Sensitivity to Synchronism of Boolean Automata Networks

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Abstract. We are interested in the way time impacts on the behaviours of interaction systems. We approach this question in a discrete framework and with a relative notion of time flow. In this paper, we focus on Boolean automata networks and the updatings of automata states in these networks. More specifically, we study how synchronous updates impact on the global behaviour of a network. Based on this, we define different sorts of network sensitivity to synchronism, which are effectively satisfied by some networks. We also relate this synchronism-sensitivity to some properties of the structure of networks and to their underlying mechanisms.

Keywords: Boolean automata network, discrete relative time, synchronism sensitivity, asynchronous transition graph, elementary transition graph.

1 Introduction

1.1 Relative time flow. Informally, the global motivation of the work we present here is to better understand the role of time in the determining of a system's behaviour. In other informal terms, it is to specify how the precise organisation in time of the execution of interdependent mechanisms influences their final output. Here, we consider a *discrete* notion of time which allows us to bypass questions about the duration of events.

It is important to start by emphasising that the notion of **time flow** that underlies the formal developments we are about to present is **relative** in two ways. First of all, it is relative to a system $\mathcal{S}$: it is **defined** with respect to $\mathcal{S}$ because it concerns only the whole set of elementary/punctual events that $\mathcal{S}$ is considered to be submitted to. Second, beyond its discrete nature, it is intrinsically relative in itself. To explain this requires to introduce the notion of **discrete synchronism**. Before that, we describe our framework.

1.2 Changes, elementary events and 'Booleanity'. Automata networks are simple mathematical objects composed of interacting entities. The entities have states that can change. They interact in the sense that their states mutually influence each other. Since these influences are supposed to be predetermined, entities are called *automata*. The local automata state changes are what defines the elementary events that the whole network is submitted to punctually.

Here, we take a fundamental point of view which makes no assumptions on what automata and automata states represent. In particular, although automata are often used to model neural and genetic networks [1, 6, 7, 9–11, 18], we are not assuming that automata in a network represent entities of the same kind, and if they do, we do not assume that their state values have the same meaning. Automata represent entities with varying states or mechanisms with varying outputs. To capture the essence of our problem requires to focus on (the possibilities of) changes rather than on the nature of changes. For this reason, we choose to concentrate on Boolean automata whose states vary between only two distinct values.

1.3 Automata, automata sets, states and configurations. By default, $\mathbf{V} = \{0, \ldots, n-1\}$ denotes a set of $n \in \mathbb{N}$ Boolean automata, also called **nodes**. We let $\mathbb{B} = \{0, 1\}$. A **configuration** of $\mathbf{V}$, or of a network $\mathcal{N}$ with automata set $\mathbf{V}$, is a Boolean vector $x \in \mathbb{B}^n$ whose component $x_i \in \mathbb{B}$ represents the **state of node** $i \in \mathbf{V}$. In this paper, special attention is paid to switches of node states starting in a given configuration, so we introduce the following notations: $\forall x = x_0 \ldots x_{n-1} \in \mathbb{B}^n$, $\forall i \in \mathbf{V}$, $\overline{x}^{\,i} = x_0 \ldots x_{i-1} \neg x_i \, x_{i+1} \ldots x_{n-1}$ and $\forall \mathrm{W} = \mathrm{W}' \uplus \{i\} \subseteq \mathbf{V}$, $\overline{x}^{\,\mathrm{W}} = \overline{(\overline{x}^{\,i})}^{\mathrm{W}'} = \overline{(\overline{x}^{\,\mathrm{W}'})}^{i}$. To switch from Boolean values to signed values, we let $\mathbf{s} : b \in \mathbb{B} \mapsto b - \neg b \in \{-1, 1\}$. Also, to compare two configurations $x, y \in \mathbb{B}^n$, we use: $\mathrm{D}(x, y) = \{ i \in \mathbf{V} \mid x_i \neq y_i \}$ and the Hamming distance $\mathrm{d}(x, y) = |\,\mathrm{D}(x, y)|$.

1.4 Networks and mechanisms. A **Boolean automata network** (BAN) $\mathcal{N}$ of size $n \in \mathbb{N}$ is a set of n Boolean functions: $\mathcal{N} = \{ \mathrm{f}_i : \mathbb{B}^n \to \mathbb{B} \mid i \in \mathbf{V} \}$ (*cf.* Fig. 1 a). f_i is the **mechanism** of $i \in \mathbf{V}$. It predetermines the possible behaviour of node i in each configuration $x \in \mathbb{B}^n$.

1.5 Structure and influences between automata. We introduce the sign of the influence of $j \in \mathbf{V}$ on $i \in \mathbf{V}$ in $x \in \mathbb{B}^n$:

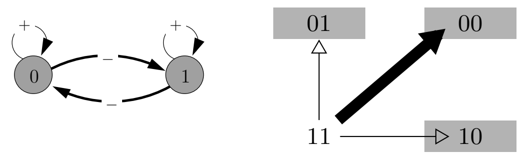
$$\mathrm{sign}_x(j, i) = \frac{\mathrm{f}_i(x) - \mathrm{f}_i(\overline{x}^{\,j})}{x_j - \overline{x}^{\,j}_j} = \mathbf{s}(x_j) \cdot (\mathrm{f}_i(x) - \mathrm{f}_i(\overline{x}^{\,j})).$$

$\mathbf{A}(x) = \{(j, i) \in \mathbf{V}^2 \mid \mathrm{sign}_x(j, i) \neq 0\}$ represents the set of influences of $\mathcal{N}$ that are effective in x. The **structure** of $\mathcal{N}$ is $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ where $\mathbf{A} = \bigcup_x \mathbf{A}(x)$ represents the set of influences of $\mathcal{N}$ (*cf.* Fig. 1 b). $\mathbf{V}^-(i)$ denotes the in-neighbourhood of $i \in \mathbf{V}$ in $\mathbf{G}$ and $\deg^-(i) = |\mathbf{V}^-(i)|$.

1.6 Monotony and signs. If $\mathrm{sign}_x(j, i)$ is constant when it is non-null, *i.e.* when $(j, i) \in \mathbf{A}(x)$, then we define its non-null value to be $\mathrm{sign}(j, i)$. In particular, for the only arc (j, i) incoming i such that $\deg^-(i) = 1$, $\mathrm{sign}(j, i)$ is defined. Generally, if an arc (j, i) is signed and positive (*i.e.* if $\mathrm{sign}(j, i) = +1$), then the state of i tends to mimic that of j. If it is negative ($\mathrm{sign}(j, i) = -1$), it tends to negate the state of j.

$\mathcal{N} = \{ \mathrm{f}_0, \mathrm{f}_1 \}$ where
$\forall x \in \mathbb{B}^n$,
$$\begin{cases} \mathrm{f}_0(x) = x_0 \wedge \neg x_1 \\ \mathrm{f}_1(x) = \neg x_0 \wedge x_1 \end{cases}$$

a.



b. c.

Fig. 1. a. A BAN of size 2. b. Its signed structure (arcs (j, i) are labelled by $\mathrm{sign}(j, i)$). c. Its ETG $\mathcal{T}$ which only differs from its ATG $\mathcal{T}_\mathrm{a}$ by 11 $\blacktriangleright$ 00. This special case of F-impact is induced by a 11-critical, Hamiltonian cycle. Its impact, precisely, consists in making reachable an unreachable attractor of $\mathcal{T}_\mathrm{a}$ (*cf.* Lemmas 2 and 3).

Here we assume that all arcs can be signed which is equivalent to assuming all f_is to be locally monotone in all components. Non-monotony is a subject that we believe to be important in itself and that we have started to study it in [12, 13]. We mention it again in §2.4.

The **sign** of a path in $\mathbf{G}$ is the product of the signs of its arcs. A positive path globally transmits "information" directly whereas a negative one transmits its negation.

1.7 Instabilities and frustrations. An node $i \in \mathbf{V}$ is **unstable** in $x \in \mathbb{B}^n$ if it belongs to the set: $\mathbf{U}(x) = \{ i \in \mathbf{V} \mid \mathrm{f}_i(x) \neq x_i \}$. It is **stable** in x if it belongs to $\overline{\mathbf{U}}(x) = \mathbf{V} \setminus \mathbf{U}(x)$. **Stable configurations** $x \in \mathbb{B}^n$ are such that $|\mathbf{U}(x)| = 0$.

Lemma 1 (loops). *$\forall i \in \mathbf{V}$, $\forall x \in \mathbb{B}^n$,*
$i \in \overline{\mathbf{U}}(x) \cap \overline{\mathbf{U}}(\overline{x}^i) \Rightarrow \mathrm{sign}(i, i) = +1$ *and* $i \in \mathbf{U}(x) \cap \mathbf{U}(\overline{x}^i) \Rightarrow \mathrm{sign}(i, i) = -1$.

An influence $(j, i) \in \mathbf{A}$ is **frustrated** if it belongs to $\textbf{FRUS}(x) = \{ (j, i) \in \mathbf{A} \mid \mathbf{s}(x_j) \cdot \mathbf{s}(x_i) = -\mathrm{sign}(j, i) \}$. Notably, it can be checked that $\exists j \in \mathbf{V}^-(i),\ (j, i) \in \textbf{FRUS}(x) \Rightarrow i \in \mathbf{U}(x)$.

1.8 Events, transitions and transition graphs. An **elementary transition** of a BAN $\mathcal{N}$ is a couple of configurations $(x, y) \in \mathbb{B}^n \times \mathbb{B}^n$, noted $x \longrightarrow y$, satisfying: $\emptyset \neq \mathrm{D}(x, y) \subseteq \mathbf{U}(x)$. In this framework, the punctual **events** experienced by a BAN $\mathcal{N}$ correspond precisely to punctual state changes of one or several of its nodes. Thus, the set of elementary transitions denoted by T represents the set of all punctual events that $\mathcal{N}$ may be submitted to in each of its configurations. Digraph $\mathcal{T} = (\mathbb{B}^n, \mathrm{T})$ is called the **elementary transition graph** (ETG). It describes the **behaviour** of $\mathcal{N}$, that is, all of its possible evolutions (*e.g.* Fig. 1 c).

The **size of an elementary transition** $x \longrightarrow y$ equals $\mathrm{d}(x, y)$. If $\mathrm{d}(x, y) = 1$ (resp. $\mathrm{d}(x, y) > 1$), then $x \longrightarrow y$ is called **asynchronous** (resp. **synchronous**) and written $x \longrightarrow\!\!\!\triangleright\ y$ (resp. $x \longrightarrow\!\!\!\blacktriangleright\ y$). The set of asynchronous transitions is noted T_a. Digraph $\mathcal{T}_\mathrm{a} = (\mathbb{B}^n, \mathrm{T}_a)$ is called the **asynchronous transition graph** (ATG). It represents only those events that $\mathcal{N}$ can undergo which involve only one local node state change at a time.

The transitive closures of $\longrightarrow$ and $\mathrel{-\!\!\triangleright}$ are denoted by $\longrightarrow\!\!\!\!\!\twoheadrightarrow$ and $\mathrel{-\!\!\triangleright\!\!\triangleright}$. Paths in a transition graph are called **derivations** and, abusing language, we speak of a *derivation* $x \longrightarrow\!\!\!\!\!\twoheadrightarrow y$ or $x \mathrel{-\!\!\triangleright\!\!\triangleright} y$.

1.9 Discrete synchronism. In agreement with our motivation introduced in §1.1, time considered in this framework is defined with respect to T. Further, if $i, j \in \mathbf{U}(x)$ and $i \neq j$, then in x, the BAN $\mathcal{N}$ may undergo both local changes $x_i \rightsquigarrow \neg x_i$ and $x_j \rightsquigarrow \neg x_j$. Considering synchronous transition $x \mathrel{-\!\!\blacktriangleright} y = \overline{x}^{\{i,j\}}$ amounts to considering the possibility that having experienced configuration x, $\mathcal{N}$ undergoes some changes that lead it directly to experience configuration y without having significantly experienced any other intermediary configuration (in which some other considerable changes could have occurred) before that. It thus amounts to considering the possibility that, in comparison to all other possible changes, changes $x_i \rightsquigarrow \neg x_i$ and $x_j \rightsquigarrow \neg x_j$ may occur close enough in time so as to disallow the occurrence of other changes. Distance in time between any two events is thus intrinsically dependent on all other events. Here, we propose to explore the sensitivity of BANs to synchronism with the ambition of better understanding the influence of time flow accounted for by the relative sequencing of events.

In works involving automata networks, synchronism has often either been considered as a founding hypothesis, as in [10] and the many studies that followed in its lead, or, on the contrary, in lines with [18], it has been disregarded altogether to the benefit of pure asynchronism. Comparisons have been made between different kinds of ways of updating node states, involving variable degrees of synchronism in both probabilistic [3, 5, 14, 17] (with cellular automata) and deterministic frameworks [2, 4, 8, 16]. In particular, for the algorithmic purpose of finding the shortest path to a stable configuration, Robert [16] compared BAN behaviours under the parallel and sequential deterministic update schedules.

Here, we focus on attractors, both stable and composite (*cf.* §2.2) and make no prior restriction on the way node states are updated. In §2.2, we classify the different impacts that the addition of a synchronous transition to the ATG of a BAN may have on the overall BAN behaviour. In §2.3, we introduce the different sorts of synchronism-sensitivities that this induces and relate them to some structural properties. And we discuss how this relates to non-monotony in §2.4. Before that, the results on which this study is based are given in §2.1.

2 Main results

2.1 Direct derivations and critical NOPE-cycles. A **cycle** of a BAN $\mathcal{N}$ is a closed directed trail of its structure $\mathbf{G}$. $\forall x \in \mathbb{B}^n$, we say that cycle $\mathbf{C} = (\mathbf{V_C}, \mathbf{A_C})$ is x**-critical** if: $\mathbf{A_C} \subset \textbf{FRUS}(x)$ (implying that $\mathbf{V_C} \subset \mathbf{U}(x)$, *cf.* §1.7) and it is **critical** if it is x-critical for some $x \in \mathbb{B}^n$ (*cf.* Fig. 1). By definition of frustrated arcs, if $\mathbf{C} = (\mathbf{V_C}, \mathbf{A_C})$ is x-critical, has length ℓ and sign $\mathbf{s}$ then $\prod_{(j,i)\in\mathbf{A_C}} -\mathrm{sign}(j,i) = (-1)^\ell \times \mathbf{s} = \prod_{(j,i)\in\mathbf{A_C}} \mathbf{s}(x_j) \cdot \mathbf{s}(x_i) = 1$. This yields:

Proposition 1. *A cycle that is critical is a* NOPE*-**cycle**, i.e. positive with an even length or negative with an odd length.*

For any configurations $x, y \in \mathbb{B}^n$, we say that x is **inclined** (resp. **unwilling**) towards y if $\mathrm{D}(x,y) \subset \mathbf{U}(x)$ (resp. $\mathrm{D}(x,y) \cap \mathbf{U}(x) = \emptyset$). Also, let $x = x(0) \longrightarrow x(1) \longrightarrow \dots x(m-1) \longrightarrow y = x(m)$ be a derivation from x to y. If $\forall t < m,\ \mathrm{D}(x(t+1), y) \subsetneq \mathrm{D}(x(t), y)$, this derivation is said to be **direct**. It performs no **reversed changes**, *i.e.* $\forall t < m,\ x(t)_i = y_i \Rightarrow \forall t < t' \leq m,\ x(t') = y_i$. It can be checked that a derivation that is not direct goes through a $x(t)$ that is unwilling towards the destination configuration y.

Proposition 2. *Let $x \in \mathbb{B}^n$ be a configuration that is inclined towards configuration $y \in \mathbb{B}^n$. If there are no asynchronous derivations from x to y then $\mathrm{D}(x,y)$ induces a* NOPE*-cycle that is x-critical. If $\mathrm{D}(x,y)$ does not induce an x-critical cycle, then there is a* direct *asynchronous transition from x to y.*

Proof. Consider the digraph $\mathbf{H} = (\mathrm{D}(x,y), \mathbf{frus}(x))$ and let $\delta : \mathrm{D}(x,y) \to \{0, 1, \dots, m-1\}$ be a topological ordering of the nodes of $\mathbf{H}$: $\forall j, i \in \mathrm{D}(x,y)$, $(j,i) \in \mathbf{frus}(x) \ \Rightarrow \ \delta(i) \leq \delta(j)$ s.t. if j and i do not belong to the same cycle in $\mathbf{H}$ (and thus do not belong to the same x-critical NOPE-cycle of $\mathbf{G}$), then $(j,i) \in \mathbf{frus}(x) \ \Rightarrow \ \delta(i) < \delta(j)$. Note that increasing the number of frustrated arcs incoming an unstable node cannot make this node stable. On this basis, letting $\mathrm{D}_t = \{ i \in \mathrm{D}(x,y),\ \delta(i) = t \}$ and $x(0) = x$, an induction on $t < m$ proves that $\forall t < m,\ x(t) \longrightarrow x(t+1) = \overline{x(t)}^{\mathrm{D}_t}$ is a transition of $\mathcal{N}$ because $\mathrm{D}_t \subset \mathbf{U}(x(t))$. Thus, $x \longrightarrow x(1) \longrightarrow \dots x(m-1) \longrightarrow y$ is a direct derivation which is asynchronous if $\mathbf{H}$ contains no NOPE-cycles. $\square$

The following consequence of this sets the backbone of the article: it shows how critical cycles are the main structural aspects of a BAN underlying its possibility to perform synchronous changes that cannot be mimicked asynchronously. First, let us say that $x \longrightarrow y$ is **sequentialisable** if it is asynchronous or if it can be broken into a derivation involving smaller transitions $x' \longrightarrow y'$, $\mathrm{d}(x', y') < \mathrm{d}(x,y)$. A synchronous transition $x \longrightarrow y$ which is not sequentialisable is called a **normal transition** and it is rather written $x \Longrightarrow y$.

Corollary 1. *If $x \Longrightarrow y$ is a normal transition, then $\mathrm{D}(x,t)$ induces a* NOPE*-cycle which is x-critical.*

As a result, in a BAN with no NOPE-cycles of size smaller than $m \in \mathbb{N}$, any synchronous change affecting no more than m node states can be totally sequentialised. We now consider the special case where the only possible critical cycles are Hamiltonian cycles.

Lemma 2. *Let $\mathcal{N}$ be a* BAN *whose critical cycles all have node set $\mathbf{V}$. Then, either $\mathcal{N}$ has a unique transition $x \Longrightarrow y$, or it has two $x \Longrightarrow y$ and $y \Longrightarrow x$. In the first case, every $i \in \overline{\mathbf{U}}(y)$ bears a positive loop $(i,i) \in \mathbf{A}$. In both cases, no asynchronous derivations can reach the endpoints of these transitions.*

Proof. Suppose that $x \Longrightarrow y$ and $x' \Longrightarrow y'$ are two normal transitions. Using Corollary 1, if $y \neq x'$, then $\mathrm{D}(x,x') \subsetneq \mathbf{U}(x) = \mathbf{V}$ and $\mathrm{D}(x',y) = \mathbf{V} \setminus \mathrm{D}(x,x') \subsetneq$

$\mathbf{U}(x') = \mathbf{V}$. In this case, $x \longrightarrow x' \longrightarrow y$ is a derivation of $\mathcal{N}$ involving smaller transitions than $x \longrightarrow y$, in contradiction with $x \longrightarrow y$ being normal. Thus, if $x \Longrightarrow y$ is not the unique normal transition of $\mathcal{N}$, then the only other one is $y \Longrightarrow x$. For any normal transition $z \Longrightarrow z' = \overline{z}^{\mathbf{V}}$, and $\forall i \in \mathbf{V}$, $z \longrightarrow \overline{z'}^{\,i}$ is a transition of $\mathcal{N}$. By hypothesis and by Corollary 1, it is sequentialisable. Since $z \Longrightarrow z'$ is not however, this implies $i \in \overline{\mathbf{U}}(\overline{z'}^{\,i})$, $\forall i \in \mathbf{V}$. Thus, the endpoint of any normal transition of $\mathcal{N}$ can be reached by no asynchronous derivations. And since $\forall i \in \mathbf{V},\ i \in \overline{\mathbf{U}}(\overline{y}^{\,i})$, any $i \in \overline{\mathbf{U}}(y)$ is such that $\mathrm{sign}(i,i) = +1$ by Lemma 1. $\square$

2.2 Impact of synchronous transitions. We call **attractors** the terminal SCCs (strongly connected components) of a transition graph (abusing language because it may be that an attractor doesn't attract anything). Attractors which are not induced by stable configurations (*i.e.* attractors that are induced by several unstable configurations) are called **composite attractors**.

In the sequel, the ATG $\mathcal{T}_\mathrm{a}$ is taken as the reference transition graph to which we consider adding a normal synchronous transition. We let $\mathcal{O}_\mathrm{a}(x) = \{\, y \in \mathbb{B}^n \,|\, x \twoheadrightarrow y \,\}$ denote the **orbit** of $x \in \mathbb{B}^n$ in $\mathcal{T}_\mathrm{a}$ and $\mathcal{B}_\mathrm{a}(x) = \{\, y \,|\, y \twoheadrightarrow x \,\}$. $\mathcal{A}_\mathrm{a}(x)$ denotes the set of attractors that x can reach in $\mathcal{T}_\mathrm{a}$. We say that a configuration x is **recurrent** when it belongs to an attractor and we denote this attractor by $[x]_\mathrm{a}$ (then, $\mathcal{A}_\mathrm{a}(x) = \{\, [x]_\mathrm{a} \,\}$). The basin of an attractor $[x]_\mathrm{a}$ is $\mathcal{B}_\mathrm{a}([x]_\mathrm{a}) = \mathcal{B}_\mathrm{a}(x) \setminus [x]_\mathrm{a}$. Non-recurrent configurations are called **transient**.

Let $\mathcal{T}_\mathrm{a}{}' = (\mathbb{B}^n, \mathrm{T}_\mathrm{a} \cup \{\, (x,y) \,\})$ denote the transition graph obtained by adding an arbitrary synchronous transition $x \rightarrow y$ to $\mathcal{T}_\mathrm{a}$. We introduce notations $\mathcal{A}(x)$, $\mathcal{B}(x)$, $\mathcal{O}(x)$ and $[x]$ relative to $\mathcal{T}_\mathrm{a}{}'$ similarly to those introduced above for $\mathcal{T}_\mathrm{a}$. We say that an attractor A of $\mathcal{T}_\mathrm{a}$ is *destroyed* by $x \rightarrow y$ if all its configurations are transient in $\mathcal{T}_\mathrm{a}{}'$. Generally, since $\mathcal{A}(x) = \mathcal{A}_\mathrm{a}(x) \cup \mathcal{A}_\mathrm{a}(y)$, the addition of $x \rightarrow y$ to $\mathcal{T}_\mathrm{a}$ can have several possible consequences on the asymptotic evolutions of $\mathcal{N}$ that go through or start on configuration x. We list them now exhaustively.

1. We say that (the addition of) $x \rightarrow y$ has **no impact** when x is transient in $\mathcal{T}_\mathrm{a}$ and $\mathcal{A}_\mathrm{a}(y) \subset \mathcal{A}_\mathrm{a}(x) = \mathcal{A}(x)$. In particular, if $x \rightarrow y$ is sequentialisable, then it shortcuts some derivations starting in x. But on the contrary, it can also deviate some derivations (when $\exists z \in \mathcal{O}_\mathrm{a}(x) \cap \mathcal{O}_\mathrm{a}(y)$ s.t. $y \twoheadrightarrow z$ is no shorter than $x \twoheadrightarrow z$).

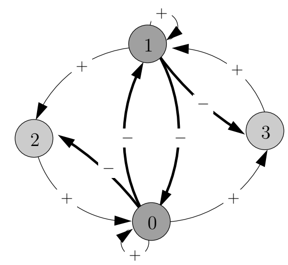
All synchronous transitions $x \rightarrow y$ that *do* have an impact on the asymptotic evolution of $\mathcal{N}$ must be normal.

2. We say that transition $x \Longrightarrow y$ has **little** or **F-impact** (*cf.* Fig. 1) if x is transient in $\mathcal{T}_\mathrm{a}$ and $\mathcal{A}(x) = \mathcal{A}_\mathrm{a}(x) \cup \mathcal{A}_\mathrm{a}(y) \neq \mathcal{A}_\mathrm{a}(x)$. Here, $x \Longrightarrow y$ causes the growth of the basins $\mathcal{B}_\mathrm{a}(A),\ A \in \mathcal{A}_\mathrm{a}(y)$, thus adding degrees of <u>f</u>reedom to the asymptotic outcomes of the evolutions that pass through x.

With addition of synchronous transitions that have no or little impact, the set of recurrent configurations of $\mathcal{T}_\mathrm{a}$ equals that of $\mathcal{T}_\mathrm{a}{}'$.

3. We say that transition $x \Longrightarrow y$ has **D-impact** (*cf.* Fig. 2) if x and y are both recurrent in $\mathcal{T}_{\mathrm{a}}$ and $\mathcal{A}_{\mathrm{a}}(y) \setminus \mathcal{A}_{\mathrm{a}}(x) \neq \emptyset$. In this case, the addition of $x \Longrightarrow y$ **d**estroys the composite attractor $[x]_{\mathrm{a}}$ by emptying it into (the basins of) the attractors $A \in \mathcal{A}_{\mathrm{a}}(y) \setminus \mathcal{A}_{\mathrm{a}}(x)$.

$$\forall x \in \mathbb{B}^n, \quad \begin{cases} \mathrm{h}_0(x) = x_2 \vee (x_0 \wedge \neg x_1) \\ \mathrm{h}_1(x) = x_3 \vee (\neg x_0 \wedge x_1) \\ \mathrm{h}_2(x) = \neg x_0 \wedge x_1 \\ \mathrm{h}_3(x) = x_0 \wedge \neg x_1 \end{cases}$$



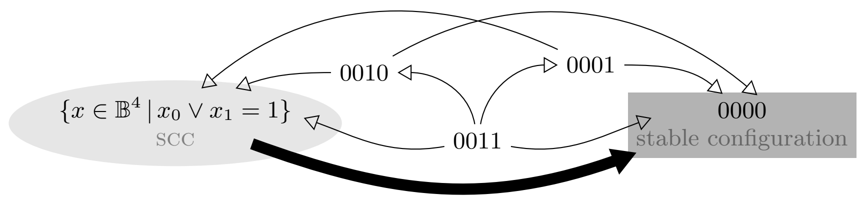


Fig. 2. Top left: mechanisms of BAN $\mathcal{N}^* = \{\, \mathrm{h}_i | i \in \mathbf{V} = \{0, \ldots, 3\}\,\}$. Top right: signed structure of $\mathcal{N}^*$. Bottom: reduced version of $\mathcal{T}_{\mathrm{a}}{}'$, which is the transition graph obtained by adding $1100 \Longrightarrow 0000$ to the ATG $\mathcal{T}_{\mathrm{a}}$ of $\mathcal{N}^*$. The shaded ellipse corresponds to a SCC which is terminal in $\mathcal{T}_{\mathrm{a}}$ but not in $\mathcal{T}_{\mathrm{a}}{}'$ nor in the ETG $\mathcal{T}$. It follows from §2.1 that this is essentially due to the positive cycle of length 2 induced by $\{\, 0, 1\,\} \subset \mathbf{V}$.

4. We say that transition $x \Longrightarrow y$ has **G-impact** (*cf.* Fig. 3) on the asymptotic evolution of $\mathcal{N}$ when, in $\mathcal{T}_{\mathrm{a}}$, x is recurrent, y is transient and $\mathcal{A}_{\mathrm{a}}(y) = \mathcal{A}_{\mathrm{a}}(x) = \{[x]_{\mathrm{a}}\}$. The addition of $x \Longrightarrow y$ to $\mathcal{T}_{\mathrm{a}}$ causes attractor $[x]_{\mathrm{a}}$ to absorb all derivations from y to $[x]_{\mathrm{a}}$ and **g**row into $[x] = [y]$ without being destroyed.

It can be checked that the four types of impacts listed above are disjoint and cover all possible cases. Let us emphasise that with the addition of (D-impact) synchronism, a recurrent configuration can become transient. Conversely, the addition of (G-impact) synchronism can turn a transient configuration into a recurrent one. Synchronism can however not create new attractors from scratch. And to merge attractors of the ATG requires more than one normal transition.

The addition of $x \Longrightarrow y$ to the ATG has no or little impact when x is transient in the ATG. To have **significant impact**, *i.e.* to change the asymptotics of $\mathcal{N}$ (rather than just some of its evolutions towards it), $x \Longrightarrow y$ needs to have G- or D-impact. Considering Hamiltonian critical cycles again as in Lemma 2, Lemma 3 below evidences that to have this sort of impact requires to embed critical cycles in a larger, structural environment.

Lemma 3. *Let $\mathcal{N}$ be a* BAN *with no normal transitions of size smaller than its size n. Then, any transition $x \Longrightarrow y$ either has no impact on the asymptotics*

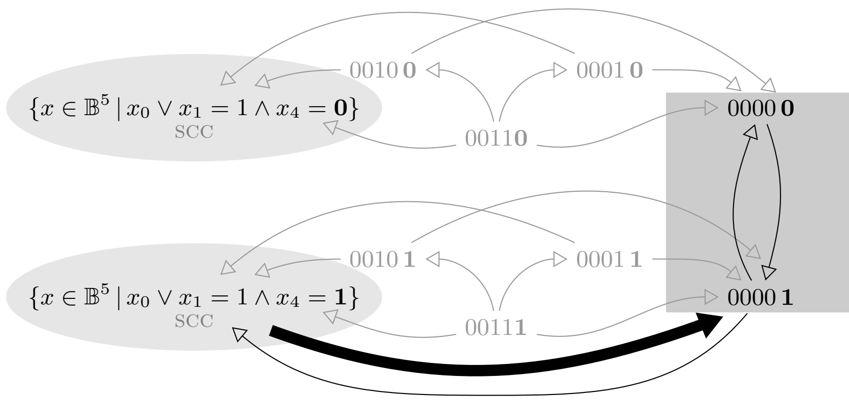


Fig. 3. Reduced version of ${\mathcal{T}_{\mathrm{a}}}'$ obtained by adding normal transition 11001 $\pmb{\longrightarrow}$ 00001 to the ATG of $\mathcal{N}$. $\mathcal{N}$ is the BAN that is obtained from $\mathcal{N}^*$ (*cf.* Fig. 2) as follows. A fifth node $i = 4 \in \mathbf{V}$ is added s.t. $\mathrm{f}_4(x) = \neg(x_0 \vee x_1 \vee x_2 \vee x_3) \wedge \neg x_4$, and we let $\mathrm{f}_0(x) = \mathrm{h}_0(x) \vee (x_4 \wedge \neg x_1 \wedge \neg x_3)$ and $\forall i \in \{1, 2, 3\}$, $\mathrm{f}_i(x) = \mathrm{h}_i(x)$. It can be checked that $\mathrm{h}_0(x) \neq \mathrm{f}_0(x) \;\Rightarrow\; x = 00001$. 11001 $\pmb{\longrightarrow}$ 00001 therefore has G-impact.

of $\mathcal{N}$ or it has F-impact. In the latter case, y is stable and has an empty basin $\mathcal{B}_{\mathrm{a}}(y) = \emptyset$ in the ATG, and all nodes of $\mathcal{N}$ have a positive loop.

Proof. Let $x \pmb{\longrightarrow} y = \overline{x}^{\mathbf{V}}$. $\forall i \in \mathbf{V}$, $x \longrightarrow \overline{y}^{\,i}$ (since $\mathrm{D}(x, \overline{y}^{\,i}) \subset \mathbf{U}(x) = \mathbf{V}$). So by Corollary 1, $\forall i \in \mathbf{V}$, $x \rightarrowtriangle\!\triangleright \overline{y}^{\,i}$. Thus, $\forall z \in \mathbb{B}^n$, $y \rightarrowtriangle z \;\Rightarrow\; x \rightarrowtriangle\!\triangleright z$. And either $\mathbf{U}(y) \neq \emptyset$ in which case $\mathcal{A}_{\mathrm{a}}(y) \subset \mathcal{A}_{\mathrm{a}}(x)$ (and $x \pmb{\longrightarrow} y$ has no impact), or y is stable and by Lemma 2, $\mathcal{B}_{\mathrm{a}}(y) = \emptyset$ and $\forall i \in \mathbf{V}$, $\mathrm{sign}(i, i) = +1$. □

2.3 Synchronism-sensitivity.

Let us say that $\mathcal{N}$ has **no (synchronism)-sensitivity** if none of its normal transitions has any impact (*cf.* §2.2), that it has **little sensitivity** if it has normal transitions with little impact, and that it has **significant sensitivity** if it has normal transitions with significant impact.

Proposition 3. *1) Synchronism-sensitivity requires the existence of a critical cycle, and thus of a NOPE-cycle.*

2) Significant sensitivity requires the existence of a critical cycle of length strictly smaller than the BAN size as well as of a negative cycle.

3) In the absence of a Hamiltonian NOPE-cycle and positive loops on all nodes, little sensitivity also requires a critical cycle of length strictly smaller than the BAN size.

Proof. 1) Synchronism-sensitivity requires a normal transition which requires a critical NOPE-cycle by Corollary 1. 2) A normal transition $x \pmb{\longrightarrow} y$ with significant impact (*cf.* §2.2) must be induced by a non-Hamiltonian critical cycle (*cf.* Lemma 3). And since x must be recurrent, there must be a negative cycle to induce the composite attractor $[x]_{\mathrm{a}}$ [15]. 3) By Lemma 3. □

2.4 Sensitivity to synchronism & non-monotony. All BANs of size 2 that have significant synchronism-sensitivity are non-monotone (*i.e.* have some f_is that are not locally monotone in all components, *cf.* §1.6). It can be checked that the smallest *monotone* BANs that have significant sensitivity have size 3. Interestingly, a close examination of their mechanisms suggests that there is a tight relationship between significant sensitivity and non-monotony (compare the h_is in Fig. 2 with the most basic non-monotone mechanism which combines two inputs with a XOR connector: $x \mapsto x_i \oplus x_j = (x_i \wedge \neg x_j) \vee (\neg x_i \wedge x_j)$). This and Fig. 4 suggest to consider a more general notion of non-monotony (by which BAN $\mathcal{N}_a$ of Fig. 4 would be considered non-monotone) that takes into account how state changes are organised relatively in time. We conjecture that all significantly sensitive BANs are non-monotone in this larger sense.

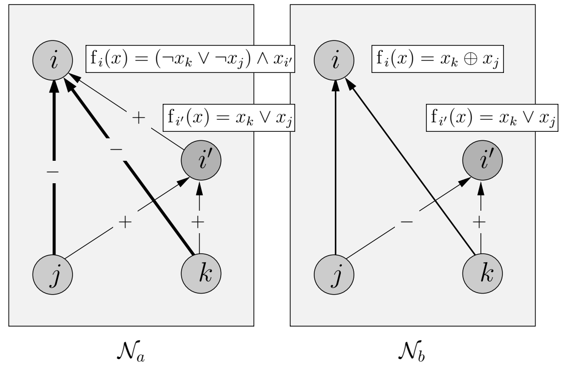


Fig. 4. In the monotone BAN $\mathcal{N}_a$, if the state change of i' occurs before that of i in all $x \in \mathbb{B}^4$, then, i takes state $\mathrm{f}_0(x_j, x_k, \mathrm{f}_{i'}(x)) = x_j \oplus x_k$. Imposing systematically the immediate precedence of i' state changes over i state changes makes the mechanism $\mathrm{f}_i \in \mathcal{N}_a$ of i non-monotone. As a result, $\mathcal{N}_a$ behaves like the non-monotone BAN $\mathcal{N}_b$.

3 Conclusion and perspectives

We have evidenced that synchronism in itself may impact significantly on the asymptotic behaviour of a network: not only can it modify transient behaviours and make attractors grow, it can also destroy composite attractors. By filtering out local instabilities that asynchronism cannot get rid of, synchronism can decrease global instability.

Notably, synchronism in this context does not need to be restrictively taken as representing *simultaneity*. As we have argued in §1.1 and §1.9, it can naturally be given a much more general meaning with respect to time. As a consequence, the importance of the results and discussions of this paper is supported by the disregard that synchronism has often received in theoretical modelling fields (based on the unlikeliness of simultaneity in real biological systems and on the accepted intuition that asynchronism guarantees a greater global stability).

Generally, the present work emphasises that time flow is a determining parameter with respect to the behaviour of interaction networks that satisfy some particular structural properties. What is more, as argued in §2.4, it seems to relate to non-monotony, that is, informally, to the fact that one automaton may receive contradicting signals emitted by the same other automaton. In the absence of such a situation, the implementation of mechanisms can be stretched out or concentrated in time with little impact on the final output behaviour of the network.

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