

On the Convergence of Elementary Cellular Automata under Sequential Update Modes

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Abstract

In this paper, we perform an theoretical analysis of the sequential convergence of elementary cellular automata that have at least one fixed point. Our aim is to establish which elementary rules always reached fixed points under some sequential update modes, regardless of the initial configuration. In this context, we classify these rules according to whether all initial configurations converge under all, some, one or none sequential update modes, depending on if they have fixed points under synchronous (or parallel) update modes.

Keywords: Elementary cellular automata, Boolean automata networks, Asynchronism, Sequential update modes, Convergence

1. Introduction

Cellular automata are simple systems that can hold great complexity [43]. They are composed of cells ordered in a grid that interact with one another following a local rule over discrete time. They were first introduced by Ulam and von Neuman [41], which were inspired in the work on formal neural networks done by McCulloch and Pitts [20] and brought . Cellular Automata have since proved to be useful in multiple areas of science, such as biology and health sciences [15, 38], social models [32, 35], parallel and distributed computation [39, 19], and physics [40, 5, 33].

In the original definition of the cellular automata model, all cells are updated simultaneously. However, issues with synchronicity have been raised from both mathematical modeling and computational perspectives [9]. Therefore, *asynchronous update modes* have been proposed such as: fully asynchronous [11], α -asynchronous [10], random sweep [36, 16] as well as others which are probabilistic in nature.

In [30, 13, 14, 29] *deterministic update modes* are studied which have the advantage of breaking the dependency on parallelism without introducing randomness. These update modes have opened interesting areas of study, such as: the sensitivity to synchronism in elementary cellular automata with respect to the number of possible resulting dynamics in [23, 3] and with respect to the longest limit cycle in [8], the complexity of the reachability problem is studied in [4], while in the context of Boolean networks (of which elementary cellular automata are particular cases), their convergence under non-deterministic and deterministic update modes are studied in [21], the robustness of their dynamical properties under different deterministic update modes are established in [2], different kinds of non-deterministic and deterministic Boolean networks are compared with regards to their attractors in [12] and in [27] limit cycles of automata networks are studied.

The uses of these update modes are varied, for example: in [18] block-sequential cellular automata are proposed as representations of physical concepts, in [22] asynchronous deterministic Boolean networks are used to portray biological systems, in [26, 31] sequential cellular automata are proposed as a solution to the density classification task and in [6] automata networks under ordered asynchronous update modes are suggested to depict different artificial and biological systems.

We have chosen to utilize deterministic update modes, among them, we have focused on sequential update modes, which consist on individual cells being updated one after another according to an established order. This allows us to exploit tools from combinatorics [34], since we are working over a ring and thus we can ignore certain sequential update modes, and discrete mathematics [1], to focus only on non equivalent update modes.

1.1. Our Contribution

In [30] it was proven that elementary cellular automata under block-sequential update modes (which include the sequential ones) have exactly the same fixed points as they do under parallel update mode. Therefore, we study the convergence of elementary cellular automata that have fixed points in parallel, knowing that “new” fixed points will not appear under sequential update modes.

Thus, in this paper we classify elementary rules according to if there exists a sequential update mode that leads all configurations to a fixed point, from where we introduce the concepts of *universal* and *covering of* update modes. For a given rule, a covering is a set of update modes such that for any initial configuration, there exists an element in the set of update mode that if the rule is applied following the order established by it the configuration reaches

a fixed point. If a set containing only one update mode is a covering, then that update mode is universal.

We start with the rules characterized in [37] as always reaching fixed points under asynchronous updating, proving that sequential update modes in particular are enough to replicate the result for all but two of these rules.

For the 50 rules that fulfill the conditions of established in [37], we provide detailed proofs for rules numbered 2, 26, and 104 according to Wolfram’s code [42], because the other rules follow similar reasoning as the ones we have chosen to showcase. Notably, we found that for Rule 106, which is a complex rule according to Wolfram’s classification [43], there is a sequential update mode that leads all configurations to a fixed point, but only when the size of the ring is even. Here, we introduce the notion of E -universal, meaning that a given rule under that update mode reaches a fixed point from all configurations belonging to a subset E . A summary of this part of the classification is provided in Table 1.

We continue our study with the 38 non-equivalent elementary cellular automata that fall outside the purview of said paper and show that five of them that can always reach fixed points, when said fixed points exist. This means that the result is weaker than the one obtained for the previous rules, since fixed points do not necessarily exist for all ring sizes, but when they do exist, there are sequential update modes that lead to them. In addition, we prove that there are rules which have fixed points that cannot be reached under any sequential update mode from certain configurations. The outline of the rules of this section is provided in Table 6.

1.2. Structure of the Paper

In Section 2, the main definitions and notations are formalized. Section 3 contains the demonstrations that justify the classification that is being proposed. In the first part, we will prove that most of the 50 rules given by the theorem in [37] can always reach fixed points under sequential update modes; while the second part focuses on elementary cellular automata outside the theorem, where additional conditions are needed to ensure the convergence to fixed points. The paper ends with Section 4, in which we discuss the results as well as some future work.

2. Preliminaries

2.1. Elementary Cellular Automata and Sequential Update Modes

In broad terms, a *cellular automata* (CA) is a collection of cells, each of which will have a state within the alphabet Q . The cells interact with

each other over discrete time according to a rule which is defined by the state of the cells in their respective neighborhood. Formally, a CA is a tuple $(\mathbb{Z}^d, Q, N, f)$ composed of

- $\mathbb{Z}^d$ the d –dimensional cellular space, also known as *grid*, which can be finite or infinite,
- Q the finite set of states that a cell can have, called the *alphabet*,
- N the neighborhood, which associates a cell of the grid with its neighbors,
- f the rule, which is defined by local functions $f_i^{N_i} \rightarrow Q$, where f_i is the i –th component of f and N_i is the neighborhood of cell i .

An *elementary cellular automata* (ECA) is a kind of CA such that the grid is 1–dimensional, the alphabet is $\mathbb{B} = \{0, 1\}$, the local function f_i is the same for the entire grid and it depends on the state of the cells $i - 1$, i itself, and $i + 1$. From where it follows that there are $2^{2^3} = 256$ distinct ECA, of which there are 88 that are not equivalent through symmetry [17].

As previously stated, the cellular space can be either infinite or finite. In this paper we have chosen to work with finite ECA, which effectively means that the grid is viewed as a 1–dimensional torus, that is, a ring of integers modulo n : $\mathbb{Z}/n\mathbb{Z}$. As such, the neighborhood of cell 0 is $\{n - 1, 0, 1\}$ and that of cell $n - 1$ is $\{n - 2, n - 1, 0\}$. Thus, a configuration x is an element of $\mathbb{B}^n$, is formally defined by a Boolean vector of dimension n and can equivalently be represented by a binary word in what follows for the sake of clarity and concision, according to the context (e.g. $x = (0, 1, 1, 0, 1) \equiv 01101$).

We can think of an update mode as a function that associates each iteration to the set of cells that will be updated in it, that is $\mu^* : \mathbb{N} \rightarrow \mathcal{P}(\llbracket n \rrbracket)$. In this paper we will work with *periodic update modes*, which means that there exists $p \in \mathbb{N}$ such that for all $t \in \mathbb{N}$, $\mu^*(t + p) = \mu^*(t)$. In other words an update mode of period p can be viewed as a finite sequence of p subsets of cells. Each iteration in which a subset of cells is updated is called a *substep*, while a *step* is the composition of the substeps having occurred over a period p .

A *sequential update mode* is a family of periodic update modes which is defined as $\mu = (\phi(\llbracket n \rrbracket))$, where $\phi(\llbracket n \rrbracket) = \{i_0\}, \dots, \{i_{n-1}\}$ is a permutation of $\llbracket n \rrbracket$, makes one and only one cell update its state at each substep so that all cells have updated their state after n substeps (i.e. after each step) depending on the order induced by ϕ . Naturally, for these update modes the period is always equal to the number of cells.

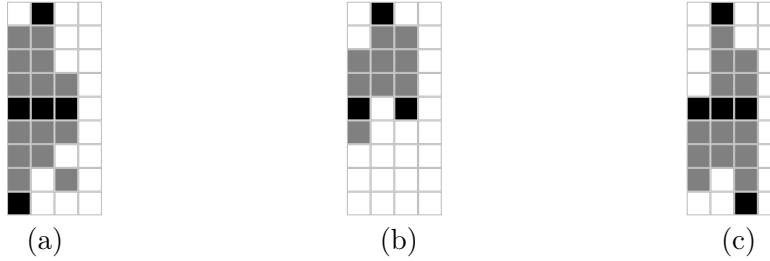


Figure 1: Space-time diagrams (time going downward) of configuration 0100 following Rule 45 under: (a) the update mode $\mu_1 = (0, 1, 2, 3)$, (b) the update mode $\mu_2 = (2, 0, 3, 1)$, (c) the update mode $\mu_3 = (3, 2, 1, 0)$. Configurations colored in gray (resp. black) represent the ones obtained at substeps (resp. steps).

Note that while the formal notation for a sequential update mode is $\mu = (\{i_0\}, \{i_1\}, \dots, \{i_{n-1}\})$, namely a sequence of n subsets of cardinality 1, we make an abuse of notation in what follows by using just a sequence of elements: $\mu = (i_0, i_1, \dots, i_{n-1})$ to facilitate reading.

Given an ECA f , if there is a set of update modes under which all configurations of a ring of size n reach a fixed point, we will call this set a *covering*. Formally, a set $S = \{\mu_0, \mu_1, \dots, \mu_k\}$ is a covering for f if for any $x \in \mathbb{B}^n$ there exists $\mu \in S$ such that there is a time t in which $f_\mu^t(x)$ is a fixed point.

If there is a covering whose cardinality is equal to 1, the update mode in such a covering is said to be *universal*, in the sense that under it all configurations are able to reach a fixed point. If for a given ECA f there exists an update mode μ that allows all $x \in X \subsetneq \mathbb{B}^n$ to reach a fixed point, then μ is X -universal for f .

An *asynchronous update mode* is an update mode in which only one cell can be updated per substep. Naturally, sequential update modes fall under this definition. However asynchronous update modes can also be non-deterministic, for instance if the cell to be updated is selected at random.

2.2. Dynamical Systems

An ECA f along with an update mode μ defines a discrete dynamical system on $\mathbb{B}^n \rightarrow \mathbb{B}^n$. More formally in our framework, let f be an ECA which is applied over a grid of size n and let μ be a periodic update mode represented as a periodical sequence of subsets of $\llbracket n \rrbracket$ such that $\mu = (B_0, \dots, B_{p-1})$. Function $f_\mu = (f, \mu)$, defined $f_\mu : \mathbb{B}^n \rightarrow \mathbb{B}^n$ defines the *discrete dynamical system* related to ECA f and update mode μ . To avoid confusion when it is useful, since ECA f is associated with its Wolfram code, we add this code

to the notation as follows: if ECA f_μ is the dynamical system of the rule having Wolfram's code w under μ , we will denote it as $f_{\mu,w}$.

Let $x \in \mathbb{B}^n$ be a configuration of f_μ . The *orbit* of x is the sequence of configurations which results from applying successively f_μ to x , they compose the set $\mathcal{O}(x)$. Since f is defined over a grid of finite size and the boundary condition is periodic, the temporal evolution of x governed by the successive applications of f_μ must lead it to eventually enter into a *limit phase*, i.e. a cyclic sequence $\mathcal{C}(x)$ such that $\forall y = f_\mu^k(x) \in \mathcal{C}(x), \exists t \in \mathbb{N}, f_\mu^t(y) = y$, with $k \in \mathbb{N}$. The orbit is then separated into two phases: The limit phase and the *transient phase* which corresponds to the finite sequence $(x^0 = x, x^1 = f(x), \dots, x^\ell = f^\ell(x))$ of length ℓ such that $\forall i \in \llbracket \ell + 1 \rrbracket, \nexists t \in \mathbb{N}, x^{i+t} = x^i$. The *limit set* of x is the set of configurations belonging to $\mathcal{C}(x)$. In other words, the limit set of x is the asymptotic orbit with respect to t of x . If the cardinality of the limit set is equal to 1, we call it a *fixed point*.

In the context of ECA, it is convenient to represent orbits by *space-time diagrams* which give a visual aspect of the latter, as illustrated in Figure 1.

For some of the proofs, we need to use the following specific notations.

Let $x \in \mathbb{B}^n$ be a configuration and $[i, j] \subseteq \llbracket n \rrbracket$ be a subset of cells. We denote by $x_{[i,j]}$ the projection of x on $[i, j]$. Since we work on ECA, such a projection defines a sub-configuration and can be of three kinds:

- $i < j$ and $x_{[i,j]} = (x_i, x_{i+1}, \dots, x_{j-1}, x_j)$,
- or $i = j$ and $x_{[i,j]} = (x_i)$,
- or $i > j$ and $x_{[i,j]} = (x_i, x_{i+1}, \dots, x_n, x_0, \dots, x_{j-1}, x_j)$.

Thus, given $x \in \mathbb{B}^n$ and $i \in \llbracket n \rrbracket$, an ECA f can be rewritten as

$$f(x) = (f(x_{[n-1,1]}), f(x_{[0,2]}), \dots, f(x_{[i,i+2]}), \dots, f(x_{[n-3,n-1]}), f(x_{[n-2,0]}).$$

Now, we introduce the notation: let x be the initial configuration, and let $i, j \in \mathbb{N}$, $x^{i,j}$ is the configuration obtained starting from x after applying a rule f for i steps and j substeps. For example, following this notation, the initial configuration x is $x = x^0 = x^{0,0}$, the configuration after the first substep is $x^{0,1}$, and the configuration after two steps and three substeps is $x^{2,3}$.

Notice that, with $k \leq n$, the word $u \in \mathbb{B}^k$ is called a *wall* for a given dynamical system if for all $a, b \in \mathbb{B}$, $f(aub) = u$. In this work we assume that walls are of size 2, i.e. $k = 2$, unless otherwise stated. Finally, a configuration x is said to be an *isle* of 1s (resp. an isle of 0s) if there exists an interval

$I = [a, b] \subsetneq \llbracket n \rrbracket$ such that $x_i = 1$ (resp. $x_i = 0$) for all $i \in I$ and $x_i = 0$ (resp. $x_i = 1$) for $i \in \{a+1, b+1\}$.

The following theorem states conditions on an ECA that guarantee its convergence to some fixed point under asynchronous updating, where a neighborhood of an asynchronous cellular automata is considered *active* if a rule f flips the state of the cell (1 to 0 or 0 to 1). Otherwise, the neighborhood is *passive*.

Theorem 1 ([37]). *Let f be an ECA. Then there exists (at least) one asynchronous update mode μ for f such that f_μ converges to fixed points if any of the following condition is verified:*

- (i) 000 (resp. 111) is passive and 010 (resp. 101) is active.
- (ii) 000, 001, 010 and 100 (resp. 011, 101, 110 and 111) are passive and 011 or 110 (resp. 001 or 100) are active.
- (iii) Either 001, 010, 100 and 101, or 010, 011, 101 and 110 are passive.

3. Results

In this section we present our results, starting with the 50 rules that fulfill the conditions of Theorem 1 and later, among the 38 rules that fall outside the purview of said theorem, we will touch upon the ones which admit fixed points.

3.1. Rules for which Theorem 1 holds.

We will begin this subsection studying the rules that can reach a fixed point from all initial configurations using only one sequential update mode, then we will show the rules that have a covering of sequential update modes. Additionally, Rule 90 will be given special attention, being the only rule that fulfills a condition of Theorem 1 that has configurations in all ring sizes that cannot reach a fixed point regardless of the sequential update mode.

In Table 1 we present a summary of the results of this section. Note that it was proven in [8] that 16 of the 50 non-equivalent rules that fulfill conditions of the theorem always reach fixed points regardless of the update mode and that 13 of them lead to fixed points under any sequential update mode. Now, we will present the results associated with the remaining 21 rules.

	I	II	III	IV	Total
All update modes are universal	0,8,128,136	4,12,36,44,72,76,78,132,140,164,200,204	$\emptyset$	$\emptyset$	16
All sequential update modes are universal	32,40,160,168	5,13,56,77,94,152,172,184,232	$\emptyset$	$\emptyset$	13
There is a sequential universal update mode	$\emptyset$	2,10,24,26,34,42,58,130,138,154,162,170	$\emptyset$	$\emptyset$	12
There is a covering of sequential update modes	$\emptyset$	50,74,104,178	18,122,146	$\emptyset$	7
There is no covering of sequential update modes	$\emptyset$	$\emptyset$	90	106	2

Table 1: Summary of the classification of the rules that fulfill at least one of the conditions stated by Theorem 1.

Rules that admit a Universal Sequential Update Mode.

Theorem 2. *The sequential update mode $\mu = (n-1, n-2, \dots, 0)$ is universal for an ECA rule R if any of the following is true:*

- 000 and 010 are passive, and 001, 010 and 110 are active.
- 000 and 011 are passive, and 001, 010, 100 and 110 are active.

Proof. Let $\mu = (n-1, n-2, \dots, 0)$. Let us study each of the rules that fulfill either condition.

- Rule 2: By definition (see Table 2), only $f_2(001) = 1$. Since we are updating the cells of the configuration one-by-one from right to left (by definition of μ), the state of all cells immediately becomes 0, unless the state of the first cell of the configuration is 1 and the last two have state equal to 0: $x^0(=x) = 1(1^{b_1}0^{a_1})1^{b_2} \dots 0^{a_{k-1}}(1^{b_k}0^{a_k})00$, such that $\sum_{i=1}^k a_i + b_i = n-3$. In this case, after the first substep, configuration $x^{0,0}$ becomes $x^{0,1} = 1(1^{b_1}0^{a_1})1^{b_2} \dots 0^{a_{k-1}}(1^{b_k}0^{a_k})(01)$. Then, after a_k substeps, $x^{0,1}$ becomes $x^{0,a_k+1} = 1(1^{b_1}0^{a_1})1^{b_2} \dots 0^{a_{k-1}}(1^{b_k}01^{a_k+2})$. Making another substep does not produce any change, since $f_2(101) = 0$, so $x^{0,a_k+2} = x^{0,a_k+1}$.

Then, by definition, the remaining substeps of the first step will make all the cells from 0 to $n - a_k - 2$ become 0 so that $x^1 = 0^{n-a_k-2}1^{a_k+2}$. Finally, the first $a_k + 2$ substeps of the second step lead the configuration to the fixed point 0^n , since $f_2(110) = 0$.

	111	110	101	100	011	010	001	000
2	0	0	0	0	0	0	1	0
10	0	0	0	0	1	0	1	0
26	0	0	0	1	1	0	1	0
34	0	0	1	0	0	0	1	0
42	0	0	1	0	1	0	1	0
58	0	0	1	1	1	0	1	0
130	1	0	0	0	0	0	1	0
138	1	0	0	0	1	0	1	0
154	1	0	0	1	1	0	1	0
162	1	0	1	0	0	0	1	0
170	1	0	1	0	1	0	1	0

Table 2: Definition of rules 2, 10, 26, 34, 42, 58, 130, 138, 154, 162 and 170.

- Rule 10: The analysis is the same as that for Rule 2, but now it includes a second type of configuration $x_1 = 10^{a_1}0^{b_1} \dots 0^{a_k}1^{b_k}01$ whose orbits have a transient part longer than zero.
- Rule 34: The analysis is the same as that for Rule 2, adding $x_2 = 10^{a_1}0^{b_1} \dots 0^{a_k}1^{b_k}10$ as another type of configuration whose orbits have a transient part longer than zero.
- Rule 42: The analysis is the same as that for Rule 2, also including x_1 from Rule 10 and x_2 from Rule 34 as configurations whose orbits have a transient part longer than zero.
- Rule 130: It is identical to Rule 2, except that now the configuration 1^n is also a fixed point, but with an empty basin of attraction.
- Rule 138: It is identical to Rule 10, except that now the configuration 1^n is also a fixed point, but with an empty basin of attraction.
- Rule 162: It is identical to Rule 34, except that now configuration 1^n is also a fixed point and configurations $y_1 = 10^{n-1}$, $y_2 = 1^20^{n-2}$, $\dots$, $y_{n-1} = 1^{n-1}0$ lead to it.
- Rule 170: It is identical to Rule 42, except that now configuration 1^n is also a fixed point. It is easy to see that the configurations lead to 1^n or 0^n depending on if the state of the first cell is 1 or 0 respectively.

- Rule 26: We need to do a case-by-case analysis:

$$\begin{aligned}
x = 1^{n-1}0 &\rightarrow f_{\mu,26}(x) = 0^n, \\
x = 1^{n-2}0^2 &\rightarrow f_{\mu,26}(x) = 0^{n-1}1 \rightarrow f_{\mu,26}^2(x) = 0^n, \\
x = 1^{n-3}0^3 &\rightarrow f_{\mu,26}(x) = 0^{n-2}1^2 \rightarrow f_{\mu,26}^2(x) = 0^n, \\
&\vdots \\
x = 1^{n-k}0^k &\rightarrow f_{\mu,26}(x) = 0^{n-k+1}1^{k-1} \rightarrow f_{\mu,26}^2(x) = 0^n,
\end{aligned}$$

Now, if we have an isle of 1s surrounded by 0s: $x = 0^{k_1}1^{k_2}0^{n-k_1-k_2}$
Then, we will have:

$$\begin{aligned}
f_{\mu,26}(x) &= 0^{k_1+k_2}10^{n-(k_1+k_2)-1}, \\
f_{\mu,26}^2(x) &= 1^{k_1+k_2+2}0^{n-(k_1+k_2+2)}, \\
f_{\mu,26}^4(x) &= 0^n.
\end{aligned}$$

The analysis is analogous for $x = 1^{k_1}0^{k_2}1^{n-k_1-k_2}$. Now, let us consider $x = 0^{a_1}1^{b_1} \dots 0^{a_k}1^{b_k}0^{a_{k+1}}$, then

$$f_{\mu,26}(x) = 0^{a_1+b_1}10^{a_2+b_2-1}1 \dots 0^{a_k+b_k-1}10^{a_{k+1}-1}.$$

If k is odd, then

$$f_{\mu,26}^2(x) = 1^{a_1+b_1+2}0^{a_2+b_2} \dots 1^{a_k+b_k}10^{a_{k+1}-2}.$$

If k is even, then

$$f_{\mu,26}^2(x) = 0^{a_1+b_1+2}1^{a_2+b_2}0^{a_3+b_3} \dots 1^{a_k+b_k}10^{a_{k+1}-2}.$$

In both cases, the number of isles decreases by half every two iterations until we are left with only one isle of 1s and one of 0s, which we already know leads to a fixed point.

- Rule 58: The analysis follows a similar reasoning as that for Rule 26.
- Rule 154: The analysis follows a similar reasoning as that for Rule 26, but in this case 1^n is a fixed point with an empty basin of attraction.

□

Theorem 3. *For Rule 24 the sequential update mode $\mu = (0, 1, \dots, n-1)$ is universal.*

Proof. The reasoning follows the lines of that for Rule 10, up to the update mode inversion. □

	111	110	101	100	011	010	001	000
104	0	1	1	0	1	0	0	0

Table 3: Definition of Rule 104.

3.1.1. Rules that admit a Covering of Sequential Update Modes

Theorem 4. *For Rule 104, if the size of the ring is even, then the sequential update mode $\mu = (n - 1, n - 2, \dots, 0)$ is E -universal.*

Proof. Note that, by definition of Rule 104, the word $w = 00$ is a wall (see Table 3). Let $\mu = (n - 1, n - 2, \dots, 0)$.

If n is odd, the update mode μ cannot lead all initial configurations to a fixed point, that is, there exists at least one initial configuration that leads to a cycle of length $T \geq 2$.

Indeed, let us consider $x = (01)1^{n-2}$, with n an odd number.

$$\begin{aligned}
f_{\mu,104}(x) &= 111(01)^{\frac{n-3}{2}}, \\
f_{\mu,104}^2(x) &= (01)^2 1^{n-4}, \\
f_{\mu,104}^3(x) &= 11111(01)^{\frac{n-5}{2}}, \\
f_{\mu,104}^4(x) &= (01)^3 1^{n-6}, \\
&\vdots \\
f_{\mu,104}^t(x) &= 1^{t+2}(01)^{\frac{n-5}{2}}, \text{ with } t \text{ an odd number,} \\
f_{\mu,104}^{t+1}(x) &= (01)^{\frac{t+3}{2}} 1^{n-(t+2)}.
\end{aligned}$$

After $t^* = n - 4$ steps, we have

$$\begin{aligned}
f_{\mu,104}^{t^*}(x) &= 1^{n-2}(01), \\
f_{\mu,104}^{t^*+1}(x) &= (01)^{\frac{n-1}{2}} 1, \\
f_{\mu,104}^{t^*+2}(x) &= 1^n, \\
f_{\mu,104}^{t^*+3}(x) &= 1(01)^{\frac{n-1}{2}}, \\
f_{\mu,104}^{t^*+4}(x) &= (10)1^{n-2}.
\end{aligned}$$

Let $y = f_{\mu,104}^{t^*+4}(x) = x^n$. We have

$$\begin{aligned}
f_{\mu,104}(y) &= 111(10)^{\frac{n-3}{2}}, \\
f_{\mu,104}^2(y) &= (10)^2 1^{n-4}, \\
&\vdots \\
f_{\mu,104}^t(y) &= 1^{t+2}(10)^{\frac{n-5}{2}}, \text{ with } t \text{ an odd number,} \\
f_{\mu,104}^{t+1}(y) &= (10)^{\frac{t+3}{2}} 1^{n-(t+2)}, \\
&\vdots \\
f_{\mu,104}^{t^*}(y) &= 1^{n-2}(10), \text{ with } t^* = n-4 \\
f_{\mu,104}^{t^*+1}(y) &= (10)^{\frac{n-1}{2}} 1, \\
f_{\mu,104}^{t^*+2}(y) &= 01^{n-1} = x.
\end{aligned}$$

Thus we have found a configuration x that does not reach a fixed point under $f_{\mu,104}$.

Now, let us consider an initial configuration $x = 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k}$, such that $\sum_{i=1}^k a_i + b_i = n \equiv 0 \pmod{2}$.

If $a_i = 1$, with $i \in \{1, \dots, k\}$, those isolated 0s will disappear (because $f_{104}(101) = 1$). If $a_i \geq 2$, with $i \in \{1, \dots, k\}$, those isles of 0 cannot decrease in size because they contain walls.

If $b_i = 1$, with $i \in \{1, \dots, k\}$, those isolated 1s will disappear (because $f_{104}(010) = 0$). Consider the case in which $b_i = 2$, with $i \in \{1, \dots, k\}$. Since $f_{104}(011) = f_{104}(110) = 1$, we could have a wall depending on the number of 0s at either side of the isle of 1s. We know that the 0s to the right of the isle of 1s will not decrease. If there is only one 0 to the left of the isle of 1s of size two, then we already know that the isolated 0s disappear and after this iteration there will be an isle of 1s of size at least 4.

Finally, in the case where $b_i \geq 3$, with $i \in \{1, \dots, k\}$, then the isle of 1s will be broken up because $f_{104}(111) = 0$:

- If b_i is even, then after the first iteration this isle of 1s will become $11(01)^{\frac{b_i-2}{2}}$ and after the second it will be 110^{b_i-2} . And from there it follows the analysis for $b_i = 2$.
- If b_i is odd, after the first iteration the isle will become $0(01)^{\frac{b_i-1}{2}}$; and after the second the isolated 1s will disappear.

Therefore, we have proven that all initial configurations of a ring of even size n arrive at a fixed point under $f_{\mu,104}$. $\square$

In Theorem 4 we chose an update mode with a simple definition, that is: $\mu = (n-1, n-2, \dots, 0)$, to prove that there exists at least one sequential update mode which is universal when the size of the ring is even. Nevertheless, when we run computer simulations for Rule 104 with a ring of size $n = 8$ we found 544 different sequential update modes that lead all configurations to a fixed point. Here we present a few examples: $\mu_1 = (0, 1, 2, 3, 4, 5, 6, 7)$, $\mu_2 = (0, 1, 3, 6, 4, 5, 2, 7)$, $\mu_3 = (0, 2, 6, 1, 3, 4, 5, 7)$ and $\mu_4 = (2, 6, 3, 0, 1, 4, 5, 7)$.

So far, the rules have had a universal sequential update mode (restricted to rings of even size, in the cases of Rules 104 and 106). In what remains of this section, we have six rules that have a covering of sequential update modes and one rule that does not have it.

Theorem 5. *For Rule 104, if the size of the ring is odd, there exists a covering of sequential update modes.*

Proof. Let $x = 0^{a_1}1^{b_1} \dots 0^{a_k}1^{b_k}$ with $\sum_{i=1}^k a_i + b_i = n$.

By definition, we know that the configuration 0^n is a fixed point, and that any configuration where $b_i = 2$ for all $i \in \{1, \dots, k\}$, is also a fixed point, since $f_{104}(110) = f_{104}(011) = 1$ (see Table ??).

To reach a fixed point from any initial configuration such as x , we can start we by dividing the isles of 1s where $b_i \geq 3$, using that $f_{104}(111) = 0$, leaving only isolated and pairs of 1s, i.e. $x' = 0^{a'_1}1^{b'_1} \dots 0^{a'_{k'}}1^{b'_{k'}}$ with $\sum_{i=1}^{k'} a'_i + b'_i = n$, where $b'_i \in \{1, 2\}$, for all $i \in \{1, \dots, k'\}$.

Then, we can update all isolated ones, using that $f_{104}(010) = 0$. This results in the configuration $x'' = 0^{c_1}1^{d_1} \dots 0^{c_\ell}1^{d_\ell}$ with $\sum_{i=1}^\ell c_i + d_i = n$, where $d_i = 2$, for all $i \in \{1, \dots, \ell\}$.

Since we have already reached a fixed point, we can now update any cells that have not yet been updating, thus completing a sequential update mode. $\square$

Theorem 6. *For Rules 18, 50, 74, 122, 146 and 178 there exists a covering of sequential update modes.*

Proof. By definition (see Table 4), we know that 0^n is a fixed point for all these rules and additionally 1^n is a fixed point for rules 146 and 178.

Let $x = 0^{a_1}1^{b_1} \dots 0^{a_k}1^{b_k}$ with $a_1, b_1, \dots, a_k, b_k \geq 0$ such that $\sum_{i=1}^k a_i + b_i = n$.

1. Rule 18: Since $f_{18}(111) = 0$ we can start by updating every other cell of the isles of 1s. The resulting configuration will only have 1s in pairs or isolated. Then we eliminate the remaining 1s using that $f_{18}(010) = f_{18}(011) = f_{18}(110) = 0$.

	111	110	101	100	011	010	001	000
18	0	0	0	1	0	0	1	0
50	0	0	1	1	0	0	1	0
74	0	1	0	0	1	0	1	0
122	0	1	1	1	1	0	1	0
146	1	0	0	1	0	0	1	0
178	1	0	1	1	0	0	1	0

Table 4: Definition of rules 18, 50, 74, 122, 146 and 178.

2. Rule 50: It is identical to Rule 18.
3. Rule 74: It is identical to Rule 18, except that $f_{74}(110) = 1$ and thus pairs of 1s must be updated from right-to-left.
4. Rule 122: The analysis is the same as for Rule 18, except that $f_{122}(011) = f_{122}(110) = 1$ which means that configurations with pairs of 1s, such as $x' = 0^{c_1}11 \dots 0^{c_{k-1}}110^{c_k}$ are also fixed points.
5. Rule 146: Since $f_{146}(011) = f_{146}(110) = 0$, we can start by updating the cells of the isles of 1s from left-to-right (or right-to-left) and given that $f_{146}(010) = 0$ the remaining 1s can be eliminated, thus arriving at 0^n .
6. Rule 178: It is identical to Rule 146.

Note that after eliminating all unwanted 1s for each of these rules we must still establish an order for updating the cells that belonged to an isle of 0s in the original configuration, in order to define a sequential update mode. $\square$

3.1.2. Rules with no Covering

Unlike the rest of the rules that fulfill one of the conditions of Theorem 1, Rules 90 and 106 does not allow all configurations to reach a fixed point, no matter which of the $n!$ sequential update mode is being used. Indeed, for Rule 90 (resp. Rule 106) for rings of size $n = 9$ or more, if n is not a power of 2 (resp. n is not even), there are configurations that cannot reach a fixed point, regardless of the sequential update mode being used. Instead, it appears to be necessary to allow for one cell to be updated more than once per step.

Theorem 7. *Let μ be any sequential update mode, and let $f_{\mu,90}$ be the discrete dynamical system in which Rule 90 evolves under the update mode μ . Then, configurations that contain only two 1s surrounded by 0s cannot reach a fixed point under sequential update modes.*

	111	110	101	100	011	010	001	000
90	0	1	0	1	1	0	1	0
106	0	1	1	0	1	0	1	0

Table 5: Definition of Rules 90 and 106.

Proof. Let μ be a sequential update mode.

Let $x = 0^{k_1}110^{k_2}$ such that $k_1 + k_2 + 2 = n$. Let i, j be the cells whose state is 1.

If the cell to the left of the pair of 1s updates first, which we will denote $\mu_{i-1} < \mu_i$, then we will have three consecutive 1s. Then we can update i which results in two isolated 1s and then we update j which leaves us with only one 1 located in $i - 1$.

Since we want the last one to disappear, we would need that $\mu_{i-2} < \mu_{i-1}$, which would mean that, in the next iteration, cell $i - 2$ will change its state to 1 and we, once again, have two consecutive cells whose state is 1.

Analogously, if instead we have that $\mu_j < \mu_{j+1}$ we obtain the two consecutive ones located in the cells $j + 1$ and $j + 1$. Thus, if we follow this strategy, we would need the cell $i - 1$ (or $j + 1$) to be updated twice before $i - 2$ (or $j + 2$) can change its state to 1.

Similarly, if we force the isle of 1s to increase to a larger size, we would need that the number of cells whose state will become 1 to be updated consecutively from left-to-right or right-to-left, depending on the side from which we want to extend the isle, since the only two neighborhoods that increase the number of 1s are 100 and 001 (see Table 5).

But given that we need every other cell to be updated consecutively within an isle for it to disappear (since $f_{90}(111) = 0$ and then $f_{90}(010) = 0$), using a similar reasoning as before, we can see that the larger isle cannot disappear.

Additionally, Rule 90 under sequential update modes cannot lead a configuration such as $x = 0^{k_1}110^{k_2}$ to the fixed points $(011)^{\frac{n}{3}}$, $(101)^{\frac{n}{3}}$ and $(110)^{\frac{n}{3}}$ which, naturally, only exist when the size of the ring is a multiple of three. Using a similar reasoning as the previous case, we can see that we would need one order to allow the size of the isle of 1s to increase and a different order to allow every third cell to be updated, thus creating pairs of 1s separated by isolated 0s.

Therefore, there does not exist a sequential update mode which would allow Rule 90 to make a configuration composed of an isle of two 1s surrounded by 0s delete all 1s and thus reach the fixed point 0^n nor the fixed

Outside Theorem		I	II	III	IV	Total
With Fixed Points	<i>E</i> -Universal	$\emptyset$	7,15	$\emptyset$	$\emptyset$	2
	<i>T</i> -Universal (conjecture)	$\emptyset$	$\emptyset$	45	$\emptyset$	1
	With <i>E</i> -Covering	$\emptyset$	23	30	$\emptyset$	2
	Without Covering	$\emptyset$	6,14,28,29,(37),38, 46,62,73,108, 134,142,156	22,60,105, 126,150	54,110	20
Without Fixed Points		$\emptyset$	1,3,9,11,19, 25,27,33,35, 41,43,51,57	$\emptyset$	$\emptyset$	13

Table 6: Summary of the classification of the rules that do not fulfill any of the conditions stated by Theorem 1

points $(011)^{\frac{n}{3}}$, $(101)^{\frac{n}{3}}$ and $(110)^{\frac{n}{3}}$. $\square$

Corollary 1. *There is no covering of sequential update modes for Rule 90.*

Proof. It is direct from Theorem 7. $\square$

Theorem 8. *For Rule 106, if the size of the ring is even, then the sequential update mode $\mu = (n-1, n-2, \dots, 0)$ is universal.*

Proof. Rule 106 allows isles of 0s to decrease in size (only at the end of the configuration and only if the state of the first cell of the configuration is 1), in which case a new isle of 1s appears of the same size as the original isle of 0s. From there, we follow a similar analysis to the one used in Theorem 4. Note that for rings of odd size, we can use the same counterexample for $f_{\mu,106}$ as we did for $f_{\mu,104}$. $\square$

Theorem 9. *There is no covering of sequential update modes for Rule 106, when the size of the ring is odd.*

Proof. It follows a similar reasoning as 7, where configurations that contain only two isolated 1s cannot reach 0^n , which is the only fixed point for this rule, when the size of the ring is odd. $\square$

3.2. Notable Rules Outside the Theorem

Theorem 1 provides a thorough characterization of ECA for which there is always, for each initial configuration, at least one asynchronous update mode that allows it to converge to a fixed point. However, we have determined additional conditions (that depend on the multiplicity of the size of the

	111	110	101	100	011	010	001	000
7	0	0	0	0	0	1	1	1
15	0	0	0	0	1	1	1	1
23	0	0	0	1	0	1	1	1
30	0	0	0	1	1	1	1	0

Table 7: Definition of rules 7, 15, 23 and 30.

ring) under which certain rules that do not fulfill any of the conditions of said theorem are still able to reach fixed points.

3.2.1. Restriction to Rings of Even Size

The rules studied in this section only have (non-isolated) fixed points if the size of the ring is even. Hence, we cannot generalize the results for these rules, but we can restrict our study to the constraints that allow fixed points to exist. Hence, given E the set of configurations of even size (over a grid of even size), the results of this section are related to E -universality and E -coverings.

Lemma 10. *For Rules 7, 15 and 23, there are two fixed points for a ring of size n , if and only if n is even: $(01)^{\frac{n}{2}}$ and $(10)^{\frac{n}{2}}$. Rule 30 has the configuration 0^n as a fixed point, in addition to the previous two.*

Proof. It is direct from the definitions if these four Rules (see Table 7). $\square$

Theorem 11. *Given a ring of even size n , $\mu = (0, 1, 2, \dots, n-2, n-1)$ is an E -universal sequential update mode for Rules 7 and 15.*

Proof. From Lemma 10, we know that $x = (01)^{\frac{n}{2}}$ and $y = (10)^{\frac{n}{2}}$ are fixed points.

Let $x = 0^n$ and $\mu = (0, 1, 2, \dots, n-2, n-1)$. Since $f_7(000) = 1$, it is easy to calculate that $f_{\mu,7}(x) = (10)^{\frac{n}{2}}$. Similarly, for $x = 1^n$ we have that $f_{\mu,7}^2(x) = (01)^{\frac{n}{2}}$.

Next, let $x = 0^a 1^b$ such that $a + b = n$, which means that a and b must either both be even or both be odd.

If they are odd, then:

$$f_{\mu,7}(x) = (01)^{\frac{a-1}{2}} 0^b 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a-1}{2}} (01)^{\frac{b-1}{2}} 01.$$

If they are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a}{2}} 0^{b-1} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a}{2}} (01)^{\frac{b}{2}}.$$

Now, if we consider $x = 0^{a_1} 1^{b_1} 0^{a_2} 1^{b_2}$ such that $a_1 + a_2 + b_1 + b_2 = n$. We know that there must be an even number of odd sequences.

1. If all are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1}{2}} 0^{b_1-1} 1 (01)^{\frac{a_2}{2}} 0^{b_2-1} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1}{2}} (01)^{\frac{b_1}{2}} (01)^{\frac{a_2}{2}} (01)^{\frac{b_2}{2}}.$$

2. If a_1, b_1 are odd, and the rest are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1-1}{2}} 0^{b_1-1} (01)^{\frac{a_2+2}{2}} 0^{b_2-1} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1-1}{2}} (01)^{\frac{b_1-1}{2}} (01)^{\frac{a_2+2}{2}} (01)^{\frac{b_2}{2}}.$$

3. If a_1, a_2 are odd, and the rest are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1-1}{2}} 0^{b_1} 1 (01)^{\frac{a_2-1}{2}} 0^{b_2} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1-1}{2}} (01)^{\frac{b_1}{2}} (01)^{\frac{a_2+1}{2}} (01)^{\frac{b_2}{2}}.$$

4. If a_1, b_2 are odd, and the rest are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1-1}{2}} 0^{b_1} 1 (01)^{\frac{a_2}{2}} 0^{b_2-1} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1-1}{2}} (01)^{\frac{b_1}{2}} (01)^{\frac{a_2}{2}} (01)^{\frac{b_2+1}{2}}.$$

5. If b_1, a_2 are odd, and the rest are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1}{2}} 0^{b_1-1} 1 (01)^{\frac{a_2-1}{2}} 0^{b_2-1} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1}{2}} (01)^{\frac{b_1+1}{2}} (01)^{\frac{a_2}{2}} (01)^{\frac{b_2-1}{2}}.$$

6. If b_1, b_2 are odd, and the rest are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1}{2}} 0^{b_1-1} 1 (01)^{\frac{a_2}{2}} 0^{b_2-1} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1}{2}} (01)^{\frac{b_1+1}{2}} (01)^{\frac{a_2}{2}} (01)^{\frac{b_2-1}{2}}.$$

7. If a_2, b_2 are odd, and the rest are even, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1-1}{2}} 0^{b_1-1} 1 (01)^{\frac{a_2-1}{2}} 0^{b_2} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1}{2}} (01)^{\frac{b_1}{2}} (01)^{\frac{a_2+b_2}{2}}.$$

8. If all are odd, then:

$$f_{\mu,7}(x) = (01)^{\frac{a_1-1}{2}} 0^{b_1} 1 (01)^{\frac{a_2-1}{2}} 0^{b_2} 1, \text{ and}$$

$$f_{\mu,7}^2(x) = (01)^{\frac{a_1+b_1}{2}} (01)^{\frac{a_2+b_2}{2}}.$$

Now, let $x = 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k}$ with $\sum_{i=1}^k a_i + b_i = n$. After the first step of $f_{\mu,7}$, all isles of 0s of even number will become sequences of (01) of half the length of the original isle, while the ones of odd length will become sequences of (01) of half the original length minus 1. At the same time, all isles of 1s will become isles of 0s whose length will depend on the parity of the isle of 0s to its left, followed by one 1 at the end.

In the next step, as we saw in the case with two isles of 1s and two of 0s, the isles of 0s belonging to $f_{\mu,7}(x)$ become sequences of (01) that follow the order established by the first isle of 0s of the original configuration. From where we obtain that $f_{\mu,7}^2(0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k}) = (01)^{\frac{n}{2}}$.

Let us start with $x = 0^{a_1} 1^{b_1} 0^{a_2}$, since $f_7(000) = 1$, the first cell becomes 1, and thus $f_{\mu,7}(x) = (10)^{\frac{a_1-1}{2}} 10^{b_1-1} (10)^{\frac{a_2+1}{2}}$ if a_1, a_2 are odd, and $f_{\mu,7}(x) = (10)^{\frac{a_1}{2}} 0^{b_1-1} (10)^{\frac{a_2+1}{2}} 1$ if a_2, b_1 are odd, $f_{\mu,7}(x) = (10)^{\frac{a_1-1}{2}} 10^{b_1-1} 1 (01)^{\frac{a_2}{2}} 1$ if a_1, b_1 are odd. If a_2 is odd (first two subcases), the result is $f_{\mu,7}(x) = (10)^{\frac{n}{2}}$, if a_2 is even, it is $f_{\mu,7}(x) = (01)^{\frac{n}{2}}$.

In the case with $x = 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k} 0^{a_{k+1}}$ with $\sum_{i=1}^k (a_i + b_i) + a_{k+1} = n$, the first sequence will be 10 instead of 01, and then the isles of 1s become isles of 0s, from where all configurations where a_{k+1} is odd result in $f_{\mu,7}(x) = (10)^{\frac{n}{2}}$, while if a_{k+1} is even, then the result is $f_{\mu,7}(x) = (01)^{\frac{n}{2}}$.

It is easy to see that if we start with $x' = 1^{b'_1} 0^{a'_1} \dots 1^{b'_k} 0^{a'_k} 1^{b'_{k+1}}$ with $\sum_{i=1}^k (a'_i + b'_i) + b'_{k+1} = n$, then $f_{\mu,7}(x')$ will start with an isle of 0s and it will end in a 1, that corresponds to the case $x = 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k}$ with $\sum_{i=1}^k a_i + b_i = n$, which we have already studied.

Therefore, any initial configuration can reach a fixed point under $f_{\mu,7}$.

The analysis for Rule 15 is very similar, except that since $f_{15}(011) = 1$, we are able to reach the fixed point after just one step. $\square$

Theorem 12. *For Rules 23 and 30, if the size of the ring is even, there exists an E-covering subset of sequential update modes.*

Proof. By definition of Rules 23 and 30, these are the fixed points for a ring of size n , if and only if n is even: $(01)^{\frac{n}{2}}$ and $(10)^{\frac{n}{2}}$. Additionally, 0^n is always a fixed point for Rule 30 ($f_{30}(000) = 0$). Now, let $x = 0^{a_1}1^{b_1} \dots 0^{a_k}1^{b_k}$, with $a_1, b_1, \dots, a_k, b_k \in \{0, \dots, n\}$ such that $\sum_{i=1}^k a_i + b_i = n$ an arbitrary initial configuration.

In this proof, a sequential update mode which leads x to a fixed point is constructed.

In the case of Rule 23, we start by updating the first cell of each isle of 1s if it is necessary to ensure that the number of cells between the first 1 of each isle is odd. This would increase the size of the corresponding isle of 0s, since $f_{23}(011) = 0$. Let L be the set of cells that need to be updated. The resulting configuration is $x' = x^{0,|L|} = 0^{a'_1}1^{b'_1} \dots 0^{a'_k}1^{b'_k}$, with $a'_1, b'_1, \dots, a'_k, b'_k \in \{0, \dots, n\}$ such that $\sum_{i=1}^k a'_i + b'_i = n$ and $a'_{i+1} + b'_i \equiv 0 \pmod{2}$, for all $i \in \{1, \dots, k-1\}$.

Then, we must update every other cell of the isles of 1s (using that $f_{23}(111) = 0$), which leaves only isolated 1s. Note that if the isle is of even size, we would have to update the last cell of the isle (since $f_{23}(110) = 0$). The result is as follows:

$$\begin{aligned} x^{0,|L|+1} &= 0^{a'_1}0101^{b'_1-2} \dots 0^{a'_k}1^{b'_k}, \\ x^{0,|L|+2} &= 0^{a'_1}(0101)01^{b'_1-4} \dots 0^{a'_k}1^{b'_k}, \\ x^{0,|L|+3} &= 0^{a'_1}(01)^301^{b'_1-6} \dots 0^{a'_k}1^{b'_k}, \\ &\vdots \\ x^{0,|L|+\left\lfloor \frac{b'_1}{2} \right\rfloor} &= 0^{a'_1}(01)^{\left\lfloor \frac{b'_1}{2} \right\rfloor} 0^{a'_2}1^{b'_2} \dots 0^{a'_k}1^{b'_k}. \end{aligned}$$

We continue as such with the rest of the isles of 1s, after which we have:

$$x'' = x^{0,|L|+\sum_{i=1}^k \left\lfloor \frac{b'_i}{2} \right\rfloor} = 0^{a'_1}(01)^{\left\lfloor \frac{b'_1}{2} \right\rfloor} 0^{a'_2}(01)^{\left\lfloor \frac{b'_2}{2} \right\rfloor} \dots 0^{a'_k}(01)^{\left\lfloor \frac{b'_k}{2} \right\rfloor}.$$

Afterwards, we update every other cell of the isles of 0s, starting with the first one:

$$\begin{aligned}
x^{0, |L| + \sum_{i=1}^k \left\lfloor \frac{b'_i}{2} \right\rfloor + 1} &= 010^{a'_1-2}0101^{b'_1-2} \dots 0^{a'_k}1^{b'_k}, \\
x^{0, |L| + \sum_{i=1}^k \left\lfloor \frac{b'_i}{2} \right\rfloor + 2} &= (01)^2 0^{a'_1-4} (0101) 01^{b'_1-4} \dots 0^{a'_k}1^{b'_k}, \\
x^{0, |L| + \sum_{i=1}^k \left\lfloor \frac{b'_i}{2} \right\rfloor + 3} &= (01)^3 0^{a'_1} (01)^3 01^{b'_1-6} \dots 0^{a'_k}1^{b'_k}, \\
&\vdots \\
x^{0, |L| + \sum_{i=1}^k \left\lfloor \frac{b'_i}{2} \right\rfloor + \left\lfloor \frac{a'_1}{2} \right\rfloor} &= (01)^{\left\lfloor \frac{a'_1}{2} \right\rfloor} (01)^{\left\lfloor \frac{b'_1}{2} \right\rfloor} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k}.
\end{aligned}$$

Continuing as such we obtain a configuration $x''' = (01)^{\frac{n}{2}}$. Finally, since we have already reached the fixed point, we proceed to update the cells that have not been updated yet and since $f_{23}(010) = 1$ and $f_{23}(101) = 0$, their respective state cannot change, and thus we have found a sequential update mode such that $f_{\mu, 23}(x) = (01)^{\frac{n}{2}}$.

Note that if the initial configuration is instead $y = 1^{a_1} 0^{b_1} \dots 1^{a_k} 0^{b_k}$, with $a_1, b_1, \dots, a_k, b_k \in \{0, \dots, n\}$ such that $\sum_{i=1}^k a_i + b_i = n$, we can instead obtain $f_{\mu, 23}(y) = (10)^{\frac{n}{2}}$ following a similar reasoning.

In the case of Rule 30, we update the last cell of each isle of 0s, using that $f_{30}(001) = 1$, if it is needed to ensure that the number of cells between the first 1 of each isle is odd. Let L be the set of cells that we need to update.

Let $x' = x^{0, |L|} = 0^{a_1} 1^{b_1} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k}$, with $a'_2, b'_2, \dots, a'_k, b'_k$ the new size of the isles, where $a'_i = a_i - 1$ and $b'_i = b_i + 1$, if the change was necessary.

Then, we update the cells of the first isle of 0s from left to right, using that $f_{30}(100) = 1$ to turn the isles of 0s into isles of 1s:

$$\begin{aligned}
x^{0, |L|} &= 0^{a_1} 1^{b_1} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k}, \\
x^{0, |L|+1} &= 10^{a_1-1} 1^{b_1} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k}, \\
x^{0, |L|+2} &= 1^2 0^{a_1-2} 1^{b_1} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k}, \\
&\vdots \\
x^{0, |L|+a_1-1} &= 1^{a_1-1} 0^1 1^{b_1} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k},
\end{aligned}$$

and since $f_{30}(101) = 0$, the last 0 of the first isle does not change and thus we have

$$x^{0, |L|+a_1} = 1^{a_1-1} 0^1 1^{b_1} 0^{a'_2} 1^{b'_2} \dots 0^{a'_k} 1^{b'_k}.$$

Then, we continue with each isle of 0s, to create a configuration such as

$$x'' = x^{0 \cdot \sum_{i=1}^k a_i} = 1^{a_1-1} 0 1^{b_1} \dots 1^{a'_k-1} 0 1^{b'_k}.$$

Note that we now have updated all cells whose state was originally 0.

Next, we update the cells corresponding to the original first isle of 1s from left to right, using that $f_{30}(011) = 1$ and $f_{30}(111) = 0$, which creates the sequence $(10)^{\lfloor \frac{b_1}{2} \rfloor}$, as follows:

$$\begin{aligned} x^{0 \cdot 1 + \sum_{i=1}^k (a_i)} &= 1^{a_1-1} 0 1^{b_1} \dots 1^{a'_k-1} 0 1^{b'_k}, \\ x^{0 \cdot 2 + \sum_{i=1}^k (a_i)} &= 1^{a_1-1} 0 1 0 1^{b_1-1} \dots 1^{a'_k-1} 0 1^{b'_k}, \\ &\vdots \\ x^{0 \cdot b_1 + \sum_{i=1}^k (a_i)} &= 1^{a_1-1} 0 (10)^{\lfloor \frac{b_1}{2} \rfloor} \dots 1^{a'_k-1} 0 1^{b'_k}. \end{aligned}$$

Then we can continue to update from left-to-right each isle of 1s of the original configuration, after which we obtain

$$f_{\mu,30}(x) = x^1 = 1^{a_1-1} 0 (10)^{\lfloor \frac{b_1}{2} \rfloor} \dots 1^{a'_k-1} 0 (10)^{\frac{b'_k}{2}}.$$

At this point we have updated all the cells whose original state was 1, and therefore, we have updated all the cells of the ring.

If we apply one step of the rule once again, following the update mode that we have previously established, we obtain $f_{\mu,30}^2(x) = (01)^{\frac{n}{2}}$.

Analogously, if the initial configuration is $y = 1^{a_1} 0^{b_1} \dots 0^{a_k} 1^{b_k}$, with $a_1, b_1, \dots, a_k, b_k \in 0, \dots, n$ such that $\sum_{i=1}^k a_i + b_i = n$, we can obtain $f_{\mu,30}^2(y) = (10)^{\frac{n}{2}}$ following a similar reasoning.

Thus, we have found a subset of sequential update modes that ensure that every configuration can reach a fixed point. $\square$

3.2.2. Restriction to Rings of Size Multiple of 3

Rule 45 is quite close to the three previous rules in the sense that it is possible to make it converge only under specific conditions. Indeed, for Rule 45 to admit fixed points, Lemma 13 below, states that the size of the grid must be a multiple of three. Thus, let $T \subsetneq \mathbb{B}^n$ the set of configuration whose size is a multiple of three.

Lemma 13. *For Rule 45 there are three fixed points for a ring of size n , if and only if n is a multiple of 3: $(001)^{\frac{n}{3}}$, $(010)^{\frac{n}{3}}$ and $(100)^{\frac{n}{3}}$.*

	111	110	101	100	011	010	001	000
45	0	0	1	0	1	1	0	1

Table 8: Definition of Rule 45.

Proof. It is easy to see that $(001)^{\frac{n}{3}}$, $(010)^{\frac{n}{3}}$ and $(100)^{\frac{n}{3}}$ are fixed points, given that $f_{45}(010) = 1$, and $f_{45}(001) = f_{45}(100) = 0$ (see Table 8). $\square$

For the following proof we need to introduce the concept of *temporal composition*. If we have two different update modes μ_1 and μ_2 and apply a Rule f under the first one and then under the second one, the resulting dynamical system will be a temporal composition of the two update modes, i.e. $f_\mu = f_{\mu_2} \circ f_{\mu_1}$. It is easy to see that if we combine two sequential update modes the resulting one will be a periodic update mode that is not sequential.

Theorem 14. *Any initial configuration of size n , with n multiple of 3, can converge to a fixed point under a periodic update mode for Rule 45.*

Proof. By Lemma 13 we know that $(001)^{\frac{n}{3}}$, $(010)^{\frac{n}{3}}$ and $(100)^{\frac{n}{3}}$ are the only fixed points of Rule 45. Furthermore, since $f_{45}(111) = f_{45}(110) = 0$ and $f_{45}(000) = 1$, we have that isles of two or more 1s and isles of three or more 0s cannot exist in a fixed point.

Let us perform a case by case analysis.

1. Let $x = 0^n$. We can define a sequential update mode

$$\mu_0 = (0, 3, 6, \dots, n-3, 1, 4, 7, \dots, n-2, 2, 5, 8, \dots, n-1)$$

which results in $f_{45, \mu_0}(0^n) = (100)^{\frac{n}{3}}$.

2. Let $x = 1^n$. We can define a sequential update mode

$$\mu_1 = (2, 5, 8, \dots, n-1, 1, 4, 7, \dots, n-2, 0, 3, 6, \dots, n-3)$$

which results in $f_{\mu_1, 45}(1^n) = (100)^{\frac{n}{3}}$.

3. Let $x = 0^a 1^b$, such that $a + b = n$.

(a) If $a \bmod 3 \equiv 0$ then $b \bmod 3 \equiv 0$, thus we can define a sequential

update mode

$$\begin{aligned}\mu_2 = (&a + 2, a + 5, a + 8, \dots, n - 1, \\ &a + 1, a + 4, a + 7, \dots, n - 2, \\ &a, a + 3, a + 6, \dots, n - 3, \\ &0, 3, 6, \dots, a - 3, \\ &1, 4, 7, \dots, a - 2, \\ &2, 5, 8, \dots, a - 1)\end{aligned}$$

which results in $f_{\mu_2, 45}(0^a 1^b) = (100)^{\frac{n}{3}}$.

- (b) If $a \bmod 3 \equiv 1$ then $b \bmod 3 \equiv 2$, thus we can define a sequential update mode

$$\begin{aligned}\mu_3 = (&a + 2, a + 5, a + 8, \dots, n - 3, \\ &a + 1, a + 4, a + 7, \dots, n - 1, \\ &a, a + 3, a + 6, \dots, n - 2, \\ &1, 4, 7, \dots, a - 3, \\ &0, 3, 6, \dots, a - 1, \\ &2, 5, 8, \dots, a - 2)\end{aligned}$$

which results in $f_{45, \mu_3}(0^a 1^b) = (010)^{\frac{n}{3}}$.

- (c) If $a \bmod 3 \equiv 2$ then $b \bmod 3 \equiv 1$, thus we can define a sequential update mode

$$\begin{aligned}\mu_4 = (&a + 2, a + 5, a + 8, \dots, n - 2, \\ &a + 1, a + 4, a + 7, \dots, n - 3, \\ &a, a + 3, a + 6, \dots, n - 1, \\ &2, 5, 8, \dots, a - 3, \\ &1, 4, 7, \dots, a - 1, \\ &0, 3, 6, \dots, a - 2)\end{aligned}$$

which results in $f_{\mu_4, 45}(0^a 1^b) = (001)^{\frac{n}{3}}$.

The procedure is analogous for $x = 1^b 0^a$.

4. Let $x = 0^{a_1} 1^{b_1} 0^{a_2} 1^{b_2}$. Let $\hat{a} = a_1 + b_1$. We need to ensure the correct distance between the two isles of 1s.
 - (a) If $(b_1 + a_2) \bmod 3 \equiv 0$ we can simply update the isles of 1s from right-to-left, and then we can update the original isles of 0s

every third cell, according to a convenient distance between the remaining 1s, following the same logic as with case 3.

$$\begin{aligned}\mu_5 = & (n-1, n-2, \dots, n-b_2, \\ & \hat{a}-1, \hat{a}-2, \dots, a_1, \hat{a}+a_2-3, \\ & \hat{a}+a_2-6, \dots, \hat{a}, \\ & \hat{a}+a_2-2, \hat{a}+a_2-5, \dots, \hat{a}+a_2-1, \\ & \hat{a}+a_2-4, \dots, a_1-3, a_1-6, \dots, 0, \\ & a_1-2, a_1-5, \dots, a_1-1, a_1-4).\end{aligned}$$

Note that if $a_1 \bmod 3 \equiv 2$, we must update the first cell of the configuration to prevent an unwanted 1 to appear. The resulting configuration will be

$$f_{\mu_5,45}(x) = (100) \lfloor \frac{a_1}{3} \rfloor 10^{b_1-1} (100) \lfloor \frac{a_2}{3} \rfloor 10^{b_2-1},$$

if $a \bmod 3 \equiv 0$, or

$$f_{\mu_5,45}(x) = (010) \lfloor \frac{a_1}{3} \rfloor 010^{b_1-1} (010) \lfloor \frac{a_2}{3} \rfloor 010^{b_2-1},$$

if $a_1 \bmod 3 \equiv 1$, or in the especial case $a_1 \bmod 3 \equiv 2$

$$f_{\mu_5,45}(x) = (001) \lfloor \frac{a_1}{3} \rfloor 0010^{b_1-1} (001) \lfloor \frac{a_2}{3} \rfloor 0010^{b_2-1}.$$

- (b) If $(b_1 + a_2) \bmod 3 \equiv 2$, we have to start with cell $\hat{a} + a_2 - 2$, then we must update cell $\hat{a} + a_2 - 1$ to correct the distance between the isles of 1s, and then we proceed with the same order as the one established in Case 4(a).

$$\begin{aligned}\mu_6 = & (\hat{a} + a_2 - 2, \hat{a} + a_2 - 1, \\ & n-1, n-2, \dots, n-b_2, \\ & \hat{a}-1, \hat{a}-2, \dots, a_1, \\ & \hat{a}+a_2-3, \hat{a}+a_2-6, \dots, \hat{a}, \\ & \hat{a}+a_2-5, \dots, \hat{a}+a_2-4, \dots, a_1-3, \\ & a_1-6, \dots, 0, \\ & a_1-2, a_1-5, \dots, a_1-1, a_1-4).\end{aligned}$$

The resulting configuration will be

$$f_{\mu_6,45}(x) = (100) \lfloor \frac{a_1}{3} \rfloor 10^{b_1-1} (100) \lfloor \frac{a_2-2}{3} \rfloor 110^{b_2},$$

if $a \bmod 3 \equiv 0$, or

$$f_{\mu_6,45}(x) = (010) \lfloor \frac{a_1}{3} \rfloor 010^{b_1-1} (010) \lfloor \frac{a_2-2}{3} \rfloor 0110^{b_2},$$

if $a_1 \bmod 3 \equiv 1$, or in the especial case $a_1 \bmod 3 \equiv 2$

$$f_{\mu_6,45}(x) = (001) \lfloor \frac{a_1}{3} \rfloor 0010^{b_1-1} (001) \lfloor \frac{a_2-2}{3} \rfloor 00110^{b_2}.$$

- (c) If $(b_1 + a_2) \bmod 3 \equiv 1$ we have to start with cell $a_1 - 2$, then we must update cell $a_1 - 1$ to correct the distance between the isles of 1s, and then we proceed with the same order as the one established in Case 4(a).

$$\begin{aligned} \mu_7 = & (a_1 - 2, a_1 - 1, \\ & n - 1, n - 2, \dots, n - b_2, \hat{a} - 1, \hat{a} - 2, \dots, a_1, \\ & \hat{a} + a_2 - 3, \hat{a} + a_2 - 6, \dots, \hat{a}, \\ & \hat{a} + a_2 - 2, \hat{a} + a_2 - 5, \dots, \hat{a} + a_2 - 1, \\ & \hat{a} + a_2 - 4, \dots, a_1 - 3, \\ & a_1 - 6, \dots, 0, a_1 - 5, \dots, a_1 - 4, a_1 - 7, \dots). \end{aligned}$$

The resulting configuration will be

$$f_{\mu_7,45}(x) = (100) \lfloor \frac{a_1-2}{3} \rfloor 110^{b_1} (100) \lfloor \frac{a_2}{3} \rfloor 10^{b_2-1},$$

if $(a_1 - 2) \bmod 3 \equiv 0$, or

$$f_{\mu_7,45}(x) = (010) \lfloor \frac{a_1-2}{3} \rfloor 0110^{b_1} (010) \lfloor \frac{a_2}{3} \rfloor 010^{b_2-1},$$

if $(a_1 - 2) \bmod 3 \equiv 1$, or in the especial case $(a_1 - 2) \bmod 3 \equiv 2$

$$f_{\mu_7,45}(x) = (001) \lfloor \frac{a_1-2}{3} \rfloor 00110^{b_1} (001) \lfloor \frac{a_2}{3} \rfloor 0010^{b_2-1}.$$

Then we create a new sequential update mode following the reasoning established in Case 1 to work with the two longer isles of 0s.

Note that this results in a temporal combination of two different sequential update modes, which results in a periodic update mode.

Extending case 4 to $x = 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k}$ and $x = 0^{a_1} 1^{b_1} \dots 0^{a_k} 1^{b_k} 0^{a_{k+1}}$ is direct and analogous for $x = 1^{b_1} 0^{a_1} \dots 1^{b_k} 0^{a_k}$ and $x = 1^{b_1} 0^{a_1} \dots 1^{b_k} 0^{a_k} 1^{b_{k+1}}$.

□

	111	110	101	100	011	010	001	000
38	0	0	1	0	0	1	1	0
46	0	0	1	0	1	1	1	0
54	0	0	1	1	0	1	1	0
60	0	0	1	1	1	1	0	0
62	0	0	1	1	1	1	1	0

Table 9: Definition of Rule 38, 46, 54, 60 and 62.

Our result is not satisfactory in the context of our study, given that the update mode we have obtained is periodic but not sequential. Therefore, in order to find candidates for sequential update modes under which Rule 45 might be able to lead all initial configurations to a fixed point, we ran computer simulations for different sized rings. These simulations considered all possible sequential update modes for rings of size 6 and 9. As a result we found that the best candidate to prove the existence of a universal sequential update mode is the one presented in the following conjecture.

Conjecture 1. *Given a ring whose size n is a multiple of three, $\mu = (0, 1, 2, \dots, n-2, n-1)$ is a T -universal update mode for Rule 45.*

Remark 1. *Unlike for Rules 7 and 15, the simulations suggest that the update mode in this case is not unique. Indeed, there are 117 sequential update modes for $n = 9$ that lead all configurations to a fixed point, and 15 sequential update modes for $n = 6$. However, the formal analysis involves a great amount of sub-cases and could not be completed in a reasonable amount of time.*

3.3. Without Covering of Sequential Update Modes

In this section we pay attention to the rules for which, even though they have fixed points, there are configurations that can never reach them, regardless of the sequential update modes being used.

Theorem 15. *For Rules 38, 46, 54, 60 and 62, configuration 0^n is the only fixed point and it is isolated under sequential update modes.*

Proof. Note that, for all these rules, 0^n is the only fixed point in parallel, therefore, it is the only fixed point under sequential update modes.

We will focus on Rule 38, the other rules follow a similar reasoning. Let μ be any sequential update mode.

	111	110	101	100	011	010	001	000
110	0	1	1	0	1	1	1	0
126	0	1	1	1	1	1	1	0

Table 10: Definition of Rule 110 and 126.

Let us consider an initial configuration $x = 0^{a_1}1^{b_1} \dots 0^{a_k}1^{b_k}$ such that $\sum_{i=1}^k a_i + b_i = n$. Given that $f_{38}(011) = f_{38}(110) = 0$ (see Table 9), we can always reduce the size of the isles of 1s until all we are left with are isolated 1s. Let ℓ_0 be a cell that contains one of the isolated 1s.

Thanks to the definition of Rule 38, we know that the isolated 1s we have left must have been updated after the cells on either side of them (otherwise its state would have to be 0, since $f_{38}(011) = f_{38}(110) = f_{38}(111) = 0$). Let us notate this by saying that $\mu_{\ell_0-1} < \mu_{\ell_0}$ and $\mu_{\ell_0} > \mu_{\ell_0+1}$.

Because $f_{38}(001) = 1$, we know that the cells to the left of each 1 will change its state to 1, until the growing isle of 1s reaches a cell ℓ_1 such that $\ell_1 - 1$ is updated before ℓ_1 , that is: $\mu_{\ell_1-1} < \mu_{\ell_1}$. Additionally, the state of cell ℓ_0 will now be 0, because $f(110) = 0$.

Since the previous statement holds for all isolated 1s, that is, all isolated 1s will create a growing isles of 1s that start on a cell ℓ_i and finish on $\ell_{i-1} - 1$, with $i \in \{0, \dots, k'\}$, with k' the number of isles. Note that the isles of 1s could merge, since $f_{38}(101) = 1$, thus the new number of isles k' could be different from k , the number of isles of the original configuration.

Thus, once we reach isolated 1s, in the next step the number of 1s must increase (by increasing the size of the isles) or stay constant (if each 1 moved exactly one cell to the left). Therefore, configuration 0^n cannot be reached. $\square$

Theorem 16. *For Rules 110 and 126, configuration 0^n is the only fixed point and it is isolated under sequential update modes.*

Proof. Note that 0^n is the only fixed point in parallel, and thus, it is the only possible fixed point under sequential update modes.

Let μ be a sequential update mode. Unlike Theorem 15, for these rules we have that $f(110) = f(011) = 1$ (see Table 10), which means that we cannot simply try to reduce the size of the isles of 1s. Instead, the best that can happen is that we use $f(111) = 0$ to break them up, leaving 1s alone or in pairs.

	111	110	101	100	011	010	001	000
134	1	0	0	0	0	1	1	0
142	1	0	0	0	1	1	1	0
150	1	0	0	1	0	1	1	0
156	1	0	0	1	1	1	0	0

Table 11: Definition of Rule 134, 142, 150 and 156.

	111	110	101	100	011	010	001	000
105	0	1	1	0	1	0	0	1

Table 12: Definition of Rules 105.

But once again, since $f(010) = 1$, those 1s cannot disappear and instead, since $f(001) = f(101) = 1$, the number of 1s will not decrease, therefore 0^n cannot be reached. $\square$

Theorem 17. *Rules 134, 142, 150 and 156 only have isolated fixed points under sequential update modes:*

- 0^n and 1^n if n is odd.
- 0^n , 1^n , $(01)^{\frac{n}{2}}$ and $(10)^{\frac{n}{2}}$ if n is even.

Proof. We know that 0^n and 1^n are fixed points, and since $f(101) = 0$ and $f(010) = 1$ (see Table 11) we know that the isolated 0s and 1s do not disappear for any of these rules, meaning that said fixed points cannot be reached.

Now, let n be even and $x \notin \{0^n, 1^n, (01)^{\frac{n}{2}}, (10)^{\frac{n}{2}}\}$.

Given that $f(000) = 0$ and $f(111) = 1$, we know that new isles cannot be created. This means that the number of isles cannot increase which implies that the only way to have $\frac{n}{2}$ isolated 1s (or 0s) is for them to have existed in the initial configuration.

Therefore, all four possible fixed points are isolated ones. $\square$

Theorem 18. *Rule 105 under sequential update modes only has fixed points if the length of the ring is a multiple of 4, and said fixed points are isolated: $(0011)^{\frac{n}{4}}$, $(0110)^{\frac{n}{4}}$, $(1001)^{\frac{n}{4}}$ and $(1100)^{\frac{n}{4}}$.*

	111	110	101	100	011	010	001	000
73	0	1	0	0	1	0	0	1

Table 13: Definition of Rule 73.

Proof. It is easy to see that $x = (0011)^{\frac{n}{4}}$, $x = (0110)^{\frac{n}{4}}$, $x = (1001)^{\frac{n}{4}}$ and $x = (1100)^{\frac{n}{4}}$ with $n \bmod 4 \equiv 0$ are fixed points, since $f_{105}(011) = f_{105}(110) = 1$ and $f_{105}(001) = f_{105}(100) = 0$ (see Table 12).

Let μ be a sequential update mode. Note that if a configuration has isles of one 1, said isles cannot contain two 1s in the next substep, since $f_{105}(100) = f_{105}(001) = 0$. Similarly, isolated 0s cannot become pairs of 0s, since $f_{105}(011) = f_{105}(110) = 1$.

Contrarily, if we have an initial configuration with isles of 1s whose length is three or more, by definition these isles cannot decrease in size from the borders, instead, in order to obtain smaller isles we need to divide them ($f_{105}(000) = 1$ and $f_{105}(111) = 0$), which will create new borders that cannot be changed. However, $f_{105}(010) = 0$ and $f_{105}(101) = 1$, meaning that any isolated 1s and 0s that appear after one step can disappear after the next one.

It follows that the only way of having pairs of 1s followed by pairs of 0s is for the initial configuration to contain them. Therefore, all four fixed points are isolated. $\square$

Corollary 2. *There does not exist a covering of sequential update modes for Rules 38, 46, 54, 60, 62, 110, 126, 134, 142, 150, 156 and 105.*

Proof. It is direct from Theorems 15, 16, 17 and 18. $\square$

So far, we have proven that there does not exist a covering by proving that the fixed points (when they exist) are isolated. In what follows we will instead prove that there are certain configurations that are unable to reach fixed points.

Theorem 19. *For Rule 73 under sequential update modes, a configuration with isolated 1s can never reach a fixed point.*

Proof. Let μ be a sequential update mode. Note that configurations with three or more consecutive 1s or isolated 1s cannot be fixed points because $f_{73}(010) = f_{73}(111) = 0$, meaning that a fixed point can only have 1s in pairs. Note that this does not mean that any configuration with two consecutive 1s surrounded by zeros is a fixed point.

	111	110	101	100	011	010	001	000
6	0	0	0	0	0	1	1	0
14	0	0	0	0	1	1	1	0
22	0	0	0	1	0	1	1	0

Table 14: Definition of Rule 6, 14 and 22.

Similarly to Theorem 18, we have that $f_{73}(011) = f_{73}(110) = 1$ and $f_{73}(001) = f_{73}(100) = 0$, which we know results in that isles of 0s and of 1s cannot increase (nor decrease) in size, which includes isolated 1s and 0s.

Therefore, configurations with isolated 1s cannot contain in their orbit configurations with isles of 1s of length two, which are the only ones that could be fixed points. $\square$

Theorem 20. *For Rules 6, 14 and 22 under sequential update modes, a configuration with isolated 1's can never reach a fixed point.*

Proof. Let $x = 0^{n-1}1$ be a configuration that only contains one cell whose state is 1.

Let μ be a sequential update mode. Since $f(010) = 1$, we know that the 1s cannot disappear. And given that $f(001) = 1$, we can see that the number of 1s will increase to the left until we reach a cell ℓ_1 such that the cell to its left updates before cell ℓ_1 , which we will notate by $\mu_{\ell_1-1} < \mu_{\ell_1}$.

We know that each ℓ_1 will be the first cell that updates before the one to its left, otherwise the isle would have stopped increasing on the previous cell with that characteristic.

On the next step, the isles move to the next cell such that $\mu_{\ell_2-1} < \mu_{\ell_2}$. This means that we have an isle of 1s moving from right to left, going from one ℓ_i to the next, resulting in cycles of length $|L|$, with $L = \{\ell \in \{0, \dots, n-1\} \mid \mu_{\ell-1} < \mu_{\ell}\}$.

Note that since $f(101) = 0$, if we have more than one isolated 1 at the start, the resulting isles cannot merge if they collide. This results in that the isle of 1s to the right (the one that would advance into the position currently occupied) stops increasing before reaching an ℓ cell. But since the isles cannot disappear, this results in cycles of length $\mathcal{O}(|L|)$.

Thus, configurations that contain isolated 1s cannot reach a fixed point. $\square$

	111	110	101	100	011	010	001	000
28	0	0	0	1	1	1	0	0
29	0	0	0	1	1	1	0	1

Table 15: Definition of rules 28 and 29.

	111	110	101	100	011	010	001	000
108	0	1	1	0	1	1	0	0

Table 16: Definition of Rule 108.

Theorem 21. *For Rules 28 and 29 under sequential update modes, a configuration that contains the word 01001 can never reach a fixed point.*

Proof. Note that, by definition (see Table 15), the word $w = 01$ is a wall for both rules, meaning that their value does not change regardless of the update mode and whatever the state of the configuration is around it.

Let μ be a sequential update mode. Now, if we have a word 01001, what we have is a single 0 surrounded by walls. Since $f_{28}(100) = f_{29}(100) = 1$ and $f_{28}(110) = f_{29}(110) = 0$ this creates a cycle within the walls of length 2. Thus, any configuration that contains 01001 is unable to reach a fixed point. $\square$

Theorem 22. *For Rule 108 under sequential update modes, a configuration can never reach a fixed point if it contains the word 0011100.*

Proof. Note that 100 and 001 are walls. Indeed, $f_{108}(000) = f_{108}(001) = f_{108}(100) = 0$ and $f_{108}(011) = f_{108}(110) = f_{108}(010) = 1$. Then, if we consider a configuration that contains the word (001)1(100), what we have is a cell whose state is 1 surrounded by two walls.

Let μ be a sequential update mode. By definition, the cells where the walls are located cannot change their state and so, after one step that sequence will change to (001)0(100) and after two it will return to its initial state. Therefore, any configuration containing (001)1(100) and/or (001)0(100) can never reach a fixed point. $\square$

Corollary 3. *There does not exist a covering of sequential update modes for Rules 73, 6, 14, 22, 28, 29 and 108.*

Proof. It is a direct result from Theorems 19, 20, 21 and 22. $\square$

	111	110	101	100	011	010	001	000
37	0	0	1	0	0	1	0	1

Table 17: Definition of Rule 37.

3.3.1. Rule 37

Rule 37 is a special case. Like Rule 45, it only has fixed points if the size of the ring is a multiple of three. Furthermore it has the same fixed point as Rule 45: $(001)_{\frac{n}{3}}$, $(010)_{\frac{n}{3}}$ and $(100)_{\frac{n}{3}}$, which can easily be checked in Table 17.

However, unlike Rule 45, the computer simulations we ran for Rule 37 say that there is no T -universal sequential update mode. Moreover, simulations performed over rings of size 6 and 9, suggest that there does not even exist a T -covering sequential update mode. Indeed, for $n = 6$ the configuration $x = 001000$ was unable to reach a fixed point regardless of the sequential update mode, and for $n = 9$ the same could be said for, among others, the configuration $x = 000010001$. Thus we have the following conjecture:

Conjecture 2. *There does not exist a covering of sequential update modes for Rule 37.*

4. Discussion

We have studied the convergence of all 88 elementary cellular automata, and have classified them into two separate groups: those that fit Theorem 1 and the ones that fall outside of it.

Out of the 50 ECA that fulfill one of the criteria given by Theorem 1, we have proven that there exists a universal sequential update mode for 13 rules, all of them belonging to class II under Wolfram’s classification.

Furthermore, we found that 7 of the 50 ECA have a covering of different sequential update mode in order to guarantee that all configurations reach a fixed point. Contrarily, for Rule 90 we found a sufficient condition under which a configuration cannot reach a fixed point under any sequential update mode. Additionally, while we did prove that there exists an E -universal sequential update mode for Rule 106, we found there does not exist a covering of sequential update modes for it. Note that this does not contradict Theorem 1, since said theorem refers to asynchronous update modes, which is a more general notion than sequential update modes.

In the case of the 38 ECA that do not fall under the purview of Theorem 1, we can easily distinguish between the 25 ECA that have fixed points

under parallel update mode, and 13 ECA that do not have any. Within the first group we have two ECA for which we were able to prove, as well as Rule 45, for which we have a conjecture, that there are sequential update modes that lead all configurations to a fixed point by demanding extra conditions over the size of the ring. Additionally, we found two ECA for which we need more than one sequential update mode to allow all configurations to arrive at a fixed point.

Moreover, we found 20 ECA for which there are configurations that can never reach a fixed point under sequential update modes, divided in three categories: twelve ECA whose fixed points are isolated, seven ECA where we characterized the configurations that are unable to reach fixed points, and finally Rule 37 for which the computer simulations we ran strongly suggest that there configurations that can never reach a fixed point under sequential update modes.

Future Work

There are other families of periodic update modes, for example: local clocks [28, 29] and block-parallel [7, 25, 24], that could create “new” fixed points for rules that do not have them under the traditional update modes, as well as for rules whose only fixed points are isolated. Indeed, more complex update modes could help us to control said rules, allowing convergence when there was none.

When we proved that there are configurations that cannot reach a fixed point under sequential update modes for Rule 90, we showed that it is necessary to update some cells two times before their neighbors are updated once. This means we would have to search for an update mode such as the ones we previously mentioned; or we would need to define a periodic update mode such as the one described for Rule 45.

Naturally, it is of interest to prove Conjecture 1, which would improve our current results by proving that Rule 45 can lead all configurations to a fixed point using only one sequential update mode, given a ring whose size is a multiple of three. As well as Conjecture 2, which would settle the question of convergence under sequential update modes for Rule 37.

Additionally, for rules with more than one universal sequential update mode, it is of interest to find out which of them takes the least amount of time to converge, as well as how does the size of the ring affects the convergence time and can we find the sequential update mode with the most efficient convergence time.

Along similar lines, for rules for which we did not find a universal sequential update mode, it begs the question of can we find a minimal cover-

ing, i.e. the smallest set of sequential update modes that is a covering and whether the size of such a covering be dependent on the size of the ring. At the same time, can we develop an algorithm that gives us a minimal covering for any given rule and size of ring?

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