Rewriting for Gray categories

Simon Forest and Samuel Mimram

May 27th 2021

Coherence

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all parallel cells are equal.

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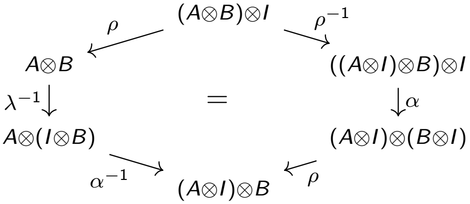
Classical example: **MacLane's coherence theorem** for monoidal categories.



Theorem (MacLane's coherence property for monoidal categories)

All morphisms made of λ, ρ, α **and their inverses** *between two objects are equal.*

Coherence tiles

Coherence tiles: the axioms allowing the coherence property

$$
\begin{array}{ccccc}
((A\otimes B)\otimes C)\otimes D & \xrightarrow{\alpha} & (A\otimes(B\otimes C))\otimes D & \xrightarrow{\alpha} & A\otimes((B\otimes C)\otimes D) \\
\alpha\downarrow & & = & & \downarrow\alpha \\
(A\otimes B)\otimes(C\otimes D) & & \xrightarrow[\alpha]{} & & A\otimes(B\otimes(C\otimes D))
\end{array}
$$

$$
\begin{array}{ccc}
(A\otimes I)\otimes B & \xrightarrow{\alpha} & A\otimes(I\otimes B) \\
\lambda\searrow & = & \swarrow\rho \\
 & A\otimes B &
\end{array}
$$

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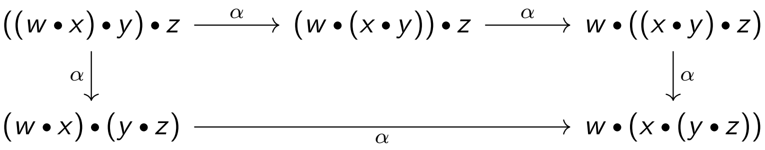

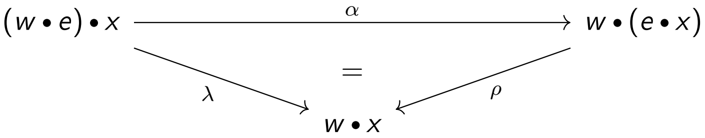

Observation: these coherence tiles are the critical branchings of a rewriting system.

$$(x \bullet e) \rightsquigarrow x \qquad (e \bullet x) \rightsquigarrow x \qquad (x \bullet y) \bullet z \rightsquigarrow x \bullet (y \bullet z)$$

Coherence in strict categories

Several weak structures can be expressed in strict categories (paradoxically!):

- pseudomonoids
- pseudoadjunctions
- Frobenius pseudoalgebras
- *etc.*

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Guiraud and Malbos developed a rewriting framework for finding coherence definitions for them.

Theorem ([G-M,08])

If a strict n-category is presented using a terminating and confluent n-polygraph, then a set of coherence conditions is given by the confluence diagrams of the critical branchings.

Coherence in strict categories

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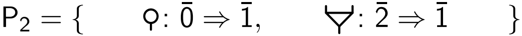
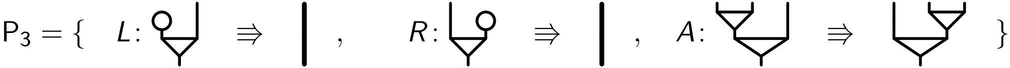
- pseudomonoids can be presented using a terminating and confluent 3-polygraph P

$$P_0 = \{*\} \qquad P_1 = \{\bar{1} : * \to *\}$$





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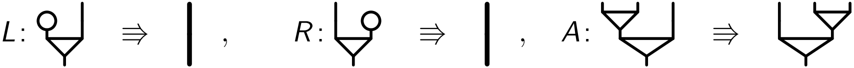
$P_3 = \{$  $\}$

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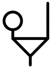
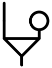
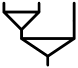
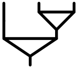
$P_3 = \{$ $L:$  $\Rrightarrow$, $R:$  $\Rrightarrow$, $A:$  $\Rrightarrow$  $\}$

- the coherence conditions derived from the critical branchings entail coherence
- these conditions are essentially the same than the ones of MacLane

Strict categories and homotopy

Strict categories are "easy" but have bad homotopical properties. Depending on the definitions:

- no good realization functor from strict categories to **Top**
- not all homotopy type can be modeled with strict categories
- vanishing Whitehead products
- *etc.*

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The most general definitions can be obtained by considering structures expressed in weak categories.

Bicategories

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A priori, weakened 2-dimensional structures should be considered in bicategories in order to obtain the most general definitions.

But actually, studying strict 2-categories is enough since

Theorem ([MacLane,85])

Every bicategory is "equivalent" to a strict 2-category.

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Is it enough to study 3-dimensional structures in strict 3-categories?

No, since

Observation

Not all tricategories are "equivalent" to strict 3*-categories.*

This is a shame since tricategories are terrible to work with.

Gray categories

However, we have the following coherence property:

Theorem ([Gordon, Power, Street, 95])

Every tricategory is "equivalent" to a Gray category.

Gray categories

Gray categories

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- unital and associative compositions
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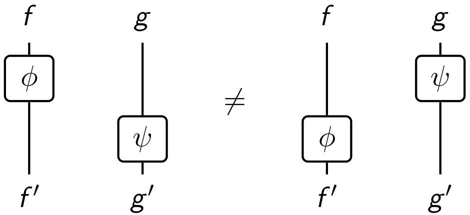

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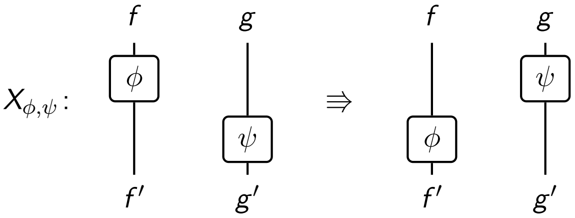

Extending rewriting theory

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Thus, we need to develop rewriting theory for an other kind of higher categories.

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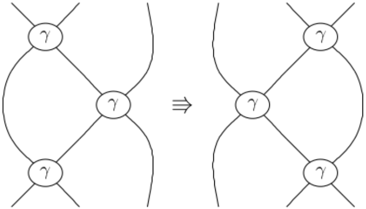
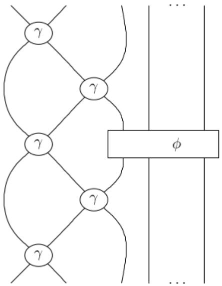
Extending rewriting theory

In fact, considering other higher categories is good since

- recent works on higher dimensional rewriting are biased towards strict categories
- strict categories are not "that" special regarding rewriting
- several shortcomings with strict categories (shapes of critical branchings, no good finiteness property)





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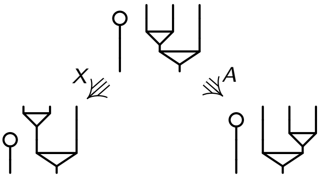

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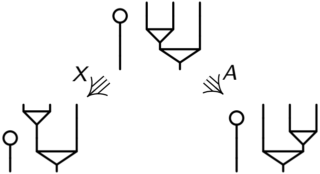

Thus, another setting is needed: **precategories**.

Content

Content

Strict categories

A **strict n-category** is an n-globular set C equipped with operations

$$\mathrm{id}^{i+1} \colon C_i \to C_{i+1}$$

and, for $i < k \leq n$,

$$(-) *_i (-) \colon C_k \times_i C_k \to C_k$$

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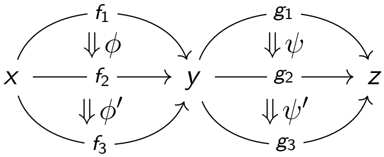
which are unital and associative, and should satisfy an exchange law



$$(\phi *_1 \phi') *_0 (\psi *_1 \psi') = (\phi *_0 \psi) *_1 (\phi' *_0 \psi')$$

Strict categories

Exchange law: alternatively described using a distributivity and a smaller exchange condition.

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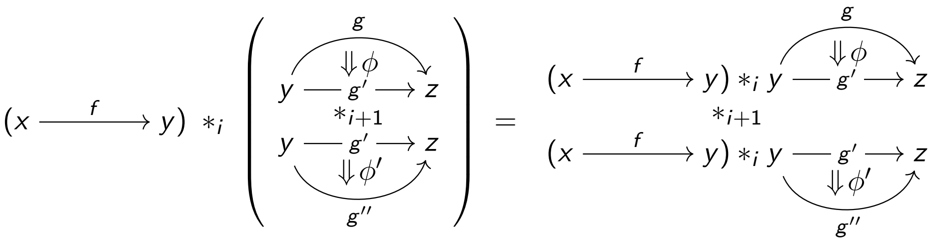
Distributivity property:



and similarly on the right.

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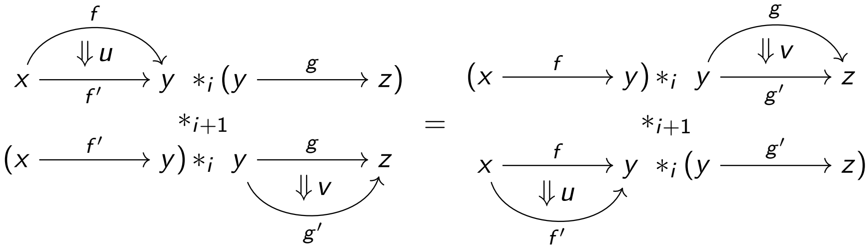
Smaller exchange property:



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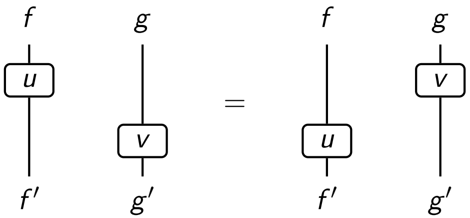

Free constructions on strict categories

By general constructions, we have

- a category $\mathbf{Cat}_n^+$ of n-cellular extensions (n-categories + generating (n+1)-cells)
- a free extension functor

$$\begin{array}{rccc} -[-]^n\colon & \mathbf{Cat}_n^+ & \to & \mathbf{Cat}_{n+1} \\ & (C, X) & \to & C[X] \end{array}$$

- a category $\mathbf{Pol}_n$ of n-polygraphs
- a free-category-on-polygraph functor

$$\begin{array}{rccc} (-)^{*,n}\colon & \mathbf{Pol}_n & \to & \mathbf{Cat}_n \\ & \mathsf{P} & \to & \mathsf{P}^* \end{array}$$

Word problem on strict categories

Given an n-polygraph P, the elements of P^* are quotients of valid terms that can be written on P:

$$\mathrm{id}^1_x, \qquad (a *_0 b) *_0 c, \qquad a *_0 (b *_0 c), \qquad (\alpha *_1 \beta) *_0 \mathrm{id}^2_d, \qquad \textit{etc.}$$

Word problem: deciding whether two terms denote the same cell in P^*.

Theorem ([Makkai,05])

The word problem for strict categories is decidable.

- however, the procedure is intricate and expensive
- arguably, rewriting algorithms on str. cat. must be as expensive

Precategories

An n-**precategory** is an n-globular set C equipped with operations

$$\mathrm{id}^{i+1}\colon C_i \to C_{i+1}$$

and, for $k, l \leq n$,

$$(-) \bullet_{k,l} (-)\colon C_k \times_{\min(k,l)-1} C_l \to C_{\max(k,l)}$$

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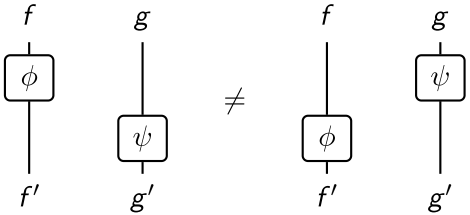
But not required to satisfy the exchange condition.



Precategories

As expected, the following property holds:

Theorem

A strict n-category is exactly an n-precategory satisfying the exchange condition.

Free constructions on precategories

By general constructions, we have

- a category $\mathbf{PCat}_n^+$ of n-cellular extensions (n-precategories + generating $(n+1)$-cells)
- a free extension functor

$$-[-]^n \colon \begin{array}{ccc} \mathbf{PCat}_n^+ & \to & \mathbf{PCat}_{n+1} \\ (C, X) & \to & C[X] \end{array}$$

- a category $\mathbf{PPol}_n$ of n-polygraphs
- a free-category-on-polygraph functor

$$(-)^{*,n} \colon \begin{array}{ccc} \mathbf{PPol}_n & \to & \mathbf{PCat}_n \\ \mathsf{P} & \to & \mathsf{P}^* \end{array}$$

Free extensions on precategories

Given an n-cellular extension (C, X), the elements of $C[X]$ are easily described: those are the sequences

$$u_1 \bullet_n \cdots \bullet_n u_k$$

where each u_i is a **whiskered generator**, *i.e.*, is of the form

$$l_n \bullet_{n-1} (\cdots (l_1 \bullet_0 g \bullet_0 r_1) \cdots) \bullet_{n-1} r_n$$

for some $l_j, r_j \in C_j$ and $g \in X$.

The case of polygraphs: given an n-polygraph P, the cells of P^* can be described as inductive sequences of whiskered generators.

Word problem on precategories

As a consequence,

Theorem

The word problem for precategories is decidable.

Indeed, the decision procedure is quite simple:

```
let test_pcat_eq c1 c2 =
  c1 = c2
```

- good sign for developing a rewriting framework on precategories

Content

Enriched definition

Higher categories can also be defined through **enrichment**.

Enriched definition

Given a monoidal category $(\mathcal{V}, 1, \otimes)$, a $\mathcal{V}$-**enriched category** is the data of

- a set C_0
- objects $C(x, y) \in \mathcal{V}$ for all $x, y \in C_0$

together with

- morphisms $i_x \colon 1 \to C(x, x)$ for $x \in C_0$
- morphisms $c_{x,y,z} \colon C(x, y) \otimes C(y, z) \to C(x, z)$ for $x, y, z \in C_0$

that are unital and associative.

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$$\begin{array}{ccc} 1 \otimes C(x, y) & \xrightarrow{i_x \otimes C(x,y)} & C(x, x) \otimes C(x, y) \\ & \underset{\lambda_{C(x,y)}}{\searrow} \quad \underset{c_{x,x,y}}{\swarrow} & \\ & C(x, y) & \end{array}$$

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$$\begin{array}{ccc} C(x,y) \otimes 1 & \xrightarrow{C(x,y) \otimes i_y} & C(x,y) \otimes C(y,y) \\ & {\scriptstyle \rho_{C(x,y)}} \searrow \quad \swarrow {\scriptstyle c_{x,y,y}} & \\ & C(x,y) & \end{array}$$

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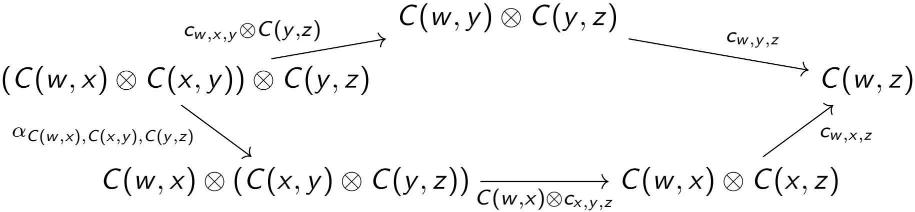

Enriched definition

Example: a strict 2-category is a category enriched over $(\mathbf{Cat}, 1, \times)$

$$C \quad = \quad f \xrightarrow{\phi} f' \qquad\qquad D \quad = \quad g \xrightarrow{\psi} g'$$

$$C \times D \quad = \quad \begin{array}{ccc} (f,g) & \xrightarrow{(f,\psi)} & (f,g') \\ {\scriptstyle (\phi,g)}\downarrow & = & \downarrow{\scriptstyle (\phi,g')} \\ (f',g) & \xrightarrow[(f',\psi)]{} & (f',g') \end{array}$$

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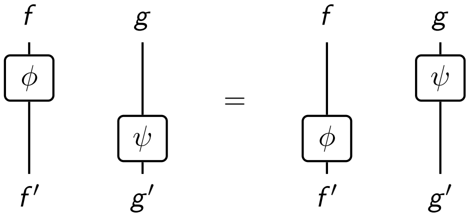

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Example: a 2-precategory is a category enriched over $(\mathbf{Cat}, 1, \square)$

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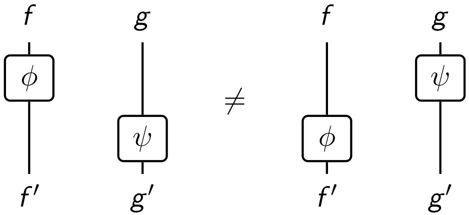

Tensor product on $\mathbf{Cat}_2$

The two previous tensor products on $\mathbf{Cat}_1$ can be easily generalized to $\mathbf{Cat}_2$

$$C \times D = \begin{array}{ccc} (f,g) & \xrightarrow{(f,\psi)} & (f,g') \\ {\scriptstyle (\phi,g)}\big\downarrow & = & \big\downarrow{\scriptstyle (\phi,g')} \\ (f',g) & \xrightarrow[(f',\psi)]{} & (f',g') \end{array} \qquad C \,\square\, D = \begin{array}{ccc} (f,g) & \xrightarrow{(f,\psi)} & (f,g') \\ {\scriptstyle (\phi,g)}\big\downarrow & \neq & \big\downarrow{\scriptstyle (\phi,g')} \\ (f',g) & \xrightarrow[(f',\psi)]{} & (f',g') \end{array}$$

Tensor product on $\mathbf{Cat}_2$

A new tensor product on $\mathbf{Cat}_2$ is given by the **Gray tensor product** $\boxtimes$

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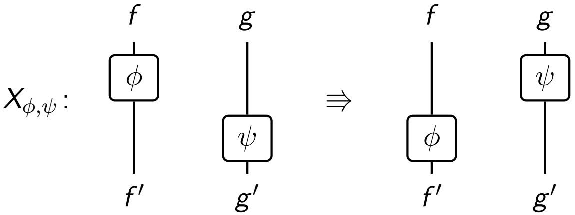

Gray categories

A **Gray category** is then a category enriched over $\mathbf{Cat}_2$ equipped with Gray tensor product.

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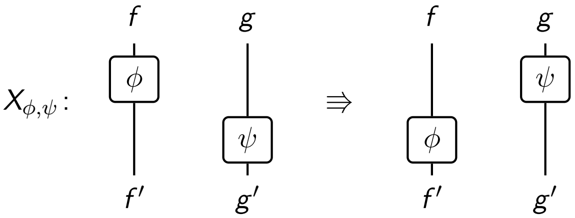
A **Gray category** is then a category enriched over $\mathbf{Cat}_2$ equipped with Gray tensor product.

Idea: it is a 3-precategory with interchange 3-cells for 2-cells with some axioms on 3-cells.



Gray categories

Elements of a Gray category:

- 0-cells and 1-cells
- 2-cells:

- 3-cells:

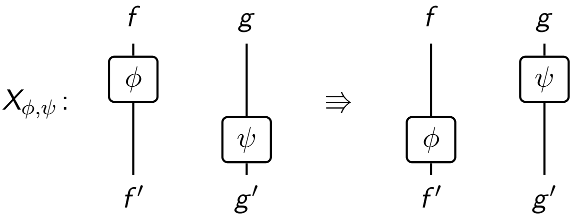
Gray categories

Elements of a Gray category:

- 0-cells and 1-cells
- 2-cells:

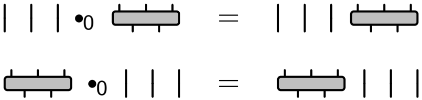
- 3-cells:

- among them, **interchangers**:



Gray categories

- composition of 2-cells with 1-cells on the left and the right



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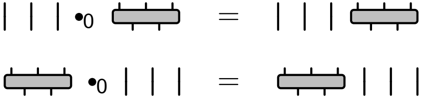
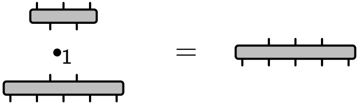


- composition: 2-cells can be composed vertically



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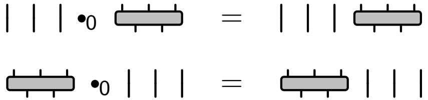

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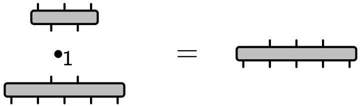
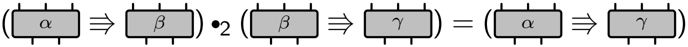
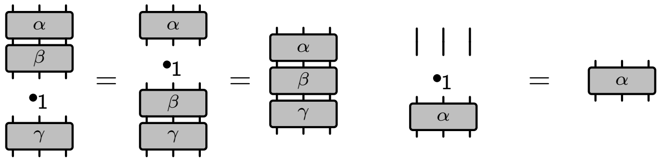


- 3-cells can be composed horizontally



Gray categories

- properties of associativity and unitality



Gray categories

Additional conditions are required:

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$$X_{\phi \bullet_1 \phi', \psi} = ((\phi \bullet_0 g) \bullet_1 X_{\phi', \psi}) \bullet_2 (X_{\phi, \psi} \bullet_1 (\phi' \bullet_0 g'))$$

Gray categories

Additional conditions are required:

- some compatibilities for $X_{-,-}$

$$X_{\phi,\psi \bullet_1 \psi'} = (X_{\phi,\psi} \bullet_1 (f' \bullet_0 \psi')) \bullet_2 ((f \bullet_0 \psi) \bullet_1 X_{\phi,\psi'})$$

Gray categories

Additional conditions are required:

- some compatibilities for $X_{-,-}$

$$X_{e \bullet_0 \phi, \psi} = e \bullet_0 X_{\phi,\psi} \qquad X_{\phi \bullet_0 f, \psi} = X_{\phi, f \bullet_0 \psi} \qquad X_{\phi, \psi \bullet_0 h} = X_{\phi,\psi} \bullet_0 h.$$

Gray categories

Additional conditions are required:

- some compatibilities for $X_{-,-}$

and others...

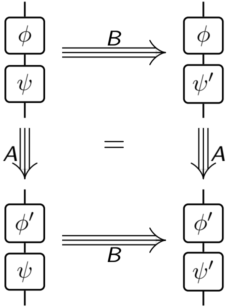
Gray categories

Additional conditions are required:

- some compatibilities for $X_{-,-}$
- an exchange law for 3-cells

$A\colon \phi \Rrightarrow \phi'$

$B\colon \psi \Rrightarrow \psi'$

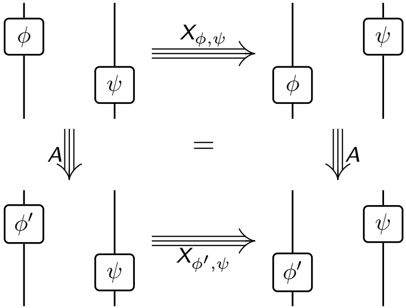


Gray categories

Additional conditions are required:

- some compatibilities for $X_{-,-}$
- an exchange law for 3-cells
- a naturality condition between 3-cells and interchangers

$A\colon \phi \Rrightarrow \phi'$



Gray presentation

In order to use rewriting methods on Gray categories, we need a notion of **presentation**.

Gray presentation

A **Gray presentation** is the data of a 4-polygraph (of precategories) P such that:

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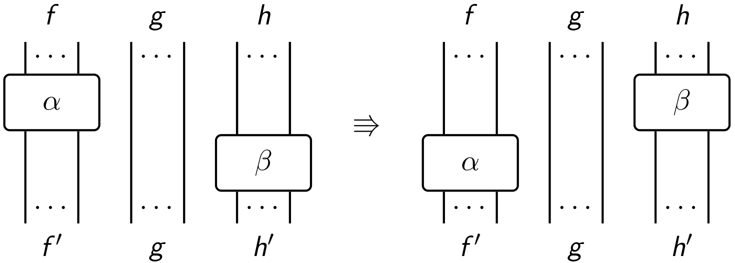
- for each $(\alpha\colon f \Rrightarrow f', g, \beta\colon h \Rrightarrow h') \in \mathsf{P}_2 \times_0 \mathsf{P}_1^* \times_0 \mathsf{P}_2$, there is a 3-generator $X_{\alpha,g,\beta}$

$$X_{\alpha,g,\beta}\colon (\alpha \bullet_0 g \bullet_0 h) \bullet_1 (f' \bullet_0 g \bullet_0 \beta) \quad \Rrightarrow \quad (f \bullet_0 g \bullet_0 \beta) \bullet_1 (\alpha \bullet_0 g \bullet_0 h')$$



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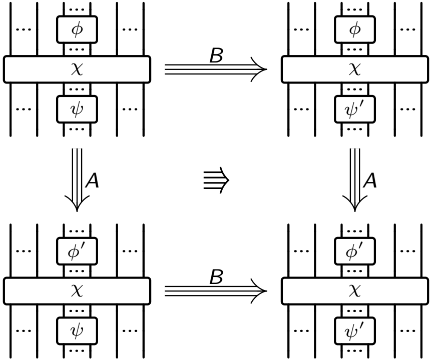
- for each $(\alpha\colon f \Rightarrow f', g, \beta\colon h \Rightarrow h') \in \mathsf{P}_2 \times_0 \mathsf{P}_1^* \times_0 \mathsf{P}_2$, there is a 3-generator $X_{\alpha,g,\beta}$
- for each instance of the axiom of Gray categories w.r.t. the gererators of P, there is a 4-generator in P_4

Gray presentation

Example: given a Gray presentation P, for each

$$A\colon \phi \Rrightarrow \phi' \qquad B\colon \psi \Rrightarrow \psi' \qquad \in \mathsf{P}_3$$

and $\chi \in \mathsf{P}_2^*$ (sufficiently composable), there is a 4-generator in P_4



Gray presentation

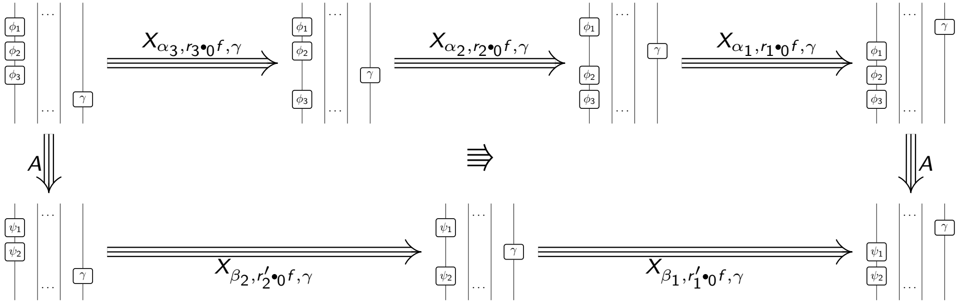
Example: given a Gray presentation P and

$$A\colon \phi_1 \bullet_1 \phi_2 \bullet_1 \phi_3 \Rrightarrow \psi_1 \bullet_1 \psi_2 \quad \in \mathsf{P}_3$$

with $\phi_i = l_i \bullet_0 \alpha_i \bullet_0 r_i$ and $\psi_i = l'_i \bullet_0 \beta_i \bullet_0 r'_i$, and

$$f \in \mathsf{P}_1^* \qquad\qquad \gamma \in \mathsf{P}_2$$

(sufficiently composable), there is a 4-generator in P_4



Example: pseudomonoids

The Gray presentation P of pseudomonoids

$$\mathsf{P}_0 = \{*\} \qquad \mathsf{P}_1 = \{\bar{1} \colon * \to *\}$$

$\mathsf{P}_2 = \{$  $: \bar{0} \Rightarrow \bar{1},$  $: \bar{2} \Rightarrow \bar{1}$ $\}$

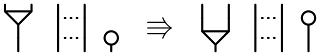
Example: pseudomonoids

The Gray presentation P of pseudomonoids

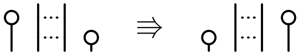
$$\mathrm{P}_3 = \mathrm{P}_3^{\mathrm{st}} \sqcup \mathrm{P}_3^{\mathrm{op}}$$

with $\mathrm{P}_3^{\mathrm{st}}$ made of generators of the form

$X_{\mu,\bar{n},\mu}$: 

$X_{\mu,\bar{n},\eta}$: 

$X_{\eta,\bar{n},\mu}$: 

$X_{\eta,\bar{n},\eta}$: 

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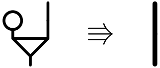
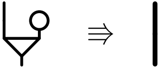
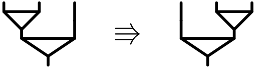
$\mathsf{P}_3^{\mathrm{op}} = \{$ L:  , R:  , A:  $\}$

Example: pseudomonoids

The Gray presentation P of pseudomonoids

$$P_4 = P_4^{st} \sqcup P_4^{coh}$$

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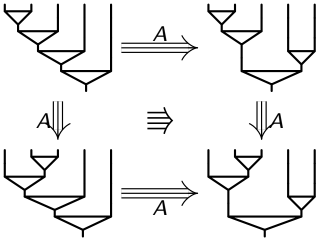
$$\mathsf{P}_4 = \mathsf{P}_4^{\mathrm{st}} \sqcup \mathsf{P}_4^{\mathrm{coh}}$$

with $\mathsf{P}_4^{\mathrm{st}}$ made of the different generators required by the definition of Gray presentation

Example:



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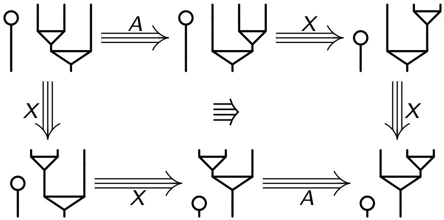


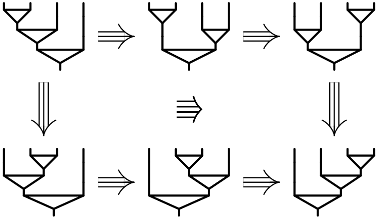
Example: pseudomonoids

The Gray presentation P of pseudomonoids

$$\mathsf{P}_4 = \mathsf{P}_4^{\mathrm{st}} \sqcup \mathsf{P}_4^{\mathrm{coh}}$$

and $\mathsf{P}_4^{\mathrm{coh}}$ made of additional generators required for coherence.

Example:



Note: these generators can involve interchange generators.

Presented category

Let P be a 4-polygraph P.

$\overline{\mathrm{P}}$: 3-precategory obtained from $(\mathrm{P}^*)_{\leq 3}$ by quotienting the 3-cells with $\sim$, where

$$F \sim G \qquad \text{for all } \Gamma \colon F \Rrightarrow G \in \mathrm{P}_4.$$

Presented category

Let C be a 3-precategory.

$C^{\top}$: 3-precategory obtained by formally inverting the 3-cells.

Presented category

Theorem

Given a Gray presentation P, *the* 3*-precategory* $\overline{\mathsf{P}}$ *is canonically a lax Gray category.*

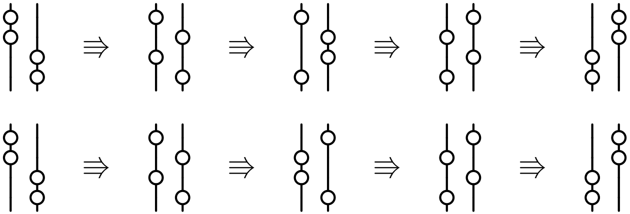
Presented category

Theorem

Given a Gray presentation P, *the* 3-*precategory* $\overline{\mathrm{P}}$ *is canonically a lax Gray category.*

The difficult part is showing that the different definitions of $X_{-,-}$ are coherent

Example for $X_{⚲,⚲}$:



Presented category

Theorem

Given a Gray presentation P, *the* 3*-precategory* $\overline{\mathsf{P}}$ *is canonically a lax Gray category.*

Corollary

Given a Gray presentation P, *the* 3*-precategory* $\overline{\mathsf{P}}^{\top}$ *is canonically a* $(3,2)$*-Gray category.*

Coherence

We want to show coherence properties:

all the ways to prove that two objects are equivalent are equal

Example for pseudomonoids:

$$
\begin{array}{ccccc}
 & & (A\otimes B)\otimes I & & \\
 & \swarrow^{\rho} & & \searrow^{\rho^{-1}} & \\
A\otimes B & & & & ((A\otimes I)\otimes B)\otimes I \\
\lambda^{-1}\downarrow & & = & & \downarrow\alpha \\
A\otimes(I\otimes B) & & & & (A\otimes I)\otimes(B\otimes I) \\
 & \searrow_{\alpha^{-1}} & & \swarrow_{\rho} & \\
 & & (A\otimes I)\otimes B & &
\end{array}
$$

Content

Coherence from rewriting

- **Rewriting system** Get a rewriting system: choose a "good" orientation for the isos of the considered structure

$$
\begin{array}{rccc}
\alpha : & (A \otimes B) \otimes C & \overset{\sim}{\to} & A \otimes (B \otimes C) \\
\lambda : & (I \otimes A) & \overset{\sim}{\to} & A \\
\rho : & (A \otimes I) & \overset{\sim}{\to} & A
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Coherence from rewriting

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In particular, we want $\to$ terminating

Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent

$$\forall (C_1, C_2) \text{ critical} \qquad \begin{array}{ccccc} & & \phi & & \\ & {\scriptstyle C_1}\swarrow & & \searrow{\scriptstyle C_2} & \\ \phi_1 & & = & & \phi_2 \\ & \searrow^{*} & & {}^{*}\swarrow & \\ & & \psi & & \end{array}$$

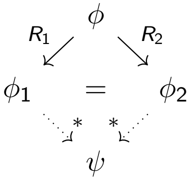
then

$$\forall (R_1, R_2) \qquad \begin{array}{ccccc} & & \phi & & \\ & {\scriptstyle R_1}\swarrow & & \searrow{\scriptstyle R_2} & \\ \phi_1 & & = & & \phi_2 \\ & \searrow^{*} & & {}^{*}\swarrow & \\ & & \psi & & \end{array}$$

Coherence from rewriting

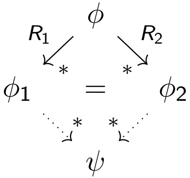
- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman's lemma**: $\rightarrow$ terminating and local confluence imply confluence

$\forall(R_1, R_2)$ **rewrite steps**



then

$\forall(R_1, R_2)$ **rewrite paths**



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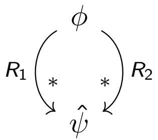
  First case: paths to a normal form $\hat{\psi}$



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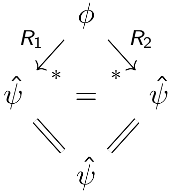

by Newman's lemma

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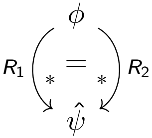

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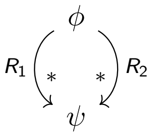

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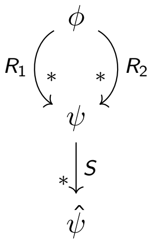

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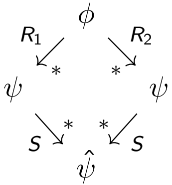

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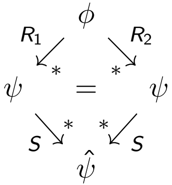

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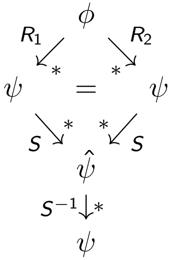

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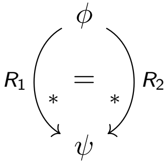

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  Third case: paths with inverses (α^{-1},λ^{-1} ...)

Coherence from rewriting

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  Third case: paths with inverses (α^{-1},λ^{-1} ...)

  $\rightarrow$ Analogous to the proof of the Church-Rosser lemma

Coherence from rewriting

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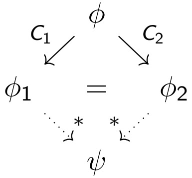
Axioms for coherence:

$\forall(C_1, C_2)$ critical



Coherence

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A Gray presentation P is **coherent** when the (3, 2)-Gray category $\overline{\mathsf{P}}^{\top}$ is coherent.

Question:

starting from a Gray presentation P, what generators need to be added in $\mathsf{P}_4^{\mathrm{coh}}$ so that the presentation becomes coherent?

Confluence

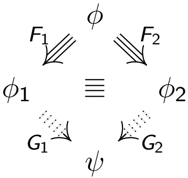
A 3-precategory C is **confluent** when, for 2-cells $\phi, \phi_1, \phi_2 \in C_2$ and 3-cells

$$F_1 \colon \phi \Rrightarrow \phi_1 \quad \text{and} \quad F_2 \colon \phi \Rrightarrow \phi_2$$

of C, there exist a 2-cell $\psi \in C_2$ and 3-cells

$$G_1 \colon \phi_1 \Rrightarrow \psi \in C_3 \quad \text{and} \quad G_2 \colon \phi_2 \Rrightarrow \psi \in C_3$$

of C such that $F_1 \bullet_2 G_1 = F_2 \bullet_2 G_2$.



Confluence

Confluence implies a Church-Rosser property:

Proposition

Given a confluent 3-precategory C, all

$$F\colon \phi \Rrightarrow \phi' \in C_3^{\top}$$

can be written

$$F = G \bullet_2 H^{-1}$$

for some $\psi \in C_2$, $G\colon \phi \Rrightarrow \psi \in C_3$ and $H\colon \phi' \Rrightarrow \psi \in C_3$.

Confluence

Criterion for coherence in $C^\top$ from confluence in C:

Proposition

Let C be a confluent 3-precategory satisfying that, for all pair of parallel 3-cells

$$F_1, F_2 \colon \phi \Rrightarrow \phi' \in C_3$$

there exists

$$G \colon \phi' \Rrightarrow \phi'' \in C_3$$

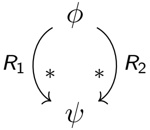
such that

$$F_1 \bullet_2 G = F_2 \bullet_2 G$$

then $C^\top$ is coherent.

Confluence

The hypothesis of the proposition can be obtained with rewriting



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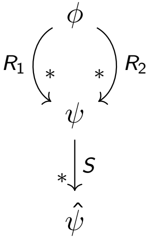

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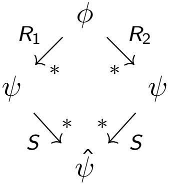

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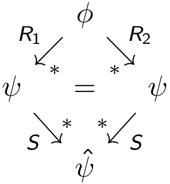


By generalized critical pair and Newman lemmas.

Rewriting system

Rewriting system: data of a 3-polygraph P together with a congruence $\equiv$ on P_3^*.

Note: a Gray presentation Q induces a rewriting system $(\mathsf{Q}_{\leq 3}, \equiv)$.

Since P^* is a 3-precategory, every $F \in \mathsf{P}_3^*$ uniquely decomposes as

$$F = S_1 \bullet_2 \cdots \bullet_2 S_k$$

where

$$S_i = \lambda_i \bullet_1 (l_i \bullet_0 A_i \bullet_0 r_i) \bullet_1 \rho_i$$

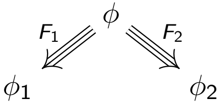
with $A_i \in \mathsf{P}_3, l_i, r_i \in \mathsf{P}_1^*, \lambda_i, \rho_i \in \mathsf{P}_2^*$.

k is called the **length** of F.

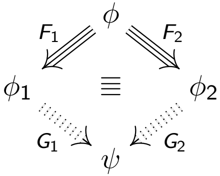
rewriting step: a 3-cell F of length 1.

Rewriting system

Given a rewriting system $(P, \equiv)$, a (local) branching



is **confluent** when there exist G_1 and G_2 such that



$(P, \equiv)$ is said **(locally) confluent** when every (local) branching is confluent.

Rewriting system

$(\mathsf{P}, \equiv)$ is said **terminating** when there is no infinite sequence of rewriting steps

$$\phi_0 \overset{F_1}{\Rrightarrow} \phi_1 \overset{F_2}{\Rrightarrow} \phi_2 \overset{F_3}{\Rrightarrow} \cdots$$

We have the following generalized version of Newman's lemma:

Proposition

If $(\mathsf{P}, \equiv)$ *is terminating and locally confluent, then it is confluent.*

Classification of branchings

Given a Gray presentation P, the local branchings

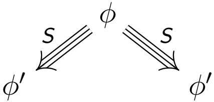
$$\phi_1 \overset{S_1}{\Lleftarrow} \phi \overset{S_2}{\Rrightarrow} \phi_2$$

can be classified into different categories

- trivial
- non-minimal
- independent
- natural
- critical

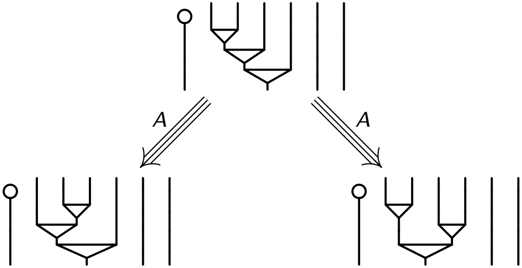
Trivial branchings

Those are the branchings involving the same rewriting steps



Non-minimal branchings

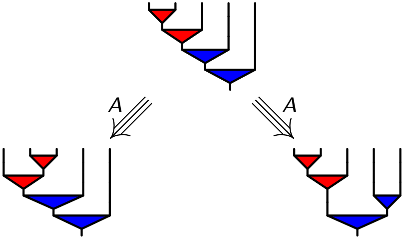
Those are the branchings with some parts that can be contextually factored out



These branchings are not interesting since they can be reduced to minimal branchings

Independent branchings

Those are the branchings that act on non-overlapping heights of the source 2-cell



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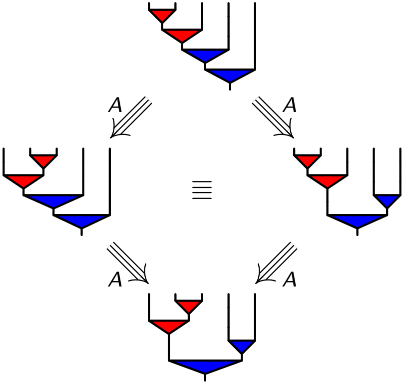
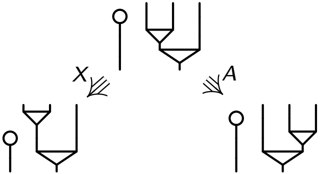


They are uninteresting since they are confluent by the generators of P_4^{st}

Natural branchings

Those are the branchings that involve an interchanger and an operational 3-generator



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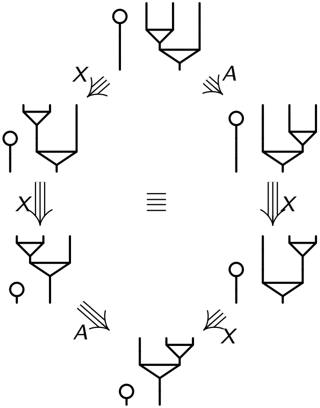
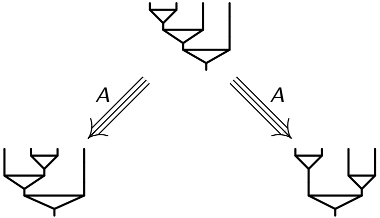


They are also uninteresting since they are confluent by the generators of P_4^{st}

Critical branchings

Those are the branchings that do not fit in other categories



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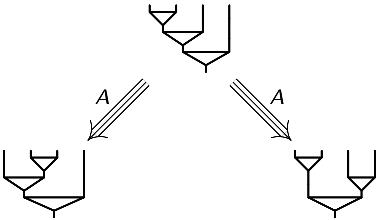

We can recover the classical critical pair lemma:

Theorem

Given a Gray presentation P*, if every critical branching is confluent, then the associated rewriting system is locally confluent.*

Coherence

We obtain a Squier-like theorem for Gray categories

Theorem

Given a terminating Gray presentation P *where every critical branching is confluent,* P *is coherent.*

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Proof.
By the critical pair lemma, P is locally confluent.

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Given a terminating Gray presentation P *where every critical branching is confluent,* P *is coherent.*

Proof.

Since P is terminating, by Newman lemma, P is confluent.

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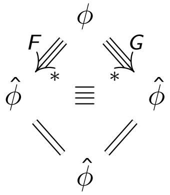
Theorem

Given a terminating Gray presentation P *where every critical branching is confluent,* P *is coherent.*

Proof.
Given $F, G \colon \phi \Rrightarrow \hat{\phi} \in \mathrm{P}_3^*$ where $\hat{\phi}$ is a normal form, we have $F = G \in \overline{\mathrm{P}}$.



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Theorem

Given a terminating Gray presentation P *where every critical branching is confluent,* P *is coherent.*

Proof.

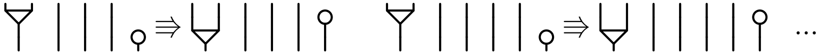
Given $F, G \colon \phi \Rrightarrow \psi \in \mathsf{P}_3^*$, there exists $H \colon \psi \Rrightarrow \hat{\psi}$, so that, by the previous case,

$$F \bullet_2 H = G \bullet_2 H \in \overline{\mathsf{P}}$$

We conclude by the earlier "confluence implies coherence" criterion for precategories.

Finite number of critical pairs

- There is an infinite number of interchangers



$X_{m,\bar{3},e}$ $X_{m,\bar{4},e}$ …

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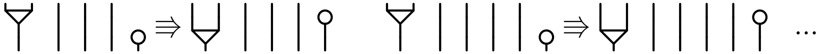

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Theorem: A finite number of operational rules (and ...) gives **a finite number of critical branchings**.

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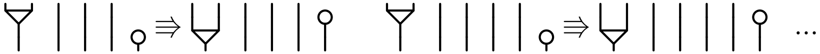

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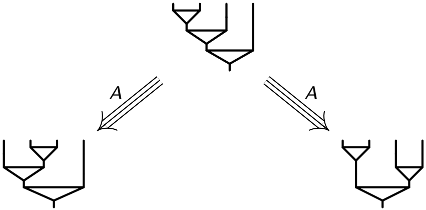
- Concerning computability

An algorithm exists to compute the critical branchings

Why finiteness ?

Three kinds of branchings:

- between two operational rules
  - *finite number of operational rules* implies *finite number of critical branchings of this kind*



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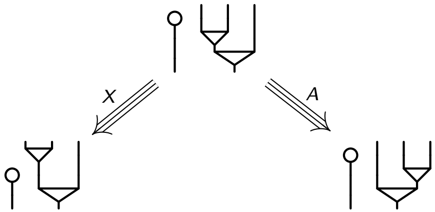
  - for n big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ can not be critical



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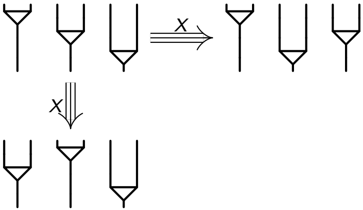
  - they are never critical and are usually “natural branchings”



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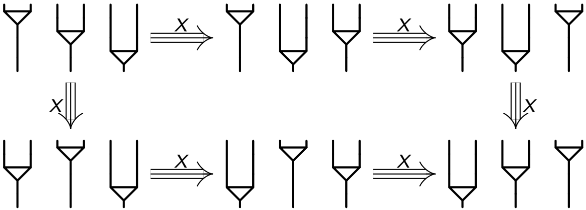

Content

Summing up

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- Start from a Gray presentation P

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Method to show coherence in Gray categories

- Start from a Gray presentation P
- Show that the rewriting system is terminating
- Find the critical branchings (an algorithm exists)
- Add a generator in P_4^{coh} for each confluence diagram
- The resulting Gray presentation is then coherent

Termination

Termination of $\Rrightarrow$:

- Taking into account operational rules and interchangers

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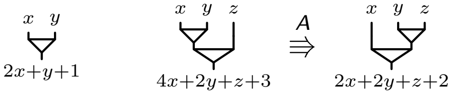
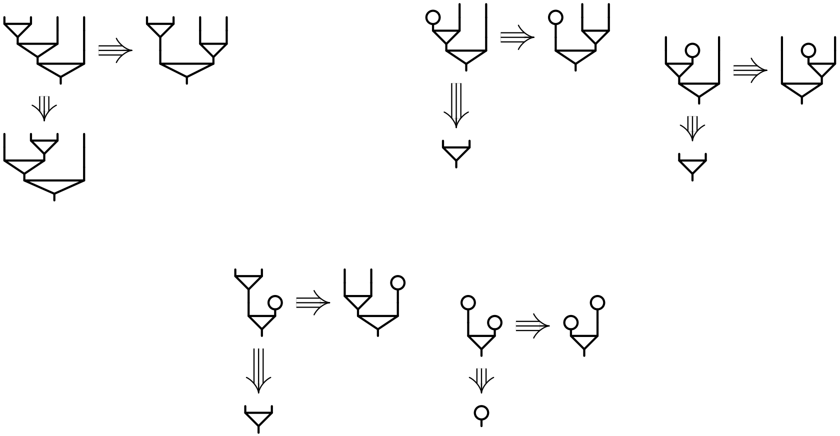
- Method for the operational rules:

  Find a measure that is left unvariant by interchangers



Example of monoids

With monoids, we find five critical pairs



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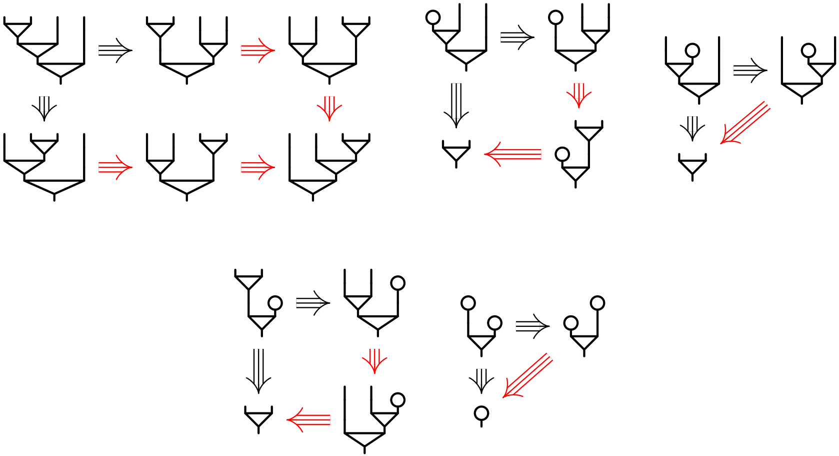
With monoids, we find five critical pairs and they are confluent



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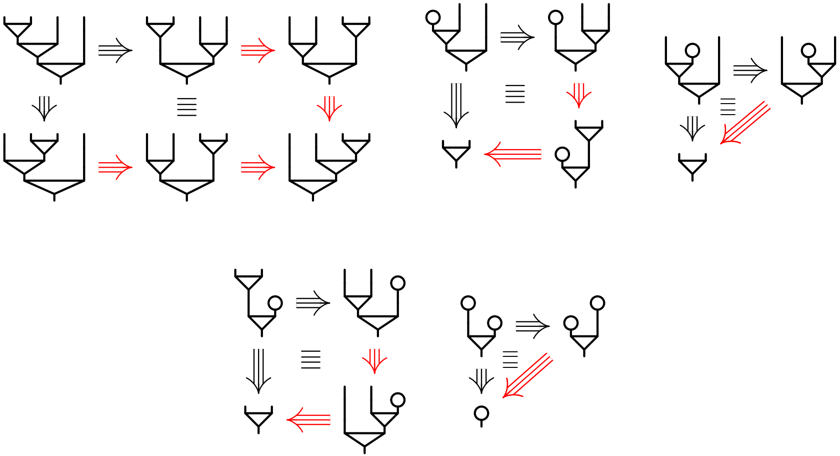

We deduce constraints on $\equiv$ for coherence

Other examples

- Adjunctions

$$P_1 = \{f, g : * \to *\}$$

$$P_2 = \{\cup, \cap\}$$

$P_3 = \{\text{zig} :$  $\Rrightarrow \; |, \quad \text{zag} :$  $\Rrightarrow \; |\}$

Other examples

- Adjunctions
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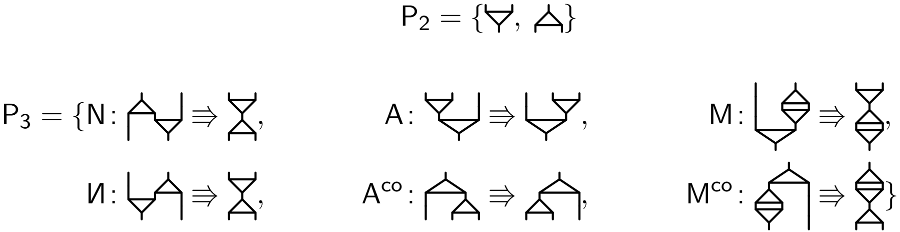
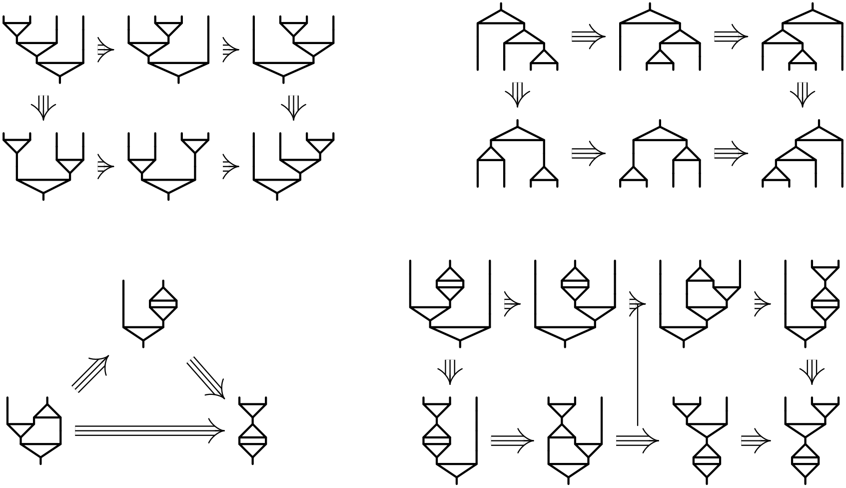
Other examples

- Adjunctions
- Self-dualities
- Frobenius monoid



Coherence relations

19 relations found by the algorithm



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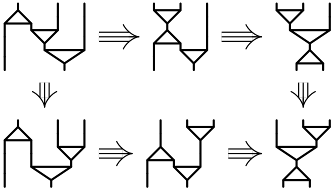

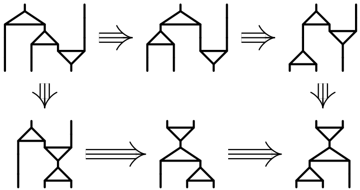

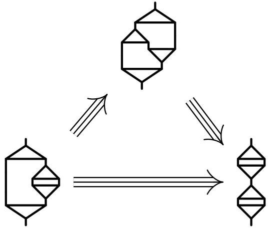

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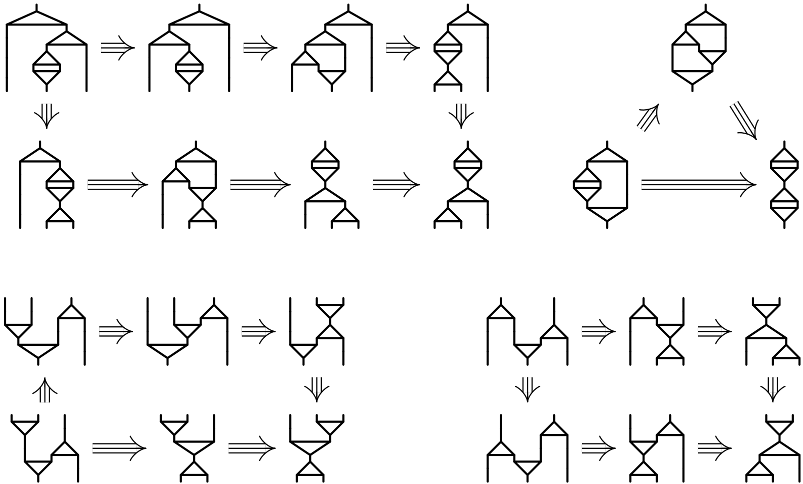

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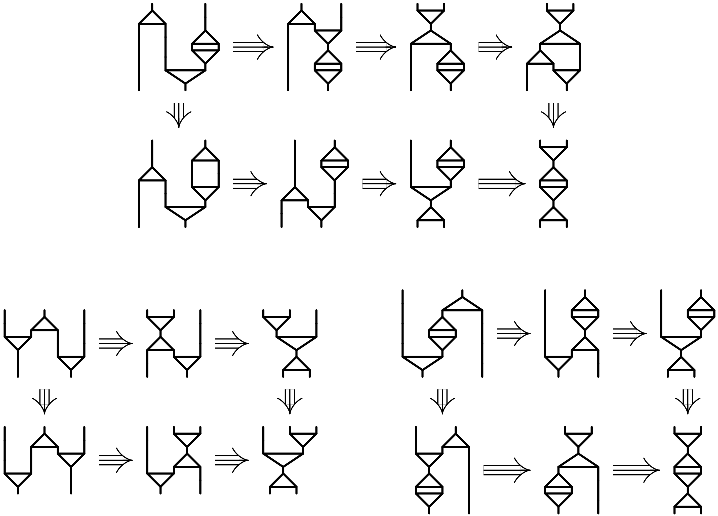

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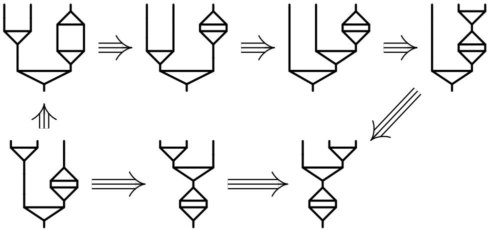

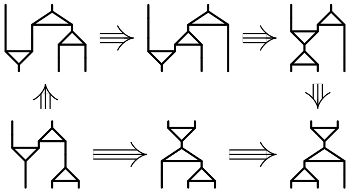

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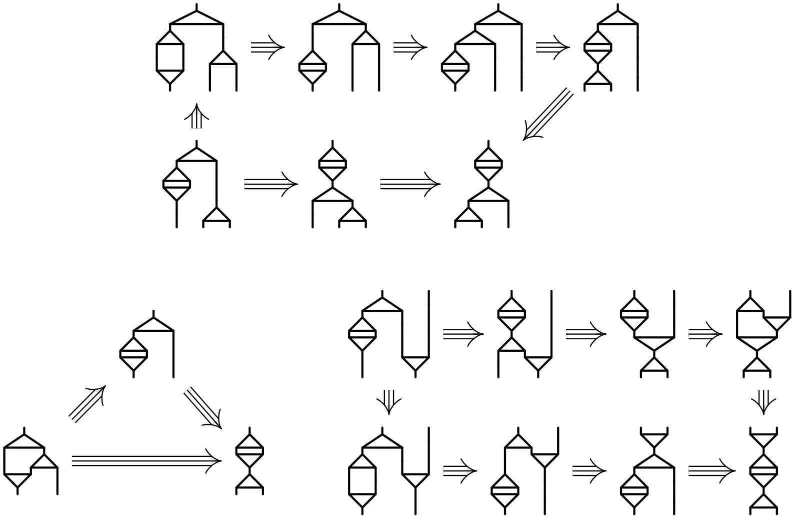

Other results

- *A coherent approach to pseudomonads*, Lack, 2000
- *Coherence for Frobenius pseudomonoids and the geometry of linear proofs*,Dunn and Vicary, 2016
- *Coherence for braided and symmetric pseudomonoids*, Verdon, 2017

Conclusion

- A rewriting system that reflects the structure of Gray categories
- Adapted tools to show coherence in this setting
- More automated method for coherence
  - Algorithm to compute the coherence conditions
- Proof of termination are still hard and tools should be developed