26/04/2024

GT DAAL
Meeting 2024

Minimization
of
Cost Register Automata
over a
Field

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Socialement engagée

LABORATOIRE
D'INFORMATIQUE
& DES SYSTÈMES

Register minimization problem

In: f rational series given as a WA , $k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant k$ registers realizing f

???

Register minimization problem

???

In: f rational series given as a WA, $k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant k$ registers realizing f

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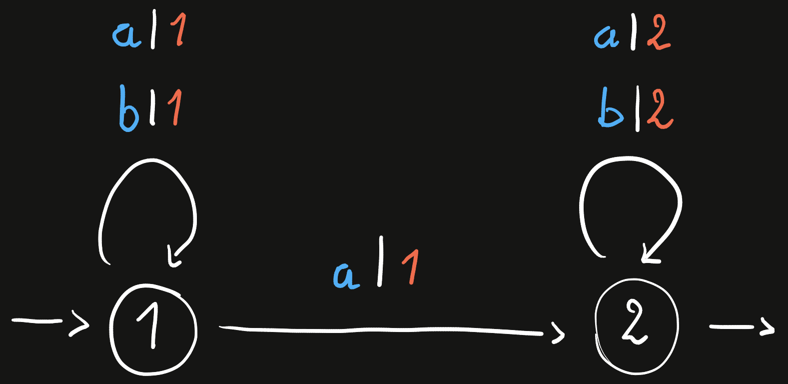
???

Weighted Automata

(WA)

on $\Sigma = \{a, b\}$ over $(\mathbb{N}, +, \times)$:



realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

Weighted Automata

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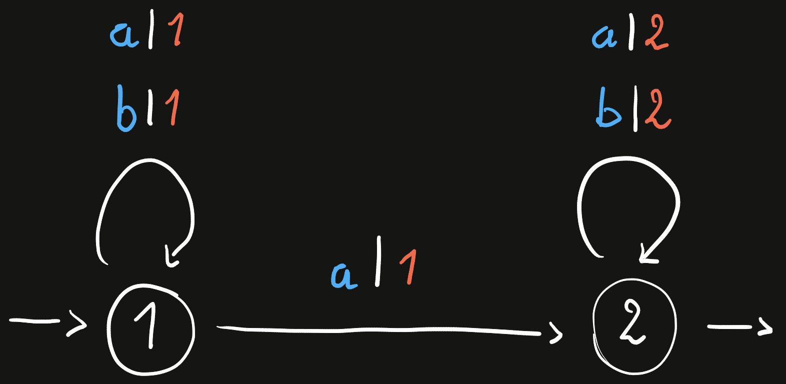

realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

aab :

$$w\left(① \xrightarrow{a|1} ① \xrightarrow{a|1} ② \xrightarrow{b|2} ② \right) = 1 \times 1 \times 2 = 2$$

Weighted Automata

(WA)

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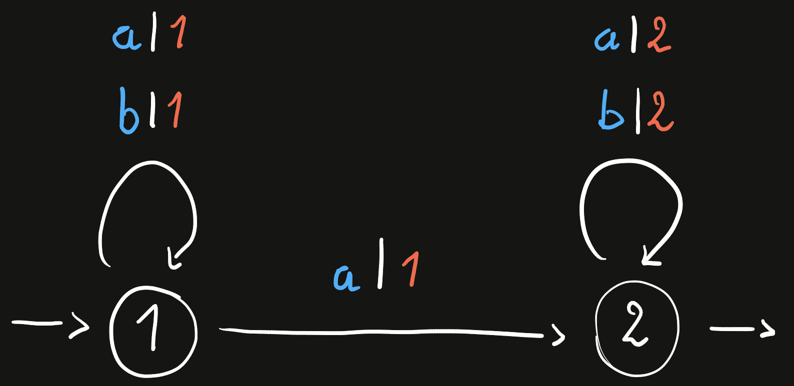

realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$a\,a\,b$:

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2} \right) = 1 \times 1 \times 2 = 2$$

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Weighted Automata

(WA)

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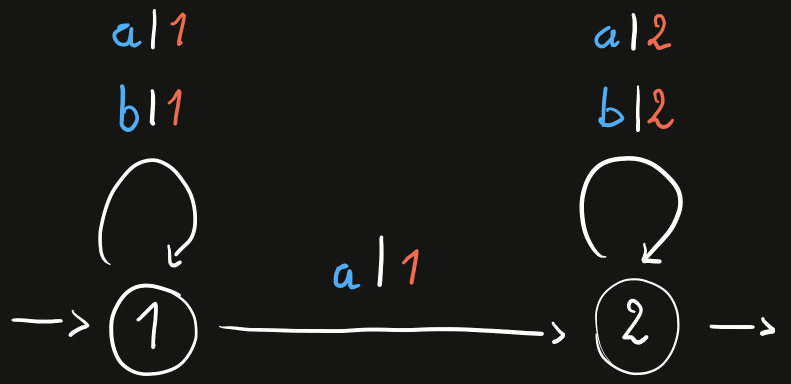

realizes a rational series $\Sigma^* \longrightarrow \mathbb{N}$

$x_2 \longmapsto x_{10}$

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$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 1 \times 2 = 2$$

$$+$$

$$w\left(\textcircled{1} \xrightarrow{a|1} \textcircled{2} \xrightarrow{a|2} \textcircled{2} \xrightarrow{b|2} \textcircled{2}\right) = 1 \times 2 \times 2 = 4$$

$aab \mapsto 6$

Weighted Automata

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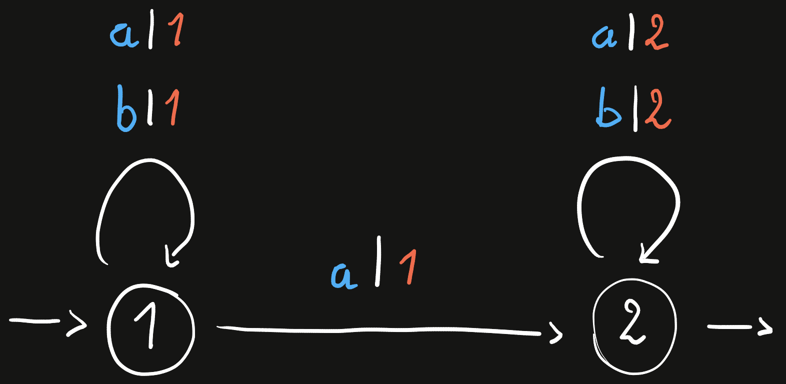

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$$aab \mapsto 6$$

Linear representation

(u, μ, ν)

initial vector

$u = (\quad\quad)$

terminal vector

$\nu = \begin{pmatrix} \\ \\ \end{pmatrix}$

transition matrices

$\mu(a) = \begin{pmatrix} & \\ & \end{pmatrix}$ $\mu(b) = \begin{pmatrix} & \\ & \end{pmatrix}$

Weighted Automata
(WA)

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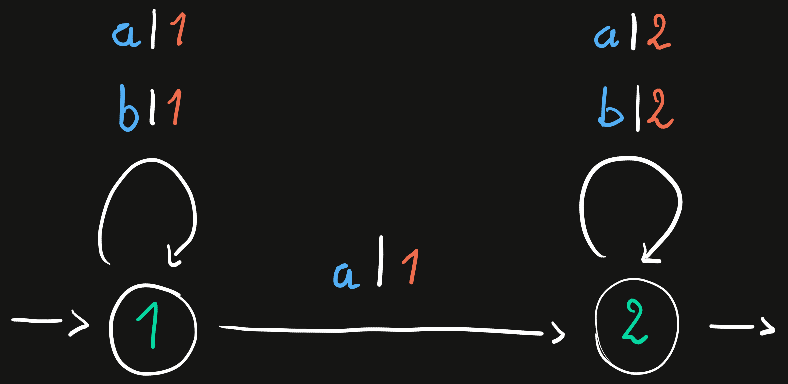

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w(① $\xrightarrow{a|1}$ ① $\xrightarrow{a|1}$ ② $\xrightarrow{b|2}$ ②) $= 1 \times 1 \times 2 = 2$

$+$

w(① $\xrightarrow{a|1}$ ② $\xrightarrow{a|2}$ ② $\xrightarrow{b|2}$ ②) $= 1 \times 2 \times 2 = 4$

---

aab $\longmapsto$ 6

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} & \end{pmatrix} \end{array}$$

terminal vector

$$v = \begin{array}{c} \begin{pmatrix} \\ \end{pmatrix} \end{array}\!\!\begin{array}{c} 1 \\ 2 \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} & \\ & \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

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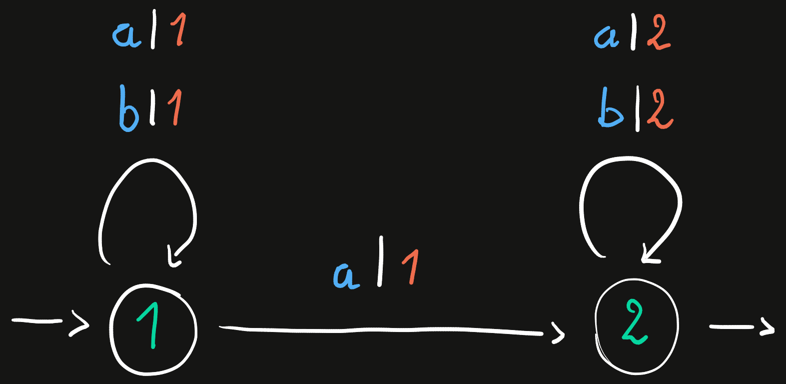

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$$aab \mapsto 6$$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

terminal vector

$$v = \begin{array}{cc} \begin{pmatrix} \; \\ \; \end{pmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{cc} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{pmatrix} \; & \; \\ \; & \; \end{pmatrix} \end{array}$$

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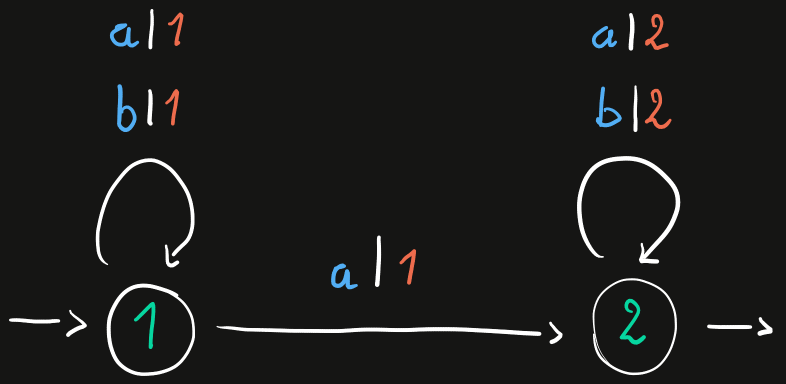

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$$aab \mapsto 6$$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ (1 & 0) \end{array}$$

terminal vector

$$v = \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{ccc} & 1 & 2 \\ 1 & & \\ 2 & & \end{array} \qquad \mu(b) = \begin{array}{ccc} & 1 & 2 \\ 1 & & \\ 2 & & \end{array}$$

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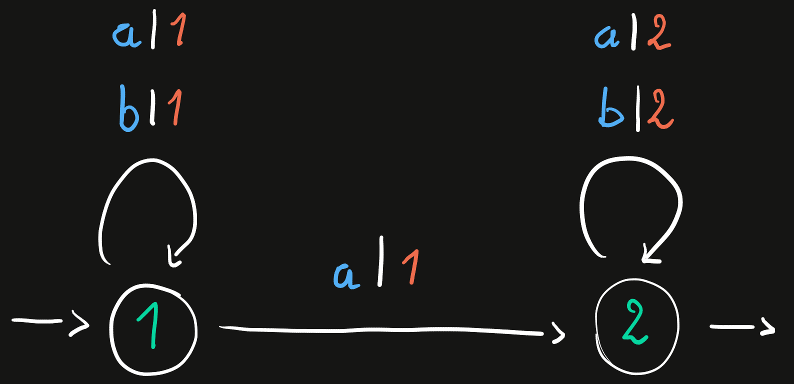

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$\text{aab} \mapsto 6$

Linear representation

(u, μ, ν)

initial vector

$$u = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$$

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$$\nu = \begin{array}{cc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

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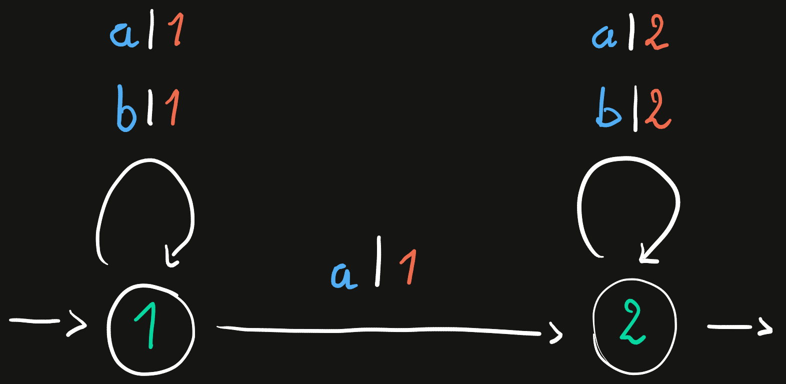

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$aab \longmapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array}$$

terminal vector

$$v = \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array}$$

transition matrices

$$\mu(a) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 0 & 2 \end{array} \qquad \mu(b) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & 0 \\ 2 & 0 & 2 \end{array}$$

Weighted Automata

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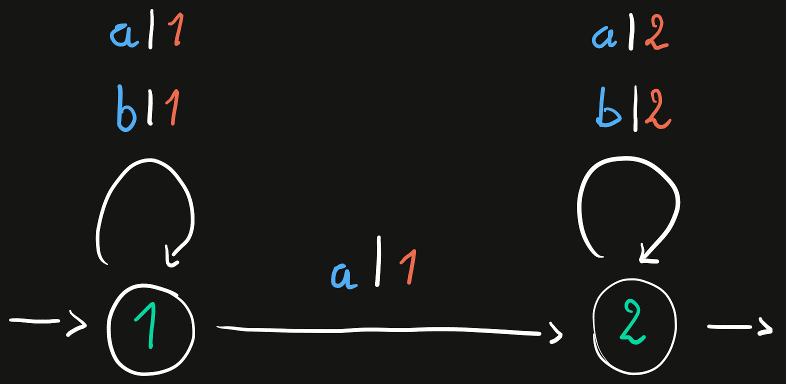

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$aab \mapsto 6$

Linear representation

(u, μ, v)

initial vector

$$u = \begin{array}{cc} \scriptstyle 1 & \scriptstyle 2 \\ \left(1\right. & \left.0\right) \end{array}$$

terminal vector

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 2 \end{array}$$

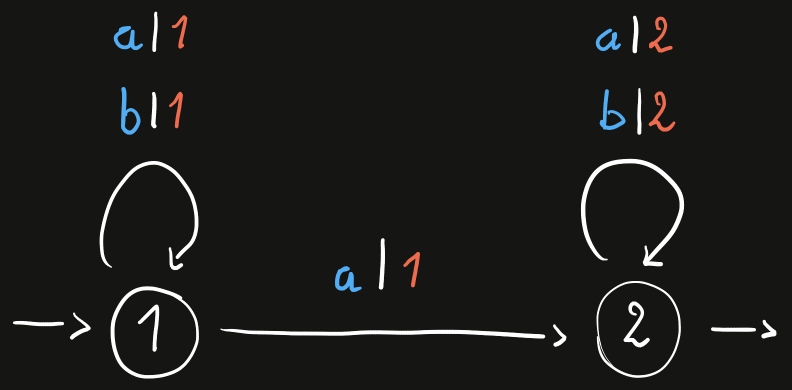
transition matrices

$$\mu(a) = \begin{array}{cc} & \begin{array}{cc} \scriptstyle 1 & \scriptstyle 2 \end{array} \\ \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 2 \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{array}{cc} \scriptstyle 1 & \scriptstyle 2 \end{array} \\ \begin{array}{c} \scriptstyle 1 \\ \scriptstyle 2 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

$$\begin{aligned} aab \longmapsto\ & u \cdot \mu(aab) \cdot v \\ &= u \cdot \mu(a) \cdot \mu(a) \cdot \mu(b) \cdot v \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 6 \end{aligned}$$

Weighted Automata

(WA)



⚠ Not always equivalent to a sequential WA

(input deterministic)

Linear representation

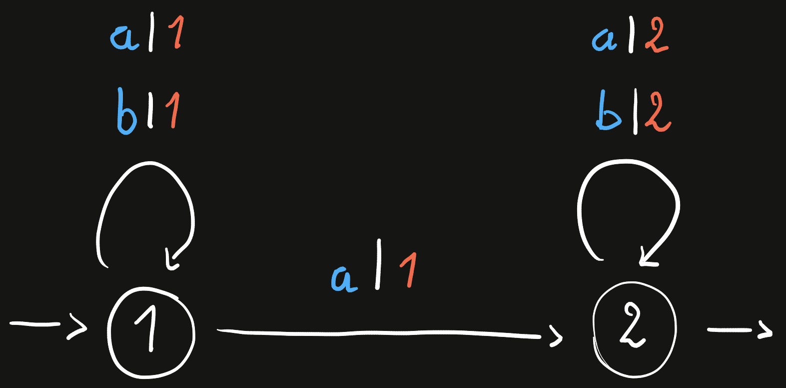
(u, μ, v)

$u = (1 \quad 0)$ $\qquad$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\qquad$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

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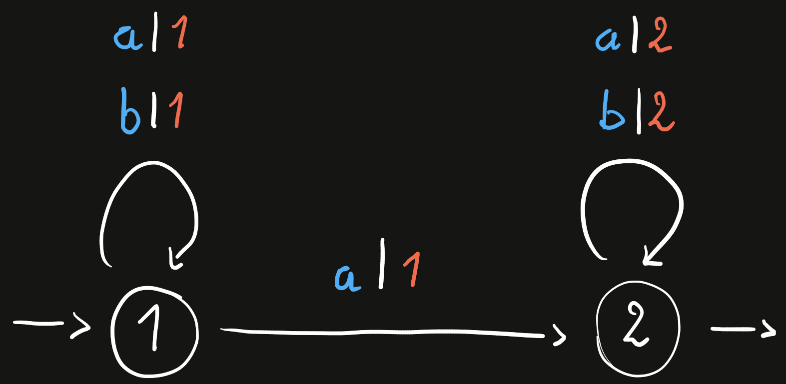
Some prop. of WA over a Field :

- Zeroness/equivalence is decidable

Weighted Automata

(WA)



Linear representation

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in poly. time

- ∃ computable minimal WA

(unique up to change of basis)

Weighted Automata

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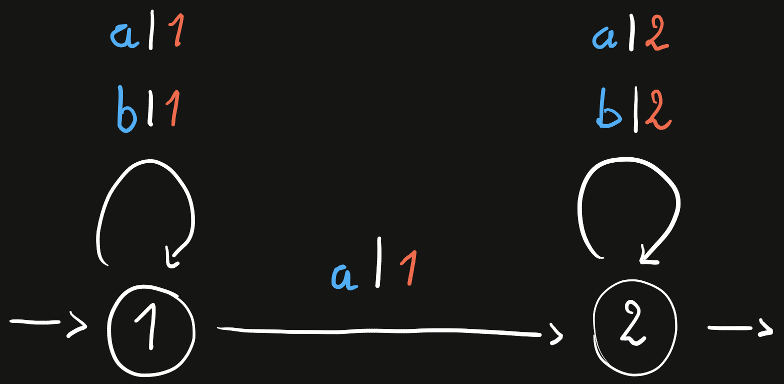

Linear representation

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$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

⚠ Not always equivalent to a sequential WA

(input deterministic)

Some prop. of WA over a Field :

- Zeroness / equivalence is decidable

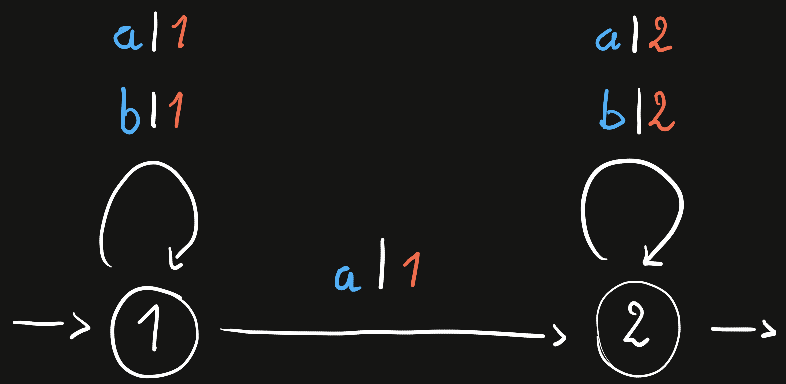
in poly. time

- ∃ computable minimal WA
  (unique up to change of basis)
- Unambiguity / Sequentiality is decidable

[Bell & Smertnig 2023]

Weighted Automata

(WA)



Linear representation

(u, μ, v)

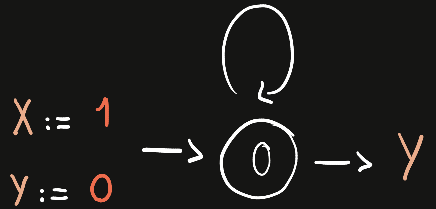
$$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Cost Register Automata

(CRA) [Alur et al. 2013]

$$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right. , \quad b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$$



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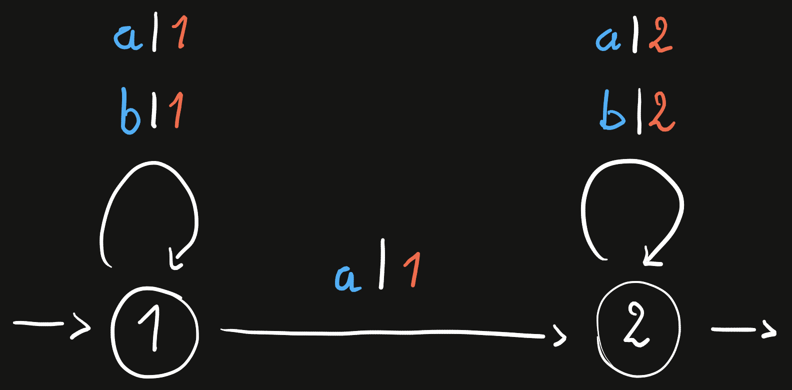

Linear representation

(u, μ, v)

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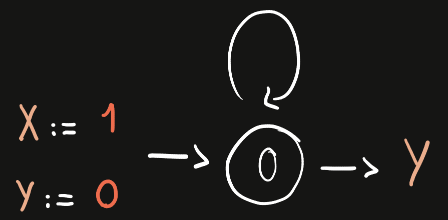


aab :

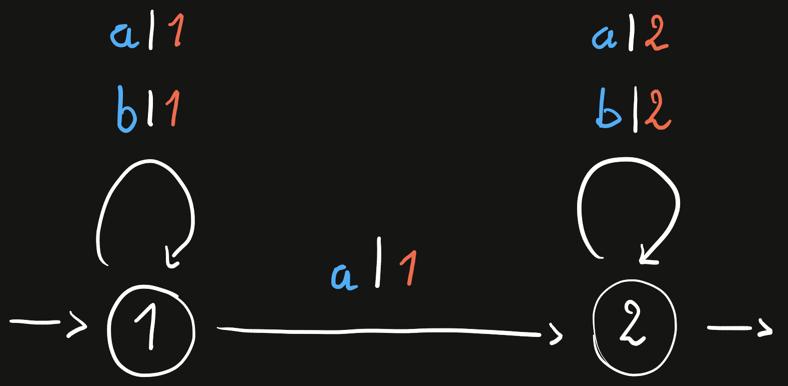
→ ⓪

X = 1

Y = 0

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

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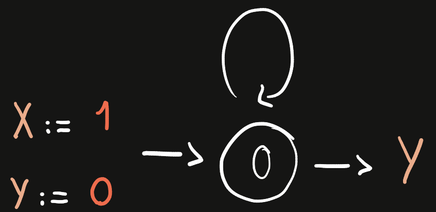


aab :

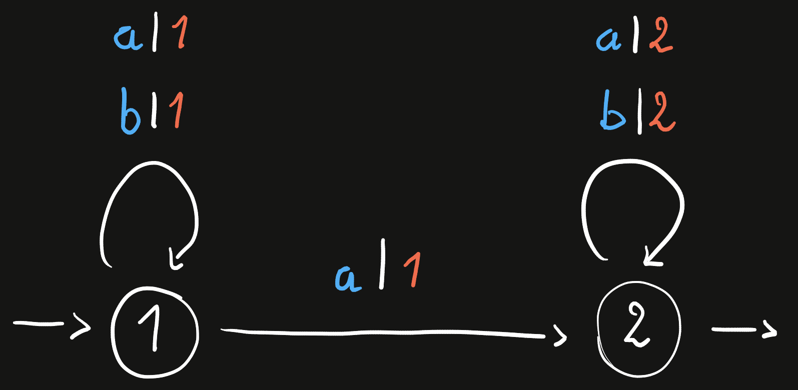
$\rightarrow 0 \xrightarrow{a} 0$

$X = 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA)

[Alur et al. 2013]

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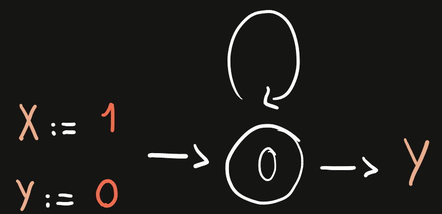

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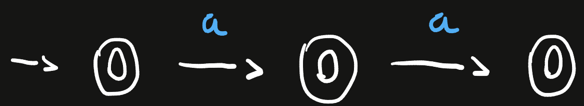
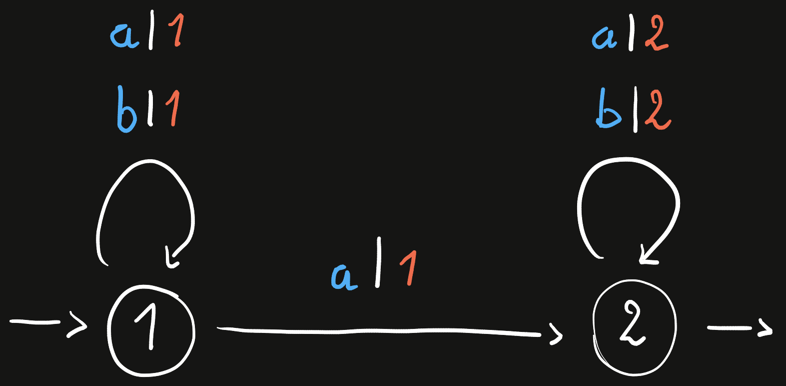


$X = 1 \longrightarrow 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1 \longrightarrow 3$

Weighted Automata

(WA)



Linear representation

(u, μ, ν)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\qquad \nu = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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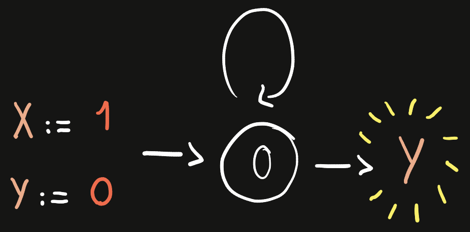


aab :

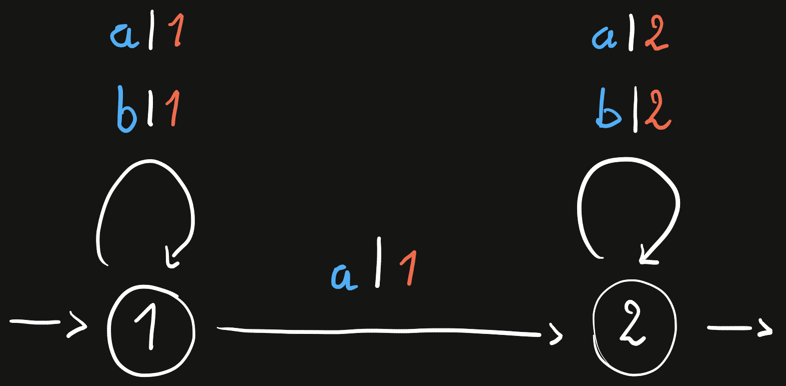
$X = 1 \longrightarrow 1 \longrightarrow 1 \longrightarrow 1$

$Y = 0 \longrightarrow 1 \longrightarrow 3 \longrightarrow 6$

$aab \mapsto 6$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cost Register Automata

(CRA) [Alur et al. 2013]

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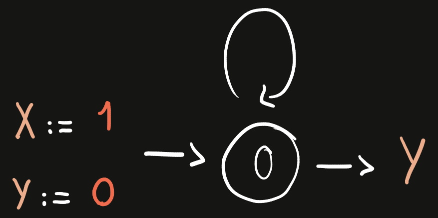


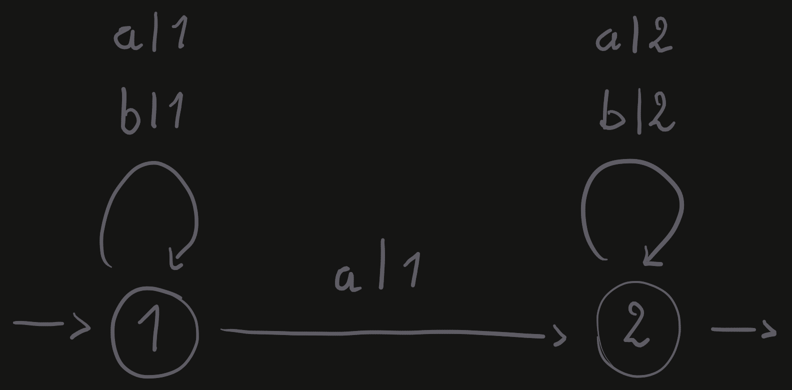
Prop.

- Linear CRA <=> WA

$(X := \alpha X + \beta Y + \gamma Z)$

Weighted Automata

(WA)



Linear representation

(u, μ, v)

$$u = \begin{array}{cc} X & Y \\ \left(1\right. & \left.0\right) \end{array} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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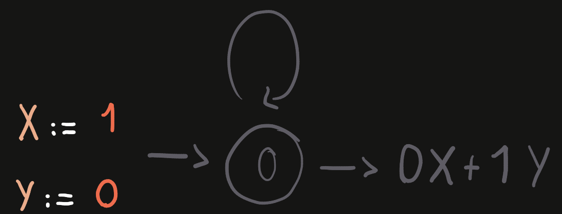

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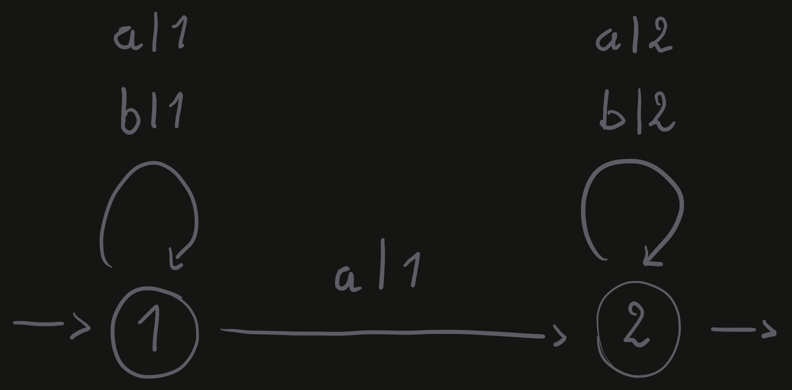

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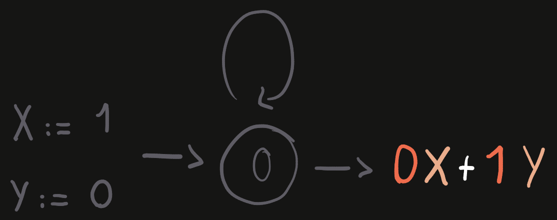

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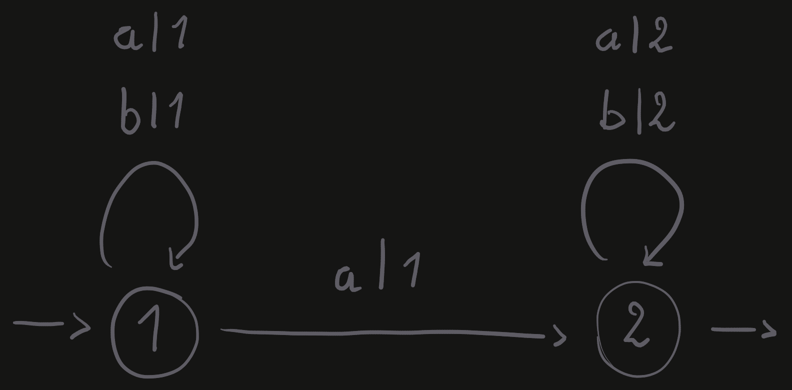

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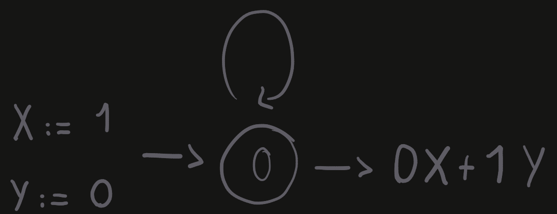

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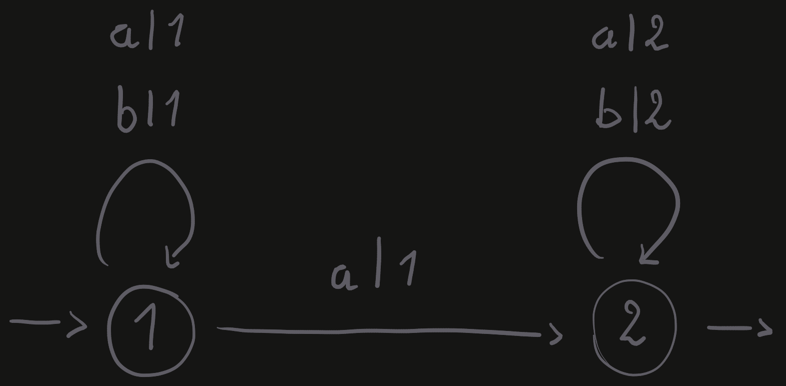
Weighted Automata

(WA)



Linear representation

(u, μ, v)

$$u = \left(\begin{array}{cc} X & Y \\ 1 & 0 \end{array}\right) \qquad v = \left(\begin{array}{cc} 0 & X \\ 1 & Y \end{array}\right)$$

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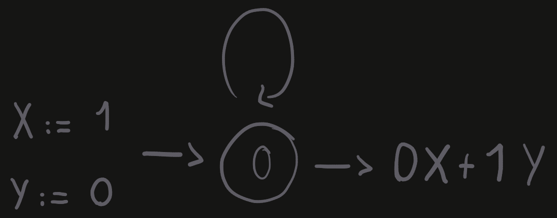

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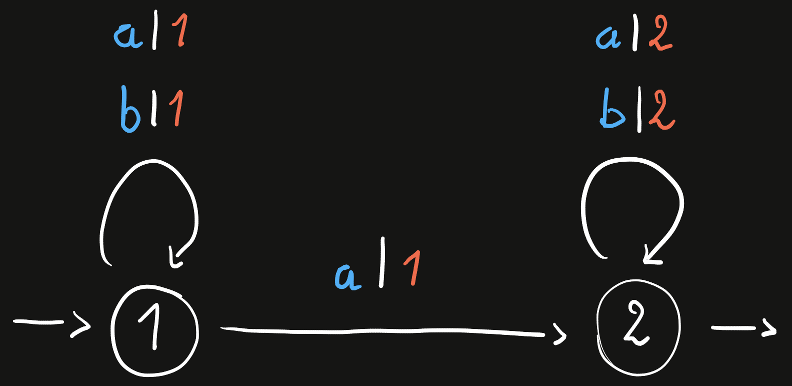

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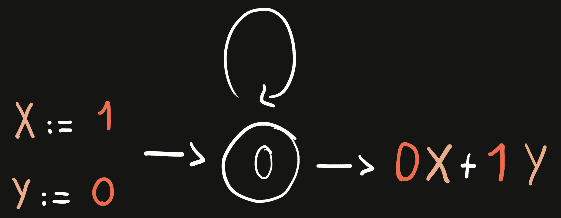

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Weighted Automata

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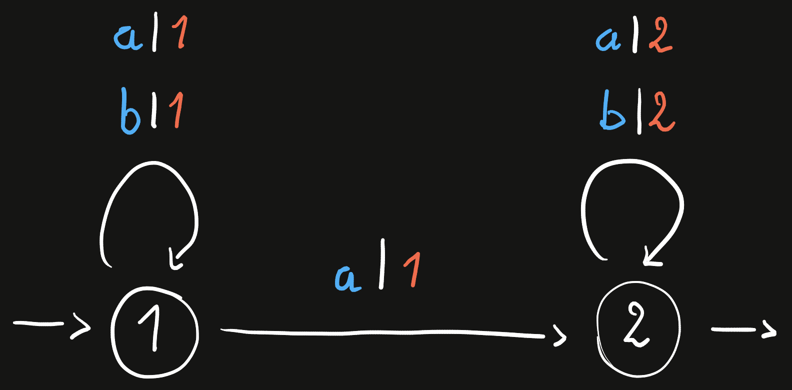

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(u, μ, v)

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$$\mu(a) = \begin{array}{cc} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \end{array} \qquad \mu(b) = \begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

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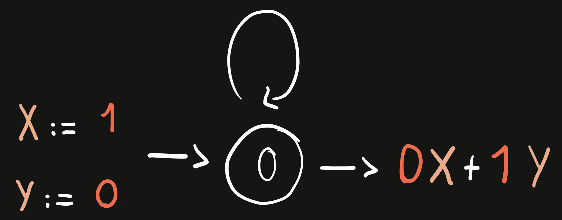

Prop.

- Linear CRA <=> WA
  ($X := \alpha X + \beta Y + \gamma Z$)
- 1 Register CRA <=> Sequential WA
  ($X := \alpha X$)

Def: Register complexity of a rational series f

= min nb. of registers needed by a CRA to realize f

Register minimization problem

In: f rational series given as a WA, $k \in \mathbb{N}$

Q?: $\exists$? CRA with $\leqslant k$ registers realizing f

Def: Register complexity of a rational series f

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Register minimization problem

In: f rational series given as a WA, $k \in \mathbb{N}$

Q?: ∃? CRA with $\leqslant k$ registers realizing f

Def: State-Register complexity of a rational series f

= set of (n,k) s.t. ∃ CRA for f with n states & k registers
& ∀ CRA for f nb. states $> n$ or nb. registers $> k$

State-Register minimization problem

In: f rational series given as a WA, $n, k \in \mathbb{N}$

Q?: ∃? CRA with $\leqslant n$ states & $\leqslant k$ registers realizing f

Let Σ finite alphabet $\mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

$\mathbb{K}$ field

Def: Invariant of $\mathcal{R}$

set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma, \forall x \in I, \; x \cdot \mu(a) \in I$

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E.g.: $u \cdot \mu(\Sigma^*)$: Reachability set

$\mathbb{K}^d$

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strongest

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$\mathbb{K}^d$

weakest

Def: I is stronger than J
if $I \subseteq J$

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Def: I is stronger than J
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Def: Linear Zariski topology
[Bell & Smertnig 2021]

closed sets: finite unions of vector subspaces of $\mathbb{K}^d$

semilinear sets

Let Σ finite alphabet
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Def: Linear Zariski topology

[Bell & Smertnig 2021]

closed sets : finite unions of vector subspaces of $\mathbb{K}^d$

semilinear sets

irreducible components

$$S = V_1 \cup V_2 \cup \cdots \cup V_n$$

Length n = nb. of components

Dimension $k = \max\limits_{1 \leq i \leq n} (\dim(V_i))$

Let Σ finite alphabet
$\mathbb{K}$ field

$\mathcal{R} = (u, \mu, v)$ d-dimensional WA on Σ over $\mathbb{K}$

Def: Semilinear Invariant of $\mathcal{R}$

semilinear set $I \subseteq \mathbb{K}^d$ s.t.

- $u \in I$
- $\forall a \in \Sigma$, $\forall x \in I$, $x \cdot \mu(a) \in I$

strongest

E.g.: $\overline{u \cdot \mu(\Sigma^*)}^{\ell}$: (Linear Hull)

$\mathbb{K}^d$

weakest

Def: I is stronger than J

if $I \subseteq J$

Def: Linear Zariski topology

[Bell & Smertnig 2021]

closed sets : finite unions of vector subspaces of $\mathbb{K}^d$

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E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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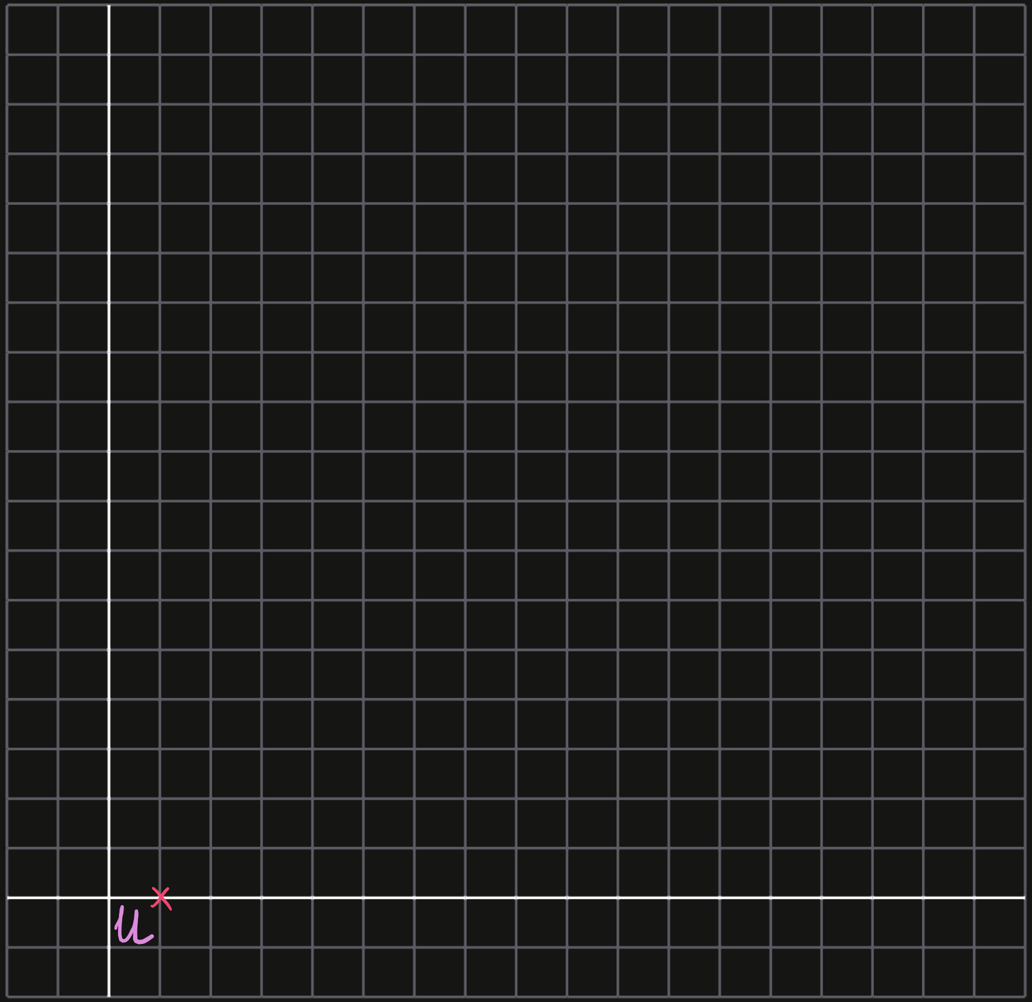

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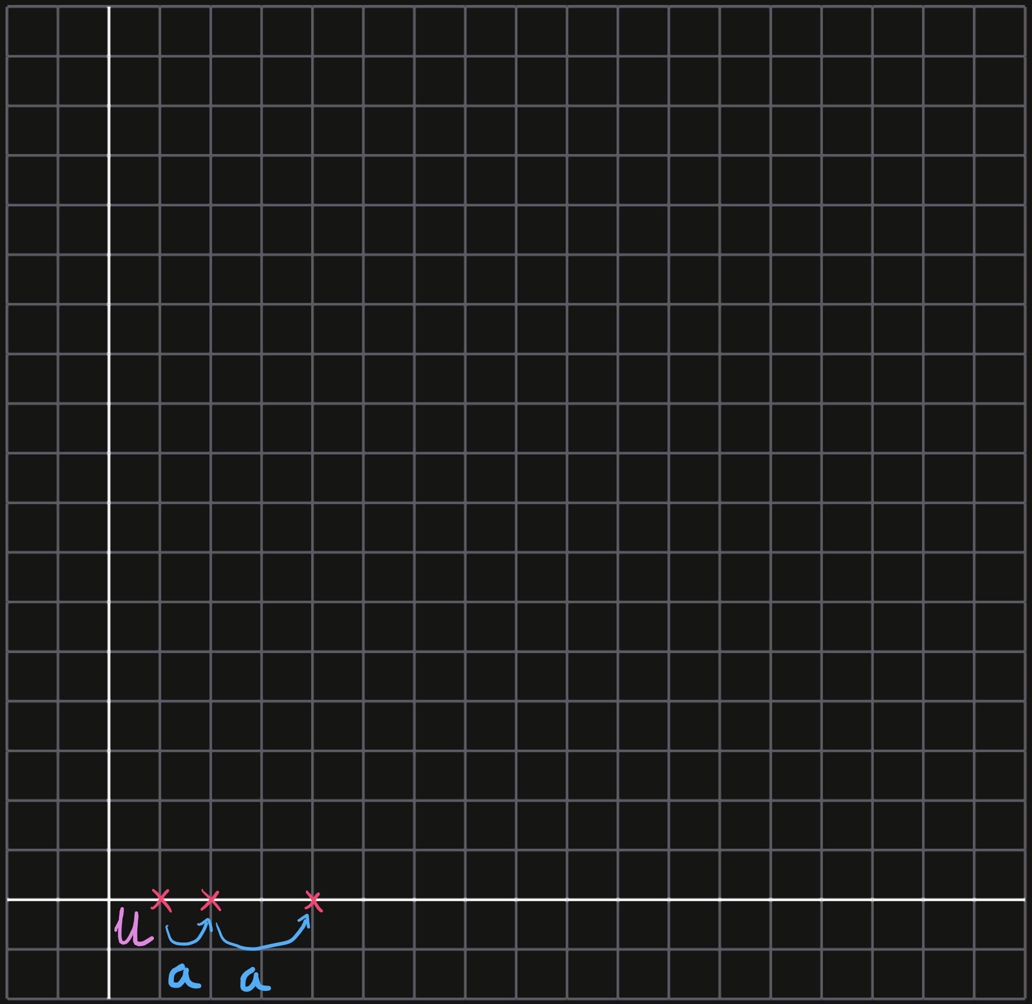

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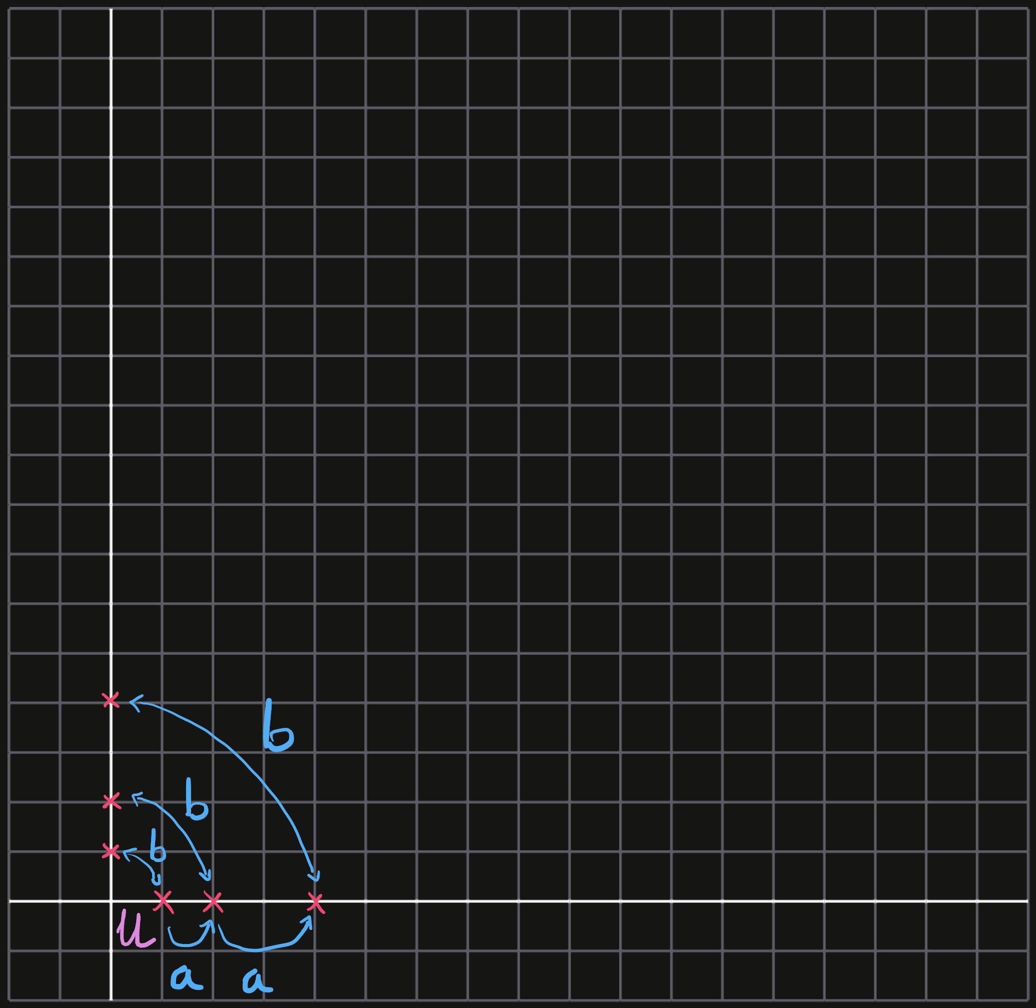

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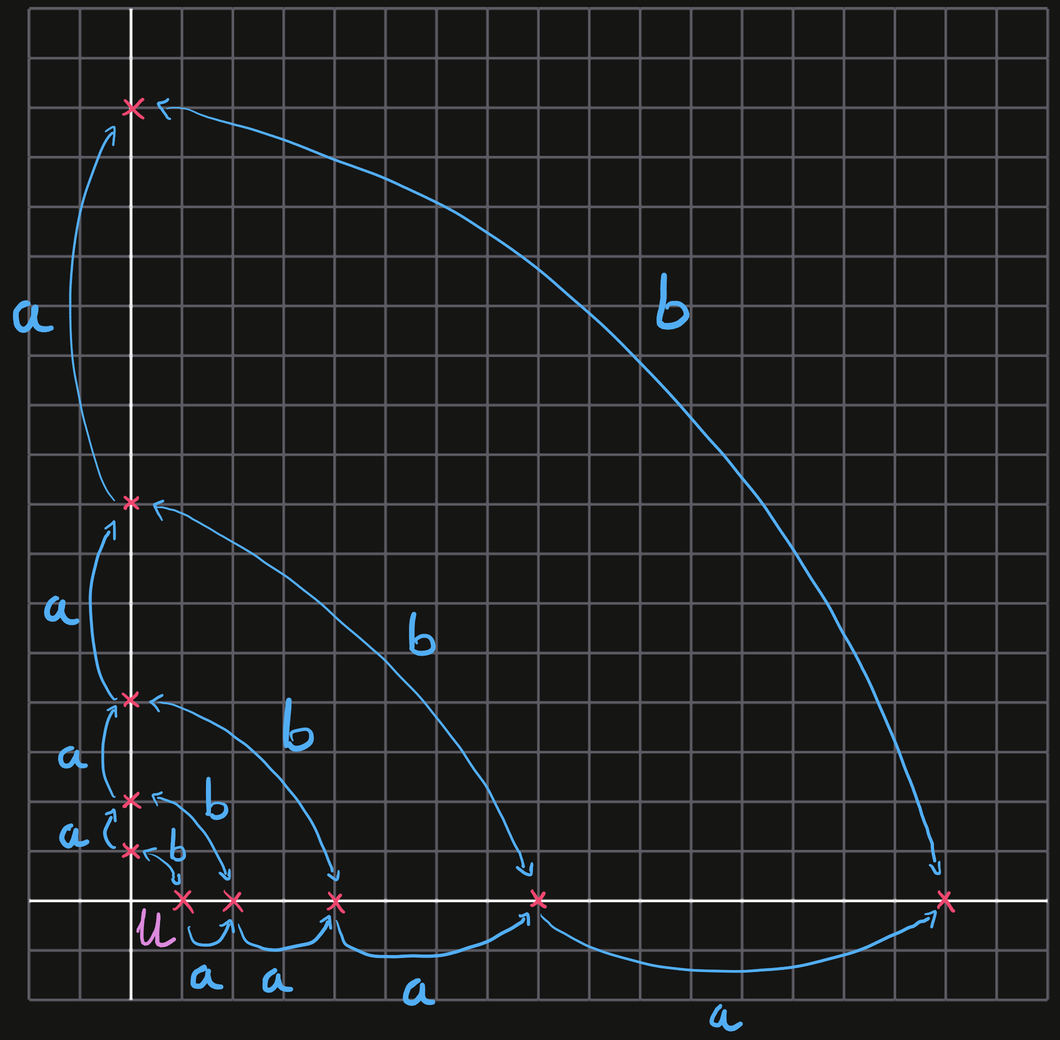


$$w \mapsto \begin{cases} 2^{|w|_a} & \text{if } |w|_b \text{ is even} \\ 0 & \text{else} \end{cases}$$

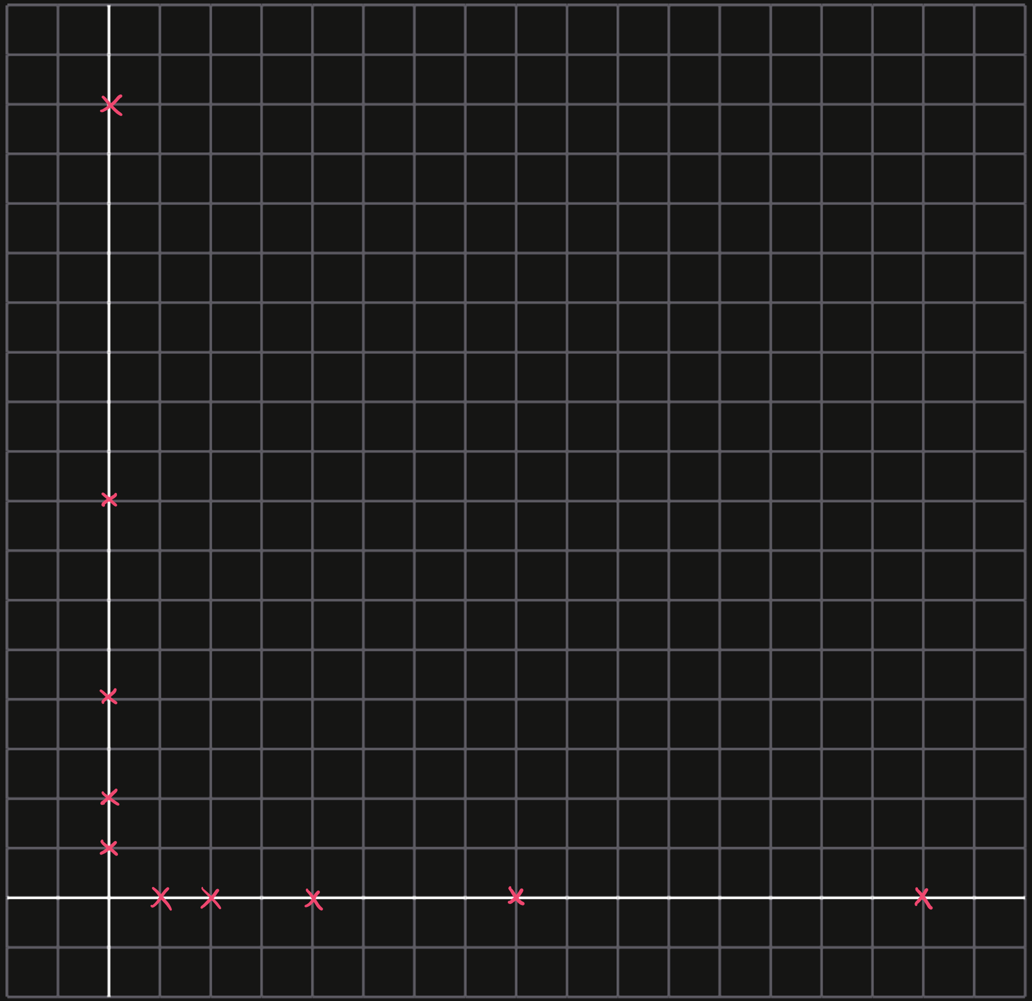
E.g. $\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = (1 \ \ 0) \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$$w \mapsto \begin{cases} 2^{|w|_a} & \text{if } |w|_b \text{ is even} \\ 0 & \text{else} \end{cases}$$

$$u\mu(\Sigma^*) = \{(2^n \ \ 0), \ n \in \mathbb{N}\} \cup \{(0 \ \ 2^n), \ n \in \mathbb{N}\}$$

E.g. $\Sigma = \{a, b\}$

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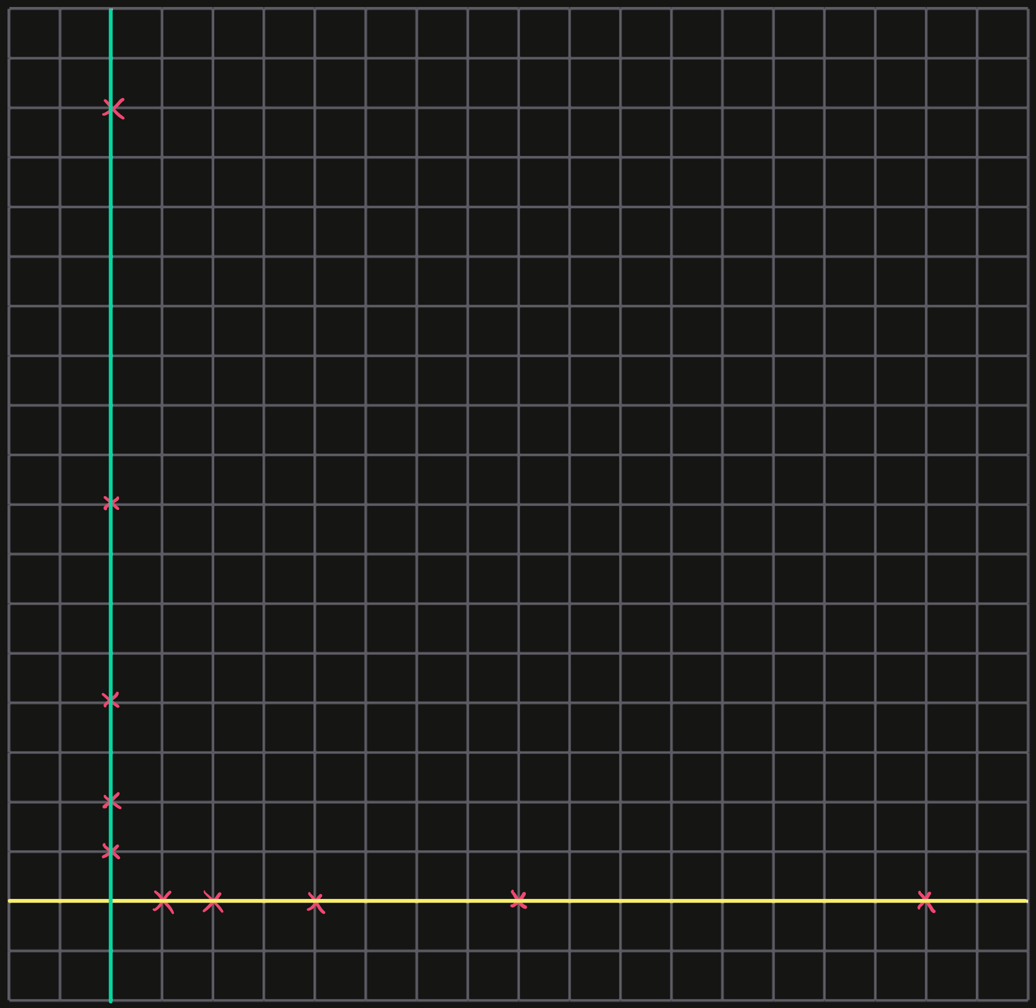

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$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$$

Length : 2

Dimension : 1

Let f be a rational series

Thm: [Bell & Smertnig 2021]

$\exists$ sequential WA for f

iff

minimal WA (u,μ,v) for f

have $\dim(\overline{u\mu(\Sigma^*)}^{\ell}) \leqslant 1$

Let f be a rational series

Our results

Thm: [Bell & Smertnig 2021]

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Thm: (Characterization)

$\exists$ CRA for f with n states & k registers

iff

minimal WA for f have a semilinear invariant I with

$\text{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Cor: Register complexity of f

$\parallel$

$\dim(\overline{u\mu(\Sigma^*)}^{\ell})$

where (u, μ, v): minimal WA for f

Our results

Let f be a rational series

Thm: [Bell & Smertnig 2021]

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have $\dim(\overline{u\mu(\Sigma^*)}^{\ell}) \leqslant 1$

↳ [Bell & Smertnig 2023]

$\overline{u\mu(\Sigma^*)}^{\ell}$ is computable

⇒ Sequential? is decidable

(Unambiguous? is decidable)

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$\|$

$\dim(\overline{u\mu(\Sigma^*)}^{\ell})$

where (u,μ,v): minimal WA for f

Let f be a rational series

Our results

Thm: [Bell & Smertnig 2021]

$\exists$ sequential WA for f

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(Unambiguous? is decidable)

in 2-EXPTIME

Thm: (Characterization)

$\exists$ CRA for f with n states & k registers

iff

minimal WA for f have a semilinear invariant I with

$\text{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

↳ I is computable in NEXPTIME

⟹ Stt-Reg min pb. is decidable in NEXPTIME

Cor: Register complexity of f

$$=$$

$$\dim(\overline{u\mu(\Sigma^*)}^{\ell})$$

where (u,μ,v): minimal WA for f

↳ $\overline{u\mu(\Sigma^*)}^{\ell}$ is computable in 2-EXPTIME

⟹ Reg min pb is decidable in 2-EXPTIME

Let f be a rational series

Proof sketch

Thm: (Characterization)

∃CRA for f with n states & k registers

iff

minimal WA for f have a semilinear invariant I with

$\text{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Let $\mathfrak{f}$ be a rational series

Proof sketch

Prop. Let $\mathcal{R}$ be a WA for $\mathfrak{f}$

$\exists$ Semilinear invariant I of $\mathcal{R}$ s.t. $\mathrm{length}(I) = n$
& $\dim(I) = k$

$\Downarrow$

$\forall\, \mathcal{R}_m$ minimal WA for $\mathfrak{f}$

$\exists$ Semilinear invariant I_m of $\mathcal{R}_m$ s.t. $\mathrm{length}(I_m) \leqslant n$
& $\dim(I_m) \leqslant k$

Thm: (Characterization)

$\exists$CRA for $\mathfrak{f}$ with n states
& k registers

iff

minimal WA for $\mathfrak{f}$ have a semilinear invariant I with

$\mathrm{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Let f be a rational series

Proof sketch

Prop. Let $\mathcal{R}$ be a WA for f

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$\Downarrow$

$\forall\ \mathcal{R}_m$ minimal WA for f

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Thm: (Characterization)

$\exists$ CRA for f with n states & k registers

iff

minimal WA for f have a semilinear invariant I with $\text{length}(I) \leqslant n$ & $\dim(I) \leqslant k$

Prop. (Invariants $\longleftrightarrow$ CRA)

$\exists$ WA for f with a semilinear invariant I s.t. $\text{length}(I) = n$ & $\dim(I) = k$ $\iff$ $\exists$ CRA for f with n states & k registers

Proof sketch

Let $\mathfrak{f}$ be a rational series

Thm: (Characterization)

$\exists$ CRA for $\mathfrak{f}$ with n states & k registers

iff

minimal WA for $\mathfrak{f}$ have a semilinear invariant I with

$\mathrm{length}(I) \leq n$ & $\dim(I) \leq k$

Prop. Let $\mathcal{R}$ be a WA for $\mathfrak{f}$

$\exists$ Semilinear invariant I of $\mathcal{R}$ s.t. $\mathrm{length}(I) = n$ & $\dim(I) = k$

$$\Downarrow$$

$\forall\, \mathcal{R}_m$ minimal WA for $\mathfrak{f}$

$\exists$ Semilinear invariant I_m of $\mathcal{R}_m$ s.t. $\mathrm{length}(I_m) \leq n$ & $\dim(I_m) \leq k$

Prop. (Invariants $\longleftrightarrow$ CRA)

$\exists$ WA for $\mathfrak{f}$ with a semilinear invariant I s.t. $\mathrm{length}(I) = n$ & $\dim(I) = k$

$\iff$

$\exists$ CRA for $\mathfrak{f}$ with n states & k registers

Invariant → CRA

$\Sigma = \{a, b\}$

$\mathbb{K} = (\mathbb{R}, +, \cdot)$

$\mathcal{R} = (u, \mu, v)$

$u = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mu(a) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$

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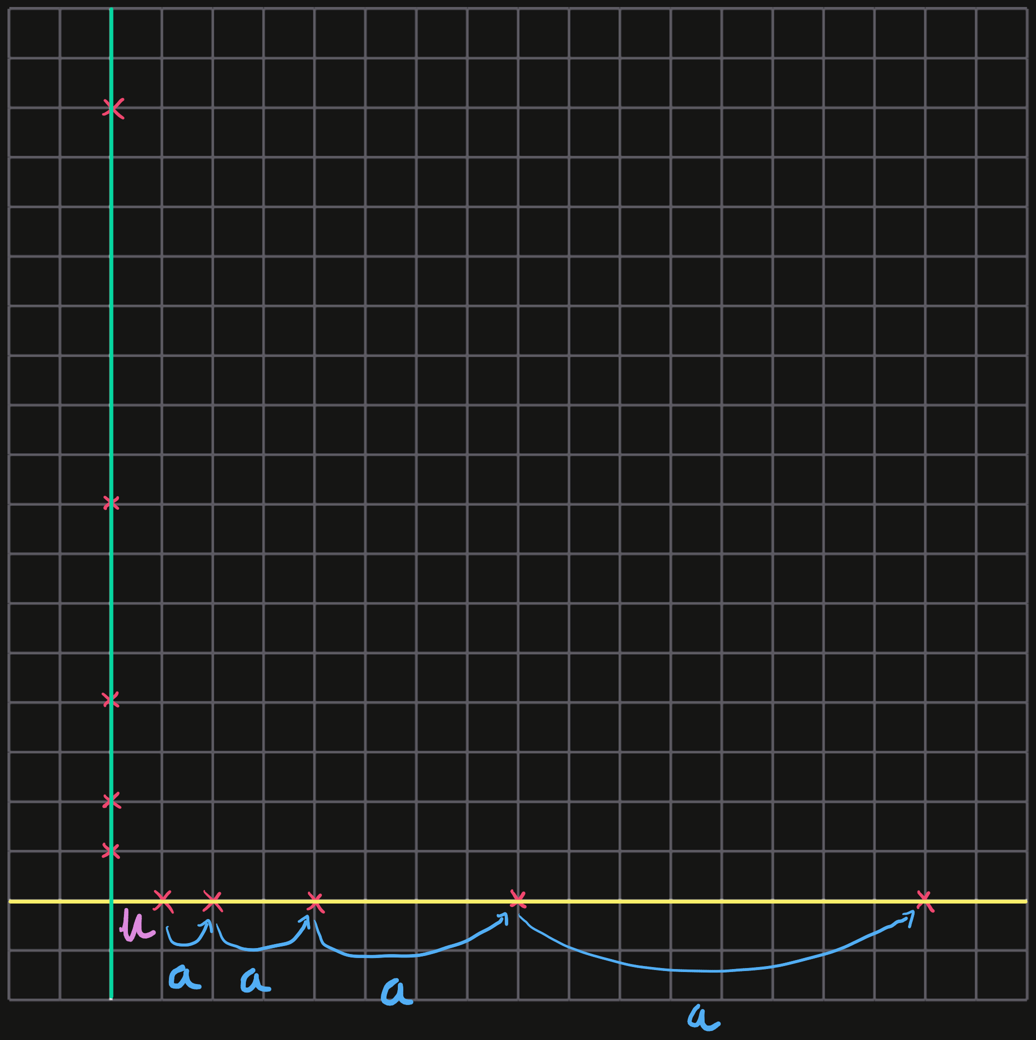

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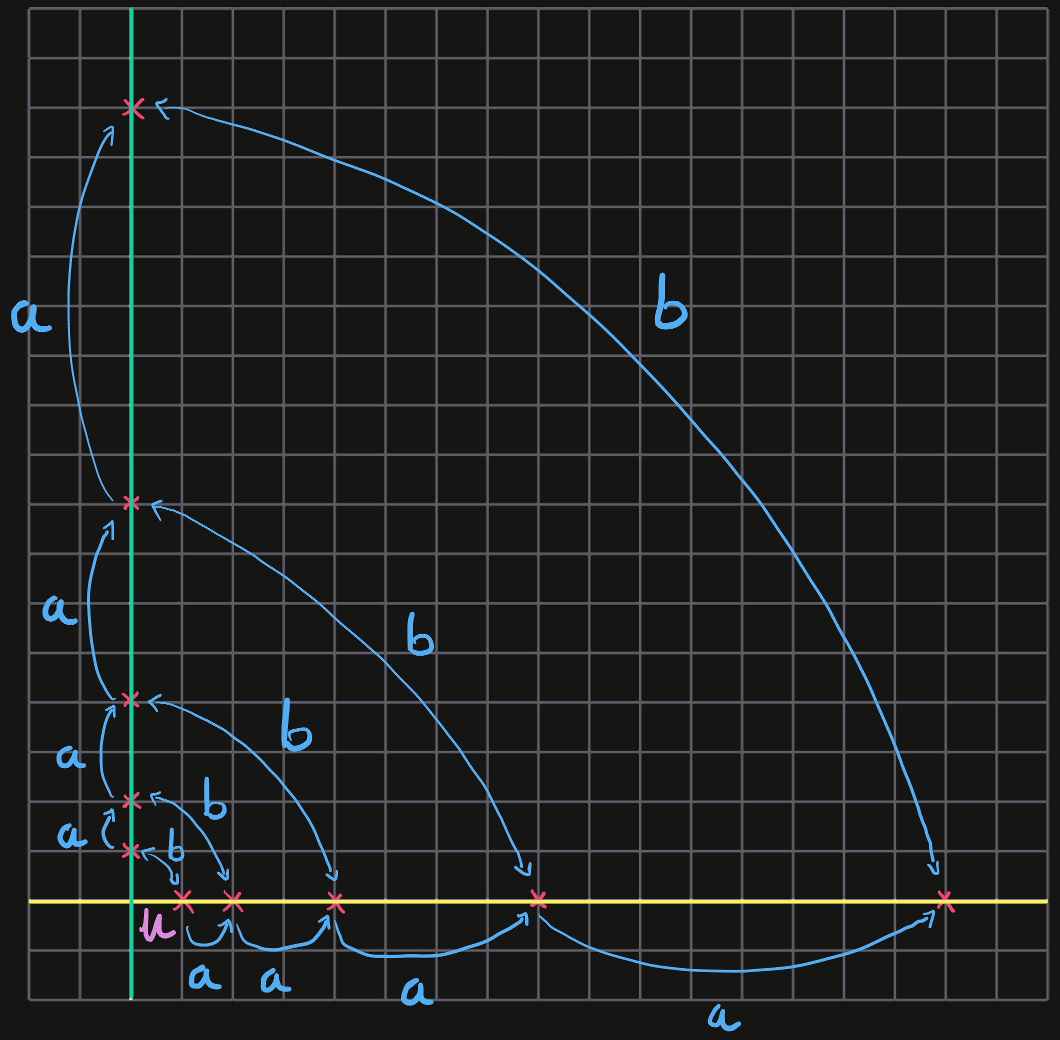

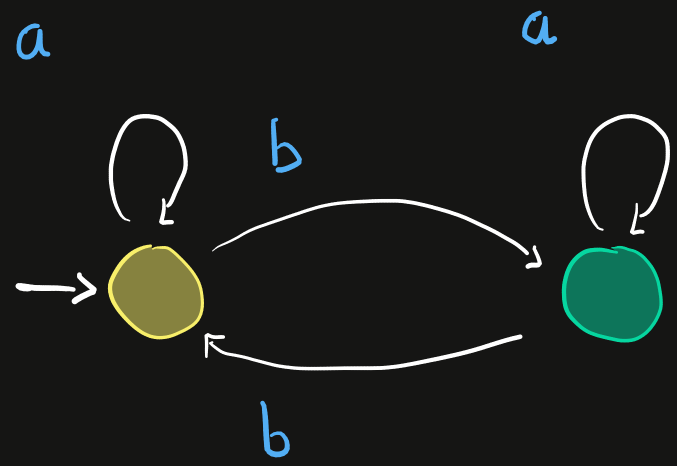

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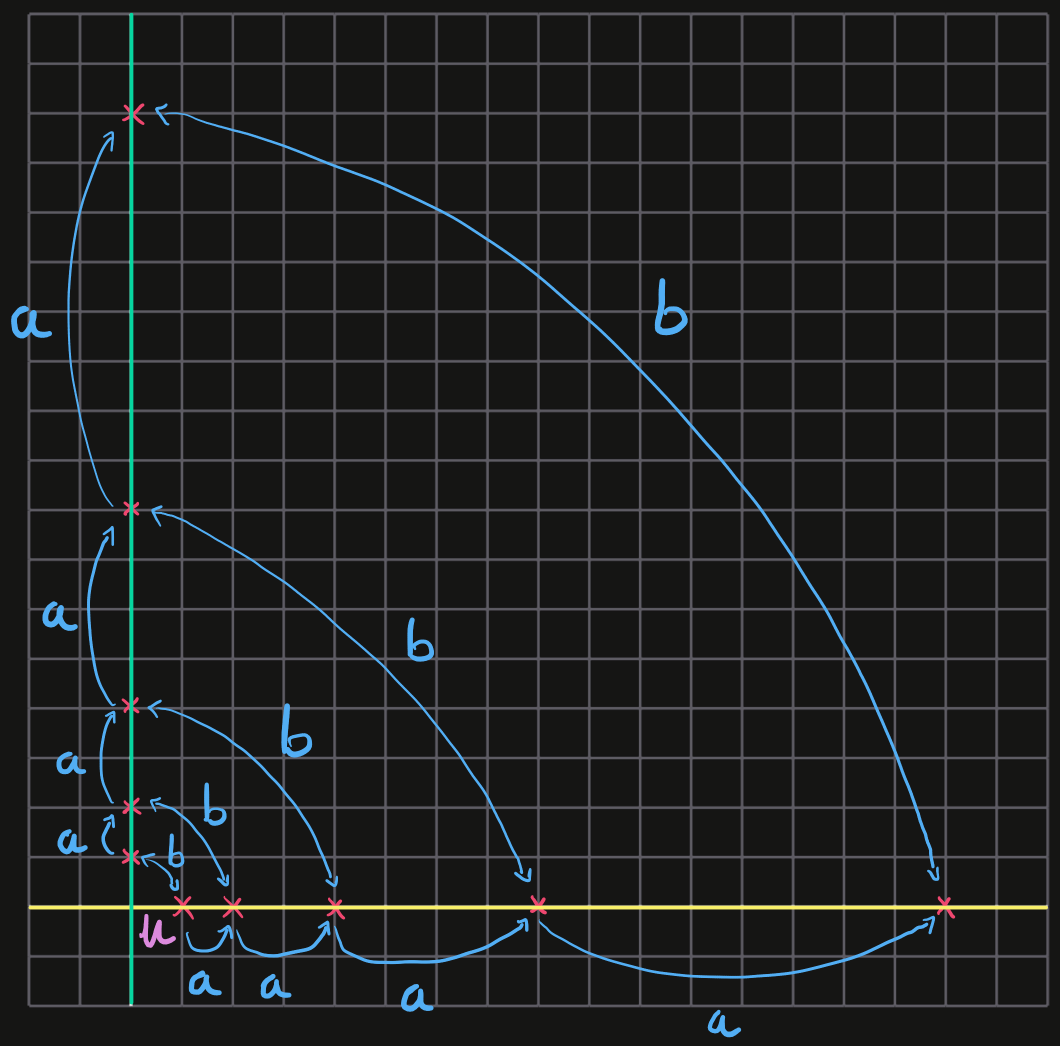

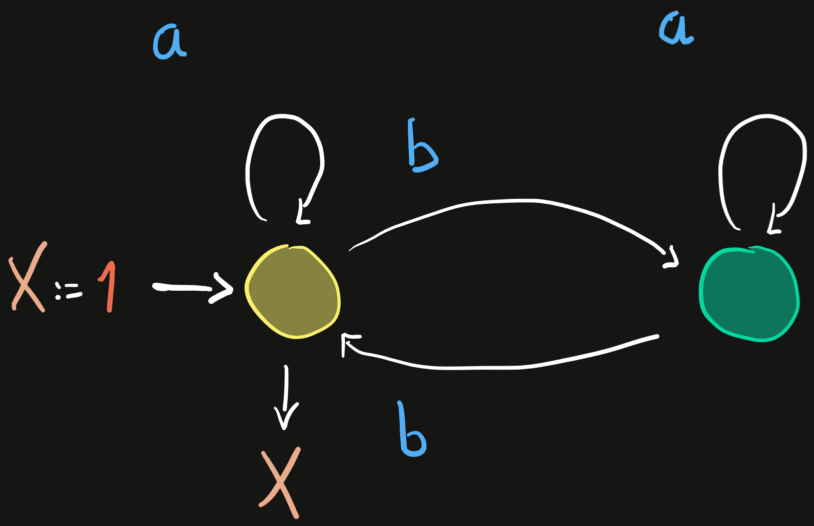

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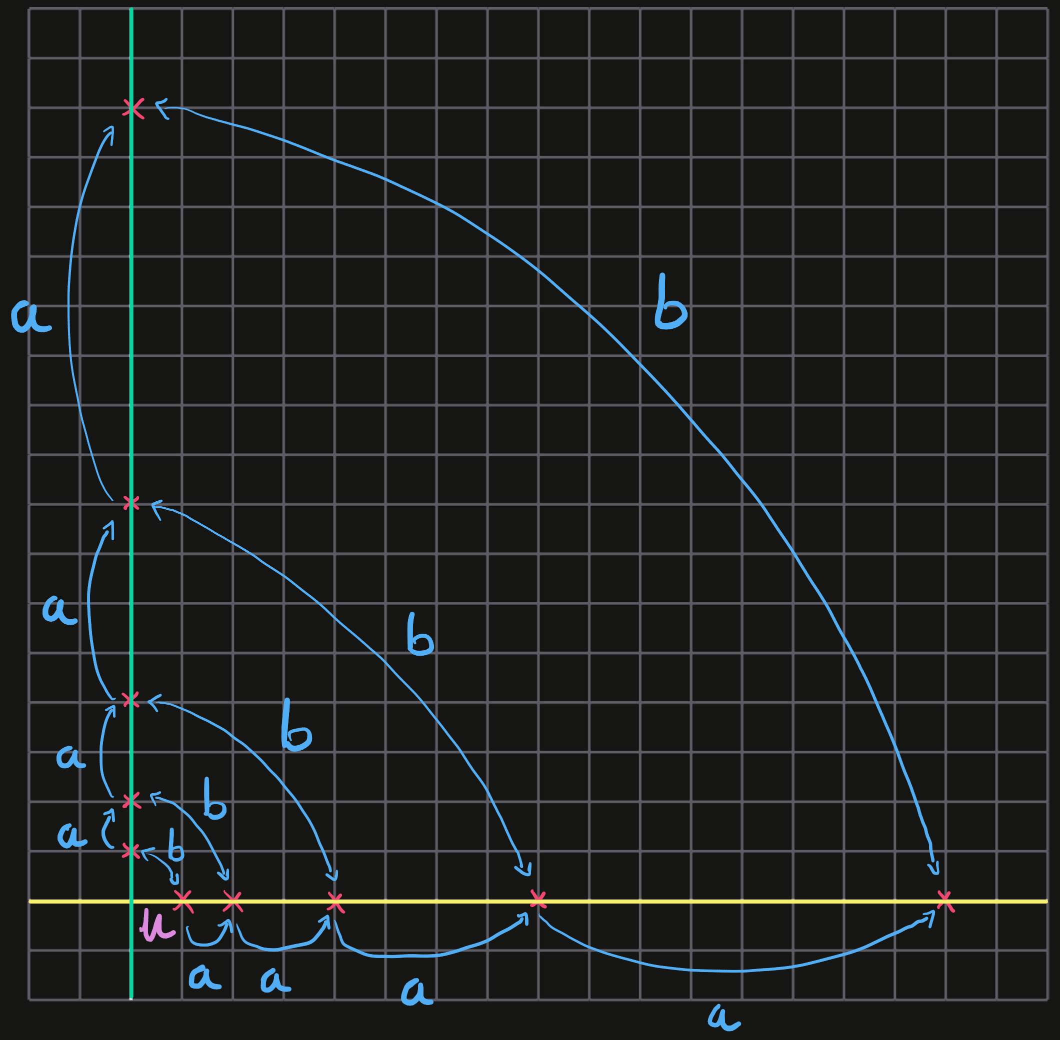

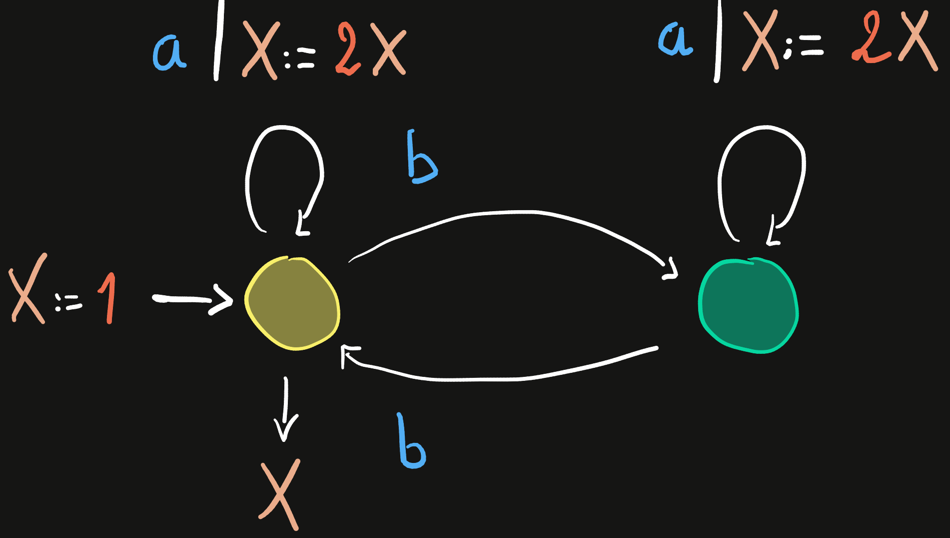

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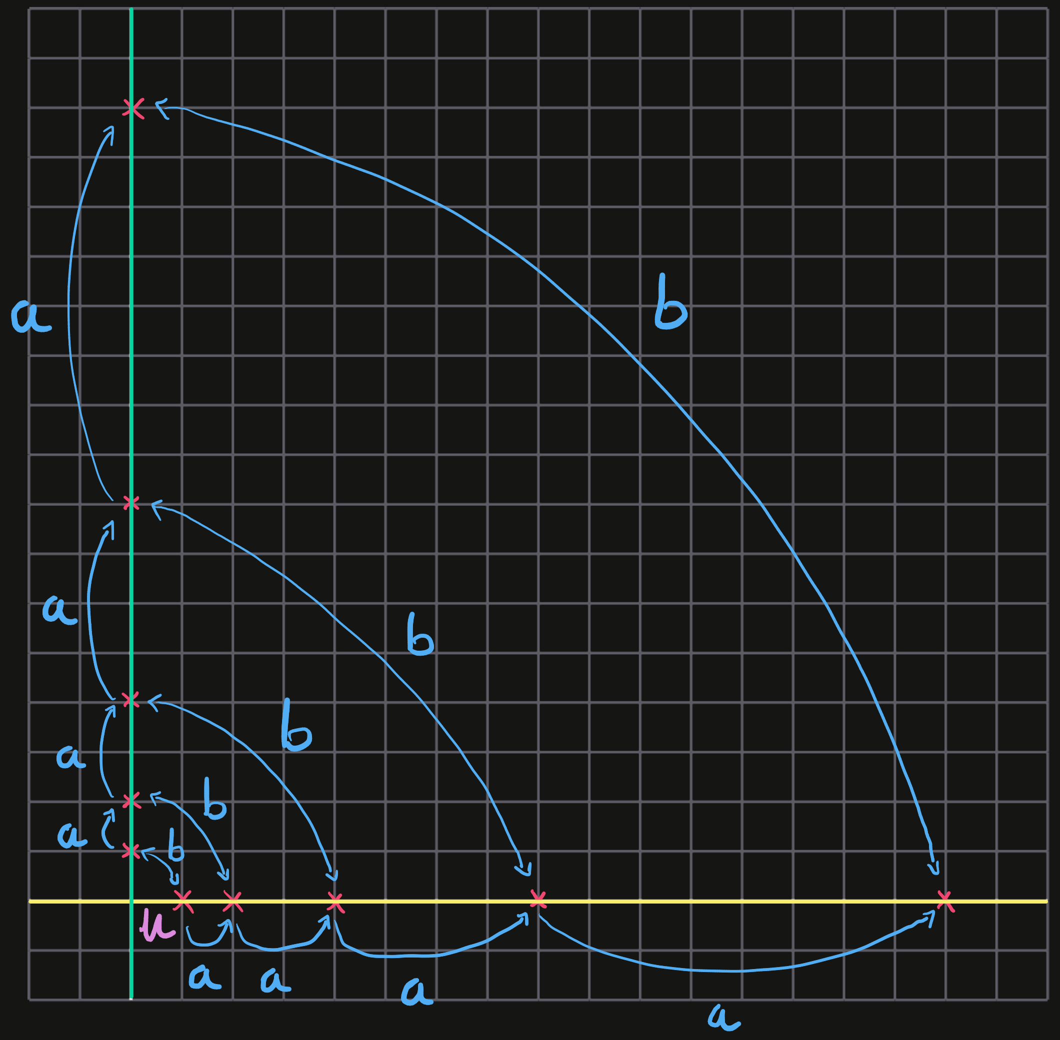

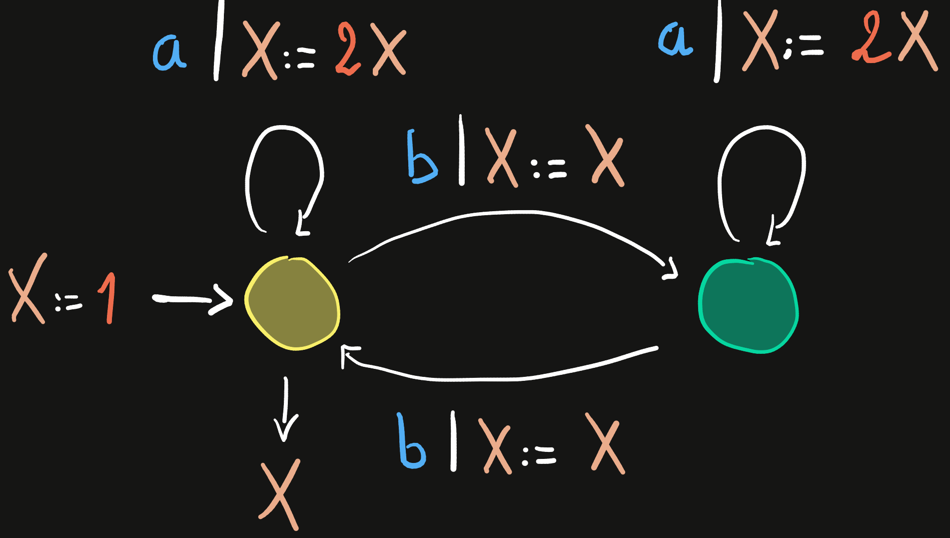

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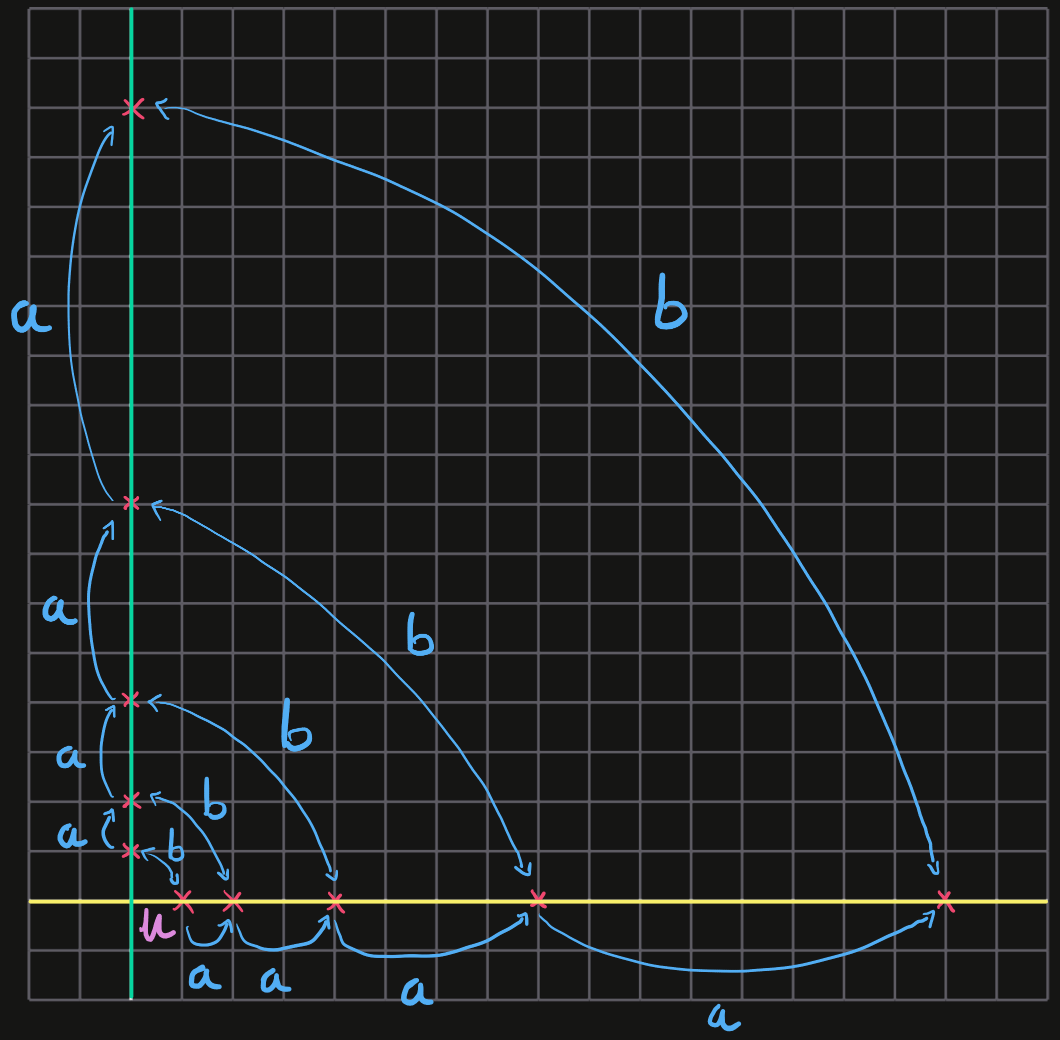
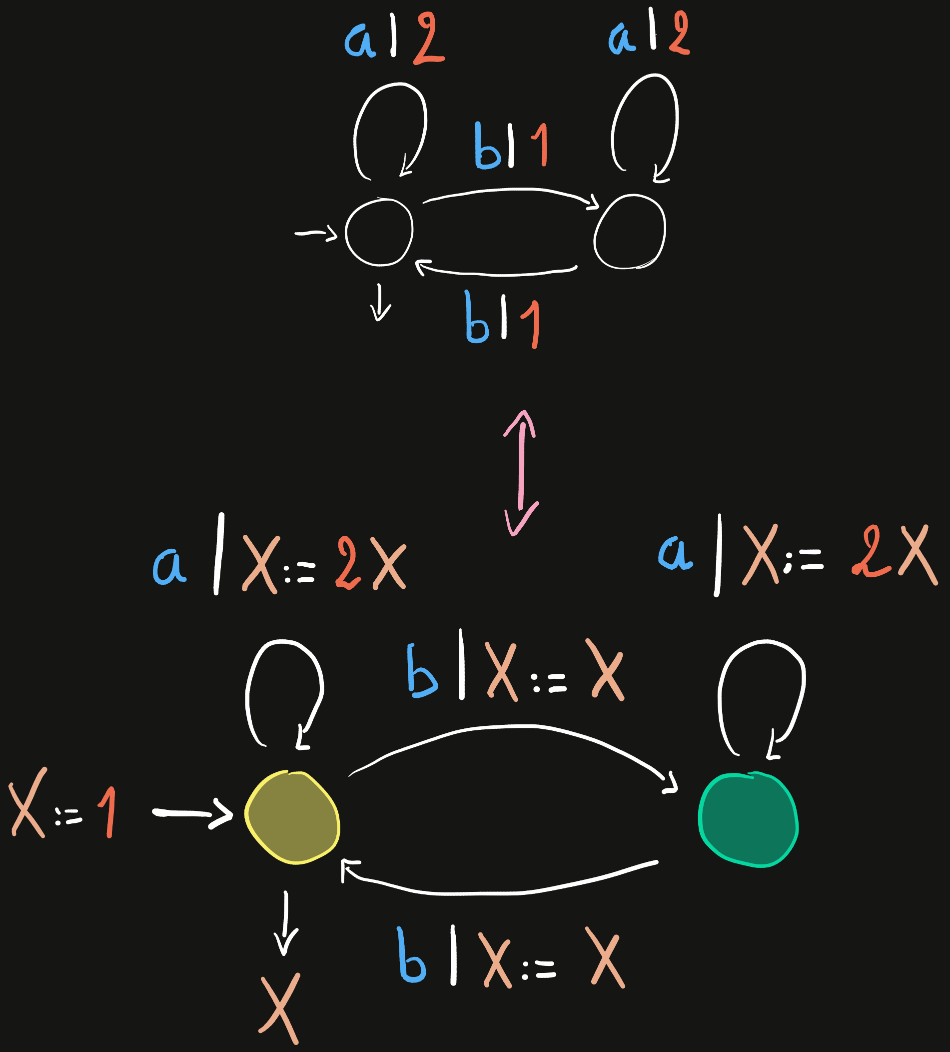




Tradeoff States / Registers

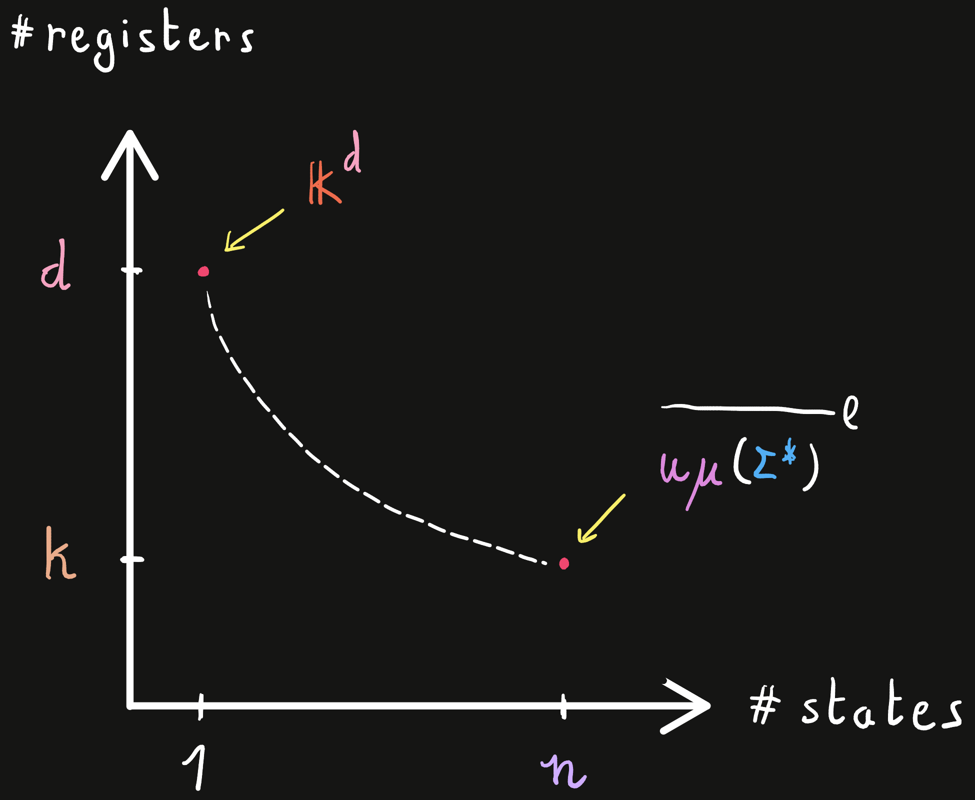
Let

$\mathfrak{R} = (u, \mu, \nu)$ be a

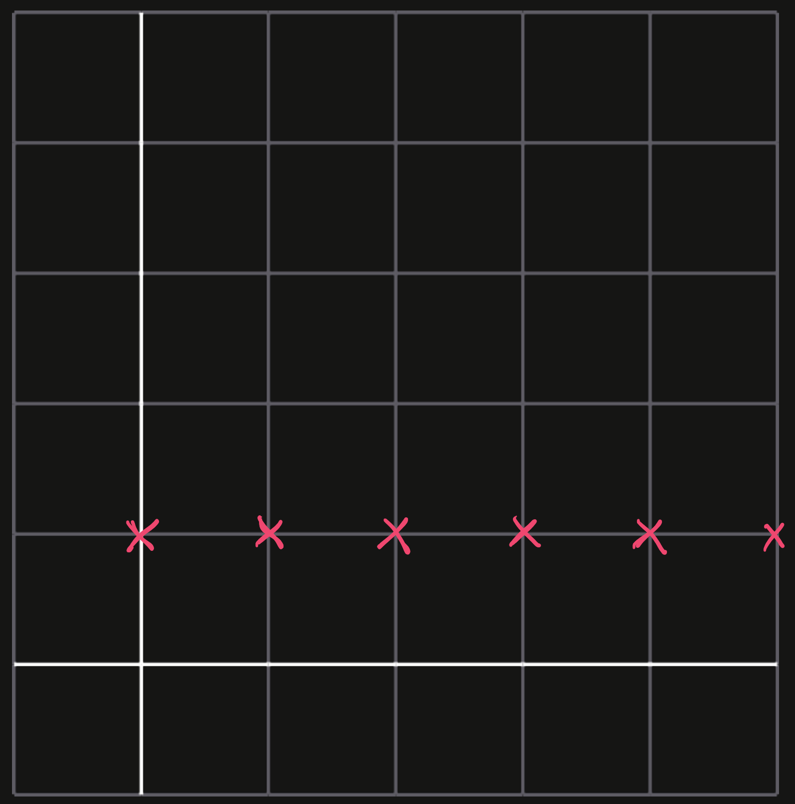
d-dimensional minimal WA

$n = \text{length}\left(\overline{u\mu(\Sigma^*)}^{\ell}\right) \lesssim$ 2-EXP in d

$k = \text{dimension}\left(\overline{u\mu(\Sigma^*)}^{\ell}\right)$



Affine CRA



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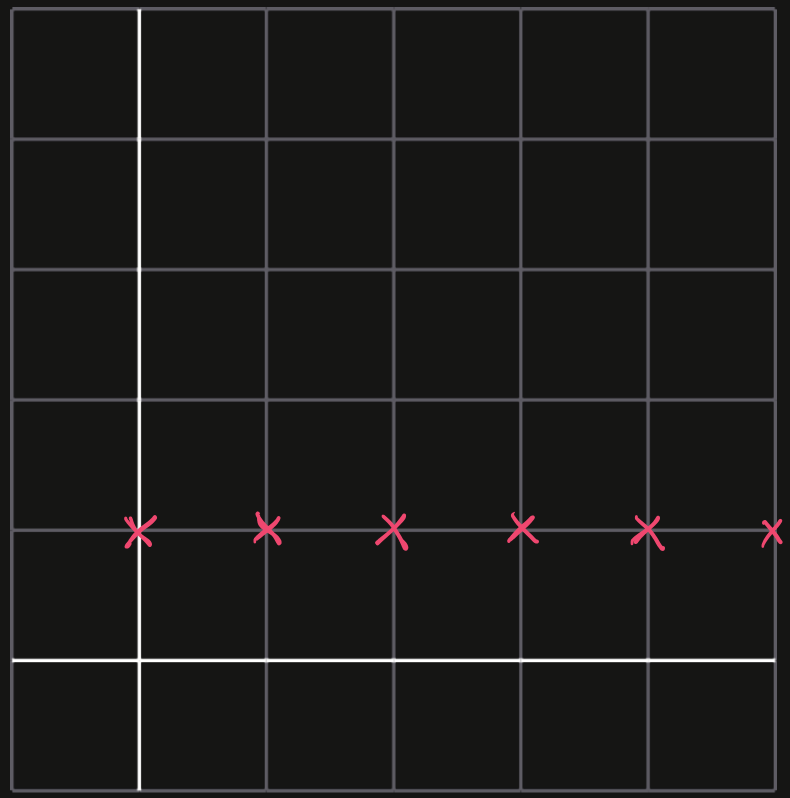

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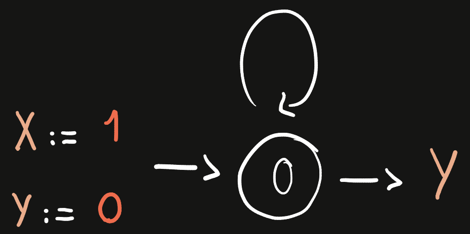
$$\overline{u\mu(\Sigma^*)}^{\ell} = \mathbb{R}^2$$

$a \left| \begin{array}{l} X := X \\ Y := X + 2Y \end{array} \right.$, $b \left| \begin{array}{l} X := X \\ Y := 2Y \end{array} \right.$



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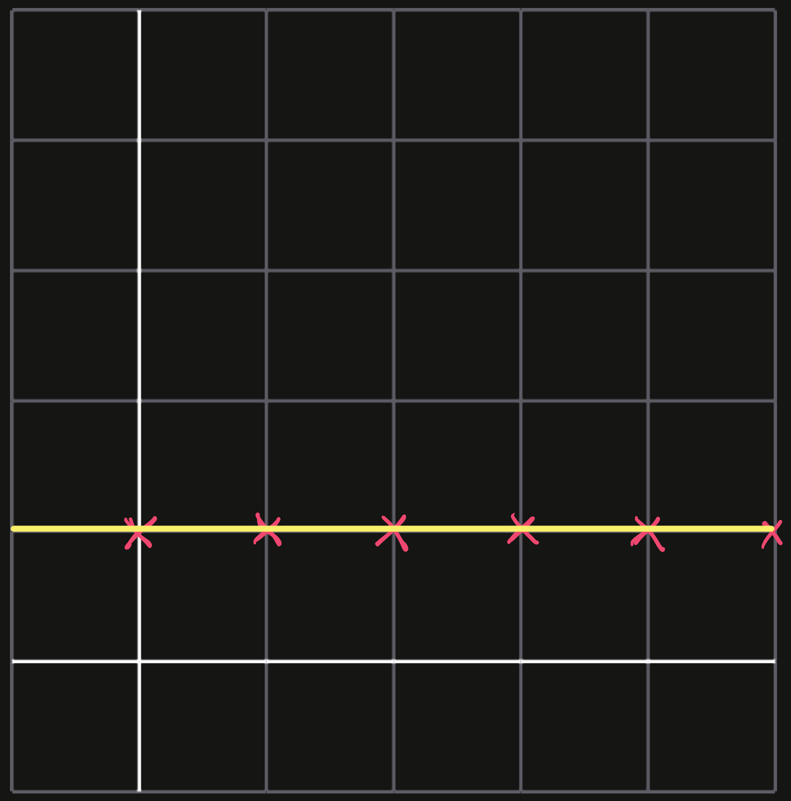

$$\overline{u\mu(\Sigma^*)}^{a} = (1\ 0) + \mathbb{R} \times \{0\}$$

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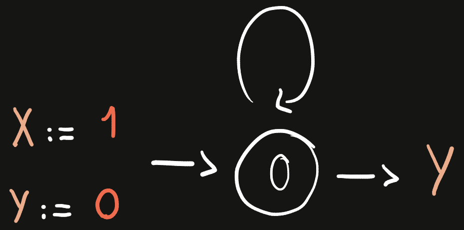

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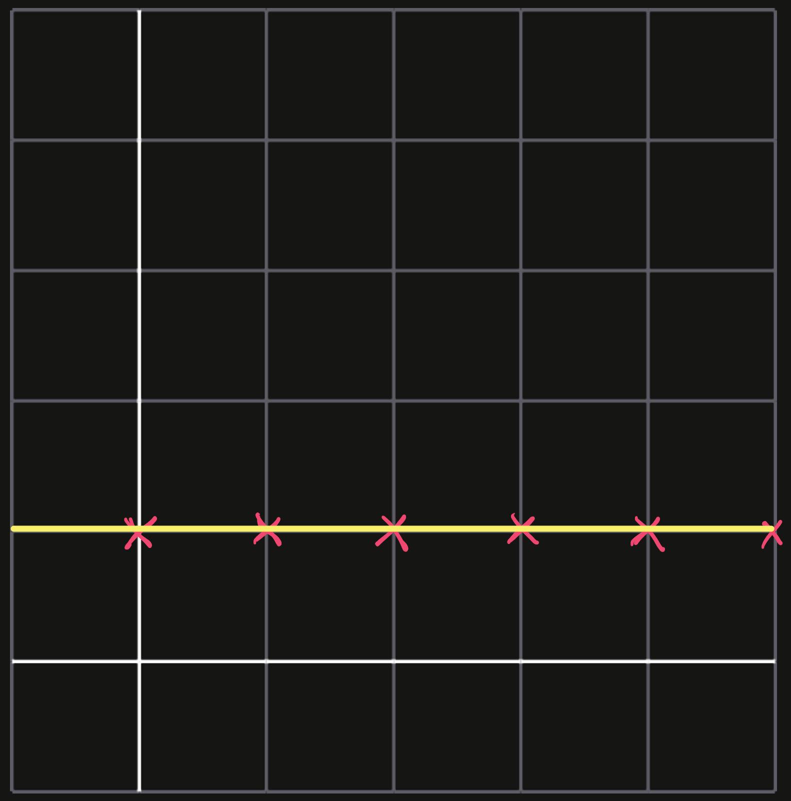
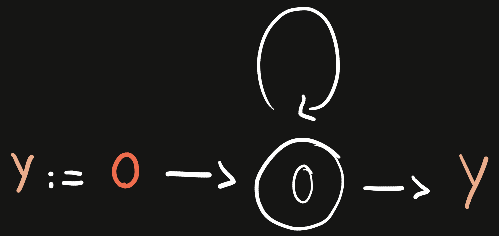


$$\overline{u\mu(\Sigma^*)}^{a} = (1\ 0) + \mathbb{R} \times \{0\}$$

$$a \mid y := 2y + 1 \ , \quad b \mid y := 2y$$



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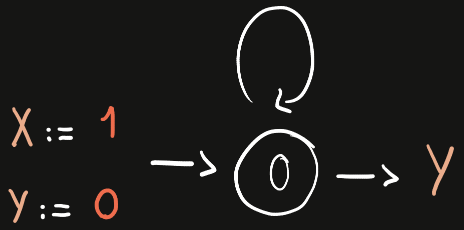

Conclusion

Semilinear / semiaffine invariants can be used to solve :

State-Register minimization problem in NEXPTIME

Register minimization problem in 2-EXPTIME

for linear / affine CRA

Sequential? & Unambiguous? too

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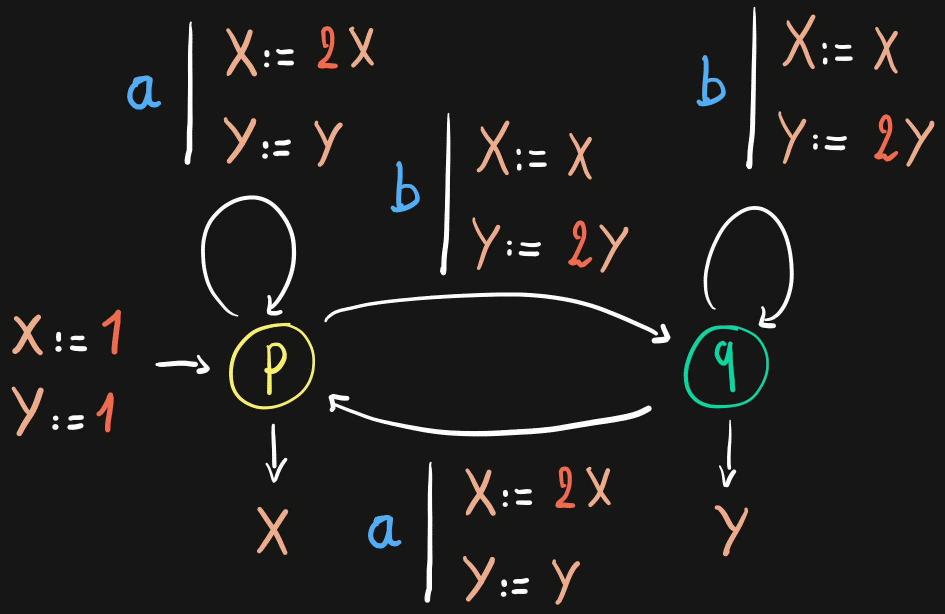
Open questions

- better complexity ?
- other classes of CRA ?
- other semirings ?

Thank you

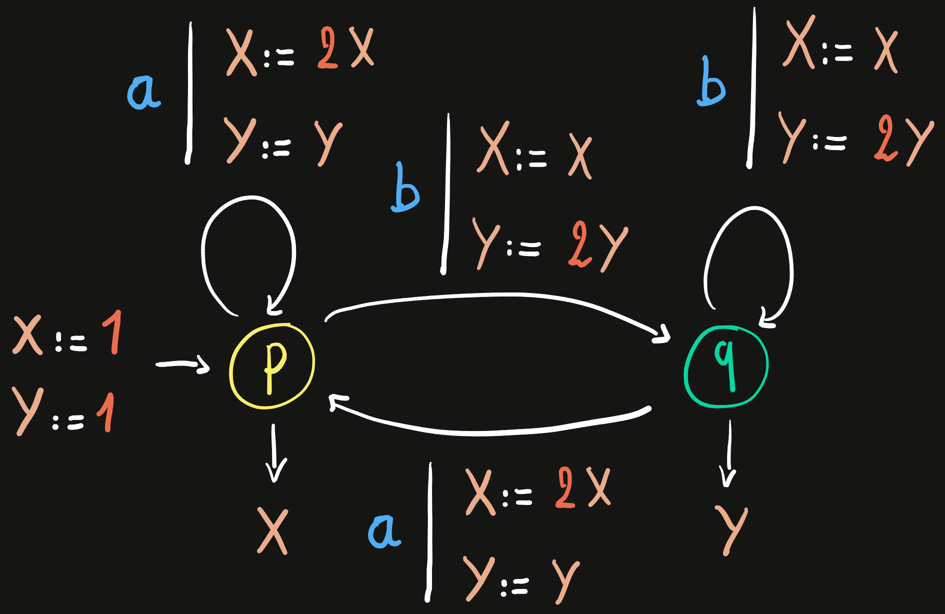
for your attention

CRA → WA



$$w\sigma \mapsto 2^{|w|_{\sigma}+1}$$

CRA → WA



$$u = \left(\begin{array}{c|c} p & q \\ \quad & \quad \end{array}\right) \qquad v = \left(\begin{array}{c|c} & \\ \hline & \end{array}\right)\begin{array}{c} p \\ q \end{array}$$

$$\mu(a) = \begin{array}{c|c|c} & p & q \\ \hline p & \quad & \quad \\ \hline q & \quad & \quad \end{array} \qquad \mu(b) = \begin{array}{c|c|c} & p & q \\ \hline p & \quad & \quad \\ \hline q & \quad & \quad \end{array}$$

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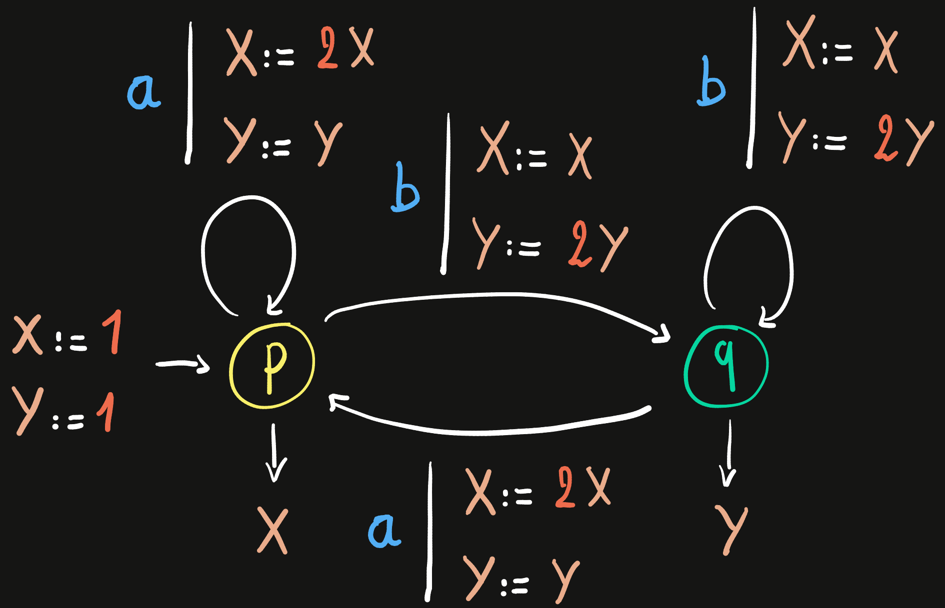

$$u = \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ (& & &) \\ x & y & x & y \end{array}$$

$$v = \begin{array}{cc|c} x & & \multirow{2}{*}{p} \\ y & & \\ \hline x & & \multirow{2}{*}{q} \\ y & & \end{array}$$

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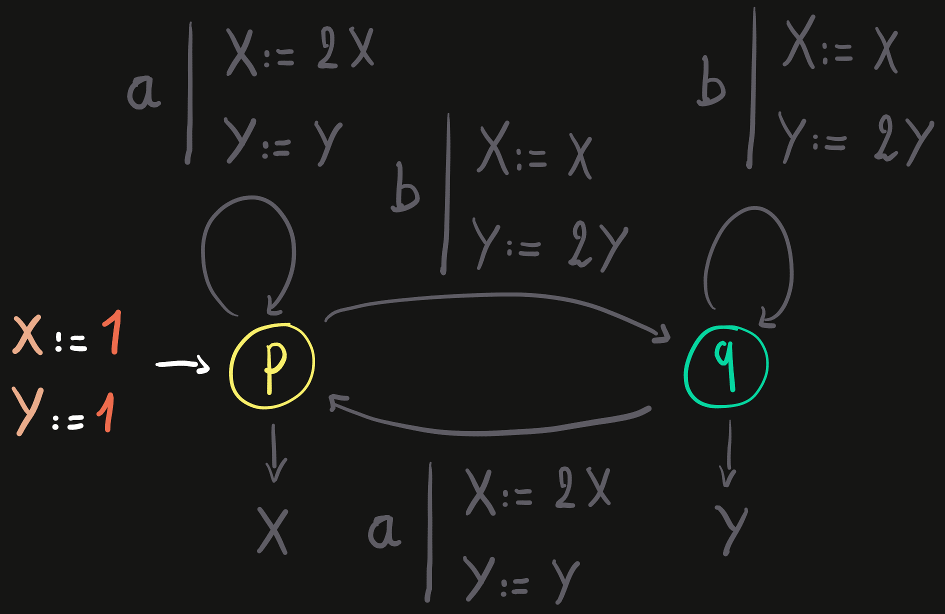

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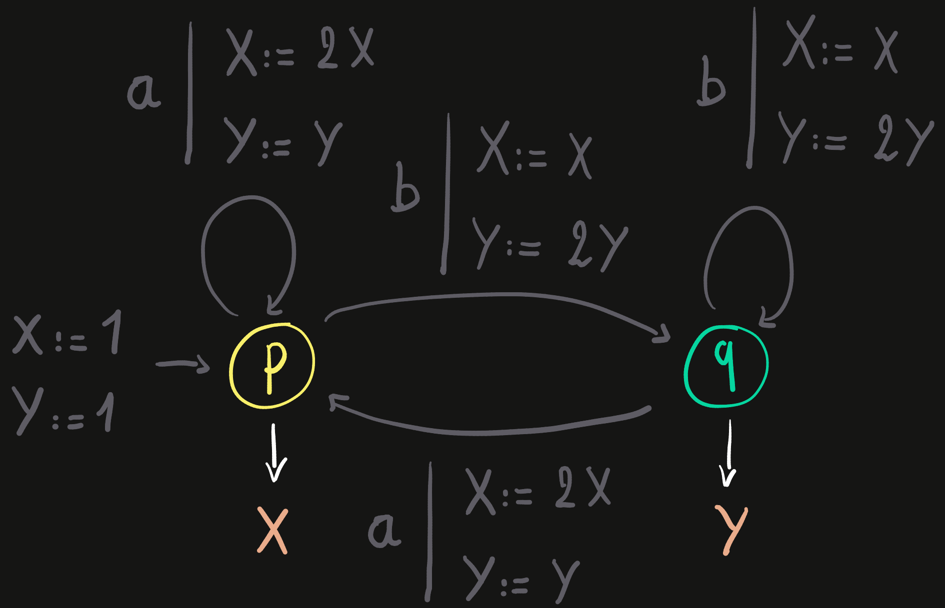

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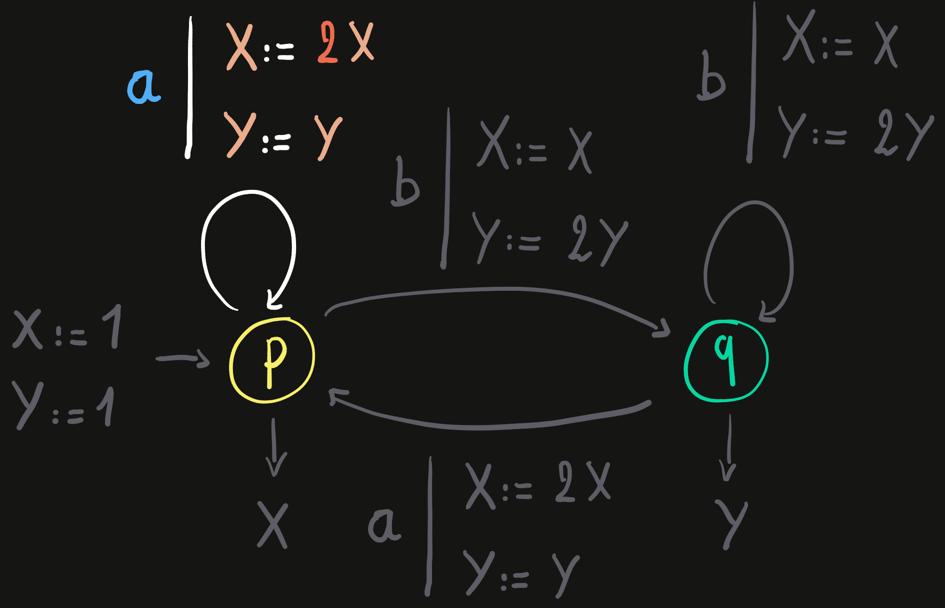

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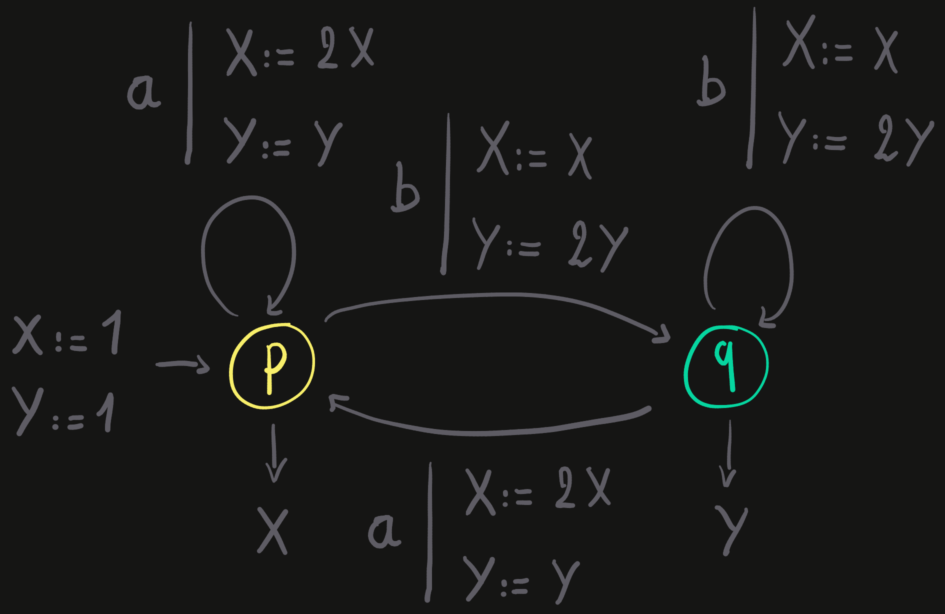

$$u = \left(\begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ 1 & 1 & 0 & 0 \\ x & y & x & y \end{array}\right) \qquad v = \begin{array}{c|c|c} x & 1 & \\ y & 0 & p \\ \hline x & 0 & \\ y & 1 & q \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array} \qquad \mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

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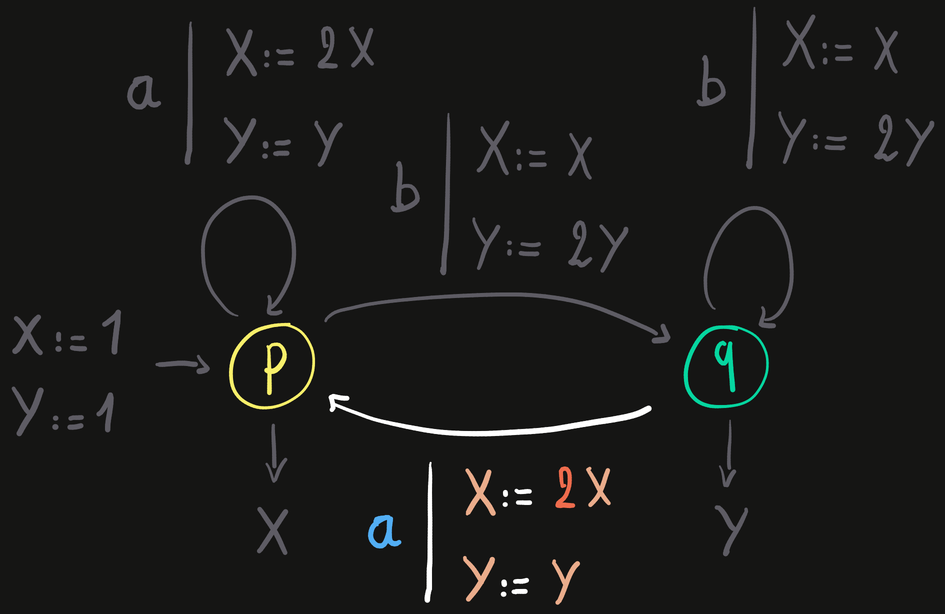


$$u = \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ 1 & 1 & 0 & 0 \\ x & y & x & y \end{array}$$

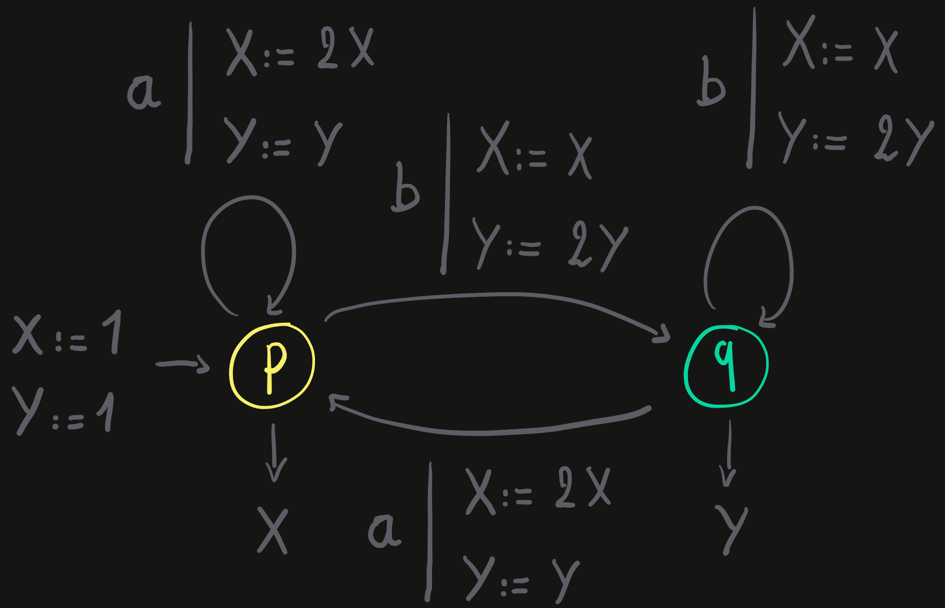
$$v = \begin{array}{c|c|c} x & 1 & \\ y & 0 & p \\ \hline x & 0 & \\ y & 1 & q \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & 2 & 0 & & & x \\ & y & 0 & 1 & & & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



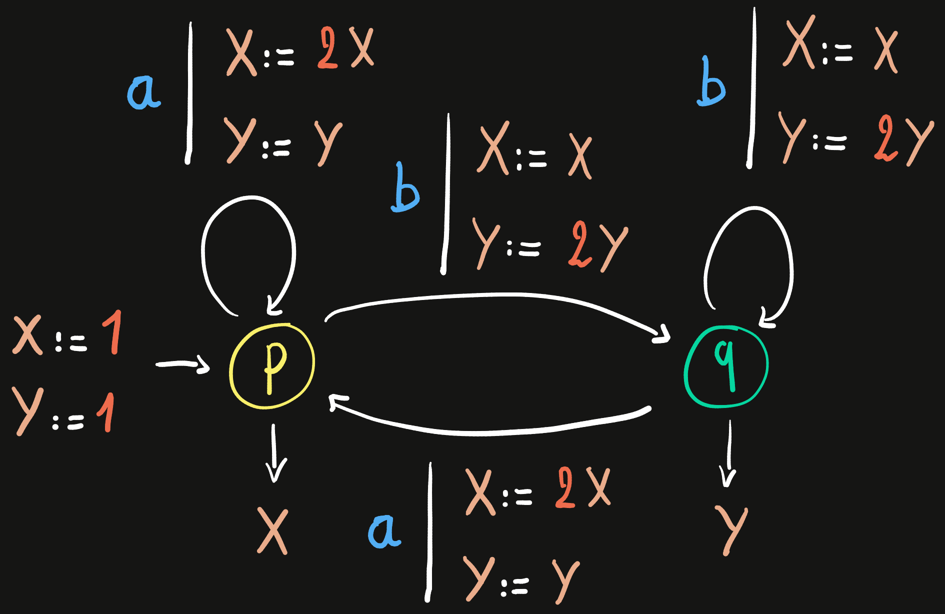
$$u = \left(\begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ 1 & 1 & 0 & 0 \\ x & y & x & y \end{array}\right) \qquad v = \begin{array}{c|c|c} x & 1 & p \\ y & 0 & \\ \hline x & 0 & q \\ y & 1 & \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & & & & & x \\ & y & & & & & y \\ \hline q & x & & & & & x \\ & y & & & & & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{cc|cc} \multicolumn{2}{c}{p} & \multicolumn{2}{c}{q} \\ 1 & 1 & 0 & 0 \\ x & y & x & y \end{array}$$

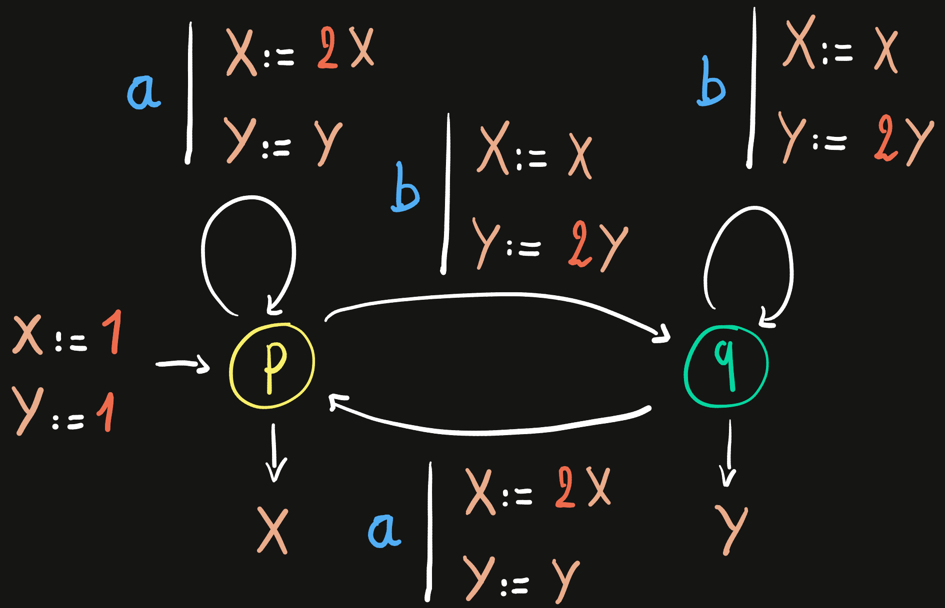
$$v = \begin{array}{c|c|c} x & 1 & \\ y & 0 & p \\ \hline x & 0 & \\ y & 1 & q \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \end{array}$$

$$\mu(b) = \begin{array}{cc|cc|cc|c} & & \multicolumn{2}{c|}{p} & \multicolumn{2}{c|}{q} & \\ & & x & y & x & y & \\ \hline p & x & 0 & 0 & 1 & 0 & x \\ & y & 0 & 0 & 0 & 2 & y \\ \hline q & x & 0 & 0 & 1 & 0 & x \\ & y & 0 & 0 & 0 & 2 & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

CRA → WA



$$u = \begin{array}{c} \begin{array}{cccc} p & & q & \end{array} \\ \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x & y & x & y \end{array} \end{array} \qquad v = \begin{array}{c|c|c} & & \\ x & 1 & p \\ y & 0 & \\ \hline x & 0 & q \\ y & 1 & \end{array}$$

$$\mu(a) = \begin{array}{cc|cc|cc|c} & & p & & q & & \\ & & x & y & x & y & \\ \hline p & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \\ \hline q & x & 2 & 0 & 0 & 0 & x \\ & y & 0 & 1 & 0 & 0 & y \end{array} \qquad \mu(b) = \begin{array}{cc|cc|cc|c} & & p & & q & & \\ & & x & y & x & y & \\ \hline p & x & 0 & 0 & 1 & 0 & x \\ & y & 0 & 0 & 0 & 2 & y \\ \hline q & x & 0 & 0 & 1 & 0 & x \\ & y & 0 & 0 & 0 & 2 & y \end{array}$$

$$w\sigma \longmapsto 2^{|w|_\sigma + 1}$$

Configurations : $(* \; * \mid 0 \; 0)$ or $(0 \; 0 \mid * \; *)$

Semilinear invariant : $\mathbb{R}^2 \times \{0\}^2 \cup \{0\}^2 \times \mathbb{R}^2$