The cartesian closed bicategory of thin spans

Pierre Clairambault\textsuperscript{1}  Simon Forest\textsuperscript{2}

\textsuperscript{1}LIS, CNRS
\textsuperscript{2}I2M, Aix-Marseille Université

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The model $\textbf{Rel}$ of $\textbf{LL}$

Objects: sets $A, B, C, \text{etc.}$

Morphisms $A \to B$: relations $R \subseteq A \times B$, i.e., sets of elements

$$a \leadsto b$$

Exponential: $!A$ is $\mathcal{M}_{\text{fin}}(A)$, the set of finite multisets on $A$

(co)Kleisli category $\textbf{Rel}_!$: morphisms $A \to B$ are morphisms $!A \to B$ of $\textbf{Rel}$, that is, sets of elements

$$[a_1, \ldots, a_n] \leadsto b$$
Interpreting programs in $\text{Rel}_I$

Since $\text{Rel}_I$ is cartesian closed, one can interpret programs inside it.

$x : \text{Bool} \vdash \text{if } x \text{ then } \text{ff} \text{ else } \text{tt} : \text{Bool}$

interpreted as

$\{ [\text{tt}] \mapsto \text{ff}, \ [\text{ff}] \mapsto \text{tt} \} \ (\subseteq \mathcal{M}_{\text{fin}}(\text{Bool} \times \text{Bool}))$
Interpreting programs in $\text{Rel}_!$

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$x : \text{Bool} \vdash \text{if } x \text{ then (if } x \text{ then } \text{ff else } \text{tt} \text{) else (if } x \text{ then } \text{tt else } \text{ff} : \text{Bool}$

interpreted as

\[
\{ [\text{tt, tt}] \rightarrow \text{ff}, \quad [\text{tt, ff}] \rightarrow \text{tt}, \quad [\text{ff, ff}] \rightarrow \text{ff} \}
\]
Interpreting programs in $\text{Rel}_!$

Since $\text{Rel}_!$ is cartesian closed, one can interpret programs inside it.

$$x : \text{Bool} \vdash \text{if } x \text{ then (if } x \text{ then } \text{ff else } \text{tt) else (if } x \text{ then } \text{tt else } \text{ff)} : \text{Bool}$$

interpreted as

$$\begin{cases} [tt, tt] \mapsto ff, & [tt, ff] \mapsto tt, & [ff, ff] \mapsto ff \end{cases}$$

Here, two different executions get identified in the interpretation.
Interpreting programs in \( \text{Rel}_! \)

Since \( \text{Rel}_! \) is cartesian closed, one can interpret programs inside it.

\[
x : \text{Bool} \vdash \text{if } x \text{ then } (\text{if } x \text{ then } \text{ff} \text{ else } \text{tt}) \text{ else } (\text{if } x \text{ then } \text{tt} \text{ else } \text{ff}) : \text{Bool}
\]

interpreted as

\[
\{ [\text{tt}, \text{tt}] \rightarrow \text{ff}, \quad [\text{tt}, \text{ff}] \rightarrow \text{tt}, \quad [\text{ff}, \text{ff}] \rightarrow \text{ff} \}
\]

Here, two different executions get identified in the interpretation.

Hence, \( \text{Rel}_! \) aggregates different executions.
Problem

How to obtain a \texttt{Rel}-style proof-relevant model of \texttt{LL}?

Some existing answers:

- Generalized species of structures

- Template games
  Melliès. "Template games and differential linear logic". 2019

We aim at providing another answer focused on \textit{effectivity}. 
Spans as generalized relations

First: we need a more quantitative structure than relations.
Spans as generalized relations

A span between two sets $A$ and $B$ is $\sigma = (\sigma, \partial^\sigma_A, \partial^\sigma_B)$ as in

\[
\begin{array}{c}
\sigma \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\partial^\sigma_A & \sigma & \partial^\sigma_B \\
\downarrow & \downarrow & \downarrow \\
A & \sigma & B
\end{array}
\]

Given $(a, b) \in A \times B$, there is a set $\sigma_{a,b}$ of witnesses above $(a, b)$.

Idea: A relation between $A$ and $B$ is a span with at most one witness above any $(a, b)$. 
Spans as generalized relations

Spans are composed using **pullbacks**: given spans $\sigma: A \to B$ and $\tau: B \to C$, $\tau \circ \sigma$ is defined by

Intuitively: a witness of $(a, b)$ and a witness of $(b, c)$ give a witness of $(a, c)$.
Spans as generalized relations

Since pullbacks are unique up to isomorphism, $\tau \odot \sigma$ is defined up to isomorphism of spans.

Given two spans $\sigma, \tau : A \to B$, a morphism between $\sigma$ and $\tau$ is $m : \sigma \to \tau$ such that

$$
\begin{array}{c}
\sigma \\
\downarrow \quad m \\
\partial^\sigma_A \\
\downarrow \\
A
\end{array}
\quad = 
\quad
\begin{array}{c}
\tau \\
\downarrow \\
\partial^\tau_A \\
\downarrow \\
\sigma
\end{array}
$$

and

$$
\begin{array}{c}
\sigma \\
\downarrow \quad m \\
\partial^\sigma_B \\
\downarrow \\
B
\end{array}
\quad = 
\quad
\begin{array}{c}
\tau \\
\downarrow \\
\partial^\tau_B \\
\downarrow \\
\sigma
\end{array}
$$

One gets a bicategory $\text{Span} = \text{Span}(\text{Set})$ of sets, spans and morphisms of spans.
Some structure on \textbf{Span}

The cocartesian structure of \textbf{Set} translates to a cartesian structure on \textbf{Span}. 
Some structure on \textbf{Span}

The cocartesian structure of \textbf{Set} translates to a cartesian structure on \textbf{Span}.

\[\top \triangleq \emptyset\] is the terminal object of \textbf{Span}.

![Diagram]

\[
\begin{array}{c}
\emptyset \\
\downarrow \\
A \\
\downarrow \\
\emptyset
\end{array}
\]
Some structure on $\text{Span}$

The cocartesian structure of $\text{Set}$ translates to a cartesian structure on $\text{Span}$.

$A \& B \triangleq A \sqcup B$ is the cartesian product on $\text{Span}$.

$$\sigma : X \rightarrow A$$

$$\tau : X \rightarrow B$$

$$\langle \sigma, \tau \rangle : X \rightarrow A \& B$$
Some structure on \textbf{Span}

The cartesian structure of \textbf{Set} translates to a monoidal structure on \textbf{Span}.

\(A \otimes B \cong A \times B\) gives a tensor product on \textbf{Span}.

\[
\begin{align*}
\sigma_A & \quad \partial^\sigma_A & \quad \partial^\sigma_{A'} \\
A & \quad \sigma & \quad A' \\
\sigma & \quad \tau_B & \quad \partial^\tau_B & \quad \partial^\tau_{B'} \\
B & \quad B & \quad B' \\
\sigma \times \tau & \quad \partial^\sigma_A \times \partial^\tau_B & \quad \partial^\sigma_{A'} \times \partial^\tau_{B'} \\
A \times B & \quad A' \times B' \\
\sigma \otimes \tau
\end{align*}
\]
A model of **LL** on spans?

We thus have a quantitative generalization of **Rel** in the form of **Span**.

Do we still have an exponential for **Span**?
A model of \textbf{LL} on spans?

We thus have a quantitative generalization of $\text{Rel}$ in the form of $\text{Span}$.

Do we still have an exponential for $\text{Span}$?

- First try: can we use $\mathcal{M}_{\text{fin}}(-)$ as exponential for $\text{Span}$?

Given $\sigma \in \text{Span}$, define

\[
\mathcal{M}_{\text{fin}}(\sigma) = \mathcal{M}_{\text{fin}}(\sigma) \rightarrow \mathcal{M}_{\text{fin}}(A) \leftarrow \mathcal{M}_{\text{fin}}(B)
\]

Problem: $\mathcal{M}_{\text{fin}}$ does not respect composition, because pullbacks are not preserved. Thus, not a functor.
A model of **LL** on spans?

We thus have a quantitative generalization of **Rel** in the form of **Span**.

Do we still have an exponential for **Span**?

▶ First try: can we use $\mathcal{M}_{\text{fin}}(\_\_)$ as exponential for **Span**? No.

▶ Second try: can we use lists as exponential?

\[
a_1, \ldots, a_n \in A \; \leadsto \; [a_1; \cdots; a_n] \in \text{List}(A)
\]
A model of **LL** on spans?

We thus have a quantitative generalization of **Rel** in the form of **Span**.

Do we still have an exponential for **Span**?

- First try: can we use $\mathcal{M}_{\text{fin}}(\text{--})$ as exponential for **Span**? No.

- Second try: can we use lists as exponential?

$$a_1, \ldots, a_n \in A \leadsto [a_1; \ldots; a_n] \in \textbf{List}(A)$$

```
\[
\text{List}(\sigma) = \text{List}(\partial^\sigma_A) \leftarrow \text{List}(A) \rightarrow \text{List}(\partial^\sigma_B) \rightarrow \text{List}(B)
\]
```
A model of $\mathbf{LL}$ on spans?

We thus have a quantitative generalization of $\mathbf{Rel}$ in the form of $\mathbf{Span}$.

Do we still have an exponential for $\mathbf{Span}$?

- First try: can we use $\mathcal{M}_{\text{fin}}(-)$ as exponential for $\mathbf{Span}$? No.
- Second try: can we use lists as exponential?

\[
\begin{align*}
  a_1, \ldots, a_n \in A & \rightsquigarrow [a_1; \cdots; a_n] \in \text{List}(A) \\
  \text{List}(\sigma) & = \text{List}(\partial^\sigma_A) \text{ or } \text{List}(\partial^\sigma_B) \\
  \text{List}(A) & \rightarrow \text{List}(B)
\end{align*}
\]

We now have a (pseudo)functor, but no \textbf{Seely equivalence}

\[
\text{see}_{A,B} : \text{List} A \otimes \text{List} B \rightsquigarrow \text{List}(A \& B) \in \mathbf{Span}.
\]
A model of \textbf{LL} on spans?

We thus have a quantitative generalization of \textbf{Rel} in the form of \textbf{Span}.

Do we still have an exponential for \textbf{Span}?  

- First try: can we use $\mathcal{M}_{\text{fin}}(-)$ as exponential for \textbf{Span}? No.

- Second try: can we use lists as exponential? Probably no.

\[ a_1, \ldots, a_n \in A \rightsquigarrow [a_1; \cdots; a_n] \in \textbf{List}(A) \]
Span is dead, long live Span!

Our definition of Span was based on the category Set of sets and functions.
Span is dead, long live Span!

Our definition of \textbf{Span} was based on the category \textbf{Set} of sets and functions.

Let \textbf{Gpd} be the 2-category of groupoids, functors and \textbf{natural transformations}. 
Span is dead, long live Span!

Our definition of Span was based on the category Set of sets and functions.

Let \( \mathbf{Gpd} \) be the 2-category of groupoids, functors and natural transformations.

We (re)define Span as \( \text{Span}(\mathbf{Gpd}) \)
Span is dead, long live Span!

Our definition of Span was based on the category Set of sets and functions.

Let Gpd be the 2-category of groupoids, functors and natural transformations.

We (re)define Span as Span(Gpd)

- 0-cells: groupoids $A, B, \ldots$
Span is dead, long live Span!

Our definition of Span was based on the category $\text{Set}$ of sets and functions.

Let $\text{Gpd}$ be the 2-category of groupoids, functors and natural transformations.

We (re)define Span as $\text{Span}(\text{Gpd})$

- 0-cells: groupoids $A, B, \ldots$
- 1-cells: spans $\sigma, \tau, \ldots$

\[ \sigma \sim \cdots \sigma \]

\[ \partial^\sigma_A \quad \partial^\sigma_B \]

\[ A \quad B \]
**Span** is dead, long live **Span**!

Our definition of **Span** was based on the category **Set** of sets and functions.

Let **Gpd** be the 2-category of groupoids, functors and natural transformations.

We (re)define **Span** as **Span**(**Gpd**)

- 0-cells: groupoids $A, B, \ldots$
- 1-cells: spans $\sigma, \tau, \ldots$
- 2-cells: pseudo-commutative triangles $(F, \phi), (G, \psi), \ldots$

\[
(F, \phi) : \sigma \Rightarrow \tau \quad \sim \quad \sigma \quad \Rightarrow \quad \tau
\]
**Span** is dead, long live **Span**!

Our definition of **Span** was based on the category **Set** of sets and functions.

Let **Gpd** be the 2-category of groupoids, functors and **natural transformations**.

We (re)define **Span** as **Span**(**Gpd**)

Idea: the isomorphisms in groupoids express **symmetries** between \(x, y \in A\) and between witnesses \(s, t \in \sigma\).
Bipullbacks

We must now give a composition which respects symmetries.

One way is to say that the middle square above is a bipullback.
Bipullbacks

Let a diagram

\[
\begin{array}{ccc}
L & P & R \\
\downarrow f^L & \downarrow & \downarrow f^R \\
M & & \\
\end{array}
\]

in \textbf{Gpd}.
Bipullbacks

It is a bipullback when every pseudocone can be decomposed along it, i.e.,

\[
\begin{align*}
\forall X & \Rightarrow \\
\forall g_L & \Rightarrow \\
\forall g_R & \Rightarrow \\
\forall \mu & \Rightarrow \exists \lambda & \Rightarrow \exists \rho & \Rightarrow \exists k & \Rightarrow
\end{align*}
\]

and additional conditions.
Supple pullbacks

Problem:
- simple and effective composition of spans $\leadsto$ pullbacks
- taking symmetries into account $\leadsto$ bipullbacks

Can we have both? Yes.
Supple pullbacks

A **supple pullback** is a pullback which is also a bipullback.

For our span model of **LL**:  
- spans will be composed by pullbacks $\leadsto$ effectivity  
- we ensure that the pullbacks appearing are all supple $\leadsto$ symmetry
Uniform groupoids

Given a groupoid $A$, a **prestrategy** $\sigma$ on $A$ is a pair $(\sigma \in \text{Gpd}, \partial^\sigma : \sigma \to A)$.

$$\sigma \xrightarrow{\partial^\sigma} A \quad \in \quad \text{Gpd}$$
Uniform groupoids

Given a groupoid $A$, a **prestrategy** $\sigma$ on $A$ is a pair $(\sigma \in \text{Gpd}, \partial^\sigma : \sigma \to A)$.

\[
\sigma : \partial^\sigma \rightarrow A \quad \in \quad \text{Gpd}
\]

Note: a prestrategy on $A \times B$ is canonically a span between $A$ and $B$.

\[
\sigma : \partial^\sigma \rightarrow A \times B \quad \sim \rightarrow \quad \partial^\sigma_A \quad \sim \quad \partial^\sigma_B
\]

\[
A \quad \sigma \quad B
\]
Uniform groupoids

Two prestrategies $\sigma, \tau$ on $A$ are said uniformly orthogonal, denoted $\sigma \perp \tau$, when the pullback is supple (i.e., is a bipullback).

![Diagram]

Given a class $S$ of prestrategies on $A$,

$$S^\perp \triangleq \{ \tau \in \text{PreStrat}(A) \mid \forall \sigma \in S, \sigma \perp \tau \}.$$
A **uniform groupoid** $\mathcal{A} = (A, \mathcal{U}_A)$ is a pair of

- a groupoid $A$,
- a class $\mathcal{U}_A$ of **prestrategies** $\sigma = (\sigma, \partial^\sigma)$ on $A$, such that

\[ \mathcal{U}_A^\perp = \mathcal{U}_A. \]
Uniform groupoids

Operations on uniform groupoids: given $\mathcal{A} = (A, U_A)$ and $\mathcal{B} = (B, U_B)$,

$\triangledown \mathcal{A} \rightleftarrows \mathcal{B} \triangleq (A, U_A)$;

$\triangledown \mathcal{A} \times \mathcal{B} \triangleq (A \times B, (U_A \otimes U_B)\perp \perp)$ where

$$U_A \otimes U_B \triangleq \{ \sigma \times \sigma' | \sigma \in U_A \text{ and } \sigma' \in U_B \}$$

$$\sigma \times \sigma' \triangleq \sigma \times \sigma' \xrightarrow{\partial \sigma \times \partial \sigma'} A \times B ;$$

$\triangledown \mathcal{A} \leftarrow_\triangledown \mathcal{B} \triangleq (A \perp \otimes B^\perp)^\perp$;

$\triangledown \mathcal{A} \rightarrow_\triangledown \mathcal{B} \triangleq A \perp \leftarrow_\triangledown \mathcal{B}$ ($= (A \times B, (U_A \otimes U_B^\perp)^\perp)$).

Note: the prestrategies of $\mathcal{U}_{A \rightarrow_\triangledown B} \subseteq \text{PreStrat}(A \times B)$ are canonically spans between $A$ and $B$. 
A bicategory of uniform groupoids?

Let \textbf{Unif} be the structure with

- 0-cells: uniform groupoids \( \mathcal{A}, \mathcal{B}, \ldots \);
- 1-cells \( \mathcal{A} \to \mathcal{B} \): uniform spans \( \sigma \in \mathcal{U}_{\mathcal{A} \to \mathcal{B}} \);
- 2-cells \( \sigma \Rightarrow \tau \): morphism of spans \( (F, \phi) : \sigma \Rightarrow \tau \in \text{Span} \).

Is it a bicategory?
A bicategory of uniform groupoids?

Let **Unif** be the structure with

- 0-cells: uniform groupoids $\mathcal{A}, \mathcal{B}, \ldots$;
- 1-cells $\mathcal{A} \rightarrow \mathcal{B}$: uniform spans $\sigma \in U_{\mathcal{A} \rightarrow \mathcal{B}}$;
- 2-cells $\sigma \Rightarrow \tau$: morphism of spans $(F, \phi): \sigma \Rightarrow \tau \in \textbf{Span}$.

Is it a bicategory?

No: the bipullback composition ensures existence, but not canonicity

$\leadsto$ missing unitality, associativity, \ldots
Idea: add more structure to make the composition canonical.
Thinness

Let $\mathcal{A} = (A, \mathcal{U}_A)$ be a uniform groupoid.

Let $\sigma \in \mathcal{U}_A$ and $\tau \in \mathcal{U}_A^\bot$.
Thinness

Let $\mathcal{A} = (A, \mathcal{U}_A)$ be a uniform groupoid.

Let $\sigma \in \mathcal{U}_A$ and $\tau \in \mathcal{U}_A^\perp$.

$\sigma$ and $\tau$ are **thinly orthogonal**, denoted $\sigma \perp \tau$, when the vertex $P$ of

![Diagram]

is **discrete** (i.e., no non-identity morphisms).
Thinness

Let $A = (A, \mathcal{U}_A)$ be a uniform groupoid.

Let $\sigma \in \mathcal{U}_A$ and $\tau \in \mathcal{U}_A^\perp$.

$\sigma$ and $\tau$ are **thinly orthogonal**, denoted $\sigma \perp \perp \tau$, when the vertex $P$ of

![Diagram]

is **discrete** (i.e., no non-identity morphisms).

Idea: $\perp \perp$ constrains the overlapping between images of $\partial^{\sigma}$ and $\partial^{\tau}$

$\leadsto$ unicity of decompositions in $A$. 
Thinness

Let $A = (A, \mathcal{U}_A)$ be a uniform groupoid.

Let $\sigma \in \mathcal{U}_A$ and $\tau \in \mathcal{U}_A^\perp$.

$\sigma$ and $\tau$ are **thinyly orthogonal**, denoted $\sigma \perp \tau$, when the vertex $P$ of

![Diagram]

is **discrete** (i.e., no non-identity morphisms).

Given $S \subseteq \mathcal{U}_A$, we write

$$S^\perp \triangleq \{ \tau \in \mathcal{U}_A^\perp \mid \forall \sigma \in S, \quad \sigma \perp \tau \}.$$
A thin $\pm$-groupoid $\mathcal{A} = (A, A_-, A_+, \mathcal{U}_A, T_A)$ is the data of

- a uniform groupoid $(A, \mathcal{U}_A)$;
A thin $\pm$-groupoid $\mathcal{A} = (A, A_-, A_+, \mathcal{U}_A, \mathcal{T}_A)$ is the data of

- a uniform groupoid $(A, \mathcal{U}_A)$;
- two subgroupoids $A_-$ and $A_+$ of $A$ with the same objects as $A$ with injections $\text{id}^-_A : A_- \hookrightarrow A$; $\text{id}^+_A : A_+ \hookrightarrow A$
A thin $\pm$-groupoid $\mathcal{A} = (A, A_-, A_+, \mathcal{U}_A, T_A)$ is the data of

- a uniform groupoid $(A, \mathcal{U}_A)$;
- two subgroupoids $A_-$ and $A_+$ of $A$ with the same objects as $A$ with injections

$$
\text{id}_A^- : \ A_- \hookrightarrow A; \quad \text{id}_A^+ : \ A_+ \hookrightarrow A
$$

- a class $T_A \subseteq \mathcal{U}_A$ of thin prestrategies, such that

$$
T_A \perp \perp = T_A \quad \text{and} \quad \text{id}_A^- \in T_A \quad \text{and} \quad \text{id}_A^+ \in T_A \perp \perp.
$$

Constructions on thin $\pm$-groupoids: $A \perp$, $A \otimes B$, $A \rightharpoonup B$, ...
A thin $\pm$-groupoid $A = (A, A_-, A_+, \mathcal{U}_A, T_A)$ is the data of

- a uniform groupoid $(A, \mathcal{U}_A)$;
- two subgroupoids $A_-$ and $A_+$ of $A$ with the same objects as $A$ with injections
  $$ \text{id}_A^- : A_- \rightarrow A; \quad \text{id}_A^+ : A_+ \rightarrow A $$
- a class $T_A \subseteq \mathcal{U}_A$ of thin prestrategies, such that
  $$ T_A \perp = T_A \quad \text{and} \quad \text{id}_A^- \in T_A \quad \text{and} \quad \text{id}_A^+ \in T_A \perp. $$

Constructions on thin $\pm$-groupoids: $A^\perp, A \otimes B, A \longrightarrow B, \ldots$
Proposition

Let $A$ be a thin $\pm$-groupoid. Given an isomorphism

$$\theta: a \rightarrow a' \in A$$

there are unique

$$\theta^- \in A_- \quad \text{and} \quad \theta^+ \in A_+$$

such that $\theta = \theta^+ \circ \theta^-$. 

By definition, we have $\text{id}^- A \in T_A \subseteq U_A$ and $\text{id}^+ A \in T_\perp \perp A \subseteq U_\perp A$. 

Existence: since $\text{id}^- A \perp \text{id}^+ A$. 

Unicity: since $\text{id}^- A \perp \perp \text{id}^+ A$. 

**Proposition**

Let \( \mathcal{A} \) be a thin \( \pm \)-groupoid. Given an isomorphism

\[
\theta : a \to a' \in \mathcal{A}
\]

there are unique

\[
\theta^- \in A_- \quad \text{and} \quad \theta^+ \in A_+
\]

such that \( \theta = \theta^+ \circ \theta^- \).

By definition, we have \( \text{id}_A^- \in T_{A} \subseteq \mathcal{U}_A \) and \( \text{id}_A^+ \in T_{A}^\perp \subseteq \mathcal{U}_A^\perp \).
Thinness

Proposition

Let $A$ be a thin $\pm$-groupoid. Given an isomorphism

$$\theta: a \to a' \in A$$

there are unique

$$\theta^- \in A_- \quad \text{and} \quad \theta^+ \in A_+$$

such that $\theta = \theta^+ \circ \theta^-$. 

By definition, we have $\text{id}_A^- \in T_A \subseteq U_A$ and $\text{id}_A^+ \in T_A^\parallel \subseteq U_A^\parallel$.

Existence: since $\text{id}_A^- \perp \text{id}_A^+$. 
Proposition

Let $\mathcal{A}$ be a thin $\pm$-groupoid. Given an isomorphism

$$\theta: a \rightarrow a' \in \mathcal{A}$$

there are unique

$$\theta^- \in \mathcal{A}_- \quad \text{and} \quad \theta^+ \in \mathcal{A}_+$$

such that $\theta = \theta^+ \circ \theta^-$. 

By definition, we have $\text{id}_A^- \in T_{\mathcal{A}} \subseteq \mathcal{U}_{\mathcal{A}}$ and $\text{id}_A^+ \in T_{\mathcal{A}}^\perp \subseteq \mathcal{U}_{\mathcal{A}}^\perp$.

Existence: since $\text{id}_A^- \perp \text{id}_A^+$.

Unicity: since $\text{id}_A^- \perp \text{id}_A^+$. 

Given a thin ±-groupoid $\mathcal{A} = (A, A_-, A_+, \mathcal{U}_A, T_A)$, a 2-cell $\phi : X \Downarrow f \to A$ is said positive on $\mathcal{A}$ when $\phi_x \in A_+$ for every $x \in X$. 
Positive 2-cells

Proposition

Given $\sigma, \tau \in T_{A \rightarrow B}$ and $(F, \phi): \sigma \Rightarrow \tau$, there exist unique

$$(F^+, \phi^+): \sigma \Rightarrow \tau \quad \text{and} \quad \mu: F^+ \Rightarrow F$$

such that

$\sigma \quad \Downarrow \quad \phi \quad \Rightarrow \quad \tau$

$\partial^\sigma \quad \Downarrow \quad \partial^\tau$

$A \times B$

$\mu$

$\phi^+$

$F^+$

$F$

$A \times B$

with $\phi^+$ positive on $A \rightarrow B$. 
The bicategory \( \text{Thin}^+ \)

We define \( \text{Thin}^+ \)

- 0-cells: the thin \( \pm \)-groupoids \( A, B, C, \ldots \);
- 1-cells \( A \to B \): the thin spans \( \sigma \in T_{A \to B} \);
- 2-cells \( \sigma \Rightarrow \tau \): the span morphisms \( (F, \phi) \) with \( \phi \) positive.
The bicategory \textbf{Thin}^+

We define \textbf{Thin}^+

- 0-cells: the thin $\pm$-groupoids $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$;
- 1-cells $\mathcal{A} \rightarrow \mathcal{B}$: the \textbf{thin spans} $\sigma \in T_{\mathcal{A} \to \mathcal{B}}$;
- 2-cells $\sigma \Rightarrow \tau$: the span morphisms $(F, \phi)$ with $\phi$ positive.

Composition of 2-cells can now be defined canonically with the positive factorization of 2-cells $\rightsquigarrow$ unitality, associativity, $\ldots$
The bicategory $\text{Thin}^+$

We define $\text{Thin}^+$

- **0-cells**: the thin $\pm$-groupoids $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$;
- **1-cells** $\mathcal{A} \rightarrow \mathcal{B}$: the thin spans $\sigma \in T_{\mathcal{A} \rightarrow \mathcal{B}}$;
- **2-cells** $\sigma \Rightarrow \tau$: the span morphisms $(F, \phi)$ with $\phi$ positive.

Composition of 2-cells can now be defined canonically with the positive factorization of 2-cells $\leadsto$ unitality, associativity, $\ldots$

**Theorem (C., F.)**

$\text{Thin}^+$ is a bicategory.
The pseudocomonad

Recall that the comonad $!: \text{Rel} \to \text{Rel}$ is derived from the monad $\mathcal{M}_{\text{fin}} : \text{Set} \to \text{Set}$.

We derive a (pseudo)comonad $!: \text{Thin}^+ \to \text{Thin}^+$ from a (pseudo)monad $\text{Fam} : \text{Gpd} \to \text{Gpd}$.
The pseudocomonad

The monad $\text{Fam} : \text{Gpd} \rightarrow \text{Gpd}$?

To $A \in \text{Gpd}$, associates $\text{Fam}(A) \in \text{Gpd}$:
The pseudocomonad

The monad $\text{Fam}: \text{Gpd} \to \text{Gpd}$?

To $A \in \text{Gpd}$, associates $\text{Fam}(A) \in \text{Gpd}$:

- objects: families $(a_i)_{i \in I}$ with $I \subseteq \text{fin} \mathbb{N}$ and $a_i \in A$;
The pseudocomonad

The monad $\text{Fam} : \text{Gpd} \rightarrow \text{Gpd}$?

To $A \in \text{Gpd}$, associates $\text{Fam}(A) \in \text{Gpd}$:

- objects: families $(a_i)_{i \in I}$ with $I \subseteq \text{fin} \ \mathbb{N}$ and $a_i \in A$;
- morphisms $(a_i)_{i \in I} \rightarrow (a'_j)_{j \in J}$: pairs $(\pi, (f_i)_{i \in I})$ where
  - $\pi$ is a bijection $I \rightarrow J$;
  - $f_i$ is a morphism $a_i \rightarrow a'_{\pi(i)}$. 
The pseudocomonad

The monad \( \text{Fam} : \text{Gpd} \to \text{Gpd} \)?

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The unit \( A \to \text{Fam}(A) \): maps \( a \in A \) to \((a)_{i \in \{0\}} \in \text{Fam}(A) \);
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The unit \( A \to \text{Fam}(A) \): maps \( a \in A \) to \((a)_{i \in \{0\}} \in \text{Fam}(A) \);

The multiplication \( \text{Fam}(\text{Fam}(A)) \to \text{Fam}(A) \): merges families of families into simply-indexed families.
The pseudocomonad

We get a pseudocomonad

\[ !: \text{Thin}^+ \to \text{Thin}^+ \]

where

\[ !\mathcal{A} := (\text{Fam}(A), \ldots) \]

for every thin \( \pm \)-groupoids \( \mathcal{A} \) and

\[ !\sigma := \text{Fam}(\partial^g_\mathcal{A}) \quad \text{Fam}(\partial^g_\mathcal{B}) \]

\[ \text{Fam}(A) \quad \text{Fam}(B) \]

for every span \( \sigma: \mathcal{A} \to \mathcal{B} \).
The Kleisli bicategory

We thus get a Kleisli bicategory $\textbf{Thin}_!^+$ with $! = \text{Fam}$, whose 1-cells $A \to B$ are of the form

$$
\sigma \\
\partial_{!A} \\
\partial_A
$$

$$
\sigma \\
\partial_B \\
\partial_B
$$

$\text{Fam}(A)$ \quad $B$

In categorical models of LL, the Kleisli category is cartesian closed.

Theorem (C., F.)

The bicategory $\textbf{Thin}_!^+$ is cartesian closed.
We thus get a Kleisli bicategory $\text{Thin}_!^+$ with $! = \text{Fam}$, whose 1-cells $\mathcal{A} \to \mathcal{B}$ are of the form

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sigma} & \mathcal{B} \\
\downarrow_{\partial^\sigma_A} & & \downarrow_{\partial^\sigma_B} \\
\text{Fam}(\mathcal{A}) & & \mathcal{B}
\end{array}
\]

In categorical models of $\text{LL}$, the Kleisli category is cartesian closed.
The Kleisli bicategory

We thus get a Kleisli bicategory $\text{Thin}_!^+$ with $! = \text{Fam}$, whose 1-cells $A \to B$ are of the form

$$\sigma \leftarrow \partial_A ! A \quad \downarrow \sigma \quad \downarrow \partial_B ! B \quad \rightarrow \text{Fam}(A) \quad \sigma$$

In categorical models of LL, the Kleisli category is cartesian closed.

**Theorem (C., F.)**

*The bicategory $\text{Thin}_!^+$ is cartesian closed.*
Examples

Notation: given $a = (a_i)_{i \in I} \in !A$ with $A \in \textbf{Gpd}$ and $I = \{ i_1, \ldots, i_n \} \subseteq_{\text{fin}} \mathbb{N}$, we write

$$a = [i_1 \cdot a_{i_1}, \ldots, i_n \cdot a_{i_n}].$$
Examples

Example 1:

\[ x : \text{Bool} \vdash \text{if } x \text{ then } \text{ff} \text{ else } \text{tt} : \text{Bool} \]

interpreted as the span (which happens to be a relation)

\[ \{ [0 \bullet \text{tt}] \rightarrow \text{ff}, \  [0 \bullet \text{ff}] \rightarrow \text{tt} \} \]
Example 2:

\[ x : \text{Bool} \vdash \text{if } x \text{ then (if } x \text{ then } \text{ff} \text{ else } \text{tt} \text{) else (if } x \text{ then } \text{tt} \text{ else } \text{ff} ) : \text{Bool} \]

interpreted as the span (which happens to be a relation)

\[
\{ [0 \bullet \text{tt}, 1 \bullet \text{tt}] \to \text{ff}, \ [0 \bullet \text{tt}, 1 \bullet \text{ff}] \to \text{tt}, \ [0 \bullet \text{ff}, 1 \bullet \text{tt}] \to \text{tt}, \ [0 \bullet \text{ff}, 1 \bullet \text{ff}] \to \text{ff} \} \]

!\text{Bool} \quad \text{Bool}
Examples

Example 2:

\[ x : \text{Bool} \vdash \text{if } x \text{ then (if } x \text{ then } \text{ff} \text{ else } \text{tt} \text{) else (if } x \text{ then } \text{tt} \text{ else } \text{ff} : \text{Bool} \]

interpreted as the span (which happens to be a relation)

\[
\{ [0 \Diamond \text{tt}, 1 \Diamond \text{tt}] \rightarrow \text{ff}, \ [0 \Diamond \text{tt}, 1 \Diamond \text{ff}] \rightarrow \text{tt}, \ [0 \Diamond \text{ff}, 1 \Diamond \text{tt}] \rightarrow \text{tt}, \ [0 \Diamond \text{ff}, 1 \Diamond \text{ff}] \rightarrow \text{ff} \}
\]

\[
\begin{array}{c}
\text{!Bool} \\
\text{Bool}
\end{array}
\]

to compare with the interpretation in \(\text{Rel}!\):

\[
\{ [\text{tt}, \text{tt}] \rightarrow \text{ff}, \ [\text{tt}, \text{ff}] \rightarrow \text{tt}, \ [\text{ff}, \text{ff}] \rightarrow \text{tt} \}.
\]
Example 3: a non-deterministic operator $\triangleright$

$$
\triangleright \mathsf{ff} \triangleright \mathsf{tt} : \mathsf{Bool}
$$

interpreted as the span (which happens to be a relation)

$$
\{ \text{inl(} \mathsf{ff} \text{)}, \text{inr(} \mathsf{tt} \text{)} \}\\
!() \quad \text{Bool}
$$
Examples

Example 4:

\[ \vdash \text{ff} \otimes \text{ff} : \text{Bool} \]

interpreted as the span

\[ \{ \text{inl(ff)}, \text{inr(ff)} \} \]

\[ !\emptyset \xrightarrow{\text{Span}} \text{Bool} \]

\( \leadsto \) two witnesses for \text{ff}.
Other works

Source of the ideas of this work:

- Concurrent games: symmetries, thinness, proofs, . . .
  Castellan, Clairambault, et al. “Games and Strategies as Event Structures”. 2017

Related works:

- Generalized species of structures

- Template games

- Infinitary intersection types
  Vial. “Infinitary intersection types as sequences: A new answer to Klop’s problem”. 2017
The end

Any questions?
Whiteboard
Recall: a common approach for exhibiting a categorical model of $\text{LL}$ is to find a Seely isomorphism

$$\text{see}_{A,B} : !A \otimes !B \to !(A \& B).$$
Seely equivalence

In $\text{Thin}^+$,

$$A \otimes B \cong (A \times B, \ldots) \quad \text{and} \quad A \& B \cong (A \sqcup B, \ldots).$$

We have the 2-categorical analogue of a Seely isomorphism, already in $\text{Gpd}$:

**Proposition**

*Given $A, B \in \text{Gpd}$, there is an adjoint equivalence of groupoids

$$\text{Fam}(A) \times \text{Fam}(B) \perp \text{Fam}(A \sqcup B).$$*

Idea: given $a = (a_i)_{i \in I}$ and $b = (b_j)_{j \in J}$, one can merge $a$ and $b$ as $c = (c_k)_{k \in K}$ with $K \cong I \sqcup J$. 
The Seely 2-cell

Recall: the Seely isomorphism

\[ \text{see}_{A,B} : !A \otimes !B \to !(A \& B) \]

is supposed to verify the equality

\[
\begin{align*}
!A \otimes !B & \xrightarrow{\text{see}_{A,B}} !(A \& B) \\
\delta_A \otimes \delta_B & \quad = \quad !!(A \& B) \cdot \\
!!A \otimes !!B & \xrightarrow{\text{see}_{!A,!B}} !(A \& !B)
\end{align*}
\]
The Seely 2-cell

The Seely equality appears here as a non-trivial 2-cell in \( \text{Gpd} \):
Cartesian structure

Definition
A bicategory $\mathcal{C}$ is **cartesian** when, for every objects $Y, Z$, there exist

an object $Y \& Z \in \mathcal{C}$ and morphisms $l: Y \& Z \to Y$ and $r: Y \& Z \to Z$

such that, for every $X$, there is an adjoint equivalence of categories

\[
\begin{array}{ccc}
\mathcal{C}(X, Y \& Z) & \cong & \mathcal{C}(X, Y) \times \mathcal{C}(X, Z)
\end{array}
\]

(+ there exists a terminal object expressed as an adjoint equivalence too).
Theorem

The bicategory $\text{Thin}_!^+$ is cartesian.
Theorem

The bicategory $\text{Thin}_!^+$ is cartesian.

Given two thin $\pm$-groupoids $A$ and $B$, we take $A & B \triangleq (A \sqcup B, \ldots)$ and

\[
\begin{align*}
\tilde{l} & = \begin{array}{c} \text{Fam}(A \sqcup B) \quad \text{id}_A \\ \eta_{A \sqcup B} \end{array} & l & = \begin{array}{c} A \sqcup B \\ \text{id}_A \\ \eta_{A \sqcup B} \end{array} \\
\tilde{r} & = \begin{array}{c} \text{Fam}(A \sqcup B) \\ \text{id}_B \end{array} & r & = \begin{array}{c} A \sqcup B \\ \eta_{A \sqcup B} \end{array}
\end{align*}
\]

for $l : A & B \to A$ and $r : A & B \to B$ in $\text{Thin}_!^+$. 

Closure

A cartesian bicategory $\mathcal{C}$ is **closed** when, for every object $Y, Z$, there exist

an object $Y \Rightarrow Z \in \mathcal{C}$ and a morphism $\text{ev}_{Y,Z} : (Y \Rightarrow Z) \& Y \to Z$

such that, for every $X \in \mathcal{C}$, there is an adjoint equivalence

$$
\begin{array}{ccc}
\mathcal{C}(X, Y \Rightarrow Z) & \perp & \mathcal{C}(X \& Y, Z) \\
\downarrow & & \downarrow \\
(-)^\dagger & & \text{ev}_{Y,Z} \circ (- \& Y)
\end{array}
$$

Theorem

The cartesian bicategory $\text{Thin}^+$ is closed.
Closure

A cartesian bicategory $\mathcal{C}$ is **closed** when, for every object $Y, Z$, there exist

an object $Y \Rightarrow Z \in \mathcal{C}$ and a morphism $\text{ev}_{Y,Z} : (Y \Rightarrow Z) \& Y \to Z$

such that, for every $X \in \mathcal{C}$, there is an adjoint equivalence

$$
\begin{array}{ccc}
\mathcal{C}(X, Y \Rightarrow Z) & \quad \bot \quad & \mathcal{C}(X \& Y, Z) \\
\downarrow & & \downarrow \\
(-)^\dagger
\end{array}
$$

**Theorem**

*The cartesian bicategory $\text{Thin}_!^+$ is closed.*
The closed structure for $\textbf{Thin}^+$

Given thin $\pm$-groupoids $\mathcal{B}, \mathcal{C}$, we take $\mathcal{B} \Rightarrow \mathcal{C} \doteq (!B \times C, \ldots)$ and

\[
\text{ev}_{\mathcal{B},\mathcal{C}}: (\mathcal{B} \Rightarrow \mathcal{C}) \& \mathcal{B} \to \mathcal{C} = \begin{array}{c}
!B \times C \\
\langle l, r, l \rangle \\
\uparrow \\
!B \times C \times !B \\
\eta_{!B \times C \times !B} \\
\downarrow \\
!((!B \times C) \sqcup !B) \\
\text{see}_{!B \times C, B} \\
\downarrow \\
C
\end{array}
\]

(writing directly $!$ for $\textbf{Fam}$).