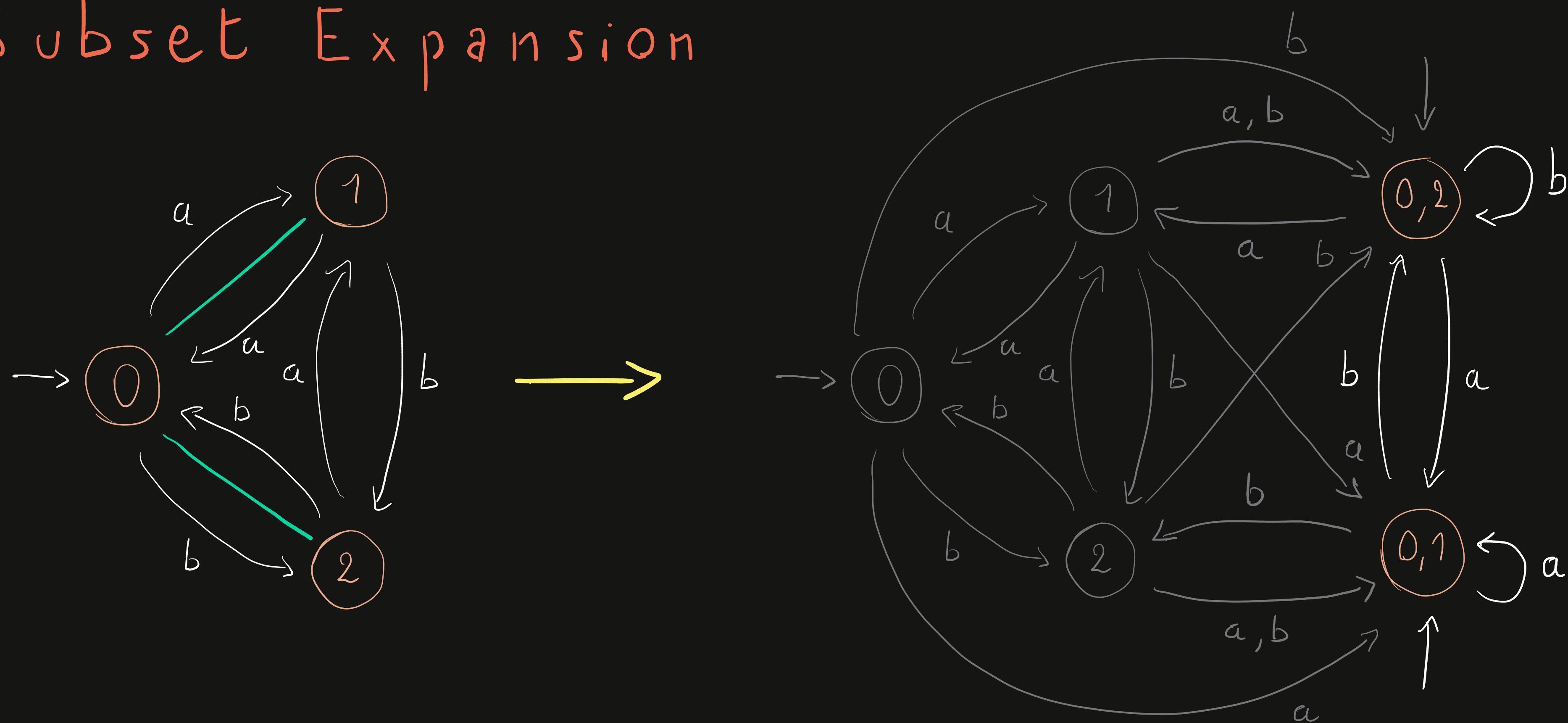

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Conclusion

- Refinement problems <=> other well-known problems

- All the solutions of Min. Refinement Problem in the Subset Expansion

Open questions

- Solutions of V-Refinement Problem always in the Subset Expansion ?

- General Min. Extension Problem ?

Thank you

for your attention