Cumulative Default Theories vs Non-cumulative Default Logics

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Abstract

From the time that [Makinson, 89] noticed the non-cumulativity of Reiter's and Łukaszewicz's default logics, many attempts have been made in order to introduce this property in full default logic. Following [Brewka, 91], most of these attempts rely on a modification of the underlying langage and/or a weakening of cumulativity such as initialy interpreted by [Makinson, 89]. In the present paper, a different approach of cumulativity in default logics is investigated. Following a proposition of [Voorbraak, 93], and using a (trivial) cumulative variant of default logic that does neither require commitment to justifications nor any extension of the langage (cf. [Risch, 95]), a criterion is proposed to distinguish cumulative Reiter's and Łukaszewicz's default theories from non-cumulative ones, for both credulous and skeptical reasonings. This offers a different insight on the way a nonmonotonic inference relation can be associated with a given default theory.

1 Introduction

Cumulativity was introduced by [Gabbay, 85] as an interesting formal option for nonmonotonicity. Roughly, a cumulative agent is supposed to be complete in the sense that, although some part of his beliefs become verified as theorems, the previous state of his beliefs and the new one remain identical. Cumulativity is associated with attractive semantics and should improve nonmonotonic theorem provers by allowing the use of lemmas. From the time that [Makinson, 89] noticed the non-cumulativity of Reiter's and Łukaszewicz's default logics, different attempts have been made in order to introduce this property in full default logic. Following [Brewka, 91], most of these attempts ([Schaub, 91], [Dix, 92], [You, Li, 94]) have been associated with a reinterpretation of default rules due to [Poole, 88]. The basic idea is to require *commitment* to justifications, so that, instead of simply reasoning with lack of given information, explicit assumptions must be done for deriving extensions. Hence, the initial meaning of a default rule is deeply modified. This modification has the advantage of both restoring cumulativity (in various forms) and providing a solution to the "broken arms" paradox (see [Brewka, 91]). However, it appears not to be suitable in *every* case. Now, it was also pointed out by [Brewka, 91] and [Schaub, 92] that cumulativity actually does not rely on commitment. However, cumulative default logics without commitment to justifications have received little attention, with the noteworthy exception of few recent works (cf. [Wilson, 93] and [Giordano, Martelli, 94]).

Another, possibly more relevant point, is that different possible "cumulativities" can be considered in default logic. There are not only different ways of defining a nonmonotonic consequence operation (e.g. skeptical, credulous), but also there are different ways of understanding what "adding a formula" means. Since default theories are not homogeneous, some authors prefer to interpret it as naturally adding a classical formula (cf. [Makinson, 89], [Dix, 92]), whereas other authors reinterpret it as adding a default (cf. [Schaub, 91], [Schaub, 92]). However, in the latter case a slightly more complicated apparatus has to be considered. Anyway, both approaches suggest changing the nature of a formula. It can be pointed out that this aspect is lacking in the current abstract studies about cumulativity (cf. [Makinson, 89], [Kraus et Al., 90]). In these studies, there is no consideration of the possible repercussions due to the change of status of formulas moved from the right to the left of the inference relation sign (which in some sense may be interpreted as turning *belief* into *knowledge*). This aspect is also lacking in the studies of cumulative default logics based on the extension of the underlying language to assertions (cf. [Brewka, 91], [Makinson, 91], [Giordano, Martelli, 94]).

In this paper, a different approach to cumulativity in default logic is investigated. Using a cumulative vari-

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ant of default logic that does neither require commitment to justifications nor any extension of the langage, and in which both non-cumulative approaches of Reiter and Lukaszewicz are embedded, a criterion is proposed to distinguish cumulative Reiter's and Lukaszewicz's default theories from non-cumulative ones, for both credulous and skeptical reasonings. Following [Voorbraak, 93], this provides a "filter" on "well formed" theories regarding cumulativity. The main interest of this approach is to allow us to characterize the family of nonmonotonic relations associated with a given default theory in a different way. Instead of considering a given nonmonotic relation *a priori*, a tool for constructing this relation is given: the type of reasoning in consideration is achieved regarding local knowledge (i.e. the defaults). In some sense, a bottom-up approach is considered here, opposed to the top-down approaches (i.e. imposing a given non-monotonic inference relation) considered so far.

Our paper is organized in the following way: in the second section, the characterizations of four variants default logics are given. The third section gives special attention to a hierarchy of cumulativities in default logics, with a discussion on previous approaches and a result conjectured by [Delgrande et Al., 94]. In the fourth section, cumulative default theories are characterized regarding credulous and skeptical reasoning.

2 Default theories

As defined by [Reiter, 80], a closed default theory is a pair (W, D) where W is a set of closed first order sentences and D a set of default rules. A default rule has the form $\frac{\alpha:\beta}{\gamma}$ where α, β and γ are closed first order sentences. α is called the prerequisite, β the justification and γ the consequent of the default. $PREREQ(D)$, $JUST(D)$ and $CONS(D)$ are respectively the sets of all prerequisites, justifications and consequents that come from defaults in a set D. Whenever one of these sets is a singleton, we may identify it with the single element it contains. For instance, we prefer to consider $PREREQ(\{\frac{\alpha:\beta}{\gamma}\})$ as an element rather than a set. The following definition shows us how the use of a default is related to its prerequisite (cf. [Schwind, 90]):

Definition 1 *[Schwind, 90] A set D of defaults is* grounded *in W iff for all $d \in D$ there is a finite sequence $d_0, \ldots, d_k$ of elements of D such that (1) $PREREQ(\{d_0\}) \in Th(W)$, (2) for $1 \leq i \leq k-1$, $PREREQ(\{d_{i+1}\}) \in Th(W \cup CONS(\{d_0, \ldots, d_i\}))$, and $d_k = d$.*

An *extension* of a default theory is usually defined as a smallest fixed point of a set of formulas. It contains W, is deductively closed, and the defaults whose consequents belong to the extension verify a property which actually allows them to be used. The manner in which this property is considered is related to the variant of Default Logic under consideration. In what follows, we move directly to the characterizations given by [Risch, 95] and [Schaub, 95] for the extensions in the sense of [Reiter, 80], [Lukaszewicz, 88],[Risch, 95], [Schaub, 91] respectively. The first are called *R-extensions* (for Reiter's extensions), the second *j-extensions* (for justified extensions), the third *g-extensions* (for guess extensions), and the fourth *c-extensions* (for constrained extensions).

Theorem 1 *Let $\Delta = (W, D)$ be a default theory. Let D' and D'' be subsets of D.*

- *$E = Th(W \cup CONS(D'))$ is a R-extension of Δ iff D' is a maximal grounded subset of D such that $(\forall\beta \in JUST(D'))$ $(\neg\beta \notin E)$, and for each default $d \in D \setminus D'$, of the form $\frac{\alpha:\beta}{\gamma}$, either $\alpha \notin E$ or $\neg\beta \in E$.*
- *$E = Th(W \cup CONS(D'))$ is a j-extension of Δ with respect to $F = JUST(D')$ iff D' is a maximal grounded subset of D such that $(\forall\beta \in JUST(D'))(\neg\beta \notin E)$.*
- *$E = Th(W \cup CONS(D'))$ is a g-extension of Δ with respect to $JUST(D'')$ iff D' is a maximal grounded subset of D'' and D'' is a maximal subset of D such that $(\forall\beta \in JUST(D''))(\neg\beta \notin Th(W \cup CONS(D'')))$. D'' is called the* support *of E.*
- *$E = Th(W \cup CONS(D'))$ is a c-extension of Δ with respect to $C = Th(W \cup JUST(D') \cup CONS(D'))$ iff D' is a maximal grounded subset of D such that $E \cup JUST(D')$ is consistent.*

Remark:

- Clearly, the only difference between R- and j-default logics is in the behavior of the defaults that do not participate in the construction of an extension. In R- default logic, the withdrawal of these defaults from the set of generating defaults has to be motivated by checking an additional condition. The only difference between j- and g-default logics concerns the maximality of the set of grounded defaults: the j-extensions are the g-extensions of Δ such that D'' is a maximal set of grounded defaults in D. Also note that the only difference between j- and c-default logics concerns the consistency of the set of justifications related to an extension. Note that default reasoning is decidable on condition that Th is defined on a decidable language.
- Whatever is the variant under consideration (either R-, j-, g- or c-), given an extension E, the set D' is called *the set of generating defaults* of E, and is also denoted by $GD(E, \Delta)$.
- There are different ways for defining a nonmonotonic consequence relation from default theories.

The most usual are the following:

(Credulous reasoning)

$W \mid\!\sim_D^{\cup} f$ iff ($\exists E$, extension of $(W, D))(f \in E)$.

(Skeptical reasoning)

$W \mid\!\sim_D^{\cap} f$ iff ($\forall E$, extension of $(W, D))(f \in E)$.

(Choice reasoning)

$W \mid\!\sim_D^{E} f$ iff (E, extension of $(W, D))(f \in E)$.

The operator C_D, associated with the corresponding form of reasoning, is defined on the basis of the previous general pattern:

$C_D(W) = \{f \mid W \mid\!\sim_{D,s} f\}$ with $s \in \{\cup, \cap, E\}$.

3 Cumulative default logics: a brief account

Following [Makinson, 89], and given A and B some sets of formulas and Cn a nonmonotonic consequence operator, cumulativity can be expressed by:

$$A \subseteq B \subseteq Cn(A) \Rightarrow Cn(A) = Cn(B).$$

That is, a cumulative agent keeps his beliefs when one of them becomes true.

Let $\Delta = (W, D)$ be a default theory. Cumulativity in default logic is interpreted by [Makinson, 89] as the adding of a classical formula to W i.e.:

$$W \subseteq W \cup \{f\} \subseteq C_D(W) \Rightarrow C_D(W) = C_D(W \cup \{f\})$$

that is:

$$f \in C_D(W) \Rightarrow C_D(W) = C_D(W \cup \{f\}).$$

In this sense, neither Reiter's default logic nor Lukaszewicz's is cumulative as shown by [Makinson, 89]: let $\Delta = (W, D)$ with $W = \emptyset, D = \{\frac{:a}{a}, \frac{a \vee b : \neg a}{\neg a}\}$. Since there are only normal defaults (i.e. defaults with justification equal to the consequent), R- and j-extensions coincide. Δ has only one extension: $E^1 = Th(\{a\})$. Since $C_D(W) = Th(\{a\})$ (no matter whether it is defined skeptically or credulously), $(a \vee b) \in C_D(W)$. But adding $\{a \vee b\}$ to W modifies $C_D(W)$ since we have to consider now $\Delta_{\{a \vee b\}} = (W \cup \{a \vee b\}, D)$ which has two extensions: $E^1_{\{a \vee b\}} = Th(\{a\})$, $E^2_{\{a \vee b\}} = Th(\{\neg a\})$.

In order to introduce cumulativity in default logic, [Brewka, 91] followed by [Makinson, 91] and [Giordano, Martelli, 94] resorts to the notion of assertional default theories. On one hand this approach is an improvement since an assertion, being a quasi-default formula, seems to correspond to a homogenization of the initial formalism. However since the justification-part of an assertion is not logicaly closed, this can lead to an unatural behaviour when considering its adding to a default theory. On the other hand with the extension of First Order formulas to assertions, a modification of how to interpret cumulativity in assertional default logic (called CDL) is achieved. Consider first this last point. The definition of cumulativity in CDL, such as proposed by Brewka is[1]: If there is a CDL extension F of a default theory (W, D) containing an assertion f, then E is a CDL extension of (W, D) containing f iff E is a CDL extension of $(W \cup \{f\}, D)$. Let us leave undefined what an assertion is for the time being. It is easy to check that the previous definition amounts to considering a set of nonmonotonic consequence relations $\mid\!\sim_D^{\mathcal{E}_g}$ such that:

(Extended Choice Reasoning "Weak Skeptical")
Let be $\mathcal{E}_g = \{E \mid E$, extension of $(W, D), g \in E\}$,
$W \mid\!\sim_D^{\mathcal{E}_g} f$ iff $(\forall E, E \in \mathcal{E}_g)(f \in E)$.

What is considered in the approach of Brewka is a generalization of *Choice Reasoning* regarding the side effect of the adding of an assertion to W on *other* extensions containing the *same* assertion[2]. Cumulativity is then considered as usual, but with the exception that it is defined regarding one nonmonotonic consequence relation for each extension. Unlike [Makinson, 89], default reasoning is here considered as a process for generating a *family* of nonmonotonic consequence relations, rather than *one* nonmonotonic logic. This is quite reasonable. But it also has to be noticed that if formulas instead of assertions are used in the definition of $\mid\!\sim_D^{\mathcal{E}_g}$, both Reiter's and Łukaszewicz's approaches are cumulative as well. Hence, regarding cumulativity, the only difference between [Brewka, 91] and both Reiter's and Lukaszewicz's default logics concerns *skeptical* reasoning.

Now let us come back to the first point, that is the extension of First Order formulas to assertions. An assertion is any expression of the form $\langle p : J \rangle$ where, roughly speaking, J is a set of formulas supporting the belief in p. Note that whereas $\langle p : J \rangle$ expresses the belief in p supported by J, at least it is not the same as the belief in p expressed by $\langle p : K \rangle$ (although this does not mean that one assertion should be stronger than the other). Consider now the question of adding an assertion to a default theory regarding cumulativity. What is usually shown is that given any extension of a default theory (W, D) containing the assertion $\langle p : J \rangle$, E is an extension of (W, D) containing $\langle p : J \rangle$ iff E is an extension of $(W \cup \{\langle p : J \rangle\}, D)$. But nothing is said in the case where instead of introducing in W the assertion $\langle p : J \rangle$ contained in a given extension of the default theory, we introduce $\langle p : K \rangle$. In other words, what happens if an expression previously considered as a certain kind of belief turns to be another kind of belief? Indeed, should the case $K = \emptyset$ be considered as a special case? Besides, this problem also concerns

[1]See Proposition 2.13, p. 191 of [Brewka, 91]

[2]Note also the dual property *(Extended Choice Reasoning "Weak Credulous")* obtained by replacing $\forall$ with $\exists$ in "Weak Skeptical". This property was not studied by Brewka.

the syntax dependency of the sets of supports: e.g. the assertions $\langle p : \{a, a \vee b\}\rangle$ and $\langle p : \{a\}\rangle$ so far are not considered as equivalent. On the other hand there is some ambiguity concerning *what* is added regarding cumulativity in the framework of [Makinson, 89]. It remains unanswered whether this ambiguity is an advantage or not. However, [Schaub, 92] noticed that since adding the assertion $\langle p : J\rangle$ eliminates all the extensions that are inconsistent with this assertion (e.g. the default theory $(\emptyset, \{\frac{: a}{a}, \frac{: \neg a}{\neg a}\})$) it appears stronger than the adding of a simple belief.

In order to avoid a modification of the language, Schaub [Schaub, 91] introduces *lemmata default rules* which, on the other hand, involve an adaptation of cumulativity. Actually, cumulativity in default logic was interpreted *a priori* from the adding of a classical formula to W by [Makinson, 89] (see above). But it is worth noting that a default is a *contextual* inference rule since its application depends on the formulas which belong to it. In other words, a default is an intermediate form between a single formula and a whole inference rule. [Schaub, 91] makes the most of this remark by reinterpreting cumulativity as the adding of a default to D. However, the form of this default, called *lemmata default rule* depends on the variant under consideration. We propose here a unique form of lemmata default rule for the variants defined above:

Definition 2 *A default proof D_f of f in E is a minimal grounded subset of $GD(E, \Delta)$ such that $W \cup \{CONS(D_f)\} \vdash f$.*

Property 1 *Let $f \in E$, and D_f be a default proof of f. Cumulativity in the sense of [Schaub, 91] holds for the above variants of default logic (R-, j-, g-, and c-) when using the following lemmata default rule:*

$$d_f = \frac{\bigwedge_{d \in D_f} CONS(\{d\}) : \top}{f}.$$

The previous results lead to consider a first partial ordering relation among cumulativities in default logics, regarding what kind of abuction (a formula or a lemmata default rule) is considered.

$$\begin{array}{ccc} \vdash f & \longrightarrow & \vdash \bigwedge_{d \in D_f} CONS(\{d\}) \rightarrow f \\ \downarrow & & \downarrow \\ \frac{\top : \top}{f} & \longrightarrow & \frac{\bigwedge_{d \in D_f} CONS(\{d\}) : \top}{f} \end{array}$$

The top element of this lattice corresponds to the strongest way of adding an element in a default theory, regarding cumulativity. In other words, if a given variant of default logic is cumulative regarding the adding of this element, then it is also cumulative regarding the adding of the weaker elements of the lattice. Besides, a second ordering relation can be established regarding what kind of reasoning is considered. :

$$Cred \longrightarrow Skep \longrightarrow Weak\ Skep \longrightarrow Choice$$

A hierarchy of cumulativities is then defined from the product of the two previous lattices. Note that whatever assertions or lemmata default rules are considered, their adding is weaker than the adding of a first order formula in W, such as considered by [Makinson, 89]. Also commitment to justifications is not suitable in all cases. Let us illustrate these points with the following two examples. The first one directly concerns a simple problem of knowledge representation regarding commitment to justifications. In the second example, it is stressed that commitment to justifications may involve an undesirable result with respect to cumulativity. Besides, it illustrates the fact that adding a formula to W has a different meaning than adding either a default or an assertion.

Example 1 Let us consider the following default theory: $\Delta = (W, D)$ with

$$\begin{aligned} W &= \{\text{HIKE}\}, \\ D &= \{\frac{\text{HIKE : GOOD-WEATHER-FORECAST}}{\text{TAKE-SUNGLASSES}}, \\ &\quad \frac{\text{HIKE : } \neg\text{GOOD-WEATHER-FORECAST}}{\text{TAKE-JACKET}}\}. \end{aligned}$$

Since constraint default logic requires commitment to justifications, Δ has the two following extensions

$$\begin{aligned} E^1 &= Th(\{\text{HIKE, TAKE-SUNGLASSES}\}), \\ E^2 &= Th(\{\text{HIKE, TAKE-JACKET}\}). \end{aligned}$$

Now we are forced to choose one of the two extensions, and hence to gamble on the weather. But it should be stressed that (1) we *do not know* anything about the weather forecast and (in lack of any actual information) we probably prefer to leave this unknown, (2) two *contrary* but not necessarily *contradictory* actions are considered (taking sunglasses or taking a jacket). Here both Reiter's and Lukaszewicz's default logics have only one extension $E = Th(\{\text{HIKE, TAKE-SUNGLASSES, TAKE-JACKET}\})$ which seems more suitable.

Example 2 Couples are invited to a party. The corresponding default theory is $\Delta_W = (W, D)$ where

$$\begin{aligned} W = \{ & \text{COUPLE}(\text{Bogart, Bacall}), \text{COUPLE}(\text{Romeo, Juliet}), \\ & \text{COUPLE}(\text{Charles, Diana}), \\ & \text{BOTH}(x, y) \leftrightarrow (\text{PRESENT}(x) \wedge \text{PRESENT}(y)), \\ & \text{EITHEROR}(x, y) \leftrightarrow (\text{PRESENT}(x) \vee \text{PRESENT}(y))\} \end{aligned}$$

$$\begin{aligned} D = \{ & \frac{\text{COUPLE}(x, y) : \text{BOTH}(x, y)}{\text{BOTH}(x, y)}, \\ & \frac{\text{COUPLE}(x, y) \wedge \text{EITHEROR}(x, y) : \neg\text{BOTH}(x, y)}{\neg(\text{BOTH}(x, y)}\}. \end{aligned}$$

Let us stress the following points: (1) W denotes *actual* knowledge; (2) the first default expresses our *a* *priori* hope that couples should come; (3) the second default expresses the idea that we may have to consider the case where only one half of a given couple is present. Since the defaults are normal, R- and j-extensions coincide. So, let us simply speak of extensions, as opposed to c-extensions. In the present state the theory has only one extension:

$$E_W = Th(W \cup \{\text{PRESENT(Bogart)}, \text{PRESENT(Bacall)}, \text{PRESENT(Romeo)}, \text{PRESENT(Juliet)}, \text{PRESENT(Charles)}, \text{PRESENT(Diana)}\}.$$

This theory is not cumulative: PRESENT(Diana) belongs to E_W as a belief, but if added to W as a fact, the new theory

$$\Delta_{W \cup \{\text{PRESENT(Diana)}\}} = (W \cup \{\text{PRESENT(Diana)}\}, D)$$

has two extensions:

$$E_{W \cup \{\text{PRESENT(Diana)}\}} = E_W,$$
$$E'_{W \cup \{\text{PRESENT(Diana)}\}} = Th(W \cup \{\text{PRESENT(Bogart)}, \text{PRESENT(Bacall)}, \text{PRESENT(Romeo)}, \text{PRESENT(Juliet)}, \neg\text{PRESENT(Charles)}, \text{PRESENT(Diana)}\}).$$

Note that in this example both constrained and Reiter's lemmata default rules corresponding to PRESENT(Diana) are the same, i.e.

$$d_{\text{PRESENT(Diana)}} = \frac{: \text{PRESENT(Charles)} \wedge \text{PRESENT(Diana)}}{\text{PRESENT(Diana)}}.$$

The only c-extension of the theory $\Delta_{D \cup \{d_{\text{PRESENT(Diana)}}\}}$ is E_W (with $d_{\text{PRESENT(Diana)}}$ as lemmata default rule). But also $\Delta_{D \cup \{d_{\text{PRESENT(Diana)}}\}}$ has E_W as the only non-contrained extension (either R- or j-). Clearly whatever is the variant considered, adding $d_{\text{PRESENT(Diana)}}$ (or the corresponding assertion (see [Giordano, Martelli, 94]) is weaker than adding PRESENT(Diana) to W. However, regarding the presence of new information (i.e. Diana *is* present, but we still know nothing about Charles) the last approach seems more realistic (for lack of being definitively optimistic about the presence of Charles). Let us note that the solutions given here to the non-cumulativity of default logics are such that they *restrict* the set of generated extensions. On the contrary, two g-extensions are generated from the beginning:

$$E_1 = Th(W \cup \{\text{PRESENT(Bogart)}, \text{PRESENT(Bacall)}, \text{PRESENT(Romeo)}, \text{PRESENT(Juliet)}, \text{PRESENT(Charles)}, \text{PRESENT(Diana)}\},$$
$$E_2 = Th(W).$$

Since $C_D(\emptyset) = Th(W)$, cumulativity trivially holds. It is shown by [Risch, 95] that g-default logic embeds both R- and j-default logics and that it is skepticaly cumulative in the sense of [Makinson, 89]. However, it is clear that this kind of cumulativity may be considered of little interest since in most cases the intersection of the g-extensions is nothing else than just $Th(W)$.

Our idea here is to use cumulative g-default logic as a technical tool in the study of cumulativity for reputed non-cumulative variants of default logic. A first step in this direction is the following result, conjectured in [Delgrande et Al., 94], which is a direct consequence of theorem 1 (since the j-extensions of any prerequisite free default theory are g-extensions):

Corollary 1 *The restriction of Lukaszewicz's variant of default logic to prerequisite free default theories is cumulative regarding skeptical reasoning.*

The next section is devoted to the complete characterization of the conditions both R- and j-default logics should satisfy to be cumulative.

4 Cumulative default theories

As told by [Voorbraak, 93], "it cannot be inferred from the rationality of cumulative monotonicity[3] that any nonmonotonic logic formalizing the non-monotonic reasoning of an ideally rational agent has to be cumulative. There is an analogy here with consistency: although an ideally rational agent only believes a consistent set of formulas, we do not have to require that the logic[4] $\mathbf{L}$ under which the beliefs are closed is consistent in the sense that for all" set of formulas Σ of the langage $\mathcal{L}_\mathbf{L}$, $Cn_\mathbf{L}(\Sigma) \neq \mathcal{L}_\mathbf{L}$. "An inconsistent set Σ will be revised before it will become accepted by a rational agent, and this revision process is not described by $\mathbf{L}$, but by operations as studied in Gärdenfors (...). Similarly, a rational agent will revise his default beliefs if they do not give rise to rational preferences and this revision process does not have to be described by the nonmonotonic consequence operation." Following this idea, our intent is to require cumulativity to be the property of a default theory rather than the property of a default logic. We get the following definition:

Definition 3 *Let $\Delta = (W, D)$, Δ is said cumulative iff cumulativity holds regarding $C_D(W)$.*

In what follows, we are interested in the characterization of which default theories in the sense of Lukaszewicz and Reiter are cumulative and which are not. First, note that whatever is the variant under consideration, g-default logic allows us to get rid of those defaults which can never be used in the generation of new extensions.

Theorem 2 *Given a default theory $\Delta = (W, D)$ and any default $d = \frac{\alpha : \beta}{\gamma}$ of D, d is not in any support of any g-extension iff $\neg\beta \in Th(W \cup \{\gamma\})$ (i.e. d is never fired).*

[3] i.e. cautious monotony

[4] i.e. a relation among formulas

So, let us consider the defaults which are involved in the construction of g-extensions but do not generate j-extensions. In what follows, $\chi \in \{R, j\}$ is used to denote either Reiter's or Lukaszewicz's approach. Meanwhile, $s \in \{\cup, \cap\}$ is used to denote either credulous or skeptical forms of reasoning. $C^{\chi}_{D,s}(W)$ is then defined regarding the type of reasoning associated with both χ and s.

Definition 4 *Let $\Delta = (W, D)$ be a default theory. $GD(C^{\chi}_{D,s}(W))$ is the set of generating defaults used in the construction of $C^{\chi}_{D,s}(W)$.*

Definition 5 *Let $\Delta = (W, D)$ be a default theory. The* difference set of defaults *for Δ, $\mathcal{DS}(\chi)\Delta$, is defined by the difference between the union of all supports of the g-extensions of Δ and $GD(C^{\chi}_{D,\cup}(W))$.*

So, the following criterions hold for cumulative default theories in the sense of Reiter and Łukaszewicz, regarding skeptical and credulous reasonings:

Theorem 3 *A default theory Δ is cumulative regarding $C^{\chi}_{D,\cap}(W)$ iff for any $\frac{\alpha:\beta}{\gamma} \in \mathcal{DS}(\chi)\Delta$, ($\gamma \notin C^{\chi}_{D,\cup}(W)$ and $\alpha \in C^{\chi}_{D,\cap}(W) \Rightarrow \neg\beta \in C^{\chi}_{D,\cap}(W \cup \{\alpha\})$).*

Theorem 4 *A default theory Δ is cumulative regarding $C^{\chi}_{D,\cup}(W)$ iff*

(i) for any $\frac{\alpha:\beta}{\gamma} \in \mathcal{DS}(\chi)\Delta$, ($\gamma \notin C^{\chi}_{D,\cup}(W)$ and $\alpha \in C^{\chi}_{D,\cup}(W) \Rightarrow \neg\beta \in C^{\chi}_{D,\cap}(W \cup \{\alpha\})$);

(ii) for any $\frac{\alpha:\beta}{\gamma} \in GD(C^{\chi}_{D,\cup}(W))$, ($\neg\beta \in C^{\chi}_{D,\cup}(W) \Rightarrow \gamma \in C^{\chi}_{D,\cup}(W \cup \{\neg\beta\})$).

Example 3 Consider $\Delta = (\emptyset, D)$ with

$$D = \{\frac{c:\neg a \wedge \neg b}{d}, \frac{:\neg a}{a \vee b}, \frac{:a}{a}\}.$$

Δ has three g-extensions: $E^1 = Th(\emptyset)$, $E^2 = Th(\{a \vee b\})$, $E^3 = Th(\{a\})$. Only E^2 and E^3 are j-extensions. Only E^3 is a R-extension.

- $C^{j}_{D,\cup}(\emptyset) = Th(\{a\})$, $C^{j}_{D,\cap}(\emptyset) = Th(\{a \vee b\})$, $\mathcal{DS}(j)(\emptyset, D) = \{\frac{c:\neg a \wedge \neg b}{d}\}$. Δ is cumulative regarding both $C^{j}_{D,\cup}$ and $C^{j}_{D,\cap}$.
- $C^{R}_{D,\cup}(\emptyset) = C^{R}_{D,\cap}(\emptyset) = Th(\{a\})$, $\mathcal{DS}(R)(\emptyset, D) = \{\frac{c:\neg a \wedge \neg b}{d}, \frac{:\neg a}{a \vee b}\}$. Δ is cumulative regarding both $C^{R}_{D,\cup}$ and $C^{R}_{D,\cap}$.

5 Conclusion

By moving down cumulativity from default logics to default theories, the type of nonmonotonic reasoning induced by a given default theory can be characterized. This gives a different insight on how to consider cumulativity in multiple extensions logics (opposed to single extension logics, e.g. preferential models). Instead of considering a given nonmonotic relation *a priori*, a tool for constructing this relation is given: the type of reasoning in consideration is achieved regarding local knowledge (i.e. the defaults). Note that, regarding the way a nonmonotonic inference relation can be associated with a given default theory, the second alternative represents a bottom-up approach opposed to the top-down approaches (i.e. imposing a given non-monotonic inference relation) considered so far. A further step would be to extend this work in the framework of a more abstract study about multiple extensions logics.

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