Dynamical Stability of Threshold Networks over Undirected Signed Graphs.

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Abstract. In this paper we study the dynamic behavior of threshold networks on undirected signed graphs. While much attention has been given to the convergence and long-term behavior of this model, an open question remains: How does the underlying graph structure influence network dynamics? While similar papers have been carried out for threshold networks (as well as for other networks) these have largely focused on unsigned networks. However, the signed graph model finds applications in various real-world domains like gene regulation and social networks. By studying a graph parameter that we call "stability index," we search to establish a connection between the structure and the dynamics of threshold network. Interestingly, this parameter is related to the concepts of frustration and balance in signed graphs. We show that graphs that present negative stability index exhibit stable dynamics, meaning that the dynamics converges to fixed points regardless of threshold parameters. Conversely, if at least one subgraph has positive stability index, oscillations in long term behavior may appear. Finally, we generalize the analysis to network dynamics under periodic update schemes and we explore the case in which the stability index is positive for some subgraph finding that attractors with superpolynomial period on the size of the network may appear.

Keywords: complex systems · discrete dynamical systems · automata networks.

1 Introduction

In this paper we study the dynamics of threshold-type functions within networks of different entities, where relationships are characterized as friendly or unfriendly. More precisely, the model is defined by a non-oriented graph $G = (V, E)$, with V denoting individuals and E representing the different relationships between them. To each of these edges a weight of +1 or -1 is assigned, signifying friendly or negative connections, respectively.

In this graph-based framework, each node holds an internal state from the set $\{-1, +1\}$. This state will evolve in time via a deterministic local transition function governed by a threshold rule. Specifically, a node adopts a +1 state if the weighted sum of its neighboring nodes' states, factoring in edge weights (+1 or -1), surpasses a predefined threshold; otherwise, the node takes on a state of -1.

The primary update scheme is parallel, where all nodes undergo simultaneous updates. However, we study a broader spectrum of update schemes, with a particular emphasis on periodic updates. Periodic updates involve a set of finite collections $\mu = \{A_1, \ldots, A_p\}$, where A_i constitutes subsets of the vertex set V. During each step, updates occur sequentially across sets, progressing from the first to the last, with parallel updates taken place within each set. Notably, a single vertex may belong to multiple sets, potentially leading to multiple updates within a single step. A distinct scenario, termed the *block sequential* update scheme, emerges when μ constitutes a partition of V, ensuring that each vertex updates only once per step. An additional common scheme is sequential, where vertices update individually according to a predefined order dictated by a permutation of the vertex set V, $\sigma \in \Sigma(|V|)$, with sets $\mu = \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$.

In this framework, this work focuses on the link between these two latter described dimensions: firstly, the structural landscape given by the structure of the graph (including the sign assignation and the underlying structure) within which dynamics unfurl. Secondly, the dynamic evolution of threshold functions under diverse update schemes in the context of signed graphs.

Threshold networks A threshold network is a tuple $\mathcal{T} = (G, W, \{-1, 1\}, b, F)$ where $G = (V, E)$ is non-oriented a graph, $W = W(G)$ is the graph's adjacency matrix (possibly with real weights), F is a collection of threshold functions $F = (F_1, \ldots, F_n)$, and $\{-1, 1\}$ emphasize the set of possible states.

Originating from McCulloch and Pitts [21], threshold networks were conceived as an initial model for the nervous system. These networks feature units interconnected by threshold functions, emulating neuron behavior [23,22,17]. In addition, Thomas and Kauffman applied similar principles to gene interaction modeling using Boolean functions [25,18]. In fact, threshold functions play a vital role in gene interaction models, with multiple studies exploring dynamics and resilience of cell cycle networks [20,6,24].

In terms of the model studied as a dynamical systems, as it shown in [12] for networks with undirected graphs, parallel iterations of symmetric configurations tend to converge to fixed points or limit cycles of period 2. In addition, symmetric networks with non-negative diagonal entries converge to fixed points during sequential iteration [9]. Both results are obtained by the analysis of decreasing energy functional similar to the spin glass model. Interestingly, this approach also provides bounds to the convergence time. In the last years, the dynamics have been also studied from a computational complexity standpoint [10,5] and also from a structural approach [24].

Signed graphs In this paper, we focus on the dynamics of threshold-type functions in non-oriented graphs with edges marked as -1 or +1. Originating from Heider's work on attitudes and cognitive organization [16], the model captures the balance theory relating to attitude changes among individuals. As presented in the seminal paper [3], the model represents relationships using graphs, with edges as +1 (friendship) or -1 (unfriendly) connections, considering symmetric relationships.

In [14], the notion of balance in signed graph is presented. A graph is *balanced* if the vertex set can be partition into two sets, with negative edges connecting them. This can be as a generalization of the concept of bipartition in unsigned graphs. In fact, as same as the latter case in which all the circuits must have an even number of edges, circuits in a signed graph play an important role for balance. If G is a signed graph, we define the sign of a cycle in G as the product of the sign of the its edges. We say that a cycle in G is even (resp. odd) it has an even (resp. odd) amount of edges. A signed graph G is balance if no cycle is negative. A dual notion is the notion of *antibalance*. We say that G is antibalanced if any even cycle in G has positive sign and any odd cycle has negative sign. In other words, a signed graph is antibalanced if even cycles have an even amount of negative edges and odd cycles have an odd amount of negative edges.

Starting from this initial standpoint, two natural questions arise: how can we measure how close is a graph to be balanced? and how difficult is to know if a graph is balanced? Regarding the first question, different measures and indices have been proposed [3,14,2]. Notably, the frustation index (resp. number) is defined on a signed graph G as the minimum amount of edges (resp. vertices) whose removal results on a balance graph. Regarding the second question, it is known that computing these two numbers is impractical (computation of both indices are linked to classical **NP**-hard problems, see [27] for a complete review on problems related to negative and positive cycles).

Involving signed graphs, several applications can be found in social dynamics, computational chemestry, physics, politial science, systems biology, among others [1]. On the other hand, signed graphs and the balance notion appear in the context of the spin glass problem in statistical physics [4]. A generalized spin network consists in a set on nodes connected with signs +1 and -1, where in each node there is a particle with spin +1 or -1. The magnetization of the network is related to minimize the energy, given by the sum of the product of contiguous spins orientation weighted with their signs. Then obtaining a magnetized material consists of reaching a spin configurations that minimizes such a quantity, which is also related with the notion of balance and the balance index, although physicists prefer to talk about frustration [19,7,26]. A graph is frustrated if there exists an even circuit with an odd number of negative connections or an odd circuit with an even number of negative connections, i.e. frustration is equivalent to antibalance in signed graphs.

1.1 Our contribution

We introduce the parameter, $\mathcal{S}(G)$, which encodes the interplay between the graph structure of the network and its dynamical behaviour. Specifically, this index, called the *stability index* is defined as $\mathcal{S}(G) = -n - d^+ + d^- + 2m - 4p$, where n and m represent the number of vertices and edges of G respectively,

d^+ and d^- denote the number of positive and negative loops, and p indicates the minimum number of edges necessary for the graph to achieve *anti-balance*. A signed graph is antibalanced if it exists a partition of the nodes (A, B) in which all the internal edges (edges connecting only nodes in A or nodes in B) are negative and all the edges between nodes in A and B are positive. This is related to the sign of the cycles as it is discussed in the paper.

In this context, we are interested on study the behavior of parallel iterations. Our analysis uncovers a pivotal connection: the stability of the network depends on the value of $\mathcal{S}(G')$ for each subgraph G. When $\mathcal{S}(G') < 0$ for each subgraph, the parallel dynamics exclusively converge to fixed points, underscoring the intimate connection between the graph's inherent structure and its dynamical behavior.

More precisely, we present two main theorems, the first one, for the synchronous update scheme:

Theorem. Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network. If for each subgraph $G' \subseteq G$ we have that $\mathcal{S}(G) < 0$ then, $\mathcal{T}$ admits only fixed points.

And then, we generalize this result for arbitrary periodic update schemes:

Theorem. Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network and let us consider a periodic update scheme $\mu = (I_1, \ldots, I_p)$. If for any $1 \leq \ell \leq p$ we have that for all subgraph $G' \subseteq G(I_\ell)$, $\mathcal{S}(G') < 0$ then, $\mathcal{T}$ admits only fixed points.

Conversely, for the synchronous update scheme, we find that if a graph G is such that $\mathcal{S}(G) \geq 0$ then dynamics admits an attractor of period 2.

Observe that the difference between *two-cycle* and attractor with period two plays a role in the theorem that is analyzed through different lemmas. Roughly speaking, a two cycle is an attractor of period 2 (a time-periodic configuration with period 2) in which any node in the network change its state. In lieu, an attractor of period 2 may have nodes that which state is fixed (they are stable).

Finally, in the context of periodic update schemes, we provide an example of a network defined on a regular topology (a cycle graph) along with an update scheme for which the assumptions of the previous theorem do not hold (there exists a subgraph with a positive stability index) and this network exhibits an attractor with superpolynomial period relative to the network's size.

1.2 Organization of the paper

In Section 3.1 we study the synchronous update scheme. In particular, we study sufficient and necessary conditions for stability depending on the stability index. In Section 3.2 we study the stability of the network under general periodic update schemes. We deduce from previous section that the main theorem for synchronous update schemes can be extended to this setting. Then, we explore what happens when the sufficient conditions are not fulfill by showing an example of a very simple network which exhibits attractors with non-polynomial period on the size of the network.

2 Preliminaries

Signed graphs A signed graph is a tuple $G = ((V, E \cup D), z)$ where G is a non-directed graph with set of nodes V, set of edges E and set of self-loops D, and $z : E \to \{-1, 1\}$ is an *assignation of signs* for each edge in G. In the rest of the paper, we will denote a signed graph without self-loops simply as (G, z). In addition, we will call n the number of nodes in G, i.e. $n = |V|$, we will call m the number of edges in G, i.e. $m = |E|$ and finally, we will call d^+ (resp. d^-) the amount of positive (resp. negative) self loops in G.

Given a signed graph G we denote by $W(G) = (w_{ij})$ its adjacency matrix. More precisely,

$$w_{ij} = \begin{cases} \varepsilon \in \{-1, 1\} & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Stability index of a graph Let (G, z) be a signed graph. We define the sign of a subgraph H of G as the product of the sign of the its edges. We say that a cycle in G is even (resp. odd) it has an even (resp. odd) amount of edges. There is a very well known property of signed graph that it is known as balance. A graph (G, z) is *balanced* if no cycle is negative. A dual notion is the notion of antibalance. We say that (G, z) is *antibalanced* if any even cycle in G has positive sign and any odd cycle has negative sign. In other words, a signed graph is antibalanced if even cycles have an even amount of negative edges and odd cycles have an odd amount of negative edges

In this context, if (G, z) is a signed graph we can always switch the sign of the edges, i.e, change any positive edge by a negative edge and vice versa. We call this graph $-G$. It is not hard to see that a graph is antibalanced if and only if $-G$ is balanced.

In addition, observe that if $z(e) = 1$ for each $e \in E$ then, a balance graph is a bipartite graph. Oftenly, in the literature, it is defined the number $\phi(G)$ as the minimum size of a set of edges $X \subseteq E$ such that $G - X$ is balanced. This parameter is called the *frustration* index of the graph. In this paper, we will work with the amount $p(G)$ (or simply p when the context is clear) which correspond to the minimum size of a set of edges $X \subseteq E$ such that $G - X$ is antibalanced. Observe that $p(G) = \phi(-G)$. We present in Figure 1 some examples of frustrated cycles.

Finally, we define the *stability index* of a signed graph (G, z, D) as the number

$$\mathcal{S}(G) = -n - d^+ + d^- + 2m - 4p.$$

In Figure 2, we show the value of alpha for some examples. We observe that in the first graph the value of p is 1 since we need to remove only an edge in order to have an antibalanced graph. Thus, $\mathcal{S} = -1$. For the second graph, we have that the cycle is antibalanced, so $p = 0$ and $\mathcal{S} = 5$. Finally, for the last one, we have that we need to remove two edges in order to have an antibalanced graph. In addition, it is easy to see that deleting only one edge does not define an antibalanced graph. Thus, we deduce $p = 2$ and $\mathcal{S} = -3$. We will see in the next

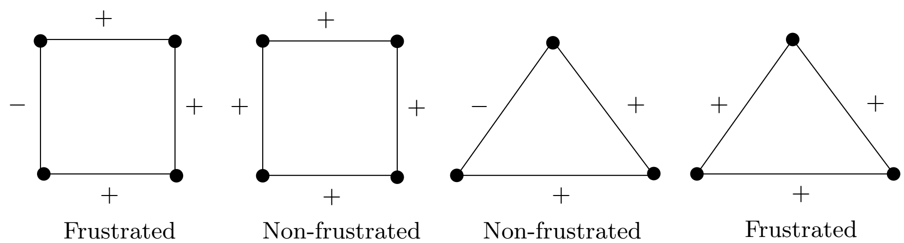

Fig. 1. Some examples of frustrated and non-frustrated graphs.

section that this graph parameter is closely related to the dynamics of threshold networks.

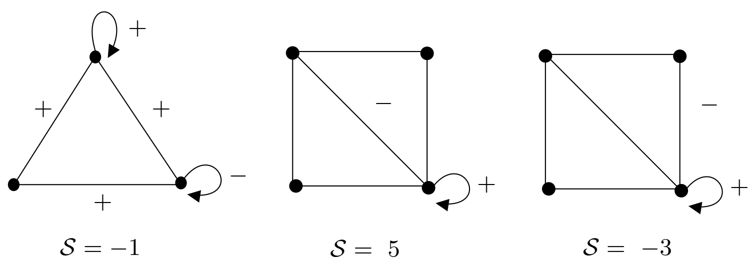

Fig. 2. Values for the stability index in different graphs. In the first graph (from left to right), we have that $p = 1$, in the second graph $p = 0$ and finally $p = 2$.

Threshold networks A threshold automata network is a tuple $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ where G is a signed graph, $W(G)$ is the signed adjacency matrix of G, $b \in \mathbb{Z}^{|V|}$ is called a *threshold vector* and the functions $F = (F_1, \ldots, F_n)$, with $F_i : \{-1, 1\}^V \to \{-1, 1\}$, are called *threshold local functions* and they are defined as:

$$x(t+1) = F_i(x(t)) = \begin{cases} 1 & \text{if } \sum\limits_{j \in V} w_{ij} x(t)_j - b_i > 0, \\ x(t)_i & \text{if } \sum\limits_{j \in V} w_{ij} x(t)_j - b_i = 0, \\ -1. & \text{otherwise} \end{cases}$$

Observe that the second case in the previous expression defines what to do in a tie case scenario, i.e. the case in which for some node i the sum of the states of its neighbors is exactly b_i. For example, in the case of the majority rule, i.e. when $b_i = 0$, a tie case scenario happens if for some node i the amount of neighbors in state 1 is the same than the amount of neighbors in state -1. Of course this can only happen if a node has even degree. In that case, the previous definition can be modeled as positive loops in each node in the network. When the majority rule is defined in this way, it is usually called *stable*. In addition, other *tie-breaking* functions can be defined (see for example [5]). A very well studied case for the majority rule, is known in the literature as *crazy spin* or *unstable*. In this case, a node changes its states if it is in a tie-case scenario instead of preserving it. More precisely, we have:

$$x(t+1) = F_i(x(t)) = \begin{cases} 1 & \text{if } \sum\limits_{j \in V} w_{ij} x(t)_j - b_i > 0, \\ -x(t)_i & \text{if } \sum\limits_{j \in V} w_{ij} x(t)_j - b_i = 0, \\ -1. & \text{otherwise} \end{cases}$$

Observe that when the degree of a node is even, this tie-breaking rule can be modeled as a negative loop. Unsurprisingly, the choice of the tie breaking rule will have a great impact in the dynamics. We illustrate this with the example in Figure 3.

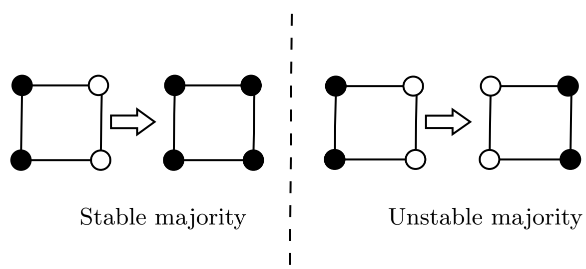

Fig. 3. Example of a stable majority rule dynamics vs unstable majority rule dynamics defined on the same graph: in the left panel, we have a stable majority dynamics which reach a fixed point. On the right panel we have an unstable majority dynamics which reach a two-cycle.

Periodic dynamics over $\mathcal{T}$ We call $x \in \{-1,1\}^V$, i.e., an assignation of states for each node in a signed graph $G = (V, E)$, a *configuration*. Let $I \subseteq V$ be a subset of vertices and $x \in \{-1,1\}^V$ an arbitrary configuration. We define the *transition function* associated to I as the function:

$$F_I(x) = \begin{cases} x_i & \text{if } i \notin I \\ F_i(x) & \text{if } i \in I, \end{cases}$$

In simple words F_I assigns a new state to any node in I according to its local threshold function. We say that nodes in I are being *updated*. Consider now $\ell \in \mathbb{N}$ and a sequence $\mu = (I_1, \ldots, I_\ell)$ such that $I_k \in \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of V. Now, given a configuration x, we define the *global transition function* of the network $F^\mu : \{-1,1\}^V \to \{-1,1\}^V$ by

$$F^{\mu}(x) = \left(F_{I_\ell} \circ F_{I_{\ell-1}} \circ, \ldots, \circ F_{I_2} \circ F_{I_1}\right)(x).$$

The sequence μ induces via F a dynamics on the set of configurations $\{-1,1\}^V$ in the following way: given an initial condition $x^0 = x$ we define $x^t = F^t(x)$ for all $t \geq 1$. We say that μ is a *periodic update scheme* for the network $\mathcal{T}$.

Generally speaking, function F^{μ} will assign a new global state for the network by sequentially applying the transition function associated to each set $I \in \mu$ according to its order. In a generic application of F nodes will be updated according to the order of the sets in μ. First, all the nodes in I_1 will be updated, then, the ones in I_2, I_3, and so on. When the nodes in the set I_p are updated, the next global state of the network has been completely computed and in order to compute the next state, the same process is repeated. We call the *size* of an update scheme a number $r > 0$ such that $|I_k| \leq r$ for each $k \in \{1, \ldots, \ell\}$.

Formally speaking, the map F^{μ} induces a dynamical system in the set of all the possible assignations of states for the nodes in G, i.e. $\{-1,1\}^V$. We call an assignation of states $x \in \{-1,1\}^V$ a *global configuration* or simply a *configuration* when the context is clear. We call an *orbit* a sequence of configurations $x_1, \ldots, x_\ell$ such that $x_{i+1} = F^{\mu}(x_i)$ for $i \in \mathbb{N}$, i.e. each term in the sequence is obtained by applying S to the previous term. Observe that, since the number of possible configurations is finite, each orbit is *eventually periodic,* i.e. there exists some $T, p \geq 0$ such that $x_{T+p} = x_T$. In simple words, this means that after some time each configuration will reach eventually a periodic orbit. Any periodic orbit of period p is called an *attractor* of period p. In the case in which $p = 1$ the attractors are called *fixed points.* Some abuse of notation is introduced in the literature and the attractors of a threshold network are also called *cycles.* In this work, we are going to refer to cycles in graphs and cycles in the dynamics without making an explicit difference whenever the context is clear. We distinguish the case in which $p = 2$. We call this special type of attractors *two-cycles.* In the case of a two-cycle in which each node changes its state, i.e. $x_i \neq F^{\mu}(x)_i$ for all $i \in V$, we call it a *total two-cycle* or simply *total-cycle* when the context is clear. We show an example of a total-cycle and a two-cyle in Figure 4. We say that a threshold network is *stable* if it has only fixed points. Otherwise, we say that the network is *unstable.* There are some important particular cases of periodic update schemes such as the case of the parallel or synchronous update scheme in which each node updates its state at the same time, i.e. $\mu = \{V\}$. In this case we simply write F instead of F^{μ}. Another interesting case is the one of the *block sequential update schemes*, in which μ is a partition of V. An important example of block sequential update scheme is the case *sequential update schemes.* In sequential update schemes each $I \in \mu$ is such that $I = \{v\}$, i.e. each set is a singleton. We can see μ in this case as a permutation of set V of the nodes of the network. We show an example of a simple dynamics under these update schemes in Figure 6. In these latter Figure, we have a conjunctive network. In this case, all the thresholds are $\theta = 0$, and thus, the nodes locally compute an AND function. In Figure 6 we show three different update schemes. A is the parallel update scheme, B is the sequential update scheme and C is a block

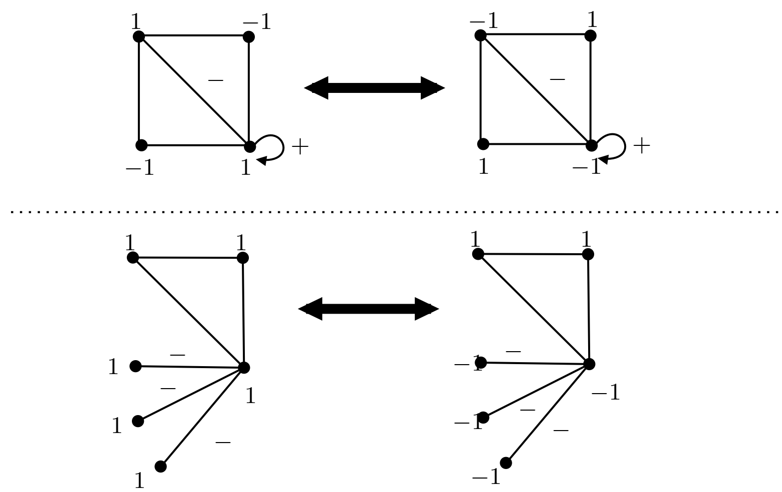

Fig. 4. Example of two majority threshold network exhibiting two-cycles and total two-cycles. In the upper panel, a majority network exhibiting a total two cycle. In the bottom panel, a majority network exhibiting a two-cycle. Observe that in this case two of the nodes in the triangle are always in state 1 and the rest of the nodes are switching states.

sequential update scheme. Observe that the dynamics induced in each case are different. In fact, A reaches a total two-cycle and B and C reach a fixed point.

Finally, let x be an attractor for $\mathcal{T}$. We introduce the following notation: we call $G'(x) = G[V']$ to the graph induced nodes that are switching states, i.e. $V'(x) = \{v \in V : x_v \neq F(x)_v\}$. Whenever the context is clear, we write only G' or V'.

3 Results

In this section we study the link between the stability index of a signed graph $G = (V, E \cup D, z)$ and the dynamics induced by a threshold network defined over G. We will study first the synchronous dynamics, i.e. the one induced by a synchronous or parallel update scheme. In particular, we give both sufficient and necessary conditions for stability. We study stability from two approaches first, the existence of only fixed points as attractors for the dynamics and also the existence of total two cycles. Then, we study the dynamics in the periodic case. In this section, we show that if we have no assumptions, long cycles (super polynomial cycles in the size of the network) may appear. Finally, for the block sequential case, we give some conditions for stability.

3.1 Synchronous or parallel dynamics

Sufficent conditions for stability. First, we cite a known result from [12,9]

Proposition 1. *Let $\mathcal{T}$ be a threshold network defined over a non-directed graph G in which its adjacency matrix $W(G)$ satsfies that $W(G)_{ij} \in \{-1, 1\}$. If $W(G)$ is non-negative definite then, the synchronous update scheme admits only fixed points.*

The latter result is based on the energy functional

$$E(x) = \frac{1}{2}x^T W x + b^T x$$

which is not-increasing. In fact, if the following functional is defined:

$$\Delta E(x^t) = E(x^{t+1}) - E(x^t) = \sum_{i=1}^{n} \delta_i - \frac{1}{2}(x^{t+1} - x^t)W(x^{t+1} - x^t),$$

where $\delta_i = (x_i^{t+1} - x_i^t)(\sum_{i=1}^{n} w_{ij}x_j^t - b_i)$, it is shown in [9,13], this difference is non-negative, i.e. the energy function is a non-increasing function. This has an important consequence in the dynamics, limiting the long-term behavior of the system to only fixed points and attractors of period 2. Moreover, if the energy delta is strictly negative, $\mathcal{T}$ admits only fixed points. In [13] the authors explore under which conditions attractors of period 2 may appear. More precisely, they explore a sufficient and necessary conditions on the interaction graph of $\mathcal{T}$ in which the latter situation holds. In this section, we present a generalization of their result to signed graphs. In order to do that, we start by stating the following technical lemma:

Lemma 1. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network. Suppose that there exists a configuration $x \in \{-1, 1\}$ such that $x_i \neq x'_i$ for all $i \in V$ where $x' = F(x)$. Then, we have that $\Delta E(x) \leq 2\mathcal{S}(G'(x))$*

We can now present the main theorem of this section:

Theorem 1. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network. If for each subgraph $G' \subseteq G$ we have that $\mathcal{S}(G) < 0$ then, $\mathcal{T}$ admits only fixed points.*

Necessary conditions for stability. Now we face the problem of studying what happens if the hypothesis of the previous lemma do not hold. More precisely, we are interested in the case in which there exist some $G' \subseteq G$ such that $\mathcal{S}(G') \geq 0$ First observe that if G is just a length 1 negative path, i.e. $E = \{i, j\}$ and $w_{ij} = -1$ then, $\mathcal{T} = (G, b)$ where $b_i = b_j = 0$ is a threshold automata network which admits a two-cycle. In fact, for $x_i = x_j = 1$ we have the cycle $(1, 1) \leftrightarrow (-1, -1)$. Generally speaking, we give in the next result a sufficient condition for the de existence of two-cycles. This condition is given in the following lemma and it is ilustrated in Figure 5

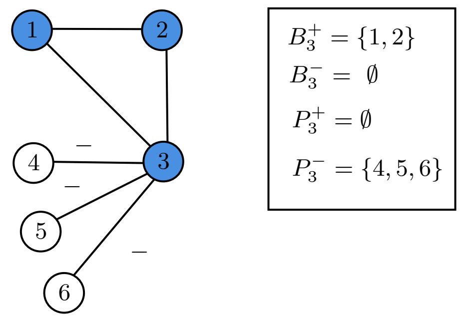

Fig. 5. An example of a network that exhibits a two cycle that is not a total cycle. Nodes in blue are in state 1 and nodes in -1 are in white. Sets B and P in Lemma 2 are given explicitely for this particular configuration. Observe that $\mathcal{S} = 1 \geq 0$ but the network does not exhibit total two-cycles.

Lemma 2. *Let $\mathcal{T} = (G = (V,E), W(G), \{-1,1\}, b, F)$ be a threshold network. $\mathcal{T}$ admits a two-cycle if and only if there exists a configuration $x \in \{-1,1\}^V$ such that, for all $u \in V$ we have:*

$$w_{uu} + (|B(x,u)^+| - |B(x,u)^-|) - (|P(x,u)^+| - |P(x,u)^-|) \geq \; 1 + |b_u|,$$

where $B(x,u)^+ = \{v \in N(u) : x_v = x_u, w_{uv} = +1\}$, $B(x,u)^- = \{v \in N(u) : x_v = x_u, w_{uv} = -1\}$, $P(x,u)^+ = \{v \in N(u) : x_v \neq x_u, w_{uv} = +1\}$ and $P(x,u)^- = \{v \in N(u) : x_v \neq x_u, w_{uv} = -1\}$.

From the latter lemma we deduce the following direct result:

Corollary 1. *Let $G = (V,E,W(G))$ be a signed graph. There exists a threshold vector $b \in \mathbb{Z}$ such that the threshold network $\mathcal{T} = (G = (V,E), W(G), \{-1,1\}, b, F)$ admits a total cycle if and only if there exists a configuration $x \in \{0,1\}^V$ such that: $\forall u \in V$,*

$$w_{uu} + (|B(x,u)^+| - |B(x,u)^-|) - (|P(x,u)^+| - |P(x,u)^-|) \geq 1,$$

where $B(x,u)^+ = \{v \in N(u) : x_v = x_u, w_{uv} = +1\}$, $B(x,u)^- = \{v \in N(u) : x_v = x_u, w_{uv} = -1\}$, $P(x,u)^+ = \{v \in N(u) : x_v \neq x_u, w_{uv} = +1\}$ and $P(x,u)^- = \{v \in N(u) : x_v \neq x_u, w_{uv} = -1\}$.

Remark 1. Observe that the condition in the previous corollary implies that a signed graph admits attractors of period 2 if and only if $b_u = 0$ for all $u \in V$. In the case in which the edges have only positive weight, this latter threshold defines the majority rule. Thus, in the case of an unsigned graph, another way to interpret the previous corollary is that if a graph does not admit attractors of period 2 for the majority rule, it will have only fixed points for any other threshold.

Theorem 2. *Let $G = (V, E, W(G))$ be a signed graph. If there exists a threshold vector $b \in \mathbb{Z}$ such that the threshold network $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ admits a total cycle, then $\mathcal{S}(G) \geq 0$. Conversely, if $\mathcal{S}(G) \geq 0$ then there exist a threshold vector $b \in \mathbb{Z}$ such that the threshold network $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ admits admits an attractor of period* 2.

Remark 2.
1. As it is shown in [10] there are some case in which $\mathcal{S}(G) \geq 0$ and there are no total cycles.
2. If there are at least one node with a negative loop then, there exists a total cycle. In fact, let us call i to a node having a negative loop. We can fix in state -1 all the nodes in $G - i$ by using the same technique that we used in the proof of the previous lemma and defining $b_i = -|N(i)_G|$. By doing this we have that $\sum\limits_{u \in V} w_{ui}x_i - b_u = -x_i - |N(i)_G| + |N(i)_G| = -x_i$. Thus, if $x_i = 1$ the node i changes its state to 1 and if it is in state -1 it changes to 1.

3.2 Periodic update schemes.

We extend the results of the previous section to the case of periodic update schemes. Remember that a periodic update scheme is a sequence $\mu = (I_1, \ldots, I_\ell)$ such that $I_k \in \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of V. From now on, for a periodic update scheme $\mu = (I_1, \ldots, I_\ell)$ we are going to call $G(I_k)$ to the subgraph induced by the set of nodes I_k

Theorem 3. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network and let us consider a periodic update scheme $\mu = (I_1, \ldots, I_p)$. If for any $1 \leq \ell \leq p$ we have that for all subgraph $G' \subseteq G(I_\ell)$, $\mathcal{S}(G') < 0$ then, $\mathcal{T}$ admits only fixed points.*

Proof. The result holds as a consequence of the fact that for each $k \in \{0, \ldots, p\}$ the nodes that may change its state are the nodes in I_k and the nodes in any other I_s with $s \neq k$ are fixed. Thus, since any subgraph $G' \subseteq G(I_k)$ satisfies $\mathcal{S}(G') < 0$ then, the energy is decreasing.

Attractors with superpolynomial period in cycle graphs We show that if there are some graph such that $\mathcal{S}(G)$ attractors with period $p > 2$ may appear even restricted to cycle graphs.

In order to illustrate this, we show the following example corresponding to an elementary cellular automaton with periodic boundary conditions. Observe that this is the same than considering a particular threshold network defined over a cycle graph. In the example of the figure, the local rule of each node is the majority rule but in tie case the nodes will switch its state. This can be represented by a negative loop in each cell.

As can be noted in the example of Table 1 in the anex, after applying a particular periodic update scheme, attractors of period 5 may appear.

Observe that in this case $\mathcal{S}(G) = -8 - 0 + 16 + 8 - 4 \times 0 = 16 > 0$ since $p = 0, n = m$ and each node has a negative loop.

We also observe that this construction can be done for each even number n such that $n > 6$. Thus, one can exhibit networks with attractors of period $n - 3$ for each n where n is even and $n > 6$.

Lemma 3. *For each $n > 6$, such that n is even, there exists a periodic update scheme μ, a cycle graph C_n and a threshold network $\mathcal{T}$ defined over C_n such that, $\mathcal{T}$ admits attractors of period $n - 3$.*

We can generalize this idea in order to show that for $N > 0$ there exists a threshold network of size N admitting attractors of super-polynomial period. In order to show this result, we use Lemma 3 together with the previous technical result. This is a classical technique used in [13,11,8]

Theorem 4. *There exists a threshold network $\mathcal{T}$ defined over a cycle graph C_N such that $\mathcal{T}$ admits attractors of super-polynomial period in N.*

4 Discussion

In this paper, we have presented a graph parameter, the stability index, which links the dynamics of a threshold network with the structure of the underlying signed graph. The sign of this parameter for subgraphs allow us to determine wether the dynamics is stable or not. However, as it is mentioned at the beginning of the article, computing this index could be very impractical (computing p is **NP**-hard in general). In this sense, a particularly interesting approach, could be the study of update schemes induced by sets of bounded size. For example, it is simple to see that if one study the family of all update schemes with at most 3 nodes, the stability for connected graphs can be characterized in terms of forbidden subgraphs (notably signed triangles). In this context an exhaustive study of different subgraphs could be a promising approach. Finally, it could be also interesting to study the structure of particular graph structures that may be of interest of some applications such as regulatory networks or social networks.

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A Full proofs

A.1 Synchronous update schemes

Lemma 1. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network. Suppose that there exists a configuration $x \in \{-1, 1\}$ such that $x_i \neq x_i'$ for all $i \in V$ where $x' = F(x)$. Then, we have that $\Delta E(x) \leq 2\mathcal{S}(G'(x))$*

Proof. We have that $\Delta E \leq \sum_{i=1}^{n} \delta_i - \frac{1}{2}(x' - x)W(x' - x)$. First, observe that for each i we have $-\delta_i \leq -2$, thus we have that $\Delta E(x) - 2n\frac{1}{2}(x' - x)W(x' - x)$. Then, by expanding the term $\frac{1}{2}(x' - x)W(x' - x)$, we get the following bound for the energy difference:

$$\Delta E \leq -2n - 2d^{+}2d^{-} + 4\varphi(x),$$

where $\varphi(x) = \sum_{i<j} -w_{ij}x_ix_j. = \sum_{i<j} \overline{w_{ij}}x_ix_j$. Observe that in the previous quadratic form, we have change the terms $y_i = (x_i' - x_i) \in \{-2, 2\}$ to $x_i \in \{-1, 1\}$ so a factor of 2 has appeared multiplying ϕ. Also, observe that if $-G$ is balanced, the configuration x^* in which vertices of the same color are connected by positive edges maximizes φ and in the general case, the amount of frustrated edges will decrease the value of φ. In fact, for x (and, actually, for any configuration) we can define the sets $E^{\pm\pm} = E^{\pm\pm}(x) = \{e = (i, j) : (x_i = x_j = \pm 1), \overline{w}_{ij} = \pm 1\}$ and $\delta^{\pm} = \delta^{\pm}(x) = \{e = (i, j) : x_i \neq x_j, \overline{w}_{ij} = \pm 1\}$. Observe that if $-G$ is balance, we have that $E^{+-} = E^{--} = \delta^{+} = \emptyset$.

Then, we have that:

$$\begin{aligned} \varphi(x) = &\sum_{E^{++}} \overline{w_{ij}}x_ix_j + \sum_{E^{+-}} \overline{w_{ij}}x_ix_j \\ &+ \sum_{E^{--}} \overline{w_{ij}}x_ix_j + \sum_{E^{-+}} \overline{w_{ij}}x_ix_j \\ &+ \sum_{\delta^{+}} \overline{w_{ij}}x_ix_j + \sum_{\delta^{-}} \overline{w_{ij}}x_ix_j \end{aligned}$$

Thus,

$$\varphi(x) = (|E^{++}| + |E^{-+}| + |\delta^{-}|) - (|E^{+-}| + |E^{--}| + |\delta^{+}|).$$

Since $m = |E^{++}| + |E^{-+}| + |\delta^{-}| + |E^{+-}| + |E^{--}| + |\delta^{+}|)$ we have that:

$$\varphi(x) = m - 2(|E^{+-}| + |E^{--}| + |\delta^{+}|).$$

Finally, as the number of frustrated edges in the configuration x is exactly $|E^{+-}| + |E^{--}| + |\delta^{+}|$ we have that $p = \phi(-G) \leq |E^{+-}| + |E^{--}| + |\delta^{+}|$ and thus,

$$\max_{x \in \{-1,1\}} \varphi(x) = m - 2p,$$

and the we deduce

$$\Delta E \leq -2n - 2d^{+} + 2d^{-} + 4m - 8p = 2\mathcal{S}(G)$$

Theorem 1. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network. If for each subgraph $G' \subseteq G$ we have that $\mathcal{S}(G) < 0$ then, $\mathcal{T}$ admits only fixed points.*

Proof. Observe that $\mathcal{T}$ can only admit attractors of period 2 and fixed points. Let us assume that $\mathcal{T}$ admits some attractor of period two. Let us define $V^{'} = \{i \in V : x_i \neq x_i'\}$ induces a subgraph $G^{'}$ of G. From the previous lemma, we have that $\Delta E = E(x') - E(x) \leq 2\mathcal{S}(G^{'}) < 0$, and thus $E(x') < E(x)$. Similarly, since x has period 2, we deduce $E(x) < E(x')$, which is a contradiction.

Lemma 2. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network. $\mathcal{T}$ admits a two-cycle if and only if there exists a configuration $x \in \{-1, 1\}^V$ such that, for all $u \in V$ we have:*

$$w_{uu} + (|B(x, u)^{+}| - |B(x, u)^{-}|) - (|P(x, u)^{+}| - |P(x, u)^{-}|) \geq\ 1 + |b_u|,$$

where $B(x, u)^{+} = \{v \in N(u) : x_v = x_u, w_{uv} = +1\}$, $B(x, u)^{-} = \{v \in N(u) : x_v = x_u, w_{uv} = -1\}$, $P(x, u)^{+} = \{v \in N(u) : x_v \neq x_u, w_{uv} = +1\}$ and $P(x, u)^{-} = \{v \in N(u) : x_v \neq x_u, w_{uv} = -1\}$.

Proof. First, let us assume that G admits a two-cycle $x \in \{-1, 1\}^V$. Let us define $B(x, u)^{+} = \{v \in N(u) : x_v = x_u, w_{uv} = +\}$, $B(x, u)^{-} = \{v \in N(u) : x_v = x_u, w_{uv} = -1\}$, $P(x, u)^{+} = \{v \in N(u) : x_v \neq x_u, w_{uv} = +1\}$ and $P(x, u)^{-} = \{v \in N(u) : x_v \neq x_u, w_{uv} = -1\}$. Since x is a total cycle, for each $u \in V$ we have that two cases:

1. if $x_u = 1$ then, $F(x)_u = -1$,
2. if $x_u = -1$ then, $F(x)_u = 1$.

If we use the definition of the local rule, we deduce that the latter cases are equivalent to the following conditions:

1. if $x_u = 1$ then, $\sum\limits_{x \in V} w_{uv}x_v - b_u > 0$,
2. if $x_u = -1$ then, $\sum\limits_{x \in V} w_{uv}x_v - b_u < 0$.

Finally, if we re-write the latter in terms of the sets P and B we deduce the following conditions:

1. if $x_u = 1$ then, $w_{uu} + (|B(x, u)^{+}| - |B(x, u)^{-}|) - (|P(x, u)^{+}| - |P(x, u)^{-}|) \geq\ 1 + b_u$,
2. if $x_u = -1$ then, $w_{uu} + (|B(x, u)^{+}| - |B(x, u)^{-}|) - (|P(x, u)^{+}| - |P(x, u)^{-}|) \geq 1 - b_u$,

And thus, the conditions hold.

Now suppose there exists some configuration x such that

$$w_{uu} + (|B(x,u)^{+}| - |B(x,u)^{-}|) - (|P(x,u)^{+}| - |P(x,u)^{-}|) \geq 1 + |b_u|$$

Then, in particular, we have that the latter condition imply that:

1. if $x_u = 1$ then, $F(x)_u = -1$,
2. if $x_u = -1$ then, $F(x)_u = 1$.

And thus, x is a two-cycle. The proposition holds.

Corollary 1. *Let $G = (V, E, W(G))$ be a signed graph. There exists a threshold vector $b \in \mathbb{Z}$ such that the threshold network $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ admits a total cycle if and only if there exists a configuration $x \in \{0, 1\}^V$ such that: $\forall u \in V$,*

$$w_{uu} + (|B(x,u)^{+}| - |B(x,u)^{-}|) - (|P(x,u)^{+}| - |P(x,u)^{-}|) \geq 1,$$

where $B(x,u)^{+} = \{v \in N(u) : x_v = x_u, w_{uv} = +1\}$, $B(x,u)^{-} = \{v \in N(u) : x_v = x_u, w_{uv} = -1\}$, $P(x,u)^{+} = \{v \in N(u) : x_v \neq x_u, w_{uv} = +1\}$ and $P(x,u)^{-} = \{v \in N(u) : x_v \neq x_u, w_{uv} = -1\}$.

Proof. First, assume that there exists b such that G admits two-cycles. Let x be a two-cycle for G. Let us define $B(x,u)^{+} = \{v \in N(u) : x_v = x_u, w_{uv} = +\}$, $B(x,u)^{-} = \{v \in N(u) : x_v = x_u, w_{uv} = -\}$, $P(x,u)^{+} = \{v \in N(u) : x_v \neq x_u, w_{uv} = +\}$ and $P(x,u)^{-} = \{v \in N(u) : x_v \neq x_u, w_{uv} = -\}$. By the previous lemma, we have that:

$$w_{uu} + (|B(x,u)^{+}| - |B(x,u)^{-}|) - (|P(x,u)^{+}| - |P(x,u)^{-}|) \geq \ 1 + |b_u|$$

And thus,

$$w_{uu} + (|B(x,u)^{+}| - |B(x,u)^{-}|) - (|P(x,u)^{+}| - |P(x,u)^{-}|) \geq \ 1$$

Conversely, if we have

$$w_{uu} + (|B(x,u)^{+}| - |B(x,u)^{-}|) - (|P(x,u)^{+}| - |P(x,u)^{-}|) \geq \ 1$$

for some configuration x we can define $b_u = 0$ for all $v \ \in V$ and thus, by the previous remark, we have that x is a two-cycle. The corollary holds.

Theorem 2. *Let $G = (V, E, W(G))$ be a signed graph. If there exists a threshold vector $b \in \mathbb{Z}$ such that the threshold network $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ admits a total cycle, then $\mathcal{S}(G) \geq 0$. Conversely, if $\mathcal{S}(G) \geq 0$ then there exist a threshold vector $b \in \mathbb{Z}$ such that the threshold network $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ admits admits an attractor of period* 2.

Proof. First, observe that for each configuration $x \in \{-1,1\}^V$ we have:

$$\sum_{u \in V}(|B(x,u)^+| - |B(x,u)^-|) - (|P(x,u)^+| - |P(x,u)^-|) =$$
$$2(|E^{++}| + |E^{-+}| + |\delta^-|) - (|E^{+-}| + |E^{--}| + |\delta^+|) = 2\varphi(x)$$

And thus, $2\varphi(x) = \sum_{u \in V}(|B(x,u)^+| - |B(x,u)^-|) - (|P(x,u)^+| - |P(x,u)^-|)$.

Now, by the previous corollary, we have that: for all $u \in V$,

$$w_{uu} + (|B(x,u)^+| - |B(x,u)^-|) - (|P(x,u)^+| - |P(x,u)^-|) \geq 1,$$

Then, we deduce that

$$d^+ - d^- + 2\varphi(x) \geq n,$$

Thus, we get

$$\mathcal{S}(G) \geq -n + d^+ - d^- + 2\varphi(x) \geq 0.$$

Conversely, let us assume $\mathcal{S}(G) \geq 0$. Then, there exists some configuration $x^* \in \{-1,1\}$ such that $\varphi(x^*) \geq \phi(x)$. Again, observe that $\varphi(x^*) = m - 2p$. Thus, we have that $\mathcal{S}(G) = -n + d^+ - d^- + 2\phi(x^*) \geq 0$. By rearranging the latter term we get that:

$$\sum_{u \in V}\big(w_{uu} + (|B(x^*,u)^+| - |B(x^*,u)^-|) - (|P(x^*,u)^+| - |P(x^*,u)^-| - 1)\big) \geq 0.$$

Thus, there must exist some subset $V' \subseteq V$ such that, for every $u \in V'$, we have that

$$w_{uu} + |B(x^*,u)^+| - |B(x^*,u)^-|) - (|P(x^*,u)^+| - |P(x^*,u)^-|) - 1 > 0.$$

Then, thanks to Corollary 1, there exists some threshold vector $b' \in \mathbb{Z}^{|V'|}$ such that b' induces a total cycle y on the subgraph V'. We are going to extend b' to a threshold vector on G named b such that, there exists some x which is an attractor of period 2 for $\mathcal{T}$. The idea is that we are going to fix in state -1 any node outside of V' by defining a large enough value for its threshold (for example, a value greater than the degree of the node) and we are going to slightly change the value of b'_i for any node in V' so it can change its state without being affected by the state of its neighbors in $G \setminus G'$. More precisely, let us define $b_i = b'_i - (|N_G(i)| - |N_{G'}(i)|)$, i.e. for each node in V' we consider the same threshold b'_i but we substract the amount of neighbors that are not in V'. In addition, we define $b_i = 2|N_G(i)|$ whenever $i \in V \setminus V'$. We are going to define $x_i = y_i$ whenever $i \in V'$ and $x_i = -1$ whenever $i \notin V'$. Observe that x is an attractor of period 2 for $\mathcal{T}$. In fact, for each node $u \in V'$ we have that:

$$\sum_{u \in V} w_{uv} x_v - b_u =$$

$$\sum_{u \in V'} w_{uv} x_v - (|N(u)|_G - |N(u)|'_G) - b'_u + (|N(u)|_G - |N(u)|'_G) =$$

$$\sum_{u \in V'} w_{uv} y_v - b'_u.$$

And for every node $u \in V \setminus V'$ we have that

$$\sum_{u \in V} w_{uv} x_v - b_u = \sum_{u \in V} w_{uv} x_v - 2|N(u)_G| < 0.$$

The corollary holds.

A.2 Periodic update schemes

Theorem 3. *Let $\mathcal{T} = (G = (V, E), W(G), \{-1, 1\}, b, F)$ be a threshold network and let us consider a periodic update scheme $\mu = (I_1, \ldots, I_p)$. If for any $1 \leq \ell \leq p$ we have that for all subgraph $G' \subseteq G(I_\ell)$, $\mathcal{S}(G') < 0$ then, $\mathcal{T}$ admits only fixed points.*

Proof. The result holds as a consequence of the fact that for each $k \in \{0, \ldots, p\}$ the nodes that may change its state are the nodes in I_k and the nodes in any other I_s with $s \neq k$ are fixed. Thus, since any subgraph $G' \subseteq G(I_k)$ satisfies $\mathcal{S}(G') < 0$ then, the energy is decreasing.

Lemma 3. *For each $n > 6$, such that n is even, there exists a periodic update scheme μ, a cycle graph C_n and a threshold network $\mathcal{T}$ defined over C_n such that, $\mathcal{T}$ admits attractors of period $n - 3$.*

Proof. By a straightforward induction argument, the example in Table 1 (which corresponds to the case $n = 8$) can be extended to any even $n > 6$.

We present now a technical lemma (see [15] for more details) that we are going to use to show the main result of this section.

Lemma 4 ([15]). *Let $m \geq 2$ and $\mathcal{P}(m) = \{p \leq m \mid p \text{ prime}\}$. If we define $\pi(m) = |\mathcal{P}(m)|$ and $\theta(m) = \sum_{p \in \mathcal{P}(m)} \log(p)$ then we have $\pi(m) \sim \frac{m}{\log(m)}$ and $\theta(m) \sim m$.*

Theorem 4. *There exists a threshold network $\mathcal{T}$ defined over a cycle graph C_N such that $\mathcal{T}$ admits attractors of super-polynomial period in N.*

Proof. Let as fix $m > 0$. Now, as a consequence of the previous lemma, we have that for each $p_i \in \mathcal{P}(m)$ there exists a threshold network $\mathcal{T}_i$ of size $2k_i$ defined over a cycle graph such that $\mathcal{T}_i$ has attractors of period p_i. In fact, it suffices to define $k_i = \frac{p_i+3}{2} \in \mathbb{N}$. Now, we are going to construct a cycle of size C_N by concatenating each cycle C_{2k_i}. In addition, we are going to define a threshold network $\mathcal{T}$ over C_s in which each node will have the same local rule than in $\mathcal{T}_\rangle$, i.e., each node in $\mathcal{T}$ is equipped with the unstable majority rule. Observe that if we define a configuration x as the concatenation of each of the attractors x_i defined over $\mathcal{T}_i$ and we consider μ as the update scheme obtained by the union of the sets in each update scheme μ_i, we have that the dynamics of each cycle es completely independent of the rest. Thus, we have a threshold network $\mathcal{T}$ on C_N admitting an attractor of period $T \geq \exp(\theta(m))$. Also observe that $N = \sum\limits_{i=1}^{\pi(m)} p_i + 3\pi(m)$. Using the latter technical lemma, we deduce that $m = \pi(m)\log m$ and that $N = \pi(m)^2 \log m$. Thus, we have that $T \geq 2^{\Omega(\sqrt{N \log N})}$. The lemma holds.

1	2	3	4	5	6	7	8	Updated nodes
-1	1	-1	-1	-1	-1	1	-1	$\{3,5\}$
-1	1	1	-1	-1	-1	1	-1	$\{1,2,7,8\}$
1	-1	1	-1	-1	-1	-1	1	$\{4,6\}$
1	-1	1	1	-1	-1	-1	1	$\{1,2,7,8\}$
-1	1	1	1	-1	-1	1	-1	$\{3,5\}$
-1	1	1	1	1	-1	1	-1	$\{1,2,7,8\}$
1	-1	1	1	1	-1	-1	1	$\{4,6\}$
1	-1	1	1	1	1	-1	1	$\{1,2,7,8\}$
-1	1	1	1	1	1	1	-1	$\{3,5\}$
-1	1	1	1	1	1	1	-1	$\{1,2,7,8\}$
1	-1	1	1	1	1	-1	1	$\{4,6\}$
1	-1	1	1	1	-1	-1	1	$\{1,2,7,8\}$
-1	1	1	1	1	-1	1	-1	$\{3,5\}$
-1	1	1	1	-1	-1	1	-1	$\{1,2,7,8\}$
1	-1	1	1	-1	-1	-1	1	$\{4,6\}$
1	-1	1	-1	-1	-1	-1	1	$\{1,2,7,8\}$
-1	1	1	-1	-1	-1	1	-1	$\{3,5\}$
-1	1	-1	-1	-1	-1	1	-1	$\{1,2,7,8\}$
1	-1	-1	-1	-1	-1	-1	1	$\{4,6\}$
1	-1	-1	-1	-1	-1	-1	1	$\{1,2,7,8\}$

Table 1. Dynamics of an attractor of period 5 obtained by applying the update scheme $\mu = (\{3,5\},\{1,2,7,8\},\{4,6\},\{1,2,7,8\}$.

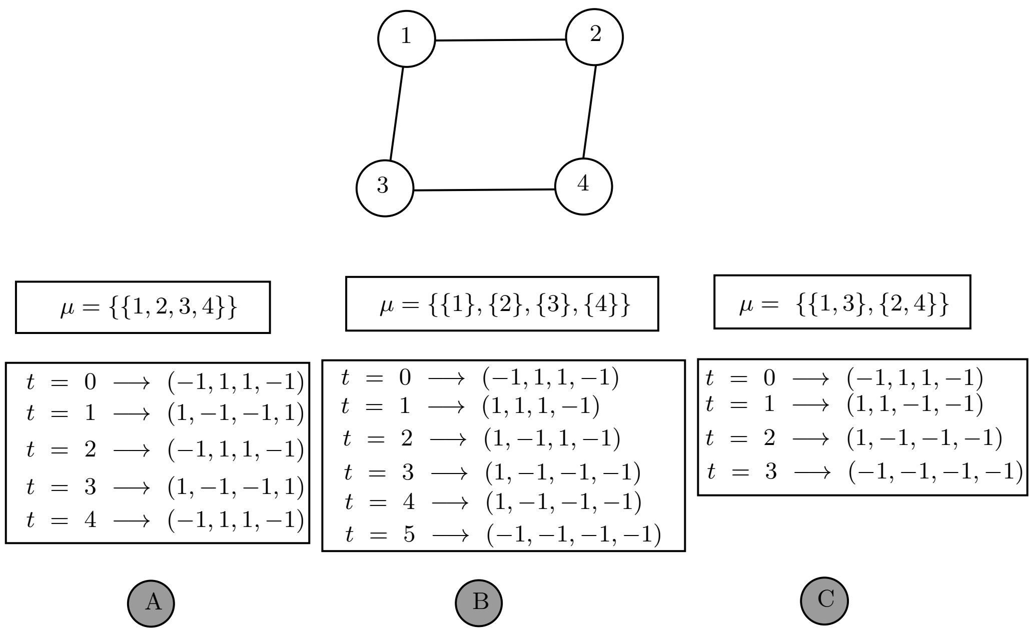

Fig. 6. Threshold network with $\theta_v = 0$ for each $v \in V$ under different update schemes. A) Parallel update scheme, B) Sequential update scheme and C) Block sequential update scheme.