Rewriting for Gray categories

Simon Forest and Samuel Mimram

May 27th 2021
Coherence

Coherence in higher categories:

all parallel cells are equal.
Coherence

Coherence in higher categories:

*all parallel cells are equal.*

Classical example: **MacLane’s coherence theorem** for monoidal categories.

\[
\begin{align*}
A \otimes B & \xrightarrow{\rho} (A \otimes B) \otimes I \\
A \otimes (I \otimes B) & \xleftarrow{\lambda^{-1}} \quad = \quad (A \otimes I) \otimes (B \otimes I) \\
(A \otimes I) \otimes B & \xrightarrow{\alpha^{-1}} \\
(A \otimes I) \otimes (B \otimes I) & \xleftarrow{\rho}
\end{align*}
\]

**Theorem (MacLane’s coherence property for monoidal categories)**

*All morphisms made of \(\lambda, \rho, \alpha\) and their inverses between two objects are equal.*
Coherence tiles

**Coherence tiles**: the axioms allowing the coherence property

\[
((A \otimes B) \otimes C) \otimes D \xrightarrow{\alpha} (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha} A \otimes ((B \otimes C) \otimes D)
\]

\[
\alpha
\]

\[
(A \otimes B) \otimes (C \otimes D) \xrightarrow{\alpha} A \otimes (B \otimes (C \otimes D))
\]

\[
\alpha
\]

\[
(A \otimes I) \otimes B \xrightarrow{\alpha} A \otimes (I \otimes B)
\]

\[
\lambda
\]

\[
A \otimes B \xrightarrow{\rho} A \otimes B
\]
Coherence tiles: the axioms allowing the coherence property

\[
\begin{align*}
((w \cdot x) \cdot y) \cdot z & \xrightarrow{\alpha} (w \cdot (x \cdot y)) \cdot z \xrightarrow{\alpha} w \cdot ((x \cdot y) \cdot z) \\
(w \cdot x) \cdot (y \cdot z) & \xrightarrow{\alpha} w \cdot (x \cdot (y \cdot z)) \\
(w \cdot e) \cdot x & \xrightarrow{\alpha} w \cdot (e \cdot x) \\
\end{align*}
\]

\[
\begin{align*}
\lambda & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Several weak structures can be expressed in strict categories (paradoxically!):

- pseudomonoids
- pseudoadjunctions
- Frobenius pseudoalgebras
- etc.

Guiraud and Malbos developed a rewriting framework for finding coherence definitions for them.

Theorem ([G-M,08])
If a strict \(n\)-category is presented using a terminating and confluent \(n\)-polygraph, then a set of coherence conditions is given by the confluence diagrams of the critical branchings.
Coherence in strict categories

Several weak structures can be expressed in strict categories (paradoxically!):

- pseudomonoids
- pseudoadjunctions
- Frobenius pseudoalgebras
- etc.

Guiraud and Malbos developed a rewriting framework for finding coherence definitions for them.

Theorem ([G-M,08])

*If a strict n-category is presented using a terminating and confluent n-polygraph, then a set of coherence conditions is given by the confluence diagrams of the critical branchings.*
Coherence in strict categories

In particular, they recover MacLane’s coherence theorem for monoidal categories:

$\text{MacLane’s coherence theorem for monoidal categories:}$
Coherence in strict categories

In particular, they recover MacLane’s coherence theorem for monoidal categories:

- a monoidal category is a pseudomonoid in an adequate 3-category
Coherence in strict categories

In particular, they recover MacLane’s coherence theorem for monoidal categories:

- a monoidal category is a pseudomonoid in an adequate 3-category
- pseudomonoids can be presented using a terminating and confluent 3-polygraph \( P \)

\[
P_0 = \{*, \} \\
P_1 = \{ \bar{1}: * \to * \} \\
P_2 = \{ \phi: 0 \Rightarrow \bar{1}, \land: 2 \Rightarrow \bar{1} \} \\
P_3 = \{ L: \quad \Rightarrow \quad , \quad R: \quad \Rightarrow \quad , \quad A: \quad \Rightarrow \quad \}
\]
Coherence in strict categories

In particular, they recover MacLane’s coherence theorem for monoidal categories:

- a monoidal category is a pseudomonoid in an adequate 3-category
- pseudomonoids can be presented using a terminating and confluent 3-polygraph $P$

\[
P_0 = \{*\} \quad P_1 = \{\bar{1}: * \to *\}
\]
\[
P_2 = \{\varnothing: \bar{0} \Rightarrow \bar{1}, \quad \forall: \bar{2} \Rightarrow \bar{1}\}
\]
\[
P_3 = \{L: \quad \Rightarrow \quad , \quad R: \quad \Rightarrow \quad , \quad A: \quad \Rightarrow \quad \}
\]

- the coherence conditions derived from the critical branchings entail coherence
Coherence in strict categories

In particular, they recover MacLane’s coherence theorem for monoidal categories:

- a monoidal category is a pseudomonoid in an adequate 3-category
- pseudomonoids can be presented using a terminating and confluent 3-polygraph $P$

\[ P_0 = \{*\} \quad P_1 = \{\bar{1}: * \to *\} \]
\[ P_2 = \{ \varphi: \bar{0} \Rightarrow \bar{1}, \quad \forall: \bar{2} \Rightarrow \bar{1} \} \]
\[ P_3 = \{ L: \quad \Rightarrow \quad, \quad R: \quad \Rightarrow \quad, \quad A: \quad \Rightarrow \quad \} \]

- the coherence conditions derived from the critical branchings entail coherence
- these conditions are essentially the same than the ones of MacLane
Strict categories and homotopy

Strict categories are “easy” but have bad homotopical properties. Depending on the definitions:

- no good realization functor from strict categories to $\text{Top}$
- not all homotopy type can be modeled with strict categories
- vanishing Whitehead products
- $\text{etc.}$
Strict categories and homotopy

Strict categories are “easy” but have bad homotopical properties. Depending on the definitions:

- no good realization functor from strict categories to $\text{Top}$
- not all homotopy type can be modeled with strict categories
- vanishing Whitehead products
- etc.

Thus, weakened structures expressed in strict categories are not the most general somehow.
Strict categories and homotopy

Strict categories are “easy” but have bad homotopical properties. Depending on the definitions:

- no good realization functor from strict categories to \( \text{Top} \)
- not all homotopy type can be modeled with strict categories
- vanishing Whitehead products
- \( \text{etc.} \)

Thus, weakened structures expressed in strict categories are not the most general somehow.

The most general definitions can be obtained by considering structures expressed in weak categories.
Bicategories

The standard definition of weak 2-dimensional categories are bicategories.
Bicategories

The standard definition of weak 2-dimensional categories are bicategories.

A priori, weakened 2-dimensional structures should be considered in bicategories in order to obtain the most general definitions.
Bicategories

The standard definition of weak 2-dimensional categories are **bicategories**.

*A priori*, weakened 2-dimensional structures should be considered in bicategories in order to obtain the most general definitions.

But actually, studying strict 2-categories is enough since

**Theorem ([MacLane,85])**

*Every bicategory is “equivalent” to a strict 2-category.*
Tricategories

The standard definition of weak 3-dimensional categories are **tricategories**.
Tricategories

The standard definition of weak 3-dimensional categories are **tricategories**.

Is it enough to study 3-dimensional structures in strict 3-categories?
Tricategories

The standard definition of weak 3-dimensional categories are \textbf{tricategories}.

Is it enough to study 3-dimensional structures in strict 3-categories?

No, since

\textbf{Observation}

\textit{Not all tricategories are “equivalent” to strict 3-categories.}
Tricategories

The standard definition of weak 3-dimensional categories are **tricategories**.

Is it enough to study 3-dimensional structures in strict 3-categories?

No, since

**Observation**

*Not all tricategories are “equivalent” to strict 3-categories.*

This is a shame since tricategories are terrible to work with.
Gray categories

However, we have the following coherence property:

Theorem ([Gordon, Power, Street, 95])

*Every tricategory is “equivalent” to a Gray category.*
Gray categories

- almost like strict 3-categories
- unital and associative compositions
- but no exchange law for 2-cells
Gray categories

- almost like strict 3-categories
- unital and associative compositions
- but no exchange law for 2-cells
Gray categories

almost like strict 3-categories
unital and associative compositions
but no exchange law for 2-cells

\[ X_{\phi,\psi} : \]

\[
\begin{array}{c}
\phi \\
\downarrow \\
\phi' \\
\end{array} 
\quad \begin{array}{c}
\psi \\
\downarrow \\
\psi' \\
\end{array} 
\quad \Rightarrow 
\begin{array}{c}
\phi \\
\downarrow \\
\phi' \\
\end{array} 
\quad \begin{array}{c}
\psi \\
\downarrow \\
\psi' \\
\end{array}
\]
Extending rewriting theory

For finding coherent definitions of Gray categories, rewriting techniques are desirable.
Extending rewriting theory

For finding coherent definitions of Gray categories, rewriting techniques are desirable.

But Gray categories are not “equivalent” to strict 3-categories, so existing tools cannot be used readily.
For finding coherent definitions of Gray categories, rewriting techniques are desirable.

But Gray categories are not “equivalent” to strict 3-categories, so existing tools cannot be used readily.

Thus, we need to develop rewriting theory for another kind of higher categories.
Extending rewriting theory

In fact, considering other higher categories is good since...
Extending rewriting theory

In fact, considering other higher categories is good since

- recent works on higher dimensional rewriting are biased towards strict categories
Extending rewriting theory

In fact, considering other higher categories is good since

- recent works on higher dimensional rewriting are biased towards strict categories
- strict categories are not “that” special regarding rewriting
Extending rewriting theory

In fact, considering other higher categories is good since

- recent works on higher dimensional rewriting are biased towards strict categories
- strict categories are not “that” special regarding rewriting
- several shortcomings with strict categories (shapes of critical branchings, no good finiteness property)
Rewriting for Gray categories

One might think:

“If strict categories were used for rewriting in strict categories,
well,
Gray categories should be used for rewriting in Gray categories.”
Rewriting for Gray categories

One might think:

“If strict categories were used for rewriting in strict categories, well, Gray categories should be used for rewriting in Gray categories.”

But the interactions between interchange cells and operational cells must be studied.
Rewriting for Gray categories

One might think:

“If strict categories were used for rewriting in strict categories, well, Gray categories should be used for rewriting in Gray categories.”

But the interactions between interchange cells and operational cells must be studied.

Thus, another setting is needed: **precategories**.
Content

Precategories

Gray categories

Rewriting

Examples
Content

Precategories

Gray categories

Rewriting

Examples
Strict categories

A strict \textit{n-category} is an \textit{n}-globular set \(C\) equipped with operations

\[ \text{id}^{i+1}: C_i \to C_{i+1} \]

and, for \(i < k \leq n\),

\[ (-) \ast_i (-): C_k \times_i C_k \to C_k \]

which are unital and associative
Strict categories

A strict $n$-category is an $n$-globular set $C$ equipped with operations

$$\text{id}^{i+1}: C_i \to C_{i+1}$$

and, for $i < k \leq n$,

$$(-) *_i (-): C_k \times_i C_k \to C_k$$

which are unital and associative, and should satisfy an exchange law

$$((\phi * \phi') * \psi) = (\phi * \psi)(\phi' * \psi')$$
Strict categories

Exchange law: alternatively described using a distributivity and a smaller exchange condition.
Strict categories

Exchange law: alternatively described using a distributivity and a smaller exchange condition.

Distributivity property:

\[(x \to y)_i \ast \begin{pmatrix} \downarrow \phi & \downarrow g' & \to & \ast_{i+1} & \to & \Downarrow \phi' & \downarrow g'' & \to & Z \\
\downarrow & y & \to & \Downarrow & g' & \to & \Downarrow & g'' & \to & \Downarrow & y & \to & \Downarrow & g' & \to & \Downarrow & g'' & \to & Z \\
\end{pmatrix} = (x \to y)_i \ast \begin{pmatrix} \downarrow \phi & \downarrow g' & \to & \ast_{i+1} & \to & \Downarrow \phi' & \downarrow g'' & \to & Z \\
\downarrow & y & \to & \Downarrow & g' & \to & \Downarrow & g'' & \to & \Downarrow & y & \to & \Downarrow & g' & \to & \Downarrow & g'' & \to & Z \\
\end{pmatrix} \]

and similarly on the right.
Strict categories

Exchange law: alternatively described using a distributivity and a smaller exchange condition.

Smaller exchange property:
Strict categories

Exchange law: alternatively described using a distributivity and a smaller exchange condition.

Smaller exchange property:
Free constructions on strict categories

By general constructions, we have

- a category $\textbf{Cat}_n^+$ of $n$-cellular extensions ($n$-categories + generating $(n+1)$-cells)
- a free extension functor

\[-[-]^n: \textbf{Cat}_n^+ \to \textbf{Cat}_{n+1}
(C, X) \to C[X]\]

- a category $\textbf{Pol}_n$ of $n$-polygraphs
- a free-category-on-polygraph functor

\[(-)^{*,n}: \textbf{Pol}_n \to \textbf{Cat}_n
P \to P^*\]
Word problem on strict categories

Given an \( n \)-polygraph \( P \), the elements of \( P^* \) are quotients of valid terms that can be written on \( P \):

\[
id_1^1, \quad (a *_0 b) *_0 c, \quad a *_0 (b *_0 c), \quad (\alpha *_1 \beta) *_0 id_2^2, \quad \text{etc.}
\]

Word problem: deciding whether two terms denote the same cell in \( P^* \).

Theorem ([Makkai,05])

The word problem for strict categories is decidable.

- however, the procedure is intricate and expensive
- arguably, rewriting algorithms on str. cat. must be as expensive
Precategories

An \textit{n-precategory} is an \textit{n}-globular set \( C \) equipped with operations

\[
\text{id}^{i+1} : C_i \to C_{i+1}
\]

and, for \( k, l \leq n \),

\[
(\cdot)_{k,l} : C_k \times_{\min(k,l)-1} C_l \to C_{\max(k,l)}
\]

which are unital, associative, and distributive.
Precategories

An \textit{n-precategory} is an \textit{n-globular set} \( C \) equipped with operations

\[ \text{id}^{i+1}: C_i \to C_{i+1} \]

and, for \( k, l \leq n \),

\[ (\_ \_ \_ k, l \_ \_ \_): C_k \times_{\min(k,l)-1} C_l \to C_{\max(k,l)} \]

which are unital, associative, and distributive.

But not required to satisfy the exchange condition.
Precategories

As expected, the following property holds:

**Theorem**

A *strict n-category is exactly an n-precategory satisfying the exchange condition.*
Free constructions on precategories

By general constructions, we have

- a category $\text{PCat}_n^+$ of $n$-cellular extensions ($n$-precategories + generating $(n+1)$-cells)
- a free extension functor

$$-[-]^n : \text{PCat}_n^+ \rightarrow \text{PCat}_{n+1}$$

$$(C, X) \rightarrow C[X]$$

- a category $\text{PPol}_n$ of $n$-polygraphs
- a free-category-on-polygraph functor

$$(-)^*, n : \text{PPol}_n \rightarrow \text{PCat}_n$$

$P \rightarrow P^*$$
Given an $n$-cellular extension $(C, X)$, the elements of $C[X]$ are easily described: those are the sequences

$$u_1 \cdot_n \cdots \cdot_n u_k$$

where each $u_i$ is a whiskered generator, i.e., is of the form

$$l_n \cdot_{n-1} \cdots (l_1 \cdot_0 g \cdot_0 r_1) \cdots) \cdot_{n-1} r_n$$

for some $l_j, r_j \in C_j$ and $g \in X$.

The case of polygraphs: given an $n$-polygraph $P$, the cells of $P^*$ can be described as inductive sequences of whiskered generators.
Word problem on precategories

As a consequence,

**Theorem**

*The word problem for precategories is decidable.*

Indeed, the decision procedure is quite simple:

```ocaml
let test_pcat_eq c1 c2 = c1 = c2
```

▶ good sign for developing a rewriting framework on precategories
Content

Precategories

Gray categories

Rewriting

Examples
Higher categories can also be defined through enrichment.
Given a monoidal category \((\mathcal{V}, 1, \otimes)\), a \(\mathcal{V}\)-enriched category is the data of

- a set \(C_0\)
- objects \(C(x, y) \in \mathcal{V}\) for all \(x, y \in C_0\)

together with

- morphisms \(i_x : 1 \to C(x, x)\) for \(x \in C_0\)
- morphisms \(c_{x,y,z} : C(x, y) \otimes C(y, z) \to C(x, z)\) for \(x, y, z \in C_0\)

that are unital and associative.
Enriched definition

Given a monoidal category \((\mathcal{V}, 1, \otimes)\), a \(\mathcal{V}\)-enriched category is the data of

- a set \(C_0\)
- objects \(C(x, y) \in \mathcal{V}\) for all \(x, y \in C_0\)

together with

- morphisms \(i_x : 1 \to C(x, x)\) for \(x \in C_0\)
- morphisms \(c_{x,y,z} : C(x, y) \otimes C(y, z) \to C(x, z)\) for \(x, y, z \in C_0\)

that are unital and associative.

\[
1 \otimes C(x, y) \xrightarrow{i_x \otimes C(x, y)} C(x, x) \otimes C(x, y) \xleftarrow{C(x, x) \otimes C(x, y)}
\]
Enriched definition

Given a monoidal category \((\mathcal{V}, 1, \otimes)\), a **\(\mathcal{V}\)-enriched category** is the data of

- a set \(C_0\)
- objects \(C(x, y) \in \mathcal{V}\) for all \(x, y \in C_0\)

together with

- morphisms \(i_x : 1 \to C(x, x)\) for \(x \in C_0\)
- morphisms \(c_{x, y, z} : C(x, y) \otimes C(y, z) \to C(x, z)\) for \(x, y, z \in C_0\)

that are unital and associative.
Given a monoidal category \((\mathcal{V}, 1, \otimes)\), a \(\mathcal{V}\)-enriched category is the data of

- a set \(C_0\)
- objects \(C(x, y) \in \mathcal{V}\) for all \(x, y \in C_0\)

together with

- morphisms \(i_x : 1 \to C(x, x)\) for \(x \in C_0\)
- morphisms \(c_{x, y, z} : C(x, y) \otimes C(y, z) \to C(x, z)\) for \(x, y, z \in C_0\)

that are unital and associative.
Enriched definition

Example: a strict 2-category is a category enriched over \((\text{Cat}, 1, \times)\)

\[
C = \begin{array}{c}
\phi \\
\end{array} f \rightarrow f' \quad D = \begin{array}{c}
\psi \\
\end{array} g \rightarrow g'
\]

\[
C \times D = \begin{array}{c}
\phi, g \\
\end{array} \begin{array}{c}
(f, g) \\
\downarrow \\
(f', g) \\
\end{array} \xrightarrow{(f', \psi)} \begin{array}{c}
\phi, g' \\
\end{array} \\
= \begin{array}{c}
\phi, g' \\
\downarrow \\
(f' , g') \\
\end{array} \begin{array}{c}
(f', g') \\
\end{array} \xrightarrow{(f', \psi)} \begin{array}{c}
(f', g') \\
\end{array}
\]
Enriched definition

Example: a strict 2-category is a category enriched over \((\mathbf{Cat}, 1, \times)\)
Example: a 2-precategory is a category enriched over \((\mathbf{Cat}, 1, \Box)\)

\[
\begin{aligned}
C &= f \overset{\phi}{\longrightarrow} f' \\
D &= g \overset{\psi}{\longrightarrow} g'
\end{aligned}
\]
Enriched definition

Example: a 2-precategory is a category enriched over \((\mathbf{Cat}, 1, \square)\)
Tensor product on $\mathbf{Cat}_2$

The two previous tensor products on $\mathbf{Cat}_1$ can be easily generalized to $\mathbf{Cat}_2$

\[
\begin{align*}
C \times D &= (\phi, g) \\
\downarrow & \\
(f', g) &\xrightarrow{(f', \psi)} (f', g')
\end{align*}
\]

\[
\begin{align*}
C \boxtimes D &= (\phi, g) \\
\downarrow & \\
(f', g) &\xrightarrow{(f', \psi)} (f', g')
\end{align*}
\]
A new tensor product on $\text{Cat}_2$ is given by the **Gray tensor product** $\boxtimes$:


tensor product on $\text{Cat}_2$

\[
C \boxtimes D = (\phi, g) \\
\downarrow \quad \chi \quad \Rightarrow \\
(f', g) \quad (f', g') \\
\downarrow \quad (f', \psi) \\
(f', g) \quad (f', g')
\]
A new tensor product on $\text{Cat}_2$ is given by the **Gray tensor product** $\boxtimes$.
Gray categories

A **Gray category** is then a category enriched over $\text{Cat}_2$ equipped with Gray tensor product.
A **Gray category** is then a category enriched over $\text{Cat}_2$ equipped with Gray tensor product.

**Idea:** it is a 3-precategory with interchange 3-cells for 2-cells with some axioms on 3-cells.
Gray categories

Elements of a Gray category:

- 0-cells and 1-cells
- 2-cells:
- 3-cells:

\[ \phi \rightarrow \psi \]
Gray categories

Elements of a Gray category:

▶ 0-cells and 1-cells
▶ 2-cells:

▶ 3-cells:

▶ among them, interchangers:
Gray categories

- composition of 2-cells with 1-cells on the left and the right

\[
\begin{array}{c}
| | | \cdot_0 \begin{array}{c} \hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{array} = | | | \begin{array}{c} \hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{array} \\
\begin{array}{c} \hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{array} \cdot_0 | | | = \begin{array}{c} \hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{array} | | | \\
\end{array}
\]
Gray categories

- composition of 2-cells with 1-cells on the left and the right

\[
\begin{array}{c}
\begin{array}{c|c|c}
 & & \bullet_0 \\
\hline
\end{array}
\begin{array}{c|c|c}
 & & \end{array}
\end{array}
= \begin{array}{c|c|c}
 & & \\
\hline
\end{array}
\begin{array}{c|c|c}
 & & \end{array}
\]

\[
\begin{array}{c|c|c}
\hline
\end{array}
\begin{array}{c|c|c}
\bullet_0 & & \\
\hline
\end{array}
\begin{array}{c|c|c}
 & & \\
\hline
\end{array}
\end{array}
= \begin{array}{c|c|c}
 & & \end{array}
\begin{array}{c|c|c}
 & & \end{array}
\]

- composition: 2-cells can be composed vertically

\[
\begin{array}{c|c|c}
\hline
\end{array}
\begin{array}{c|c|c}
\bullet_1 & & \\
\hline
\end{array}
\begin{array}{c|c|c}
 & & \\
\hline
\end{array}
\end{array}
= \begin{array}{c|c|c}
 & & \\
\hline
\end{array}
\begin{array}{c|c|c}
 & & \\
\hline
\end{array}
\]

Gray categories

- composition of 2-cells with 1-cells on the left and the right

\[
\begin{align*}
\begin{array}{c|c|c}
| & | & \bullet_0 \end{array} & = & \begin{array}{c|c|c}
| & | & \end{array} \\
\begin{array}{c|c|c}
\end{array} & \bullet_0 & \begin{array}{c|c|c}
| & | & \end{array} & = & \begin{array}{c|c|c|c}
| & | & | & | \\
\end{array}
\end{align*}
\]

- composition: 2-cells can be composed vertically

\[
\begin{align*}
\begin{array}{c|c|c}
\end{array} & \bullet_1 \\
\begin{array}{c|c|c}
\end{array} & = & \begin{array}{c|c|c|c}
\end{array}
\end{align*}
\]

- 3-cells can be composed horizontally

\[
\left(\begin{array}{c|c|c}
\alpha & \Rightarrow & \beta \end{array}\right) \bullet_2 \left(\begin{array}{c|c|c}
\beta & \Rightarrow & \gamma \end{array}\right) = \left(\begin{array}{c|c|c}
\alpha & \Rightarrow & \gamma \end{array}\right)
\]
Gray categories

- properties of associativity and unitality

\[
\begin{align*}
\alpha \beta \cdot 1 \gamma &= \alpha \beta \\
\alpha \beta \gamma &= \alpha \beta \\
\alpha \beta \gamma &= \alpha
\end{align*}
\]
Gray categories

Additional conditions are required:
Gray categories

Additional conditions are required:

- some compatibilities for $X_{\_\_}$
Additional conditions are required:

- some compatibilities for $X_{\_,\_}$

\[ X_{\phi \cdot 1 \phi', \psi} = ((\phi \cdot 0 \ g) \cdot 1 \ X_{\phi', \psi}) \cdot 2 \ (X_{\phi, \psi} \cdot 1 \ (\phi' \cdot 0 \ g')) \]
Gray categories

Additional conditions are required:

- some compatibilities for $X_{-, -}$

$$
X_{\phi, \psi_1 \psi'} = (X_{\phi, \psi} \cdot_1 (f' \cdot_0 \psi')) \cdot_2 ((f \cdot_0 \psi) \cdot_1 X_{\phi, \psi'})
$$
Gray categories

Additional conditions are required:

▶ some compatibilities for $X_{\cdot, \cdot}$

$$X_{e \bullet_0 \phi, \psi} = e \circ_0 X_{\phi, \psi} \quad X_{\phi \bullet_0 f, \psi} = X_{\phi, f \circ_0 \psi} \quad X_{\phi, \psi \circ_0 h} = X_{\phi, \psi \bullet_0 h}.$$
Gray categories

Additional conditions are required:

- some compatibilities for $X_{-, -}$

and others...
Gray categories

Additional conditions are required:

- some compatibilities for $X_{-, -}$
- an exchange law for 3-cells

\[
\begin{align*}
A : & \quad \phi \Rightarrow \phi' \\
B : & \quad \psi \Rightarrow \psi'
\end{align*}
\]
Gray categories

Additional conditions are required:

- some compatibilities for $X_{-,-}$
- an exchange law for 3-cells
- a naturality condition between 3-cells and interchangers

$A: \phi \Rightarrow \phi'$
Gray presentation

In order to use rewriting methods on Gray categories, we need a notion of presentation.
A **Gray presentation** is the data of a 4-polygraph (of precategories) $P$ such that:

1. For each $(\alpha : f \Rightarrow f', g, \beta : h \Rightarrow h') \in P_2 \times P_0 \times P_2$,
2. For each instance of the axiom of Gray categories w.r.t. the generators of $P$, there is a 4-generator in $P$.
A **Gray presentation** is the data of a 4-polygraph (of precategories) $P$ such that:

- for each $(\alpha: f \Rightarrow f', g, \beta: h \Rightarrow h') \in P_2 \times_0 P_1^* \times_0 P_2$, there is a 3-generator $X_{\alpha,g,\beta}$

$$X_{\alpha,g,\beta}: (\alpha \circ_0 g \circ_0 h) \cdot_1 (f' \circ_0 g \circ_0 \beta) \Rightarrow (f \circ_0 g \circ_0 \beta) \cdot_1 (\alpha \circ_0 g \circ_0 h')$$
A **Gray presentation** is the data of a 4-polygraph (of precategories) $P$ such that:

- for each $(\alpha: f \Rightarrow f', g, \beta: h \Rightarrow h') \in P_2 \times_0 P_1^* \times_0 P_2$, there is a 3-generator $X_{\alpha,g,\beta}$
- for each instance of the axiom of Gray categories w.r.t. the generators of $P$, there is a 4-generator in $P_4$
Gray presentation

Example: given a Gray presentation $P$, for each

$$A: \phi \Rightarrow \phi' \quad B: \psi \Rightarrow \psi' \quad \in P_3$$

and $\chi \in P_2^*$ (sufficiently composable), there is a 4-generator in $P_4$
Gray presentation

Example: given a Gray presentation $\mathcal{P}$ and

\[ A: \phi_1 \cdot_1 \phi_2 \cdot_1 \phi_3 \Rightarrow \psi_1 \cdot_1 \psi_2 \in \mathcal{P}_3 \]

with $\phi_i = l_i \cdot_0 \alpha_i \cdot_0 r_i$ and $\psi_i = l_i' \cdot_0 \beta_i \cdot_0 r_i'$, and

\[ f \in \mathcal{P}_1^* \quad \gamma \in \mathcal{P}_2 \]

(sufficiently composable), there is a 4-generator in $\mathcal{P}_4$
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

\[
P_0 = \{\ast\} \quad P_1 = \{\bar{1}: \ast \to \ast\}
\]

\[
P_2 = \{\varphi: \bar{0} \Rightarrow 1, \quad \forall: \bar{2} \Rightarrow 1\}
\]
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

$$P_3 = P^s_3 \sqcup P^o_3$$

with $P^s_3$ made of generators of the form

$$X_{\mu, \bar{n}, \mu} : \triangledown \Rightarrow \triangledown$$

$$X_{\eta, \bar{n}, \mu} : \circ \Rightarrow \circ$$

$$X_{\mu, \bar{n}, \eta} : \triangledown \Rightarrow \circ$$

$$X_{\eta, \bar{n}, \eta} : \circ \Rightarrow \circ$$
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

$$P_3 = P^\text{st}_3 \sqcup P^\text{op}_3$$

with $P^\text{st}_3$ made of generators of the form

$$X_{\mu, \overline{n}, \mu}: \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array} \quad \Rightarrow \quad \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array}$$

$$X_{\eta, \overline{n}, \mu}: \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array} \quad \Rightarrow \quad \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array}$$

and

$$P^\text{op}_3 = \{ \quad X_{\mu, \overline{n}, \eta}: \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array} \quad \Rightarrow \quad \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array} \quad , \quad X_{\eta, \overline{n}, \eta}: \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array} \quad \Rightarrow \quad \begin{array}{c} \begin{array}{c} \circ \end{array} \end{array} \quad \}$$
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

$$P_4 = P_{4}^{\text{st}} \sqcup P_{4}^{\text{coh}}$$
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

\[ P_4 = P^\text{st}_4 \sqcup P^\text{coh}_4 \]

with $P^\text{st}_4$ made of the different generators required by the definition of Gray presentation.

Example:
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

$$P_4 = P_{4}^{st} \sqcup P_{4}^{coh}$$

with $P_{4}^{st}$ made of the different generators required by the definition of Gray presentation

Example:
Example: pseudomonoids

The Gray presentation $P$ of pseudomonoids

$$P_4 = P_{4}^{st} \sqcup P_{4}^{coh}$$

and $P_{4}^{coh}$ made of additional generators required for coherence.

Example:

![Diagram](image)

Note: these generators can involve interchange generators.
Let $P$ be a 4-polygraph $P$.

$\overline{P}$: 3-precategory obtained from $(P^*)_{\leq 3}$ by quotienting the 3-cells with $\sim$, where

$$F \sim G \quad \text{for all } \Gamma : F \Rightarrow G \in P_4.$$
Presented category

Let $C$ be a 3-precategory.

$C^\top$: 3-precategory obtained by formally inverting the 3-cells.
Presented category

**Theorem**

*Given a Gray presentation $\mathcal{P}$, the 3-precategory $\mathcal{P}$ is canonically a lax Gray category.*
Presented category

Theorem
Given a Gray presentation $P$, the 3-precategory $\overline{P}$ is canonically a lax Gray category.

The difficult part is showing that the different definitions of $X_{-,-}$ are coherent

Example for $X_{\circ\circ}$:
Presented category

Theorem
Given a Gray presentation $P$, the 3-precategory $\overline{P}$ is canonically a lax Gray category.

Corollary
Given a Gray presentation $P$, the 3-precategory $\overline{P^T}$ is canonically a $(3, 2)$-Gray category.
Coherence

We want to show coherence properties:

*all the ways to prove that two objects are equivalent are equal*

Example for pseudomonoids:
Coherence from rewriting

**Rewriting system** Get a rewriting system: choose a “good” orientation for the isos of the considered structure

\[
\alpha : \ (A \otimes B) \otimes C \ \sim \rightarrow \ A \otimes (B \otimes C)
\]
\[
\lambda : \ (I \otimes A) \ \sim \rightarrow \ A
\]
\[
\rho : \ (A \otimes I) \ \sim \rightarrow \ A
\]
Coherence from rewriting

Rewriting system  Get a rewriting system: choose a “good” orientation for the 
isos of the considered structure

\[ \alpha : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C) \]

\[ \lambda : (I \otimes A) \rightarrow A \]

\[ \rho : (A \otimes I) \rightarrow A \]

In particular, we want \( \rightarrow \) terminating
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent

∀(C₁, C₂) critical

∀(R₁, R₂)

then
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence

\[
\forall (R_1, R_2) \text{ rewrite steps } \phi_1 = \phi_2 \\
\phi \quad \phi \quad R_1 \quad R_2 \\
\psi
\]

then

\[
\forall (R_1, R_2) \text{ rewrite paths } \phi_1 = \phi_2 \\
\phi \quad \phi \quad R_1 \quad R_2 \\
\psi\]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  First case: paths to a normal form $\hat{\psi}$
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: terminating and local confluence imply confluence
- **Coherence**
  
  First case: paths to a normal form $\hat{\psi}$

  ![Diagram](image)

  by Newman’s lemma
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence
- **Coherence**
  
  First case: paths to a normal form \( \hat{\psi} \)
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  Second case: paths to an arbitrary object $\psi$
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  
  Second case: paths to an arbitrary object $\psi$

![Diagram of the coherence theorem]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: terminating and local confluence imply confluence
- **Coherence**
  
  Second case: paths to an arbitrary object $\psi$

\[
R_1 \quad \phi \quad R_2
\]

\[
\psi \quad * \quad * \quad \psi
\]

\[
S \quad * \quad * \quad S
\]

\[
\hat{\psi}
\]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  Second case: paths to an arbitrary object $\psi$

![Diagram of paths and rewrite rules](image)
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**

  Second case: paths to an arbitrary object $\psi$

\[
\psi \xrightarrow{R_1} \phi \xrightarrow{*} \psi \xrightarrow{R_2} \psi \xrightarrow{S} \hat{\psi} \xrightarrow{S^{-1}} \psi
\]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  
  Second case: paths to an arbitrary object \( \psi \)

\[
\frac{\phi}{R_1 \ast R_2} = \psi
\]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: terminating and local confluence imply confluence
- **Coherence**
  - Third case: paths with inverses ($\alpha^{-1}, \lambda^{-1} ...$)
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  - Third case: paths with inverses \((\alpha^{-1}, \lambda^{-1}, \ldots)\)
  
  → Analogous to the proof of the Church-Rosser lemma
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence
- **Coherence**

Axioms for coherence:

\[
\forall (C_1, C_2) \text{ critical} \\
\phi_1 = \phi_2
\]
A 3-precategory $C$ is **coherent** when, for all parallel $F, G \in C_3$, $F = G$. 

**Coherence**

A Gray presentation $P$ is **coherent** when the $(3, 2)$-Gray category $P^\top$ is coherent.

**Question:** Starting from a Gray presentation $P$, what generators need to be added in $P_{coh}$ so that the presentation becomes coherent?
A 3-precategory $C$ is **coherent** when, for all parallel $F, G \in C_3$, $F = G$.

A Gray presentation $P$ is **coherent** when the $(3, 2)$-Gray category $\overline{P}^T$ is coherent.
Coherence

A 3-precategory $C$ is **coherent** when, for all parallel $F, G \in C_3$, $F = G$.

A Gray presentation $P$ is **coherent** when the $(3, 2)$-Gray category $\overline{P}^T$ is coherent.

Question:

starting from a Gray presentation $P$, what generators need to be added in $P_{4}^{coh}$ so that the presentation becomes coherent?
Confluence

A 3-precategory $C$ is **confluent** when, for 2-cells $\phi, \phi_1, \phi_2 \in C_2$ and 3-cells

$$F_1 : \phi \Rightarrow \phi_1 \quad \text{and} \quad F_2 : \phi \Rightarrow \phi_2$$

of $C$, there exist a 2-cell $\psi \in C_2$ and 3-cells

$$G_1 : \phi_1 \Rightarrow \psi \in C_3 \quad \text{and} \quad G_2 : \phi_2 \Rightarrow \psi \in C_3$$

of $C$ such that $F_1 \bullet_2 G_1 = F_2 \bullet_2 G_2$.

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{
\phi \ar@{=>}[rr]^{F_1} & & \phi_1 \\
\phi_2 \\
\psi \ar@{=>}[uu]_{G_1} & & \ar@{=>}[uu]_{G_2}
}
\end{array}
\end{array}
\]
Confluence implies a Church-Rosser property:

**Proposition**

Given a confluent 3-precategory $C$, all

$$F : \phi \Rightarrow \phi' \in C_3^T$$

can be written

$$F = G \bullet_2 H^{-1}$$

for some $\psi \in C_2$, $G : \phi \Rightarrow \psi \in C_3$ and $H : \phi' \Rightarrow \psi \in C_3$. 
Confluence

Criterion for coherence in $C^\top$ from confluence in $C$:

**Proposition**

Let $C$ be a confluent 3-precategory satisfying that, for all pair of parallel 3-cells $F_1, F_2 : \phi \Rightarrow \phi' \in C_3$

there exists $G : \phi' \Rightarrow \phi'' \in C_3$

such that $F_1 \cdot_2 G = F_2 \cdot_2 G$

then $C^\top$ is coherent.
Confluence

The hypothesis of the proposition can be obtained with rewriting

\[
\begin{array}{c}
\phi \\
R_1 \\
R_2 \\
\psi
\end{array}
\]
The hypothesis of the proposition can be obtained with rewriting
The hypothesis of the proposition can be obtained with rewriting
The hypothesis of the proposition can be obtained with rewriting

\[ R_1 \leftarrow \phi \rightarrow R_2 \]
\[ \psi \quad = \quad \psi \]
\[ S \leftarrow \ast \rightarrow \ast \rightarrow S \]
\[ \hat{\psi} \]

By generalized critical pair and Newman lemmas.
**Rewriting system**

**Rewriting system**: data of a 3-polygraph $P$ together with a congruence $\equiv$ on $P^*_3$.

Note: a Gray presentation $Q$ induces a rewriting system $(Q_{\leq 3}, \equiv)$.

Since $P^*$ is a 3-precategory, every $F \in P^*_3$ uniquely decomposes as

$$F = S_1 \bullet_2 \cdots \bullet_2 S_k$$

where

$$S_i = \lambda_i \bullet_1 (l_i \bullet_0 A_i \bullet_0 r_i) \bullet_1 \rho_i$$

with $A_i \in P_3, l_i, r_i \in P^*_1, \lambda_i, \rho_i \in P^*_2$.

$k$ is called the **length** of $F$.

**rewriting step**: a 3-cell $F$ of length 1.
Rewriting system

Given a rewriting system \((P, \equiv)\), a (local) branching

\[
\begin{array}{c}
F_1 \xrightarrow{\phi_1} \Phi \xrightarrow{\phi} F_2 \\
\Phi \xrightarrow{\phi_2} \\
\end{array}
\]

is **confluent** when there exist \(G_1\) and \(G_2\) such that

\[
\begin{array}{c}
F_1 \xrightarrow{\phi} \equiv \xrightarrow{\equiv} F_2 \\
\Phi \xrightarrow{\phi_1} \equiv \xrightarrow{\equiv} \Phi \xrightarrow{\phi_2} \\
\Phi \xrightarrow{\phi_2} G_2 \xleftarrow{\psi} G_1 \\
\end{array}
\]

\((P, \equiv)\) is said **(locally) confluent** when every (local) branching is confluent.
Rewriting system

(P, ≡) is said **terminating** when there is no infinite sequence of rewriting steps

\[ \phi_0 \xrightarrow{F_1} \phi_1 \xrightarrow{F_2} \phi_2 \xrightarrow{F_3} \cdots \]

We have the following generalized version of Newman’s lemma:

**Proposition**

*If (P, ≡) is terminating and locally confluent, then it is confluent.*
Classification of branchings

Given a Gray presentation $P$, the local branchings $\phi_1$ and $\phi_2$ can be classified into different categories:

- trivial
- non-minimal
- independent
- natural
- critical
Trivial branchings

Those are the branchings involving the same rewriting steps

\[
\begin{align*}
\phi & \quad \phi' \\
S & \quad S \\
\phi' & \quad \phi'
\end{align*}
\]
Non-minimal branchings

Those are the branchings with some parts that can be contextually factored out

These branchings are not interesting since they can be reduced to minimal branchings
Independent branchings

Those are the branchings that act on non-overlapping heights of the source 2-cell
Independent branchings

Those are the branchings that act on non-overlapping heights of the source 2-cell

They are uninteresting since they are confluent by the generators of $P_{4}^{st}$
Natural branchings

Those are the branchings that involve an interchanger and an operational 3-generator.
Natural branchings

Those are the branchings that involve an interchanger and an operational 3-generator

They are also uninteresting since they are confluent by the generators of $P_{4}^{st}$
Critical branchings

Those are the branchings that do not fit in other categories
Critical branchings

Those are the branchings that do not fit in other categories

We can recover the classical critical pair lemma:

**Theorem**

*Given a Gray presentation $P$, if every critical branching is confluent, then the associated rewriting system is locally confluent.*
Coherence

We obtain a Squier-like theorem for Gray categories

Theorem
Given a terminating Gray presentation $P$ where every critical branching is confluent, $P$ is coherent.
We obtain a Squier-like theorem for Gray categories

Theorem

*Given a terminating Gray presentation $P$ where every critical branching is confluent,* $P$ *is coherent.*

Proof.

By the critical pair lemma, $P$ is locally confluent.
Coherence

We obtain a Squier-like theorem for Gray categories

**Theorem**

*Given a terminating Gray presentation $P$ where every critical branching is confluent, $P$ is coherent.*

Proof.

Since $P$ is terminating, by Newman lemma, $P$ is confluent.
Coherence

We obtain a Squier-like theorem for Gray categories

**Theorem**

*Given a terminating Gray presentation $P$ where every critical branching is confluent, $P$ is coherent.*

**Proof.**

Given $F, G : \phi \Rightarrow \hat{\phi} \in P_3^*$ where $\hat{\phi}$ is a normal form, we have $F = G \in \overline{P}$.
Coherence

We obtain a Squier-like theorem for Gray categories

**Theorem**

*Given a terminating Gray presentation* $P$ *where every critical branching is confluent, $P$ is coherent.*

**Proof.**

Given $F, G : \phi \Rightarrow \psi \in P^*$, there exists $H : \psi \Rightarrow \hat{\psi}$, so that, by the previous case,

$$F \bullet_2 H = G \bullet_2 H \in \overline{P}$$

We conclude by the earlier “confluence implies coherence” criterion for precategories.
Finite number of critical pairs

- There is an infinite number of interchangers

\[
\begin{align*}
\begin{array}{c}
\forall \ U & \Rightarrow \forall \ U \\
X_{m,\bar{3},e} & \Rightarrow \forall \ U \\
X_{m,\bar{4},e} & \Rightarrow \forall \ U \\
X_{m,\bar{n},e} & \text{ for all } n
\end{array}
\end{align*}
\]
Finite number of critical pairs

- There is an infinite number of interchangers

\[
\begin{align*}
\begin{array}{c}
\text{X}_{m,3,e} \\
\Rightarrow
\end{array} & \begin{array}{c}
\text{X}_{m,4,e} \\
\Rightarrow
\end{array} & \begin{array}{c}
\text{X}_{m,n,e}
\end{array}
\end{align*}
\]

- So potentially an infinite number of critical branchings

Theorem: A finite number of operational rules (and ...) gives a finite number of critical branchings.

Concerning computability
An algorithm exists to compute the critical branchings.
Finite number of critical pairs

- There is an infinite number of interchangers

\[ \square \quad \Rightarrow \quad \square \quad \quad \Rightarrow \quad \square \quad \quad \Rightarrow \quad \square \quad \ldots \]

\[ X_{m,3,e} \quad \quad \quad \quad \quad \quad \quad \quad X_{m,4,e} \quad \ldots \quad \quad \quad \quad \quad \quad \quad \quad X_{m,n,e} \text{ for all } n \]

- So potentially an infinite number of critical branchings

- In fact, no!

**Theorem:** A finite number of operational rules (and ...) gives a finite number of critical branchings.

(operational = that are not interchangers)
Finite number of critical pairs

- There is an infinite number of interchangers
  
  \[
  \begin{align*}
  \vdash & \Rightarrow \vdash \quad \vdash & \Rightarrow \vdash \\
  X_{m,3,e} & \quad X_{m,4,e} & \quad \cdots \\
  \\
  X_{m,n,e} & \text{for all } n
  \end{align*}
  \]

- So potentially an infinite number of critical branchings
- In fact, no!

**Theorem:** A finite number of operational rules (and ...) gives a finite number of critical branchings.

(operational = that are not interchangers)

- Concerning computability

  **An algorithm exists to compute the critical branchings**
Why finiteness?
Three kinds of branchings:

▶ between two operational rules
  ▶ *finite number of operational rules* implies *finite number of critical branchings of this kind*

▶ between an operational rule and an interchanger
  ▶ for *n* big enough, branchings with an operational rule and *X*_\(\alpha\), *n*_\(\beta\) can not be critical

▶ between two interchangers
  ▶ they are never critical and are usually "natural branchings"
Why finiteness?

Three kinds of branchings:

▷ between two operational rules
  ▶ finite number of operational rules implies finite number of critical branchings of this kind

▷ between an operational rule and an interchanger
  ▶ for $n$ big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ can not be critical
**Why finiteness?**

Three kinds of branchings:

- between two operational rules
  - finite number of operational rules implies finite number of critical branchings of this kind

- between an operational rule and an interchanger
  - for $n$ big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ can not be critical

- between two interchangers
  - they are never critical and are usually “natural branchings”
Why finiteness? 
Three kinds of branchings:

▶ between two operational rules
  ▶ finite number of operational rules implies finite number of critical branchings of this kind

▶ between an operational rule and an interchanger
  ▶ for $n$ big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ can not be critical

▶ between two interchangers
  ▶ they are never critical and are usually “natural branchings”
Content

Precategories

Gray categories

Rewriting

Examples
Summing up

Method to show coherence in Gray categories

- Start from a Gray presentation $P$
Summing up

Method to show coherence in Gray categories

- Start from a Gray presentation $P$
- Show that the rewriting system is terminating
Summing up

Method to show coherence in Gray categories

- Start from a Gray presentation $P$
- Show that the rewriting system is terminating
- Find the critical branchings (an algorithm exists)

Add a generator in $P$ for each confluence diagram

The resulting Gray presentation is then coherent
Summing up

Method to show coherence in Gray categories

- Start from a Gray presentation $P$
- Show that the rewriting system is terminating
- Find the critical branchings (an algorithm exists)
- Add a generator in $P_{4}^{\text{coh}}$ for each confluence diagram
Summing up

Method to show coherence in Gray categories

- Start from a Gray presentation $P$
- Show that the rewriting system is terminating
- Find the critical branchings (an algorithm exists)
- Add a generator in $P^\text{coh}_4$ for each confluence diagram
- The resulting Gray presentation is then coherent
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
- We can reduce the problem to operational rules

**Theorem:** (under reasonable conditions on the 2-generators) rewriting using only interchangers terminates.
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
- We can reduce the problem to operational rules

**Theorem**: (under reasonable conditions on the 2-generators) rewriting using only interchangers terminates.

- *Normal forms for planar connected string diagrams*, Delpeuch and Vicary, 2018
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
- We can reduce the problem to operational rules

**Theorem**: (under reasonable conditions on the 2-generators) rewriting using only interchangers terminates.

- Normal forms for planar connected string diagrams, Delpeuch and Vicary, 2018

- Method for the operational rules:
  
  *Find a measure that is left unvariant by interchangers*

\[
\begin{align*}
  &x \quad y \\
  &\quad \downarrow \quad \downarrow \\
  &\quad \quad \quad 2x + y + 1
\end{align*} \quad \quad \quad \begin{align*}
  &x \quad y \quad z \\
  &\quad \downarrow \quad \downarrow \quad \downarrow \\
  &\quad \quad \quad 4x + 2y + z + 3 \quad \quad \Rightarrow \quad \quad \quad 2x + 2y + z + 2
\end{align*}
\]
Example of monoids

With monoids, we find five critical pairs
Example of monoids

With monoids, we find five critical pairs and they are confluent.
Example of monoids

With monoids, we find five critical pairs and they are confluent

We deduce constraints on $\equiv$ for coherence
Other examples

▶ Adjunctions

\[
P_1 = \{f, g : \ast \to \ast\}
\]

\[
P_2 = \{\cup, \cap\}
\]

\[
P_3 = \{\text{zig} : \bigcup \implies |, \quad \text{zag} : \bigcap \implies |\}
\]
Other examples

- Adjunctions
- Self-dualities

\[ P_1 = \{ f : \ast \to \ast \} \]
\[ P_2 = \{ \cup, \cap \} \]
\[ P_3 = \{ \text{zig}: \bigcup \Rightarrow \mid, \quad \text{zag}: \bigcap \Rightarrow \mid \} \]
Other examples

- Adjunctions
- Self-dualities
- Frobenius monoid

\[ P_2 = \{ \forall, \exists \} \]

\[ P_3 = \{ N: \begin{array}{c}
\exists \\
\forall 
\end{array} \Rightarrow \begin{array}{c}
\exists \\
\forall 
\end{array}, \\
A: \begin{array}{c}
\exists \\
\forall 
\end{array} \Rightarrow \begin{array}{c}
\exists \\
\forall 
\end{array}, \\
M: \begin{array}{c}
\exists \\
\forall 
\end{array} \Rightarrow \begin{array}{c}
\exists \\
\forall 
\end{array} \}
\]

\[ \forall: \begin{array}{c}
\exists \\
\forall 
\end{array} \Rightarrow \begin{array}{c}
\exists \\
\forall 
\end{array}, \\
A^c: \begin{array}{c}
\exists \\
\forall 
\end{array} \Rightarrow \begin{array}{c}
\exists \\
\forall 
\end{array}, \\
M^c: \begin{array}{c}
\exists \\
\forall 
\end{array} \Rightarrow \begin{array}{c}
\exists \\
\forall 
\end{array} \}
\]
Coherence relations

19 relations found by the algorithm
Coherence relations

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\end{array}
\end{align*}
\]
Coherence relations
Coherence relations
Coherence relations
Coherence relations
Other results

- A coherent approach to pseudomonads, Lack, 2000
- Coherence for Frobenius pseudomonoids and the geometry of linear proofs, Dunn and Vicary, 2016
- Coherence for braided and symmetric pseudomonoids, Verdon, 2017
Conclusion

- A rewriting system that reflects the structure of Gray categories
- Adapted tools to show coherence in this setting
- More automated method for coherence
  - Algorithm to compute the coherence conditions
- Proof of termination are still hard and tools should be developed