A computational method for left adjointness

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Usual questions in category theory

Given a category $\mathcal{C}$, examples of things that we want to know:

- is $\mathcal{C}$ complete or cocomplete?
- is $\mathcal{C}$ closed?

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, examples of things that we want to know:

- does F preserve limits or colimits?
- is F part of an adjunction?

Towards automation

Goal: automate or assist with some reasonings for solving these questions.

This requires:

- good computational representations
- efficient algorithms
- interaction with the user in case of partial decidability

Tools exist for higher categories (Globular, Homotopy.io, Opetopy, *etc.*) but not as many for simple categories.

Computational representations

Presentations as computational representations of algebraic structures.

Computational representations

Presentations as computational representations of algebraic structures.

Example: one can consider a category C with

- objects u, v, w
- generating arrows $a\colon u \to v$, $b\colon v \to w$ and $c\colon u \to v$

$$
\begin{tikzcd}
u \arrow[dd, "a"'] \arrow[dr, "c"] & \\
& w \\
v \arrow[ur, "b"'] &
\end{tikzcd}
$$

Computational representations

Presentations as computational representations of algebraic structures.

Also a category D with

- objects x, y, z
- generating arrows $d\colon x \to y$ and $e\colon y \to z$

$$\begin{tikzcd} u \arrow[dd, "a"'] \arrow[dr, "c"] & \\ & w \\ v \arrow[ur, "b"'] & \end{tikzcd} \qquad\qquad x \xrightarrow{d} y \xrightarrow{e} z$$

Computational representations

Presentations as computational representations of algebraic structures.

Then one can consider the functor F such that

$$F(u) = x \qquad F(v) = y \qquad F(w) = z$$
$$F(a) = d \qquad F(b) = e \qquad F(c) = d * e$$

$$\begin{tikzcd}
u \arrow[d, "a"'] \arrow[dr, "c"] & & & & \\
& w \arrow[rr, "F"] & & x \arrow[r, "d"] & y \arrow[r, "e"] & z \\
v \arrow[ur, "b"'] & & & &
\end{tikzcd}$$

Computational representations

Presentations as computational representations of algebraic structures.

Such data can be given to a computer.

```
A := category {
  obj := {u,v,w},
  arr := {a : u => v, b : v => w, c : u => w}
}
B := category {
  obj := {x,y,z},
  arr := {d : x => y, e : y => z}
}
F := functor A => B {
  u -> x, v -> y, w -> z,
  a -> d, b -> e, c -> d * e
}
```

Computational representations

Presentations as computational representations of algebraic structures.

$$
\begin{tikzcd}
u \arrow[dd, "a"'] \arrow[dr, "c"] & & & & & \\
& w \arrow[rr, "F"] & & x \arrow[r, "d"] & y \arrow[r, "e"] & z \\
v \arrow[ur, "b"'] & & & & &
\end{tikzcd}
$$

One can ask questions like

- is C complete?
- is F limit-preserving?
- *etc.*

Computational representations

Presentations as computational representations of algebraic structures.

$$
\begin{tikzcd}
u \arrow[dd, "a"'] \arrow[dr, "c"] & & & & \\
 & w \arrow[rr, "F"] & & x \arrow[r, "d"] & y \arrow[r, "e"] & z \\
v \arrow[ur, "b"'] & & & &
\end{tikzcd}
$$

But C, D and F are very artificial objects that might not be of interest.

What about "real" categories: categories of sets, groups, *etc.* and functors between them?

Another notion of presentation

Idea: large categories can also be presented in another sense.

$\rightsquigarrow$ notion of *locally presentable categories*

- category of sets
- category of groups, rings, monoids
- category of sheaves and presheaves
- *etc.*

Outline

Locally presentable categories

Computational descriptions of functors

Method for left adjointness

Applications

Playing a game

Proof of the criterion

Outline

Locally presentable categories: idea

Often, we deal with categories whose object can be **presented**.

Locally presentable categories: idea

Take $\mathbf{Gph}$, the category of graphs.

Every graph can be presented as $\langle S_V, S_A \mid E \rangle$ where

- S_V is a set of generating vertices
- S_A is a set of generating arrows
- E is a set of equations between sources and targets of arrows, and objects

$$x \xrightarrow{a} y \xrightarrow{b} z$$

$$\langle \emptyset, \{a, b\} \mid \partial^+(a) = \partial^-(b) \rangle$$

Locally presentable categories: idea

Take $\mathbf{Grp}$, the category of groups.

Every group G can be presented as $\langle S \mid E \rangle$ where

- S is a set of generators
- E is a set of equations

Free commutative group on two elements

$$\langle \{a, b\} \mid ab = ba \rangle$$

Locally presentable categories: idea

Take $\mathbf{Cat}$, the category of small categories.

Every group C can be presented as $\langle S_O, S_M \mid E \rangle$ where

- S_O is a set of generating objects
- S_M is a set of generating morphisms
- E is a set of equations on objects and morphisms.

$\mathbb{N}$ seen as a category with one object

$$\langle \emptyset, \{1\} \mid \partial^+(1) = \partial^-(1) \rangle$$

Locally presentable categories: idea

The notion of **locally finitely presentable categories** describes such theories.

It encompasses a lot of very common categories.

Locally finitely presentable categories

The abstract definition: a category is locally finitely presentable when

1. it is locally small
2. it has all colimits
3. its class of objects which can be finitely presented is essentially small
4. every objects is a directed colimits of finitely presentable objects

A more concrete definition

Proposition (Adámek, Rosický)

A locally presentable category is the category of models of an ***essentially algebraic theory****.*

Essentially algebraic theory $\mathbb{T}$: data of

- sorts
- operations between sorts
- equations that should be satisfied

A more concrete definition

Example: the ess. alg. theory of monoids.

1 sort:

$$\mathbf{M}$$

2 generating operations:

$$e\colon 1 \to \mathbf{M} \qquad c\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$$

A more concrete definition

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2 generating operations:

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Note: the domains of the operations are limit cones over the only sort.

A more concrete definition

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1 sort:

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2 generating operations:

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satisfying the equations

$$c(e(x), y) = y \qquad c(x, e(y)) = x \qquad c(c(x, y), z) = c(x, c(y, z))$$

A more concrete definition

Example: the ess. alg. theory of monoids.

1 sort:

$$\mathbf{M}$$

2 generating operations:

$$e\colon 1 \to \mathbf{M} \qquad c\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$$

A model $\mathcal{M}$ is then the data of

- a set $\mathcal{M}(\mathbf{M})$,
- functions $\mathcal{M}(e)\colon 1 \to \mathcal{M}(\mathbf{M})$ and $\mathcal{M}(c)\colon \mathcal{M}(\mathbf{M}) \times \mathcal{M}(\mathbf{M}) \to \mathcal{M}(\mathbf{M})$ satisfying the equations.

Proposition

The category of models (i.e., monoids) is a locally finitely presentable category.

A more concrete definition

Example: the ess. alg. theory of small categories.

2 sorts:

$$\mathbf{C_0} \quad \text{and} \quad \mathbf{C_1}$$

4 operations:

$$\mathrm{id}\colon \mathbf{C}_0 \to \mathbf{C}_1 \qquad \partial^-\colon \mathbf{C}_1 \to \mathbf{C}_0 \qquad \partial^+\colon \mathbf{C}_1 \to \mathbf{C}_0 \qquad c\colon \mathbf{C}_1 \times_0 \mathbf{C}_1 \to \mathbf{C}_1$$

together with equations

$$\partial^\epsilon(\mathrm{id}(x)) = x \qquad c(\mathrm{id}(x), g) = g \qquad c(f, \mathrm{id}(y)) = f \qquad c(c(f,g),h) = c(f, c(g,h))$$

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Example: the ess. alg. theory of small categories.

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A model $\mathcal{M}$ is then the data of

- two sets $\mathcal{M}(\mathbf{C}_0)$ and $\mathcal{M}(\mathbf{C}_1)$,
- functions $\mathcal{M}(\mathrm{id})\colon \mathcal{M}(\mathbf{C}_0) \to \mathcal{M}(\mathbf{C}_1)$ and *etc.* satisfying the equations.

Proposition

The category of models (i.e., small categories) is a locally presentable category.

A more concrete definition

So there are a lot of locally finitely presentable categories:

- category of sets
- categories of groups, rings, *etc.*
- categories of presheaves, sheaves
- categories of strict n-categories, (algebraic) weak n-categories
- *etc.*

Another description of locally presentable categories

Consider again the ess. alg. theory of monoids:

1 sort:

$$\mathbf{M}$$

2 generating operations:

$$e\colon 1 \to \mathbf{M} \qquad\qquad c\colon \mathbf{M}^2 \to \mathbf{M}$$

Let's build a category out of this.

Another description of locally presentable categories

$$\mathbf{M}$$

Start with sorts as objects.

Another description of locally presentable categories

$$\mathbf{1} \qquad\qquad \mathbf{M} \qquad\qquad \mathbf{M}^2$$

Add objects for the domains of the operations.

$$e\colon 1 \to \mathbf{M} \qquad\qquad c\colon \mathbf{M}^2 \to \mathbf{M}$$

Another description of locally presentable categories

$$\mathbf{1} \xrightarrow{e} \mathbf{M} \xleftarrow{c} \mathbf{M^2}$$

Add the arrows for these operations.

Another description of locally presentable categories

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "e"] & \mathbf{M} & \mathbf{M^2} \arrow[l, shift right=3, "\pi_L"'] \arrow[l, "c" description] \arrow[l, shift left=3, "\pi_R"]
\end{tikzcd}
$$

Add arrows for the cone projections.

Another description of locally presentable categories

$$\mathbf{1} \xleftarrow{\;e\;} \mathbf{M} \underset{\pi_R}{\overset{\pi_L}{\underset{c}{\rightrightarrows}}} \mathbf{M^2}$$

Reverse all arrows.

Another description of locally presentable categories

$$\begin{tikzcd} \mathbf{1} & \mathbf{M} \arrow[l, "e"'] \arrow[r, shift left=2, "\pi_L"] \arrow[r, "c" description] \arrow[r, shift right=2, "\pi_R"'] & \mathbf{M^2} \end{tikzcd}$$

A model $\mathcal{M}$ of $\mathbf{T}$ is then a particular **presheaf** on the above category C, *i.e.*, a functor

$$X\colon C^{\mathrm{op}} \to \mathbf{Set}$$

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A model $\mathcal{M}$ of $\mathbf{T}$ is then a particular **presheaf** on the above category C, *i.e.*, a functor

$$X \colon C^{\mathrm{op}} \to \mathbf{Set}$$

Which presheaf $X \in \widehat{C}$ are actual models, *i.e.*, monoids?

- $X(\mathbf{1})$ must be a terminal set
- $(X(\mathbf{M^2}), X(\pi_L), X(\pi_R))$ must be the product of $X(\mathbf{M})$ and $X(\mathbf{M})$
- the equations of monoids must hold: $X(c)(X(e)(x), y) = y$, *etc.*

These conditions can be expressed through **orthogonality conditions**.

Orthogonality

Let $\mathcal{C}$ be a category, $g\colon A \to B$ and $X \in \mathcal{C}$.

X is **orthogonal** to g when, for all $h\colon A \to X$, there is a unique $\bar{h}\colon B \to X$ such that $h = \bar{h} \circ g$.

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "\forall h"'] & & B \arrow[dl, dotted, "\exists! \bar{h}"] \\ & X & \end{tikzcd}$$

Orthogonality

Let $O^{\mathcal{C}} \subseteq \mathcal{C}_1$ be a chosen set of **orthogonality morphisms**.

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There is then a canonical inclusion functor

$$J\colon \mathcal{C}^{\perp} \to \mathcal{C}.$$

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Proposition (Adámek, Rosický)

If $\mathcal{C}$ is loc. fin. presentable, the canonical inclusion functor $J \colon \mathcal{C}^{\perp} \to \mathcal{C}$ has a left adjoint $(-)^{\perp}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\perp} \arrow[l, shift left=2, "J"] \end{tikzcd}$$

Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

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Example for monoids:
$$\mathbf{1} \xleftarrow{e} \mathbf{M} \overset{\pi_R}{\underset{\pi_L}{\overset{c}{\rightrightarrows}}} \mathbf{M^2}$$

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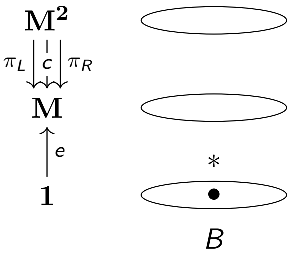
Let B be the presheaf freely generated from one element $*$ in $B(\mathbf{1})$.



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Let B be the presheaf freely generated from one element $*$ in $B(\mathbf{1})$.

Let X in $\widehat{C}$. Then, $X(\mathbf{1})$ is a terminal set when X is orthogonal to $\emptyset \to B$

$$\begin{tikzcd} \emptyset \arrow[rr, "\emptyset"] \arrow[dr, "\forall H"'] & & B \arrow[dl, dotted, "\exists! \bar{H}"] \\ & X & \end{tikzcd}$$

Indeed, $\widehat{C}(B, X) \simeq X(\mathbf{1})$, so that the condition says $X(\mathbf{1}) \simeq \{*\}$.

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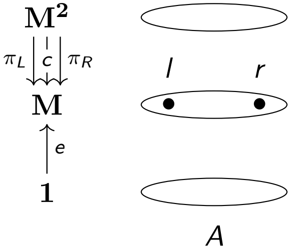
Let

- $A \in \widehat{C}$ freely gen. from two element l, r in $B(\mathbf{M})$

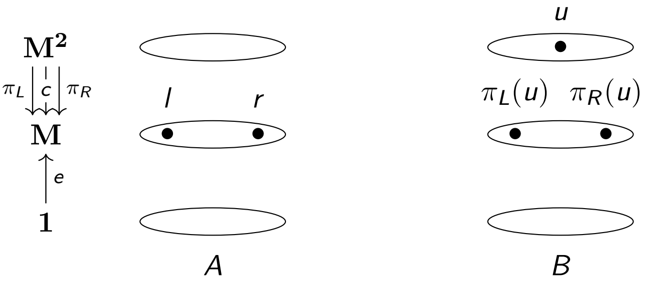


Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Let

- $A \in \widehat{C}$ freely gen. from two element l, r in $B(\mathbf{M})$
- $B \in \widehat{C}$ freely gen. from an element $u \in B(\mathbf{M^2})$



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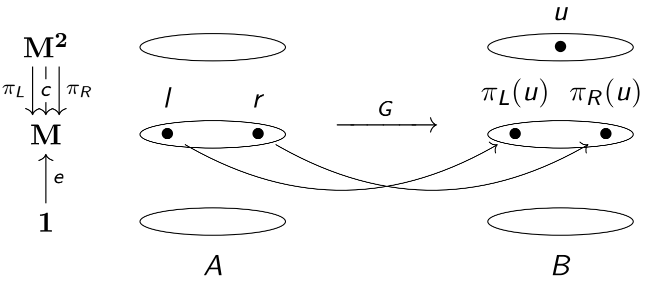
- $A \in \widehat{C}$ freely gen. from two element l, r in $B(\mathbf{M})$
- $B \in \widehat{C}$ freely gen. from an element $u \in B(\mathbf{M^2})$
- $G \colon A \to B$ such that $G(l) = \pi_L(u)$ and $G(r) = \pi_R(u)$.



Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

Let

- $A \in \widehat{C}$ freely gen. from two element l, r in $B(\mathbf{M})$
- $B \in \widehat{C}$ freely gen. from an element $u \in B(\mathbf{M^2})$
- $G\colon A \to B$ such that $G(l) = \pi_L(u)$ and $G(r) = \pi_R(u)$.

$(X(\mathbf{M^2}), X(\pi_L), X(\pi_R))$ is a product iif X is orthogonal to $G\colon A \to B$.

Indeed, $\widehat{C}(A, X) \simeq X(\mathbf{M}) \times X(\mathbf{M})$ and $\widehat{C}(B, X) \simeq X(\mathbf{M^2})$.

Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

The equations of monoids can also be expressed as orthogonality conditions.

$$A^L \xrightarrow{G^L} B^L \qquad A^R \xrightarrow{G^R} B^R \qquad A^A \xrightarrow{G^A} B^A$$

Thus, $\mathbf{Mon} \simeq \widehat{C}^{\perp}$ for a set $O^C \subseteq \widehat{C}_1$ of orthogonality morphisms.

$$C \quad = \qquad \begin{tikzcd} \mathbf{1} & \mathbf{M} \arrow[l, "e"'] \arrow[r, shift left=2, "\pi_L"] \arrow[r, "c" description] \arrow[r, shift right=2, "\pi_R"'] & \mathbf{M^2} \end{tikzcd}$$

Orthogonality conditions

The restrictions on presheaves can be expressed as orthogonality conditions.

More generally,

Proposition

Every loc. fin. pres. category $\mathcal{C}$ can be described as

$$\mathcal{C} \simeq \widehat{C}^{\perp}$$

for some $C \in \mathbf{Cat}$ and $O^C \subseteq (\widehat{C})_1$.

Summary

- A lot of categories of interest are locally presentable categories.
- Such categories can be seen as orthogonality classes of presheaf categories.

Outline

Describe functors

$$\mathcal{F} \colon \quad \mathcal{C} \quad \to \quad \mathcal{D}$$

Goal: describe (some) functors between two loc. pres. categories $\mathcal{C}$ and $\mathcal{D}$.

We will need to filter some out.

Describe functors

$$F\colon \quad \widehat{C}^{\perp} \quad \to \quad \widehat{D}^{\perp}$$

First, we use the characterization: $\mathcal{C} \simeq \widehat{C}^{\perp}$ and $\mathcal{D} \simeq \widehat{D}^{\perp}$.

Describe functors

$$\bar{F}'\colon \quad \widehat{C} \quad \to \quad \widehat{D}^{\perp}$$

Then, let's actually define a functor $\bar{F}'$ on a larger domain.

In good cases, F can then be recovered by precomposition with $J\colon \widehat{C}^{\perp} \to \widehat{C}$.

Describe functors

$$\bar{F}\colon \quad \widehat{C} \quad \to \quad \widehat{D}$$

Also, let's actually define a functor $\bar{F}$ on a larger domain.

In good cases, $\bar{F}'$ can be recovered by post-composition with $(-)^{\perp}$.

Describe functors

$$\tilde{F}\colon \quad C \quad \to \quad \widehat{D}$$

Then, let's actually only define $\bar{F} \circ y$ where y is the Yoneda embedding

$$y\colon c \mapsto \mathrm{Hom}(-, c)$$

Describe functors

$$\tilde{F}\colon \quad C \quad \to \quad \widehat{D}$$

If $\bar{F}$ is nice enough, it can be recovered using a **left Kan extension**:

$$\begin{tikzcd}
\widehat{C} \arrow[drr, "\bar{F}"] & & \\
C \arrow[u, "y"] \arrow[rr, "\tilde{F}"'] \arrow[r, phantom, "\Uparrow \alpha"] & & \widehat{D}
\end{tikzcd}$$

Describe functors

$$\tilde{F}\colon \quad C \quad \to \quad \widehat{D}$$

Under some finiteness hypothesis on C, D and $\tilde{F}$, the latter can be described computationally.

Describe functors

Summary: nice functors $\mathcal{F}$ between presentable categories $\mathcal{C} \simeq \widehat{C}^{\perp}$ and $\mathcal{D} \simeq \widehat{D}^{\perp}$ can be described computationally by a functor

$$\tilde{F} \colon C \to \widehat{D}$$

and recovered using the diagram

$$\begin{tikzcd}
\widehat{C}^{\perp} \arrow[d, "J"'] \arrow[drr, "F"] & & \\
\widehat{C} \arrow[r, "\bar{F}"] & \widehat{D} \arrow[r, "(-)^{\perp}"'] & \widehat{D}^{\perp} \\
C \arrow[u, "y"] \arrow[ur, "\tilde{F}"'] & &
\end{tikzcd}$$

Kan extensions

What is actually a Kan extension doing?

Some intuition with a particular case but essential for the following.

Kan extensions

$$
\begin{tikzcd}
\widehat{C} \\
C \arrow[u, "y"] \arrow[r, "\tilde{F}"'] & \widehat{D}
\end{tikzcd}
$$

Given $\tilde{F}\colon C \to \widehat{D}$ and $y\colon C \to \widehat{C}$ the Yoneda embedding,

Kan extensions

$$\begin{tikzcd}
\widehat{C} \arrow[dr, "F"] & \\
C \arrow[u, "y"] \arrow[r, "\tilde{F}"'] \arrow[ur, phantom, "\Uparrow \alpha"] & \widehat{D}
\end{tikzcd}$$

a left Kan extension of $\tilde{F}$ along y is a pair (F, α) which is universal in some sense.

Kan extensions

$$\begin{tikzcd} \widehat{C} \arrow[dr, "F"] & \\ C \arrow[u, "y"] \arrow[r, "\tilde{F}"'] \arrow[ur, phantom, "\Uparrow \alpha"] & \widehat{D} \end{tikzcd}$$

Concretely:

$$F(X) = \int^{c \in C} \tilde{F}(c) \otimes X(c)$$

Idea: for each $e \in X(c)$, there is one copy of $\tilde{F}(c)$ in $F(X)$, adequately glued to other copies.

Kan extensions

$$\begin{tikzcd}
\widehat{C} \arrow[dr, "F"] & \\
C \arrow[u, "y"] \arrow[r, "\tilde{F}"'] \arrow[ur, phantom, "\Uparrow \alpha"] & \widehat{D}
\end{tikzcd}$$

Even more concretely:

$$F(X) = \Big(\coprod_{c \in C, e \in X(c)} \tilde{F}(c) \Big) / \sim$$

where

$$(c', e', \tilde{F}(g)(u)) \sim (c, X(g)(e), u)$$

for every $g\colon c \to c' \in C$, $e' \in X(c')$, $u \in \tilde{F}(c)$.

Note: under finiteness conditions, this is computable.

Examples of functor descriptions

Taking

- $\mathbf{Set} \simeq \widehat{1}^{\perp}$ with $O^{\mathbf{Set}} = \emptyset$
- $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{1 \coprod 1}^{\perp}$ with $O^{\mathbf{Set} \times \mathbf{Set}} = \emptyset$

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- $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{1 \coprod 1}^{\perp}$ with $O^{\mathbf{Set} \times \mathbf{Set}} = \emptyset$

the functor

$$\mathcal{F}\colon \qquad (X, Y) \in \mathbf{Set} \times \mathbf{Set} \qquad \mapsto \qquad X \in \mathbf{Set}$$

can be described by $\tilde{F}\colon 1 \coprod 1 \to \widehat{1}$ where $\tilde{F}(0_L) = \{*\}$ and $\tilde{F}(0_R) = \emptyset$.

$$\begin{tikzcd}
\mathbf{Set} \times \mathbf{Set} \arrow[dr, "\mathcal{F}"] & \\
1 \coprod 1 \arrow[u, "y"] \arrow[r, "{[\{*\},\emptyset]}"'] \arrow[ur, phantom, "\Uparrow \alpha"] & \mathbf{Set}
\end{tikzcd}$$

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- $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{1 \coprod 1}^{\perp}$ with $O^{\mathbf{Set} \times \mathbf{Set}} = \emptyset$

the functor

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$$\begin{tikzcd}
\mathbf{Set} \times \mathbf{Set} \arrow[drr, "\mathcal{F}"] & & \\
1 \coprod 1 \arrow[u, "y"] \arrow[rr, "{[\{*\},\emptyset]}"'] \arrow[urr, phantom, "\Uparrow \alpha"] & & \mathbf{Set}
\end{tikzcd}$$

Idea: in $\mathbf{Set} \times \mathbf{Set}$, $0_L \rightsquigarrow (\{*\}, \emptyset)$, $0_R \rightsquigarrow (\emptyset, \{*\})$

Examples of functor descriptions

Taking

- $\mathbf{Set} \simeq \widehat{1}^{\perp}$ with $O^{\mathbf{Set}} = \emptyset$
- $\mathbf{Mon} \simeq \widehat{C}^{\perp}$ with $O^{\mathbf{Mon}} = \{G^T, G^P, G^L, G^R, G^A\}$ and

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Examples of functor descriptions

Taking

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the free monoid functor

$$\mathcal{F}\colon \qquad S \in \mathbf{Set} \qquad \mapsto \qquad S^* \in \mathbf{Mon}$$

can be described by $\tilde{F}\colon 1 \to \widehat{C}$ where $\tilde{F}(0) = y(\mathbf{M})$.

$$\begin{tikzcd} \mathbf{Set} \arrow[dr] \arrow[drr, "\mathcal{F}"] & & \\ \mathbf{1} \arrow[u, "y"] \arrow[r, "y(\mathbf{M})"'] \arrow[ur, phantom, "\Uparrow \alpha"] & \widehat{C} \arrow[r, "(-)^{\perp}"'] & \mathbf{Mon} \end{tikzcd}$$

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Idea:

- in $\mathbf{Set}$, $0 \rightsquigarrow \{*\}$
- in $\mathbf{Mon}$, $y(\mathbf{M})$ corresponds to the free monoid $\{*\}^*$

Outline

Problem

Given a functor

$$\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$$

described by a functor

$$\tilde{F}\colon C \to \widehat{D}$$

how can we check that $\mathcal{F}$ is a left adjoint?

A solution

Proposition (Adámek, Rosický)

A functor $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ between loc. fin. pres. cat. is a left adjoint if and only if it preserves all small colimits.

So: when is $\mathcal{F}$ preserving all small colimits?

A solution

Using $\mathcal{C} \simeq \widehat{C}^{\perp}$ and $\mathcal{D} \simeq \widehat{D}^{\perp}$,

Theorem

If the functor $(-)^{\perp} \circ \bar{F} \colon \widehat{C} \to \widehat{D}^{\perp}$ *sends the elements of* O^{C} *to isomorphisms, then* $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ *preserves all colimits (and thus is a left adjoint).*

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- C, D, O^C, O^D can be described to a computer
- the images of $G \colon A \to B \in O^C$ by the functor $\bar{F} \colon \widehat{C} \to \widehat{D}$ can be computed

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$$\bar{F}(A) = \Big(\coprod_{c \in C, e \in X(c)} \tilde{F}(c) \Big) / \sim$$

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- C, D, O^C, O^D can be described to a computer
- the images of $G \colon A \to B \in O^C$ by the functor $\bar{F} \colon \widehat{C} \to \widehat{D}$ can be computed
- checking that a functor $G' \colon A' \to B' \in \widehat{D}$ is sent to an isomorphism by $(-)^{\perp}$ can be done by **playing a game**

Outline

Non-example

Consider the functor

$$\begin{array}{rccc} \mathcal{F}\colon & \mathbf{Set} \times \mathbf{Set} & \to & \mathbf{Set} \\ & (X, Y) & \mapsto & X \times Y \end{array}$$

It is not a left adjoint. Let's see where the criterion fails.

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First, let's get a description for $\mathcal{F}$:

- $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{\mathbf{1} \coprod \mathbf{1}}$

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But, $\mathcal{F}$ cannot be expressed by $\tilde{F}\colon \mathbf{1}\coprod\mathbf{1} \to \widehat{\mathbf{1}}$.

Indeed,

- $0_L \rightsquigarrow (\{*\},\emptyset)$, $\qquad 0_R \rightsquigarrow (\emptyset,\{*\})$
- $(\{*\},\emptyset)$ and $(\emptyset,\{*\})$ are mapped to $\emptyset$ by $\mathcal{F}$.
- but $\tilde{F}=\emptyset$ describes the functor $(X,Y)\mapsto\emptyset$.

Non-example

Another try: we add a (useless) product in the description of $\mathbf{Set} \times \mathbf{Set}$

- $\mathbf{Set} \simeq \widehat{\mathbf{1}}$
- $\mathbf{Set} \times \mathbf{Set} \simeq \widehat{C}^{\perp}$

where

$$C = \begin{tikzcd} & p & \\ 0_L \arrow[ur, "\pi_L"] & & 0_R \arrow[ul, "\pi_R"'] \end{tikzcd}$$

Idea: $\quad 0_L \rightsquigarrow (\{*\}, \emptyset), \qquad 0_R \rightsquigarrow (\emptyset, \{*\}), \qquad p \rightsquigarrow (\{*\}, \{*\})$

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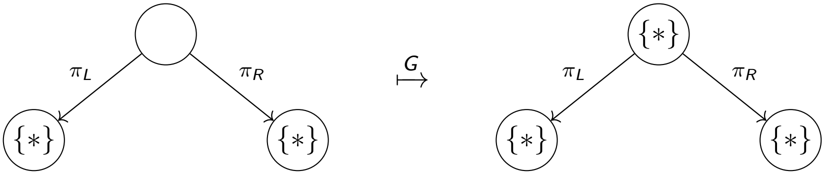
where

$$C = \begin{tikzcd} & p & \\ 0_L \arrow[ur, "\pi_L"] & & 0_R \arrow[ul, "\pi_R"'] \end{tikzcd}$$

and where we require orthogonality to $G \colon A \to B$:



i.e., given $X \in \widehat{C}^{\perp}$, $X(p)$ must be the product of $X(0_L)$ and $X(0_R)$.

Non-example

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where

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Now, we can describe $\mathcal{F} \colon (X, Y) \mapsto X \times Y$ with

$$\begin{array}{rccc} \tilde{F} \colon & C & \to & \widehat{\mathbf{1}} \\ & 0_L & \mapsto & \emptyset \\ & 0_R & \mapsto & \emptyset \\ & p & \mapsto & \{*\} \end{array}$$

Non-example

$\mathcal{F}\colon (X, Y) \mapsto X \times Y$ is not a left adjoint (coproducts are not preserved), so the criterion should not be satisfied.

We thus check that $(-)^{\perp} \circ \bar{F}\colon \widehat{C} \to \widehat{D}^{\perp}$ does not map $G\colon A \to B$ to an isomorphism.

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$$\begin{tikzcd}
& {\phantom{\{*\}}} \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & & & & \{*\} \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\
\{*\} & & \{*\} & \xrightarrow{G} & \{*\} & & \{*\}
\end{tikzcd}$$

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$$\begin{tikzcd}
& \bar{F}(A) \arrow[rrrr, "\bar{F}(G)"] & & & & \bar{F}(B) & \\
& & & & & \{*\} \arrow[u, dashed] & \\
\emptyset \arrow[uur, dashed] & & \emptyset \arrow[uul, dashed] & & \emptyset \arrow[ur] & & \emptyset \arrow[ul]
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$$\emptyset \xrightarrow{\quad \bar{F}(G) \quad} \{*\}$$

A small application

We recover the following well-known property using our criterion:

Proposition

Every functor $F \colon \mathbf{Set} \to \mathcal{D}$ of the form $F(X) = \coprod_X B$ is a left adjoint.

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Indeed,

- functors as above are described by functors $\mathbf{1} \to \widehat{D}$,
- $\mathbf{Set} \simeq \widehat{\mathbf{1}}^{\perp}$ with an empty set of orthogonality morphisms

so that our criterion is verified automatically.

A bigger example

Let's show that this functor is a left adjoint:

$$\begin{aligned} \mathcal{F}\colon \quad \mathbf{Cat} &\to \mathbf{Set} \\ D &\mapsto D_0 \end{aligned}$$

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Proposition

The functor $\mathcal{F}$ is presented by $\tilde{F}$.

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Let's show that this functor is a left adjoint:

$$\begin{array}{rccc} \mathcal{F}\colon & \mathbf{Cat} & \to & \mathbf{Set} \\ & D & \mapsto & D_0 \end{array}$$

Let's compute whether $O^C = \{G^P, G^L, G^R, G^A\}$ is sent to isomorphisms by $\bar{F}\colon \widehat{C} \to \mathbf{Set}$

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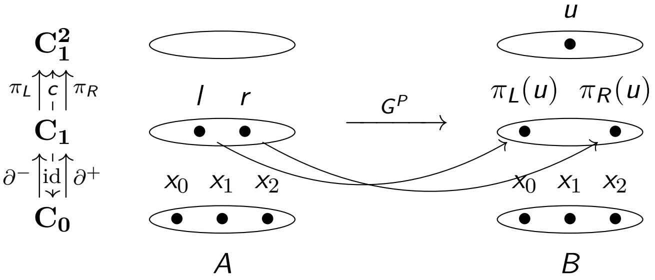

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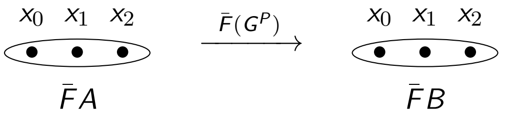

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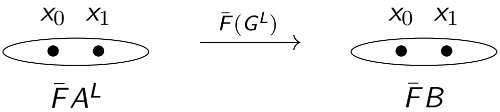

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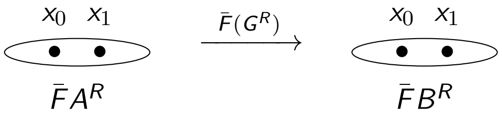

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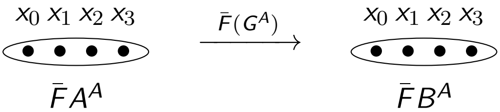

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Proposition

The functor $\mathcal{F}$ is a left adjoint.

Product functors

Product functors can be given as inputs to the criterion:

Proposition

Given $\mathcal{C} \simeq \widehat{C}^{\perp}$ and $A \in \mathcal{C}$, the functor

$$X \mapsto A \times X$$

can be described by a functor $C \to \widehat{C}$.

Thus, our criterion can be used to show that functors $A \times (-) \colon \mathcal{A} \to \mathcal{C}$ are left adjoints.

A criterion for closedness?

A category $\mathcal{C}$ is **closed** when, for every $A, B \in \mathcal{C}$, there is B^A such that

$$\mathrm{Hom}(A \times X, B) \simeq \mathrm{Hom}(X, B^A)$$

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are left adjoint for all $A \in \mathcal{C}$.

This suggests that closedness could be a computable property by the earlier criterion.

Problem: the above quantification on A is infinite.

Future work: how can we change that?

Example

We can use the criterion to show that $\mathbf{2} \times (-) \colon \mathbf{Cat} \to \mathbf{Cat}$ is a left adjoint where $\mathbf{Cat} \simeq \widehat{C}^{\perp}$ with

$$C = \begin{tikzcd} \mathbf{C_0} \arrow[r, shift left=2, "\partial^+"] \arrow[r, shift right=2, "\partial^-"'] & \mathbf{C_1} \arrow[l, "\bar{\mathrm{id}}" description] \arrow[r, shift left=2, "\bar{\pi}_R"] \arrow[r, "\bar{c}" description] \arrow[r, shift right=2, "\bar{\pi}_L"'] & \mathbf{C_1^2} \end{tikzcd}$$

Indeed, by computation, we check that every orthogonality morphism is sent to an isomorphism.

Outline

The reflection construction

Recall the adjunction

$$\begin{tikzcd} \widehat{D} \arrow[r, bend left, "(-)^{\perp}"] \arrow[r, phantom, "\perp"] & \widehat{D}^{\perp} \arrow[l, bend left, "J"] \end{tikzcd}$$

Given $H\colon X \to Y$, we have

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ JX^{\perp} \arrow[r, "JH^{\perp}"'] & JY^{\perp} \end{tikzcd}$$

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How to compute whether $H^{\perp}$ is an isomorphism?

The reflection construction

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First: given $X \in \widehat{D}$, what is $\eta_X\colon X \to X^{\perp}$?

Idea: if X is not orthogonal, η_X is adding and merging the elements as required.

The reflection construction

Let $G\colon A \to B \in O^D$ be an orthogonality morphism.

The reflection construction

Let $G\colon A \to B \in O^D$ be an orthogonality morphism.

If some liftings are missing, as in

$$\begin{tikzcd} B \arrow[dr, dashed, "??"] & \\ & X \\ A \arrow[uu, "G"] \arrow[ur, "H"'] & \end{tikzcd}$$

we correct that using a pushout:

$$\begin{tikzcd} B \arrow[r, dashed] & X' \\ A \arrow[u, "G"] \arrow[r, "H"'] & X \arrow[u, dashed] \end{tikzcd}$$

The reflection construction

Let $G\colon A \to B \in O^D$ be an orthogonality morphism.

If some liftings are non-unique, as in

$$
\begin{tikzcd}
B \arrow[dr, shift left, "\bar{H}_1"] \arrow[dr, shift right, "\bar{H}_2"'] & \\
& X \\
A \arrow[uu, "G"] \arrow[ur, "H"'] &
\end{tikzcd}
$$

we correct that using a coequalizer:

$$
\begin{tikzcd}
B \arrow[r, shift left, "\bar{H}_1"] \arrow[r, shift right, "\bar{H}_2"'] & X \arrow[r, dashed] & X'
\end{tikzcd}
$$

The reflection construction

η_X is then the transfinite composition

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X^{\perp}$$

The game

Given $H\colon X \to Y \in \widehat{D}$, how can we check that $H^{\perp}\colon X^{\perp} \to Y^{\perp}$ is an isomorphism?

Idea: progressively apply the moves of the reflection procedure until an isomorphism is obtained.

The game

$$H\colon X \to Y \in \widehat{D}$$

Four possible moves

The game

$$H\colon X \to Y \in \widehat{D}$$

Four possible moves

- add elements to X using a pushout with $G \in O^D$

$$H'\colon X' \to Y$$

The game

$$H \colon X \to Y \in \widehat{D}$$

Four possible moves

- add elements to X using a pushout with $G \in O^D$
- merge elements in X using a coequalizer of liftings of $G \in O^D$

$$H' \colon X' \to Y$$

The game

$$H \colon X \to Y \in \widehat{D}$$

Four possible moves

- add elements to X using a pushout with $G \in O^D$
- merge elements in X using a coequalizer of liftings of $G \in O^D$
- add elements to Y using a pushout with $G \in O^D$

$$H' \colon X \to Y'$$

The game

$$H\colon X \to Y \in \widehat{D}$$

Four possible moves

- add elements to X using a pushout with $G \in O^D$
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- add elements to Y using a pushout with $G \in O^D$
- merge elements in Y using a coequalizer of liftings of $G \in O^D$

$$H'\colon X \to Y'$$

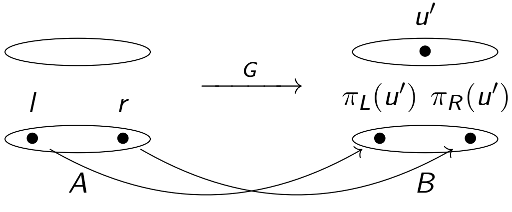
Play the game

Consider the category D where

$$D = \begin{tikzcd} e \\ b \arrow[u, shift left, "\pi_r"'] \arrow[u, shift right, "\pi_l"] \end{tikzcd}$$

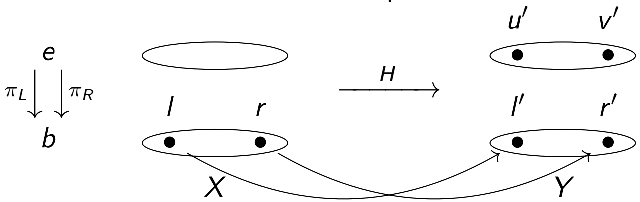
and with $O^D = \{G\colon A \to B\} \subseteq \widehat{D}$ with

$$\begin{tikzcd} e \arrow[d, shift right, "\pi_R"'] \arrow[d, shift left, "\pi_L"] \\ b \end{tikzcd}$$



Play the game

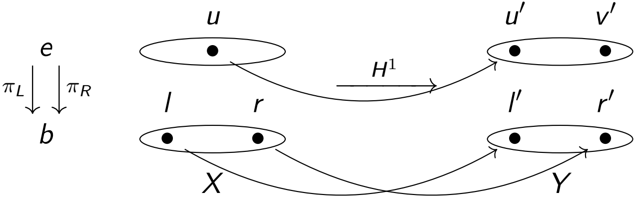
Show that $H\colon X \to Y \in \widehat{D}$ is sent to an isomorphism:



with $l' = \pi_l(u') = \pi_l(v')$ and $r' = \pi_r(u') = \pi_r(v')$

Play the game

Show that $H \colon X \to Y \in \widehat{D}$ is sent to an isomorphism:

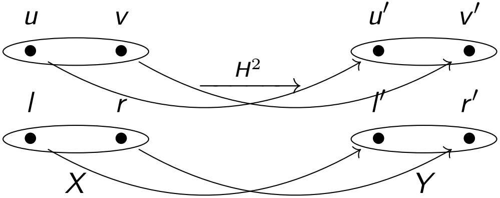


First, create a preimage for u'.

Play the game

Show that $H \colon X \to Y \in \widehat{D}$ is sent to an isomorphism:

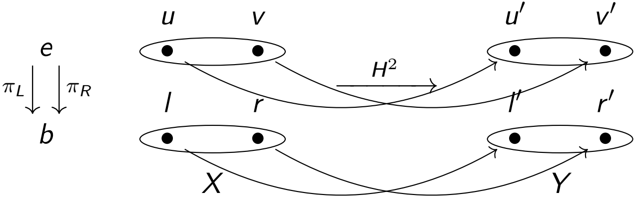
$$\begin{tikzcd} e \arrow[d, shift right, "\pi_L"'] \arrow[d, shift left, "\pi_R"] \\ b \end{tikzcd}$$



Then, create a preimage for v'.

Play the game

Show that $H \colon X \to Y \in \widehat{D}$ is sent to an isomorphism:

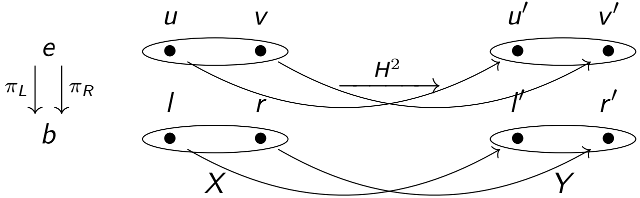


Then, create a preimage for v'.

We thus get an isomorphism.

Play the game

Show that $H\colon X \to Y \in \widehat{D}$ is sent to an isomorphism:



Then, create a preimage for v'.

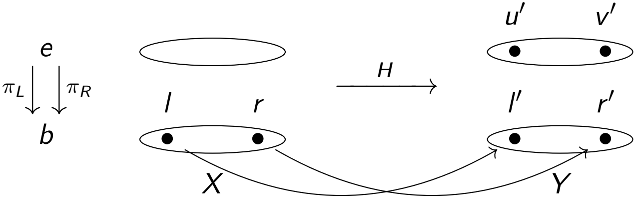
We used a "greedy strategy": add/merge when required and possible.

Proposition

The greedy strategy can decide whether $H^{\perp}$ is an isomorphism for finite $H\colon X \to Y \in \widehat{D}$.

Play the game

Another strategy:



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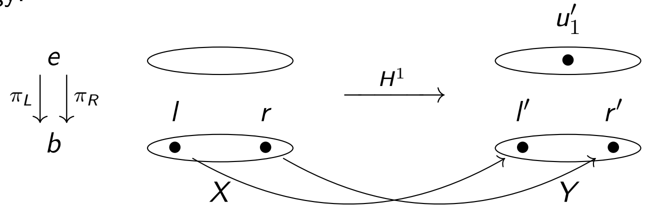
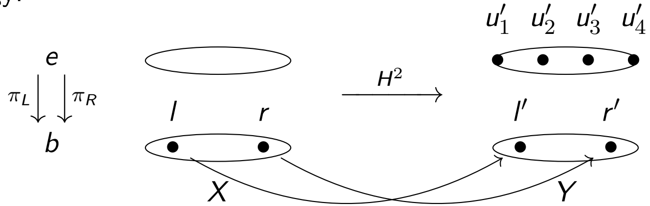


First, merge u' and v', since they lift the same morphism.

Play the game

Another strategy:

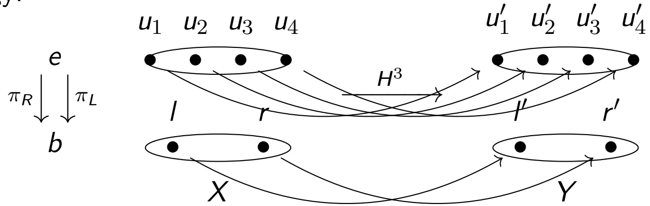


Then, create all the possible liftings in Y.

$$u'_1 = (l', r') \qquad u'_2 = (l', l') \qquad u'_3 = (r', r') \qquad u'_4 = (r', l')$$

Play the game

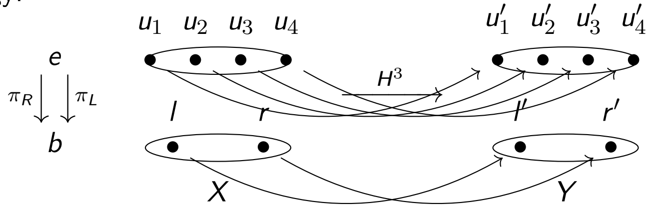
Another strategy:



Then, create all the possible liftings in X.

Play the game

Another strategy:



Then, create all the possible liftings in X.

We thus get an isomorphism.

Play the game

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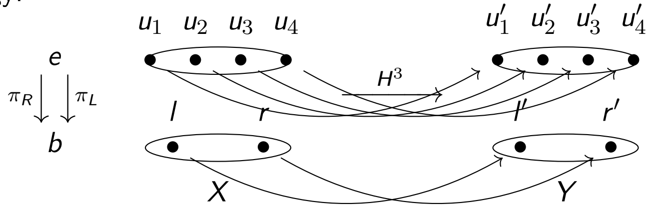

Then, create all the possible liftings in X.

We used an "exhaustive strategy": add/merge whenever possible.

Proposition

The exhaustive strategy can decide whether $H^{\perp}$ is an isomorphism for finite $H\colon X \to Y \in \widehat{D}$.

Strategies in general

Winning the game can answer positively whether a morphism is sent to an isomorphism.

However,

- greedy strategies can be too stupid and miss some winnable games
- exhaustive strategies might not terminate

Future work: characterize the categories D and sets O^D for which these strategies terminate.

In any case: one can enter “manual mode” and provide a winning play.

Outline

Colimit preservation

Recall the definition of F:

$$\begin{tikzcd}
\widehat{C}^{\perp} \arrow[d, "J"'] \arrow[rrd, "F"] & & \\
\widehat{C} \arrow[r, "\bar{F}"] & \widehat{D} \arrow[r, "(-)^{\perp}"'] & \widehat{D}^{\perp} \\
C \arrow[u, "y"] \arrow[ur, "\tilde{F}"'] & &
\end{tikzcd}$$

Proposition

The functor $\bar{F} \colon \widehat{C} \to \widehat{D}$ preserves colimits.

Proof.

Indeed we have

$$\bar{F}(\mathrm{colim}_i\, X_i) \simeq \int^{c \in C_0} \tilde{F}(c) \otimes (\mathrm{colim}_i\, X_i)(c)$$

□

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Proof.

Indeed we have

$$\bar{F}(\mathrm{colim}_i\, X_i) \simeq \mathrm{colim}_i\Big(\int^{c \in C_0} \tilde{F}(c) \otimes X_i(c)\Big) \simeq \mathrm{colim}_i\, \bar{F}(X_i)$$

$\square$

Colimit preservation

$$
\begin{tikzcd}
\widehat{C}^{\perp} \arrow[d, "J"'] \arrow[dr, "F"] & \\
\widehat{C} \arrow[r, "\bar{F}'"'] & \widehat{D}^{\perp}
\end{tikzcd}
$$

Knowing that $\bar{F}' \triangleq (-)^{\perp} \circ \bar{F}$ is preserving colimits, when F is?

Colimit preservation

$$\begin{tikzcd}
\widehat{C}^{\perp} \arrow[d, "J"'] \arrow[dr, "F"] & \\
\widehat{C} \arrow[r, "\bar{F}'"'] & \widehat{D}^{\perp}
\end{tikzcd}$$

Proposition (A-R)

The colimits in $\widehat{C}^{\perp}$ are the reflection of the ones computed in $\widehat{C}$:

$$\operatorname{colim}_i^{\widehat{C}^{\perp}} A_i \simeq (\operatorname{colim}_i^{\widehat{C}} J(A_i))^{\perp}$$

Thus, the unit of the reflection gives a canonical morphism

$$\eta\colon \operatorname{colim}_i^{\widehat{C}} JA_i \to J(\operatorname{colim}_i^{\widehat{C}^{\perp}} A_i)$$

Colimit preservation

$$\begin{tikzcd} \widehat{C}^{\perp} \arrow[d, "J"'] \arrow[dr, "F"] & \\ \widehat{C} \arrow[r, "\bar{F}'"'] & \widehat{D}^{\perp} \end{tikzcd}$$

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Thus, the unit of the reflection gives a canonical morphism

$$\bar{F}'\eta \colon \bar{F}'(\mathrm{colim}_i^{\widehat{C}} JA_i) \to \bar{F}'J(\mathrm{colim}_i^{\widehat{C}^{\perp}} A_i)$$

Colimit preservation

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Colimit preservation

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Thus, the unit of the reflection gives a canonical morphism

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Colimit preservation

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Thus, the unit of the reflection gives a canonical morphism

$$\bar{F}'\eta\colon \operatorname{colim}_i^{\widehat{D}^{\perp}}(FA_i) \to F(\operatorname{colim}_i^{\widehat{C}^{\perp}} A_i)$$

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Proposition

The functor $F\colon \widehat{C}^{\perp} \to \widehat{D}^{\perp}$ preserves colimits (and is a left adjoint) if and only if $\bar{F}'\eta_{\mathrm{colim}_i^{\widehat{C}} JA_i}$ is an isomorphism for all diagrams $i \mapsto A_i$ in $\widehat{C}^{\perp}$.

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Corollary

If $\bar{F}'\eta$ is an isomorphism, then F preserves colimits (and is a left adjoint).

Theorem

Suppose now that, for every orthogonality morphism $G \in O^C$, $\bar{F}(G)$ is an isomorphism.

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If some liftings are missing for X, as in

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B \arrow[dr, dashed, "??"] & \\
& X \\
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we correct that using a pushout:

$$\begin{tikzcd} B \arrow[r, dashed] & X' \\ A \arrow[u, "G"] \arrow[r, "H"'] & X \arrow[u, dashed] \end{tikzcd}$$

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...and we obtain the pushout

$$\begin{tikzcd} \bar{F}B \arrow[r, dashed] & \bar{F}X' \\ \bar{F}A \arrow[u, "\bar{F}(G)"] \arrow[r, "\bar{F}(H)"'] & \bar{F}X \arrow[u, dashed] \end{tikzcd}$$

where $\bar{F}(G)$ is an isomorphism. Thus, $\bar{F}X \simeq \bar{F}X'$.

Theorem

Suppose now that, for every orthogonality morphism G, $\bar{F}(G)$ is an isomorphism.

If liftings are non-unique, as in

$$\begin{tikzcd}
B \arrow[dr, shift left, "\bar{H}_1"] \arrow[dr, shift right, "\bar{H}_2"'] & \\
& X \\
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$$

we correct that using a coequalizer:

$$
\begin{tikzcd}
B \arrow[r, shift left, "\bar{H}_1"] \arrow[r, shift right, "\bar{H}_2"'] & X \arrow[r, dashed] & X'
\end{tikzcd}
$$

Theorem

Suppose now that, for every orthogonality morphism G, $\bar{F}(G)$ is an isomorphism.

If liftings are non-unique, as in

$$\begin{tikzcd} B \arrow[dr, shift left, "\bar{H}_1"] \arrow[dr, shift right, "\bar{H}_2"'] & \\ & X \\ A \arrow[uu, "G"] \arrow[ur, "H"'] & \end{tikzcd}$$

...and we obtain the coequalizer:

$$\begin{tikzcd} \bar{F}B \arrow[r, shift left, "\bar{F}(\bar{H}_1)"] \arrow[r, shift right, "\bar{F}(\bar{H}_2)"'] & \bar{F}X \arrow[r, dashed] & \bar{F}X' \end{tikzcd}$$

with $\bar{F}(\bar{H}_1) \circ \bar{F}(G) = \bar{F}(\bar{H}_2) \circ \bar{F}(G)$, thus $\bar{F}(\bar{H}_1) = \bar{F}(\bar{H}_2)$ and $\bar{F}X \simeq \bar{F}X'$

Theorem

Thus, $\bar{F}\eta_X$ is a transfinite composition of isomorphism

$$\bar{F}X = \bar{F}X_0 \xrightarrow{\sim} \bar{F}X_1 \xrightarrow{\sim} \bar{F}X_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \bar{F}X^{\perp}$$

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Theorem

If, for all $G \in O^C$, $\bar{F}(G)$ is an isomorphism, then $\bar{F}\eta$ is an isomorphism.

Theorem

Thus, $\bar{F}\eta_X$ is a transfinite composition of isomorphism

$$\bar{F}X = \bar{F}X_0 \xrightarrow{\sim} \bar{F}X_1 \xrightarrow{\sim} \bar{F}X_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \bar{F}X^{\perp}$$

Theorem

If, for all $G \in O^C$, $\bar{F}(G)$ is an isomorphism, then $\bar{F}\eta$ is an isomorphism.

Corollary

With the same hypothesis, F preserves colimits and is a left adjoint.

The end

Thank you!