Cumulative Default Logics Revisited

Vincent Risch*

The University of Western Ontario
Department of Computer Science
Middlesex College
London, Ontario, Canada N6A 5B7
e-mail : risch@csd.uwo.ca

Abstract

Following [Brewka, 91], most attempts for introducing cumulativity in full default logic are associated with the consistency of the set of justifications used for deriving an extension. But this modification of the original formalism of Reiter is not always suitable. Actually, it is known that cumulativity does not rely on commitment to justifications. However, it is only recently that cumulative default logics without commitment to justifications has received a real attention[1]. In the present paper, a cumulative variant of default logic that does not require commitment to justifications is introduced for skeptical reasoning in the sense of [Makinson, 89]. It can be noted that both Reiter's and Lukaszewicz's approaches are embedded in it, and that no extension of the underlying language is required (contrary to [Brewka, 91]). Indeed, can be distinguished the default theories in the sense of Lukaszewicz that are cumulative from those which are not using this framework. By reconsidering the approach of [Schaub, 91] the present work also tries to focus on the difference between two kinds of cumulativity allowed by default logics.

1 Introduction

Cumulativity was introduced by [Gabbay, 85] as an interesting formal option for nonmonotonicity. Roughly, a cumulative agent is supposed to be complete in the sense that, although some part of his beliefs become verified as theorems, the previous state of his beliefs and the new one remain identical. Cumulativity is associated with attractive semantics and should improve nonmonotonic theorem provers by allowing the use of lemmas. From the time that [Makinson, 89] noticed the non-cumulativity of Reiter's and Lukaszewicz's default logics, different attempts have been made in order to introduce this property in full default logic. Following [Brewka, 91], most of these attempts ([Schaub, 91], [Dix, 92], [You, Li, 94]) have been associated with a reinterpretation of default rules

*Supported by a grant from the *Natural Science and Engineering Research Council of Canada*.

[1] While finishing this paper (corresponding to a part of his PhD thesis, see [Risch, 93]), the author became aware of the works of [Wilson, 93] and [Giordane, Martelli, 94]. The first one describes a variant of default logic close to the one presented here, but with a different concern. The second one has just become available at the same time that the present paper was submitted.

due to [Poole, 88]. The basic idea is to require *commitment* to justifications, so that, instead of simply reasoning with lack of given information, explicit assumptions must be done for deriving extensions. Hence, the initial meaning of a default rule is deeply modified. This modification has the advantage of both restoring cumulativity (in various forms) and providing a solution to the "broken arms" paradox (see [Brewka, 91]). However, it appears not to be suitable in *every* case. Now, it was also pointed out by [Brewka, 91] and [Schaub, 92] that cumulativity actually does not rely on commitment. However, cumulative default logics without commitment to justifications has received little attention, with the noteworthy exception of some very recent work (cf. [Wilson, 93] and [Giordane, Martelli, 94]).

Different possible "cumulativities" can be considered in default logic. There are not only different ways of defining a nonmonotonic consequence operation (e.g. skeptical, credulous), but also there are different ways of understanding what "adding a formula" means. Since default theories are not homogeneous, some authors prefer to interpret it as naturally adding a classical formula (cf. [Makinson, 89], [Dix, 92]), whereas other authors reinterpret it as adding a default (cf. [Schaub, 91], [Schaub, 92]). However, in the latter case a slightly more complicated apparatus has to be considered. Anyway, both approaches suggest changing the nature of a formula. It can be pointed out that this aspect is lacking in the current abstract studies about cumulativity (cf. [Makinson, 89], [Kraus & Al., 90]). In those studies, formulas change their status from *belief* to *knowledge* without any consideration to the possible repercussions due to this change. This aspect also is also lacking in the studies of cumulative default logics based on the extension of the underlying language to assertions (cf. [Brewka, 91], [Makinson, 91], [Giordane, Martelli, 94]).

In this paper, a different approach to cumulativity in default logic is investigated. Following [Dix, 92], it focuses on a skeptical approach, but it also should be easily extendible to a credulous approach à la [Schaub, 92]. No extension of the first order language is required. Since no commitment between justifications is required, both Reiter's and Lukaszewicz's approaches are embedded in it, and can be easily characterized. Indeed, it distinguish those default theories in the sense of Lukaszewicz which are cumulative from those which are not. Following [Voorbrak, 93], this provides a "filter" on "well formed" theories regarding cumulativity. Although the semantical aspect is not studied, it is noteworthy that fixed-point theories (see Reiter's and Lukaszewicz's formal presentation) can probably be described inside a preferential framework (that is the semantic counterpart of cumulativity)[2].

Our paper is organized in the following way: in the second section, basic abstract properties about cumulativity are briefly recalled. In the third section, the characterizations of three main default logics are given. The fourth section gives special attention to two different approaches of cumulativity in default logics. In the fifth section, commitment to justifications is discussed. In section six, guess default logic is introduced and its basic properties are given.

2 Cumulativity

Definition 1 ([Makinson, 89], [Kraus & Al., 90]) *Let $\mid\!\sim$ be a nonmonotonic consequence relation, A a set of formulas, f and g any formulas;* Cut *and* Cautious Monotony *are defined by:*

$$\textit{(Cut)} \quad \frac{A \cup \{f\} \mid\!\sim g, \quad A \mid\!\sim f}{A \mid\!\sim g}$$

$$\textit{(Cautious Monotony)} \quad \frac{A \mid\!\sim f, \quad A \mid\!\sim g}{A \cup \{f\} \mid\!\sim g}$$

[2]But this was already noticed by [Siegel, Schwind, 93]

Property 1 ([Makinson, 89], [Kraus & Al., 90]) *Cut and Cautious Monotony together can be expressed jointly by* Cumulativity*:*

$$\textit{If } A \mid\!\sim f \textit{ then } (A \mid\!\sim g \textit{ iff } A \cup \{f\} \mid\!\sim g).$$

Definition 2 *A set of nonmonotonic consequences derived from a set A of formulas is defined by*

$$C(A) = \{f \mid A \mid\!\sim f\}.$$

Remark: Following [Makinson, 89], Cut and Cautious Monotony can be expressed infinitistically by:

$$\begin{array}{ll} \textit{(Cut)} & A \subseteq B \subseteq C(A) \Rightarrow C(B) \subseteq C(A) \\ \textit{(Cautious Monotony)} & A \subseteq B \subseteq C(A) \Rightarrow C(A) \subseteq C(B) \end{array}$$

So, we get:

$$\textit{(Cumulativity)} \quad A \subseteq B \subseteq C(A) \Rightarrow C(A) = C(B)$$

That is, a cumulative agent keeps his beliefs when one of them becomes true.

3 Default theories

As defined by [Reiter, 1980], a closed default theory is a pair (W, D) where W is a set of closed first order sentences and D a set of default rules. A default rule has the form $\frac{\alpha : \beta}{\gamma}$ where α, β and γ are closed first order sentences. α is called the prerequisite, β the justification and γ the consequent of the default. $PREREQ(D)$, $JUST(D)$ and $CONS(D)$ are respectively the sets of all prerequisites, justifications and consequents that come from defaults in a set D. Whenever one of these sets is a singleton, we may identify it with the single element it contains. For instance, we prefer to consider $PREREQ(\{\frac{\alpha : \beta}{\gamma}\})$ as an element rather than a set. The following definition shows us how the use of a default is related to its prerequisite (cf. [Schwind, 90]):

Definition 3 ([Schwind, 90]) *A set D of defaults is* grounded *in W iff for all $d \in D$ there is a finite sequence $d_0, \ldots, d_k$ of elements of D such that (1) $PREREQ(\{d_0\}) \in Th(W)$, (2) for $1 \leq i \leq k-1$, $PREREQ(\{d_{i+1}\}) \in Th(W \cup CONS(\{d_0, \ldots, d_i\}))$, and $d_k = d$.*

An *extension* of a default theory is usually defined as a smallest fixed point of a set of formulas. It contains W, is deductively closed, and the defaults whose consequents belong to the extension verify a property which actually allows them to be used. The manner in which this property is considered is related to the variant of Default Logic under consideration. In what follows, we directly give the characterizations previously obtained by Camilla Schwind and the author for the extensions in the sense of [Reiter, 1980], [Lukaszewicz, 88], [Schaub, 91] respectively. The first are called *R-extensions* (for Reiter's extensions), the second *j-extensions* (for justified extensions), and the third *c-extensions* (for constrained extensions).

Theorem 1 *Let $\Delta = (W, D)$ be a default theory.*

- *[Schwind, Risch, 91], [Risch, 93] E is an R-extension for Δ iff there is a D' a grounded subset of D, such that $E = Th(W \cup CONS(D'))$, and $\forall d \in D, d = \frac{\alpha : \beta}{\gamma}$:*

(i) If $d \in D'$ *then* $\alpha \in E$ *and* $\neg\beta \notin E$,

(ii) If $d \notin D'$ *then* $\alpha \notin E$ *or* $\neg\beta \in E$.

- *[Risch, 91]* E *is a j-extension with respect to* F *for* Δ *iff there is a* D'*, a maximal grounded subset of* D*, such that* $E = Th(W \cup CONS(D'))$, $F = JUST(D')$, *and* $\forall d \in D'$, $d = \dfrac{\alpha : \beta}{\gamma}$:

  (i) If $d \in D'$ *then* $\alpha \in E$ *and* $\neg\beta \notin E$.

- *[Risch, 93]* E *is a c-extension with respect to* C *for* Δ *iff there is a* D'*, a maximal grounded subset of* D*, such that* $E = Th(W \cup CONS(D'))$, $C = Th(W \cup JUST(D') \cup CONS(D'))$, *and* $\forall d \in D'$, $d = \dfrac{\alpha : \beta}{\gamma}$:

  (i') If $d \in D'$ *then* $\alpha \in E$ *and* $E \cup JUST(D')$ *is consistent.*

Since the belonging of prerequisites to a given extension is obviously covered by the groundedness of the corresponding set of defaults, we get:

Corollary 1 *Let* $\Delta = (W, D)$ *be a default theory. Let* $D' \subseteq D$.

- $E = Th(W \cup CONS(D'))$ *is a j-extension of* Δ *with respect to* $F = JUST(D')$ *iff* D' *is a maximal grounded subset of* D*, such that* $(\forall\beta \in JUST(D'))(\neg\beta \notin E)$.
- $E = Th(W \cup CONS(D'))$ *is a c-extension of* Δ *with respect to* $C = Th(W \cup JUST(D') \cup CONS(D'))$ *iff* D' *is a maximal grounded subset of* D*, such that* $E \cup JUST(D')$ *is consistent.*

Remark:

- Clearly, both conditions *(i)* and *(ii)* imply the maximality of D'. Hence, the only difference between Reiter's and Lukaszewicz's approach is in the behavior of the defaults that do not participate in the construction of an extension. These defaults have to verify condition *(ii)* in Reiter's default logic. The present characterization sheds light on the intuition which stands behind Lukaszewicz's variant. In the latter, we are never allowed to revise a justification used for deriving the consequent of a default. According to Lukaszewicz, it is only in this precise way that we may speak of "justification" in a correct sense. Indeed, note that the only difference between Lukaszewicz's and Schaub's approaches concerns the consistency of the set of justifications related to an extension. Note that default reasoning is decidable on condition that *Th* is defined on a decidable language.

- There are different ways for defining a nonmonotonic consequence relation from default theories. The most usual are the following:

  $$\begin{array}{ll} \textit{(Credulous reasoning)} & W \mid\!\sim_{D,\cup} f \text{ iff } (\exists E)(E, \text{ extension of } (W,D))(f \in E) \\ \textit{(Skeptical reasoning)} & W \mid\!\sim_{D,\cap} f \text{ iff } (\forall E)(E, \text{ extension of } (W,D))(f \in E) \end{array}$$

  The operator C_D, associated with the corresponding form of reasoning, is defined on the base of the previous general pattern:

  $$C_D(W) = \{f \mid W \mid\!\sim_{D,s} f\} \text{ with } s \in \{\cup, \cap\}.$$

4 Different approaches of cumulativity in default logic

Let $\Delta = (W, D)$ be a default theory. Cumulativity in default logic is interpreted by [Makinson, 89] as the adding of a classical formula to W i.e.:

$$\textit{(Cumulativity)} \quad W \subseteq W \cup \{f\} \subseteq C_D(W) \Rightarrow C_D(W) = C_D(W \cup \{f\})$$

that is:

$$f \in C_D(W) \Rightarrow C_D(W) = C_D(W \cup \{f\}).$$

In this sense, neither Reiter's default logic nor Lukaszewicz's one is cumulative as shown by [Makinson, 89]: let $\Delta = (W, D)$ with $W = \emptyset, D = \{\frac{: a}{a}, \frac{a \vee b : \neg a}{\neg a}\}$. Since there are only normal defaults (i.e. defaults with justification equal to the consequent), R- and j-extensions coincide. Δ has only one extension: $E^1 = Th(\{a\})$. Since $C_D(W) = Th(\{a\})$ (no matter whether it is defined skeptically or credulously), $a \vee b \in C_D(W)$. But adding $\{a \vee b\}$ to W modifies $C_D(W)$ since we have to consider now $\Delta_{\{a \vee b\}} = (W \cup \{a \vee b\}, D)$ which has two extensions: $E^1_{\{a \vee b\}} = Th(\{a\})$, $E^2_{\{a \vee b\}} = Th(\{\neg a\})$.

In order to introduce cumulativity in default logic, [Brewka, 91] followed by [Makinson, 91], and [Giordane, Martelli, 94] resorts to the notion of assertional default theories, but this involves a modification of the underlying language. On the one hand such a modification is an improvement since an assertion, being a quasi-default formula, corresponds to an homogenization of the initial formalism. But on the other hand this transformation (1) may involve a limitation of the scope of cumulativity as defined in [Makinson, 89], (2) leads to a non-intuitive behavior when considering the abduction of a belief to the theory. An assertion is any expression of the form $\langle p : J \rangle$ where, roughly speaking, J is a set of formulas supporting the belief in p. Note that whereas $\langle p : J \rangle$ expresses the belief in p supported by J, at least it is not the same as the belief in p expressed by $\langle p : K \rangle$ (although this does not mean that one assertion should be stronger than the other). Consider point (1), that is cumulativity in assertional default logics. What is usually shown is that given any extension of a default theory (W, D) containing the assertion $\langle p : J \rangle$, E is an extension of (W, D) containing $\langle p : J \rangle$ iff E is an extension of $(W \cup \{\langle p : J \rangle\}, D)$. But nothing is told in the case where instead of introducing in W the assertion $\langle p : J \rangle$ contained in a given extension of the default theory, we introduce $\langle p : K \rangle$. In other words, what happens if an expression previously considered as a certain kind of belief turns to be an other kind of belief? Indeed, should the case $K = \emptyset$ be considered as a special case? On the other hand there is some ambiguity concerning *what* is added regarding cumulativity in the framework of [Makinson, 89]. It remains unanswered whether this ambiguity is an advantage or not. Point (2) is a consequence of a remark due to [Schaub, 92]. Since adding the assertion $\langle p : J \rangle$ eliminates all the extensions that are inconsistent with the assertion, it appears stronger than the abduction of a simple belief, which seems to go against the original intuition.

In order to avoid a modification of the language, [Schaub, 91] introduces *lemmata default rules* which, on the other side, involve an adaptation of cumulativity. Actually, cumulativity in default logic was interpreted *a priori* from the adding of a classical formula to W by [Makinson, 89] (see above). But it is noteworthy that a default is a *contextual* inference rule since its application depends on the formulas which belong to it. In other words, a default is an intermediate form between a single formula and a whole inference rule. [Schaub, 91] makes the most of this remark by reinterpreting cumulativity as the adding of a default to D, although this turns out to require a slightly more complicated apparatus. Given a default theory $\Delta = (W, D)$, (1) let us define a new operator $\cup_D$ by:

$$(W, D) \cup_D \{f\} =_{\text{def}} (W, D \cup \{d_f\})$$

where d_f is a default corresponding to the formula f; (2) C_D is now extended to a new operator C on whole default theories:

$$C((W,D)) = \{f \mid (W,D) \mid\!\sim_D f\}$$

Now, we are able to redefine the usual cumulativity by the following *D-cumulativity*:

$$\textit{(D-cumulativity)} \quad f \in C(\Delta) \Rightarrow C(\Delta) = C(\Delta \cup_D \{d_f\})$$

In what follows the term "cumulativity" is used to refer to the usual cumulativity[3], as opposed to "D-cumulativity". Note that whereas D-cumulativity corresponds to the abduction of a belief, usual cumulativity should now be interpreted as the abduction of a fact.

5 Cumulativity versus commitment to justifications

Schaub's variant is D-cumulative in the following way:

Definition 4 *Let* $\mid\!\sim_D$ *be the* credulous *nonmonotonic relation defined from* $\Delta = (W,D)$ *by*

$$(W,D) \mid\!\sim_D f \text{ iff } (\exists E)(E, \text{ c-extension of } (W,D))(f \in E).$$

The set of nonmonotonic consequences derived from (W,D) *is:* $C((W,D)) = \{f \mid (W,D) \mid\!\sim_D f\}$.

Definition 5 ([Schaub, 91]) *Let* $\Delta = (W,D)$ *be a default theory, and* E *be a c-extension of* Δ. *Let* $f \in E$, *and* D_f *be a minimal set of defaults such that* $W \cup \{CONS(\{d\}) \mid d \in D_f\} \vdash f$. *A* lemmata default rule *for* f *is the following default:*

$$d_f = \frac{: \bigwedge_{d\in D_f} JUST(\{d\}) \wedge \bigwedge_{d \in D_f} CONS(\{d\})}{f}.$$

Property 2 ([Schaub, 91]) *D-cumulativity holds for* $\mid\!\sim_D$.

Example 1[Makinson, 89] Let $\Delta = (W,D)$ with $W = \emptyset, D = \{\frac{:a}{a}, \frac{a \vee b : \neg a}{\neg a}\}$. Δ has only one c-extension: $E^1 = Th(\{a\})$. Since $a \vdash a \vee b$, we may add the lemmata default rule $\frac{:a}{a \vee b}$ to D. $\Delta_{\{\frac{:a}{a\vee b}\}} = (W, D \cup \{\frac{:a}{a \vee b}\})$ has the same c-extension as $\Delta = (W,D)$: $E^1_{\{\frac{:a}{a\vee b}\}} = Th(\{a\})$.

Remark:

- [Dix, 92] establishes *usual* cumulativity for a *skeptical* relation based on a twofold application of a fixed-point operator on default theories with commitment to justifications (that is, constrained default theories), whereas [You, Li, 94] establishes usual cumulativity for a *credulous* relation with respect to prerequisite-free constraint default theories.
- Actually, it is easy to show that D-cumulativity as defined above also holds for Lukaszewicz's default logic. [Schaub, 92] also defines D-cumulativity for Reiter's default theories. In this case, a lemmata default rule is any non-singular default

$$d_f = \frac{: JUST(\{d_1\}), \ldots, JUST(\{d_n\})}{f},$$

where f and D_f are defined as above, with $D_f = \{d_1, \ldots, d_n\}$.

[3] or, in other words, "W-cumulativity"...

- Note that there are other ways of defining D-cumulativity. For instance, it is easy to prove that D-cumulativity holds for all the above variants of default logic when using the following lemmata default rule:
$$d_f = \frac{: \bigwedge_{d \in D_f} CONS(\{d\})}{f},$$
where f and D_f are defined as in definition 5.

Commitment to justifications is not suitable in all cases. Let us illustrate this with the following two examples. The first one directly concerns a simple problem of knowledge representation. In the second example, it is stressed that commitment to justifications may involve an undesirable result with respect to cumulativity.

- Let us consider the following default theory: $\Delta = (W, D)$ with $W = \{\text{hike}\}$, $D = \{\frac{\text{hike} : \text{good-weather-forecast}}{\text{take-sunglasses}}, \frac{\text{hike} : \neg\text{good-weather-forecast}}{\text{take-jacket}}\}$. Since constraint default logic requires commitment to justifications, Δ has the two following extensions $E^1 = Th(\{\text{hike}, \text{take-sunglasses}\})$, $E^2 = Th(\{\text{hike}, \text{take-jacket}\})$. Now we are forced to choose one of the two extensions, and hence to gamble on the weather. But it should be stressed that (1) we *do not know* anything about the weather forecast and (in lack of any actual information) we probably prefer to leave this unknown, (2) two *contrary* but not necessarily *contradictory* actions are considered (taking sunglasses or taking a jacket). Here both Reiter's and Lukaszewicz's default logics have only one extension $E = Th(\{\text{hike}, \text{take-sunglasses}, \text{take-jacket}\})$ which seems more suitable.

- Couples are invited to a party. The corresponding default theory is $\Delta_W = (W, D)$ where
$$\begin{aligned} W &= \{\text{COUPLE}(\text{Bogart}, \text{Bacall}), \text{COUPLE}(\text{Romeo}, \text{Juliet}), \text{COUPLE}(\text{Charles}, \text{Diana})\} \\ D &= \{\frac{\text{COUPLE}(x, y) : \text{PRESENT}(x) \wedge \text{PRESENT}(y)}{\text{PRESENT}(x) \wedge \text{PRESENT}(y)}, \\ &\quad \frac{\text{COUPLE}(x, y) \wedge (\text{PRESENT}(x) \vee \text{PRESENT}(y)) : \neg(\text{PRESENT}(x) \wedge \text{PRESENT}(y))}{\neg(\text{PRESENT}(x) \wedge \text{PRESENT}(y))}\}. \end{aligned}$$
Let us stress the following points: (1) W denotes *actual* knowledge; (2) the first default expresses our *a priori* hope that couples should come; (2) the second default expresses the idea that we may have to consider the case where only one half of a given couple is present. Since the defaults are normal, R- and j-extensions coincide. So, let us simply speak of extensions, as opposed to c-extensions. In the present state the theory has only one extension:
$$\begin{aligned} E_W = \; & Th(W \cup \{\text{PRESENT(Bogart)}, \text{PRESENT(Bacall)}, \text{PRESENT(Romeo)}, \text{PRESENT(Juliet)}, \\ & \text{PRESENT(Charles)}, \text{PRESENT(Diana)}\}). \end{aligned}$$
This theory is not cumulative: PRESENT(Diana) belongs to E_W as a belief, but if added to W as a fact, the new theory
$$\Delta_{W \cup \{\text{PRESENT(Diana)}\}} = (W \cup \{\text{PRESENT(Diana)}\}, D)$$
has two extensions:
$$\begin{aligned} E_{W \cup \{\text{PRESENT(Diana)}\}} &= E_W, \\ E'_{W \cup \{\text{PRESENT(Diana)}\}} &= Th(W \cup \{\text{PRESENT(Bogart)}, \text{PRESENT(Bacall)}, \\ & \quad \text{PRESENT(Romeo)}, \text{PRESENT(Juliet)} \\ & \quad \neg\text{PRESENT(Charles)}, \text{PRESENT(Diana)}\}) \end{aligned}$$

whereas the only c-extension of the corresponding theory $\Delta_{D\cup\{\frac{:\mathrm{PRESENT(Charles)} \wedge \mathrm{PRESENT(Diana)}}{\mathrm{PRESENT(Diana)}}\}}$ is E_W (with $\frac{:\mathrm{PRESENT(Charles)} \wedge \mathrm{PRESENT(Diana)}}{\mathrm{PRESENT(Diana)}}$ as lemmata default rule). However, regarding the presence of new information (i.e. Diana *is* present, but we still know nothing about Charles) the noncumulative version seems more realistic (for lack of being definitively optimistic about the presence of Charles). Maybe, it could be argued that a different formulation of the theory Δ_W would allow the expected extensions in a "D-cumulative" way. Let us point out that on the one hand cumulativity does not change anything about the formulation of a default theory; on the other hand, it is here defined in order to *restrict* the set of generated extensions.

6 Guess-default logic

We introduce now our variant, called *guess default logic*. Let us stress the following intuitive ideas:

- W denotes *actual* knowledge (corresponding to the "world" of [Reiter, 1980]);
- any prerequisite denotes a *local context* in which a corresponding default should be applied;
- any justification corresponds to a *plausible assumption* related both to a context and derived consequences of defaults;
- any consequence of a default is considered as a *guess* depending on both a context and the plausible assumptions.

Our idea is to get sets of guesses from maximal sets of plausible assumptions. So, we first try to put forward as many plausible assumptions as possible provided that they are consistent with the corresponding conclusions attached to them (i.e. the consequents of the corresponding default rules). In this way, pre-sets of guesses are obtained which are sound with respect to plausible assumptions. In order to get the actual guess, we keep only those which hold regarding the current contexts. More formally:

Definition 6 *Let $\Delta = (W, D)$ be a default theory. Let D'' and D' be any sets of defaults such that $D'' \subseteq D' \subseteq D$. $E = Th(W \cup CONS(D''))$ is a g-extension*[4] *of Δ with respect to $JUST(D')$ iff D' is a maximal subset of D such that $(\forall \beta \in JUST(D'))(\neg\beta \notin Th(W \cup CONS(D')))$ and D'' is a maximal grounded subset of D'.*

Definition 7 *Let $\Delta = (W, D)$ be a default theory. Let $E = Th(W \cup CONS(D''))$ be a g-extension of Δ with respect to $JUST(D')$.*

- *The set $JUST(D')$ is called* the set of plausible assumptions supporting *E, and D' itself is called the* support *of E.*
- *The set $PREREQ(D'')$ is called* the set of actual contexts of *E, and D'' itself is called the set of* generating defaults *of E.*

[4] for guess-extension...

Example 2 The default theory $(\emptyset, \{\frac{c:\neg a \wedge \neg b}{d}, \frac{:\neg a}{a \vee b}, \frac{:a}{a}\})$ has three g-extensions:

$$\begin{aligned} E^1 &= Th(\emptyset) \text{ under the plausible assumption } \{\neg a \wedge \neg b\}, \\ E^2 &= Th(\{a \vee b\}) \text{ under the plausible assumption } \{\neg a\}, \\ E^3 &= Th(\{a\}) \text{ under the plausible assumption } \{a\}. \end{aligned}$$

Note that $E^1 \subseteq E^2 \subseteq E^3$.

Example 3 The default theory $(\{a\}, \{\frac{a:b}{b}, \frac{c:d}{e}, \frac{a:\neg e}{b}\})$ has two g-extensions:

$$\begin{aligned} E^1 &= Th(\{a, b\}) \text{ under the plausible assumptions } \{b, d\}, \\ E^2 &= Th(\{a, b\}) \text{ under the plausible assumptions } \{b, \neg e\}. \end{aligned}$$

Since $E^1 = E^2$, it can also be considered that Δ has only one g-extension but with possibly two different sets of defaults as supports.

A g-extension is defined regarding a pre-extension for which the justification conditions hold. In this way, any g-extension is contained in a g-extension that could appear after the abduction of a previously derived formula. This is precisely the key point of our approach: by giving up the idea of systematically getting maximal extensions we are able to recover cumulativity in default reasoning.

Definition 8 *Let be* $\mid\!\sim_D$ *be the* skeptical *nonmonotonic relation defined from* $\Delta = (W, D)$ *by*

$$W \mid\!\sim_D f \text{ iff } (\forall E)(E, \text{ g-extension of } (W, D))(f \in E).$$

The set of nonmonotonic consequences derived from W *is:* $C_D(W) = \{f \mid W \mid\!\sim_D f\}$.

Given a default theory Δ, and in order to establish that cumulativity holds for $\mid\!\sim_D$, it is necessary to consider how adding a formula f to W affects the behaviour of defaults. In the following lemma, we show that adding $\{f\}$ to W actually does not change the supports initially obtained from Δ.

Lemma 1 *Let* $\Delta = (W, D)$ *be a default theory, and* f *a formula. Let us call* $\mathcal{D}_W$ *and* $\mathcal{D}_{W \cup \{f\}}$ *the sets of supports of the g-extensions of* (W, D) *and* $(W \cup \{f\}, D)$ *respectively. We have:*

$$f \in C_D(W) \Rightarrow (\mathcal{D}_W = \mathcal{D}_{W \cup \{f\}}).$$

Proof : Let D' be any element of $\mathcal{D}_W$. So D' is a maximal subset of D such that $(\forall \beta \in JUST(D'))(\neg\beta \notin Th(W \cup CONS(D'))$. Now, since $f \in C_D(W)$, $Th(W \cup CONS(D')) = Th(W \cup \{f\} \cup CONS(D'))$, hence $D' \in \mathcal{D}_{W \cup \{f\}}$, i.e. $\mathcal{D}_W \subseteq \mathcal{D}_{W \cup \{f\}}$. Conversely, any $D'_f \in \mathcal{D}_{W \cup \{f\}}$ is a maximal subset of D such that $(\forall \beta \in JUST(D'_f))(\neg\beta \notin Th(W \cup \{f\} \cup CONS(D'_f))$. Since $f \in C_D(W)$, $Th(W \cup \{f\} \cup CONS(D'_f)) = Th(W \cup CONS(D'_f))$, hence $D'_f \in \mathcal{D}_W$, i.e. $\mathcal{D}_{W \cup \{f\}} \subseteq \mathcal{D}_W$. *Q.E.D.*

Property 3 *Cautious Monotony holds for* $\mid\!\sim_D$.

Proof : Let $C_D(W)$ be the intersection of all the g-extensions of a default theory $\Delta = (W, D)$, as defined above, and let f be a formula of $C_D(W)$. We would like to show $C_D(W) \subseteq C_D(W \cup \{f\})$. So given any default $d = \frac{\alpha : \beta}{\gamma}$ such that $\gamma \in C_D(W)$ we have to show that $\gamma \in C_D(W \cup \{f\})$. Let us assume, on the contrary, that this does not hold, i.e. $\gamma \notin C_D(W \cup \{f\})$. Following lemma 1, the support to which d belongs does not change, hence this means that $\{d\}$ is grounded in W but is no longer grounded in $W \cup \{f\}$. This is a contradiction since (by monotonicity of Th) a set of grounded default in W cannot decrease when adding f to W. *Q.E.D.*

Property 4 *Cut holds for* $\mid\!\sim_D$.

Proof : Similar to the proof of property 3: as previously, let f be a formula of $C_D(W)$. We would like to show $C_D(W \cup \{f\}) \subseteq C_D(W)$. That is, for any default $d = \frac{\alpha : \beta}{\gamma}$ such that $\gamma \in C_D(W \cup \{f\})$ we have to show that $\gamma \in C_D(W)$. Let us assume, on the contrary, that this does not hold, i.e. $\gamma \notin C_D(W)$. Again, by lemma 1, the support to which d belongs does not change hence this means that $\{d\}$ is grounded in $W \cup \{f\}$, but is not grounded in W. Since $f \in C_D(W)$, this turns out to be a contradiction. *Q.E.D.*

Corollary 2 $\mid\!\sim_D$ *is cumulative.*

Example 4 [Makinson, 89] Let $\Delta = (W, D)$ with $W = \emptyset, D = \{\frac{: a}{a}, \frac{a \vee b : \neg a}{\neg a}\}$. Δ has two g-extensions:

$$
\begin{aligned}
E^1 &= Th(\{a\}) \text{ under the plausible assumption } \{a\}, \\
E^2 &= Th(\emptyset) \text{ under the plausible assumption } \{\neg a\}.
\end{aligned}
$$

The only guess generated under the assumption $\{\neg a\}$ are the tautologies (i.e. the content of W) since the corresponding context does not hold. Since $C_D(\emptyset) = \emptyset$, cumulativity trivially holds. Note that $\Delta_{\{a \vee b\}} = (W \cup \{a \vee b\}, D)$ still has two g-extensions:

$$
\begin{aligned}
E^1_{\{a \vee b\}} &= Th(\{a\}) \text{ with } \{a\} \text{ as support}, \\
E^2_{\{a \vee b\}} &= Th(\{\neg a\}) \text{ with } \{\neg a\} \text{ as support}.
\end{aligned}
$$

$\Delta_{\{a\}} = (W \cup \{a\}, D)$ keeps $E^1_{\{a\}}$ as the only g-extension. Because there is more precise actual knowledge in W, fewer conjectures are possible. We have $E^i \subseteq E^i_{\{a \vee b\}}$, for $i \in \{1, 2\}$, and $E^1_{\{a \vee b\}} \subseteq E^1_{\{a\}}$. The lack of an extension $E^2_{\{a\}}$ can be considered as an extreme attempt to preserve consistency, when trying to apply the default $\frac{a \vee b : \neg a}{\neg a}$ facing the actual knowledge $W \cup \{a\}$.

The previous example concerning couples behaves in the same way. From the beginning two g-extensions are generated, associated with possible guesses regarding the actual knowledge of W. Actually, the process involved by the generation of the g-extensions of a default theory corresponds to the opposite way for obtaining the c-extensions of this theory. Instead of removing the "too many" extensions, we try to initially generate the "missing" ones in order to establish cumulativity. Since we do not require commitment to the justification, we avoid any restriction on the reasoning of a cumulative agent. Indeed, we now have a cumulative approach of default logic in which two main non-cumulative approaches are embedded:

Theorem 2 *Let $\Delta = (W, D)$ be a default theory, and let us consider $E = Th(W \cup CONS(D''))$ with $D'' \subseteq D$. E is a j-extension of Δ iff E is a g-extension of Δ such that D'' is a maximal set of generating defaults.*

Proof :

($\Rightarrow$) straightforward, using corollary 1.

($\Leftarrow$) Let $E = Th(W \cup CONS(D''))$ be a g-extension with D' as support, such that $D'' \subseteq D' \subseteq D$. That is, D'' is any maximal grounded subset of D', D' is maximal in D such that $(\forall \beta \in JUST(D'))(\neg\beta \notin Th(W \cup CONS(D')))$, and finally, D'' is a maximal set of generating defaults. Let us assume that E is not a j-extension. Hence, from corollary 1, D'' cannot be a maximal subset of D such that both following conditions hold:

- D'' is grounded in W, *and*
- $(\forall \beta \in JUST(D''))(\neg\beta \notin Th(W \cup CONS(D'')))$.

Since $D'' \subseteq D'$, this is a contradiction!

Q.E.D.

Example 5 Let be $D = \{\frac{c : \neg a \wedge \neg b}{d}\}$ and $D' = \{\frac{: \neg a}{a \vee b}, \frac{: a}{a}\}$. The default theory $(\emptyset, D)$ has only one g-extension which also is a j-extension: $E = Th(\emptyset)$. The default theory$(\emptyset, D \cup D')$ has three g-extensions: $E^1 = E = Th(\emptyset)$, $E^2 = Th(\{a \vee b\})$, $E^3 = Th(\{a\})$. Only E^2 and E^3 are j-extensions. Note that $E^1 \subseteq E^2 \subseteq E^3$.

Hence:

- any j-extension is a g-extension;
- any R-extension is a g-extension (since following theorem 1 it is also a j-extension).

Guess default logic is cumulative whereas Lukaszewicz's default logic is not in general. Now, considering a given default theory $\Delta = (W, D)$, let us say that Δ itself is cumulative iff cumulativity holds regarding $C_D(W)$. In what follows, we are interested in the characterization of which among the default theories in the sense of Lukaszewicz are cumulatives from which are not. First, let us consider the defaults which are involved in the construction of g-extensions but do not generate j-extensions.

Definition 9 *Let $\Delta = (W, D)$ be a default theory. Let $\mathcal{E}_j$ and $\mathcal{E}_g$ be respectively the set of the j-extensions and the set of the g-extensions of Δ. The* difference set of defaults *for Δ, $\mathcal{DS}(\Delta)$, is defined by the union of all the $D' \setminus D''$ such that, for any $E \in \mathcal{E}_g \setminus \mathcal{E}_j$, D' is the support of E and D'' is the set of generating defaults of E.*

So, the following criteria holds for cumulative default theories in the sense of Lukaszewicz:

Property 5 *Let $\Delta = (W, D)$ be a default theory and let $C_D(W)$ be defined skeptically regarding the j-extensions of Δ. Let $\mathcal{DS}(\Delta)$ be the difference set of defaults for Δ. The default theory Δ understood in the sense of Lukaszewicz is cumulative iff for any default $d \in \mathcal{DS}(\Delta)$, $d = \frac{\alpha : \beta}{\gamma}$, $(\alpha \in C_D(W) \Rightarrow \neg\beta \in C_D(W \cup \{\alpha\}))$.*

Proof : First note that if $\mathcal{DS}(\Delta) = \emptyset$, the property obviously holds. Hence assume $\mathcal{DS}(\Delta) \neq \emptyset$.

($\Rightarrow$) Let us assume Δ to be cumulative. Consider $d \in \mathcal{DS}(\Delta)$, $d = \frac{\alpha : \beta}{\gamma}$, with $\alpha \in C_D(W)$ and $\neg\beta \notin C_D(W \cup \{\alpha\})$. But in that case, since Δ is cumulative, also $\neg\beta \notin C_D(W)$. That is d is involved in the generation of each j-extension of Δ, which contradicts $d \in \mathcal{DS}(\Delta)$.

($\Leftarrow$) Assume that for any default $d \in \mathcal{DS}(\Delta)$, $d = \frac{\alpha : \beta}{\gamma}$, $(\alpha \in C_D(W) \Rightarrow \neg\beta \in C_D(W \cup \{\alpha\}))$. That is, adding any formula f of $C_D(W)$ to W never fires d because either $\alpha \notin C_D(W)$, or $\alpha \in C_D(W)$, but in that case $\neg\beta \in C_D(W \cup \{\alpha\})$. Since no extra default can be fired by adding f to W, $C_D(W) = C_D(W \cup \{f\})$.

Q.E.D.

7 Conclusion

A cumulative variant of default logic has been introduced. On the one hand this variant matches the initial definition of cumulativity for default logic (see [Makinson, 89]), on the other hand Reiter's and Lukaszewicz's approaches are embedded in it. Indeed this variant can be used in order to select cumulative default theories in the sense of Lukaszewicz. This work is in progress and a lot of questions still require further study: it should be easy to show that guess default logic is D-cumulative, but actually we would even try to define a pure abstract link between usual W-cumulativity and D-cumulativity. It also remains to characterize cumulative Reiter default theories inside our framework. A semantical characterization (for instance in the frameworks of [Besnard, Schaub, 92] or [Voorbrak, 93]) would certainly provide an essential tool for allowing a comparison with other approaches. Finally, it will be essential for us also to provide a comparison with the work of [Giordane, Martelli, 94] which has just appeared.

Acknowledgements. The author is gratefully indebted to Christine Froidevaux, Torsten Schaub, and Camilla Schwind for their helpful comments on earlier versions of the present paper. He especially thanks Robert E. Mercer for valuable arguments, discussions and feedback.

References

[Besnard, Schaub, 92] Besnard P., Schaub T., *Possible Worlds Semantics for Default Logics*. Proceedings of the 9th Canadian Conference on Artificial Intelligence, AI-92. Glasgow, Hadley (Ed), p. 148–155.

[Brewka, 91] Brewka G., *Cumulative Default Logic: in Defense of Nonmonotonic Inference Rule*. Artificial Intelligence, 50, p. 183–205.

[Dix, 92] Dix J., *Default Theories of Poole-Type and a Method for Constructing Cumulative Versions of Default Logic*. Proceedings of the 10th European Conference on Artificial Intelligence, ECAI-92. Neumann (Ed). John Wiley & Sons, p. 289–293.

[Gabbay, 85] Gabbay D., *Theoretical foundations for non-monotonic reasoning in expert systems*. in K.R. Apt (Ed), Logics and Models of Concurrent Systems, Springer Verlag.

[Giordane, Martelli, 94] Giordane L., Martelli A., *On Cumulative Default Logics*. Artificial Intelligence, 66, p. 161–179.

[Kraus & Al., 90] Kraus S., Lehman D., Magidor M., *Nonmonotonic Reasoning, Preferential Models and Cumulative Logics*. Artificial Intelligence, 44, p. 167–207.

[Lukaszewicz, 88] Lukaszewicz W., *Considerations on Default Logic - An Alternative Approach*. Computational Intelligence 4, p. 1–16.

[Makinson, 89] Makinson D., *A General Theory of Cumulative Inference*. Non-Monotonic Reasoning. Proceedings of the second international workshop. Siekmann (Ed). Lecture Notes in Artificial Intelligence 346, Springer Verlag, p. 1–18.

[Makinson, 91] Makinson D., *unpublished manuscript*. Quoted by Brewka in his paper of 91.

[Poole, 88] Poole D., *A Logical Framework for Default Reasoning*. Artificial Intelligence, 36, p. 27–47.

[Reiter, 1980] Reiter R., *A Logic for Default Reasoning*. Artificial Intelligence, 13, 1, p. 81–132.

[Risch, 91] Risch V., 1991, *Démonstration à Base de Tableaux Sémantiques pour la Logique des Défauts au sens de Lukaszewicz*. Proceedings of the 8th conference AFCET "Reconnaissance des Formes et Intelligence Artificielle", Lyon, France, p. 355–361.

[Risch, 93] Risch V., *Les Tableaux Analytiques au Service des Logiques de Défauts*. PhD thesis, GIA, Faculté des Sciences de Luminy, Université Aix-Marseille II, France.

[Schaub, 91] Schaub T., *On Commitment and Cumulativity in Default Logics*. Proceedings of the European Conference on Symbolic and Quantitative Approaches for Uncertainty. Kruse, Siegel (Ed). Lecture Notes in Computer Science 548, Springer Verlag, p. 305–309.

[Schaub, 92] Schaub T., *On Constrained Default Theories*. Proceedings of the 10th European Conference on Artificial Intelligence, ECAI-92. Neumann (Ed). John Wiley & Sons, p. 304–308.

[Schwind, 90] Schwind C., *A Tableau-based Theorem Prover for a Decidable Subset of Default Logic*. Proceedings of the 10th International Conference on Automated Deduction, CADE 10, Springer Verlag, p. 541–546.

[Schwind, Risch, 91] Schwind C., Risch V., *A Tableau-Based Characterization for Default Logic*. Proceedings of the European Conference on Symbolic and Quantitative Approaches for Uncertainty. Kruse, Siegel (Ed). Lecture Notes in Computer Science 548, Springer Verlag, p. 310–317.

[Siegel, Schwind, 93] Siegel P., Schwind C., *A Modal Logic Based Theory for Nonmonotonic Reasoning*. Journal of Applied Nonstandard Logics, 1.

[Voorbrak, 93] Voorbrak F., *Preference-based Semantics for Nonmonotonic Logics*. Proceedings of 13th International Joint Conference on Artificial Intelligence, IJCAI-93. Bajcsy (Ed). Morgan Kaufmann Publishers, p. 584–589.

[Wilson, 93] Wilson N., *Default Logic and Dempster-Shafer Theory*. Proceedings of the European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty. Clarke, Kruse, Moral (Ed). Lecture Notes in Computer Science 747, Springer Verlag, p. 372–379.

[You, Li, 94] You J., Li L., *Two Cumulative Results On J- and PJ-Default Logics*. Proceedings of the 10th Canadian Conference on Artificial Intelligence, AI-94. Elio (Ed), p. 219–226.