Coherence of Gray categories via rewriting

Simon Forest and Samuel Mimram

July 12\textsuperscript{th} 2018
Coherence

We want to show coherence properties:

\[ \text{all the ways to prove that two objects are equivalent are equal} \]

Think: MacLane’s coherence theorem

\[
\rho \quad \rho^-
\]

\[
\alpha^{-} \quad \alpha \quad \alpha^{-}
\]

**Coherence:** all morphisms made of \( \alpha, \lambda, \rho \) and their inverses between two objects are equal
Coherence

- Structural isomorphisms of a monoidal category

\[ \alpha : (A \otimes B) \otimes C \overset{\sim}{\to} A \otimes (B \otimes C) \]
\[ \lambda : (I \otimes A) \overset{\sim}{\to} A \]
\[ \rho : (A \otimes I) \overset{\sim}{\to} A \]

- These isos satisfy axioms that imply coherence

\[ ((A \otimes B) \otimes C) \otimes D \overset{\alpha}{\to} (A \otimes (B \otimes C)) \otimes D \overset{\alpha}{\to} A \otimes ((B \otimes C) \otimes D) \]
\[ (A \otimes B) \otimes (C \otimes D) \overset{\alpha}{\to} A \otimes (B \otimes (C \otimes D)) \]

\[ (A \otimes I) \otimes B \overset{\alpha}{\to} A \otimes (I \otimes B) \]

Idea: such coherence conditions can be obtained by orienting the isos and considering the associated rewriting system
Coherence from rewriting

- **Rewriting system**
  
  Get a rewriting system: choose a “good” orientation for the isos of the considered structure

\[
\alpha : \ (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)
\]

\[
\lambda : \ (I \otimes A) \rightarrow A
\]

\[
\rho : \ (A \otimes I) \rightarrow A
\]
Coherence from rewriting

- **Rewriting system**
  
  Get a rewriting system: choose a “good” orientation for the isos of the considered structure

  \[ \alpha : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C) \]
  
  \[ \lambda : (I \otimes A) \rightarrow A \]
  
  \[ \rho : (A \otimes I) \rightarrow A \]

  In particular, we want \( \rightarrow \) terminating
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent

\[ \forall (C_1, C_2) \text{ critical} \]

\[ \phi_1 \overset{*}{=} \phi_2 \]

\[ \psi \]

\[ \forall (R_1, R_2) \]

\[ \phi_1 \overset{*}{=} \phi_2 \]

\[ \psi \]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence

∀(R₁, R₂) rewrite steps

∀(R₁, R₂) rewrite paths
Coherence from rewriting

- Rewriting system
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  First case: paths to a normal form $\hat{\psi}$

\[
R_1 \left( \begin{array}{c} \phi \\ * \\ \hat{\psi} \\ * \\ \phi \end{array} \right) R_2
\]
Coherence from rewriting

- **Rewriting system**

- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent

- **Newman’s lemma**: → terminating and local confluence imply confluence

- **Coherence**
  First case: paths to a normal form $\hat{\psi}$

\[
\hat{\psi} = \quad \phi
\]

\[
\hat{\psi} \quad R_1 \quad \cdots \quad R_2 \quad \hat{\psi}
\]

by Newman’s lemma
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence
- **Coherence**
  
  First case: paths to a normal form \( \hat{\psi} \)

\[
R_1 \left( \begin{array}{c}
\phi \\
* \\
\hat{\psi}
\end{array} \right) \xRightarrow{=} \left( \begin{array}{c}
* \\
\hat{\psi}
\end{array} \right) R_2
\]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence
- **Coherence**
  
  Second case: paths to an arbitrary object \( \psi \)

![Diagram](image)
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**

  Second case: paths to an arbitrary object $\psi$
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  
  Second case: paths to an arbitrary object $\psi$

![Diagram showing the coherence relationship between $\phi$, $R_1$, $R_2$, $S$, and $\hat{\psi}$]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence
- **Coherence**

  Second case: paths to an arbitrary object \( \psi \)

\[
\begin{align*}
\phi & \quad \psi = \psi \\
R_1 & \quad R_2 \\
S & \quad \hat{\psi} \\
\psi & \quad S
\end{align*}
\]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**

  Second case: paths to an arbitrary object $\psi$

```
\[
\begin{array}{c}
R_1 & \phi & R_2 \\
\leftarrow & * & \rightarrow \\
\psi & = & \psi \\
& \leftarrow & \rightarrow \\
S & * & S \\
& \psi & \psi \\
& \downarrow & \\
S & \rightarrow & \psi
\end{array}
\]
```
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: \( \rightarrow \) terminating and local confluence imply confluence
- **Coherence**

  Second case: paths to an arbitrary object \( \psi \)

\[ \phi \]
\[ R_1 \quad = \quad R_2 \]
\[ * \quad \psi \]
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  Third case: paths with inverses ($\alpha^-, \lambda^- ...$)
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**
  Third case: paths with inverses \((\alpha^-, \lambda^- \ldots)\)
  → Analogous to the proof of the Church-Rosser lemma
Coherence from rewriting

- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent
- **Newman’s lemma**: → terminating and local confluence imply confluence
- **Coherence**

Axioms for coherence:

\[ \forall (C_1, C_2) \text{ critical} \]

\[ \phi_1 = \phi_2 \]

\[ C_1 \quad \phi \quad C_2 \]

\[ * \quad * \]

\[ \psi \]
Coherence of monoidal categories is a special case of the coherence of monoids in a 2-category
Algebraic structures in higher categories

- Coherence of monoidal categories is a special case of the coherence of monoids in a 2-category
- For \textit{strict}-categories, it is well-known how to do rewriting using \textit{polygraphs}
Algebraic structures in higher categories

- Coherence of monoidal categories is a special case of the coherence of monoids in a 2-category
- For strict-categories, it is well-known how to do rewriting using polygraphs
- What we would like: adapt these techniques and results to weak-categories
Coherence of monoidal categories is a special case of the coherence of monoids in a 2-category

For strict-categories, it is well-known how to do rewriting using polygraphs

What we would like: adapt these techniques and results to weak-categories

In dimension $n \geq 3$, weak categories are hard!
Coherence of monoidal categories is a special case of the coherence of monoids in a 2-category

For \textit{strict}-categories, it is well-known how to do rewriting using polygraphs

What we would like: adapt these techniques and results to \textit{weak}-categories

In dimension $n \geq 3$, weak categories are \textbf{hard}!

An easier step: semi-strict categories in dimension 3

\textbf{Gray categories}
Known results

- A coherent approach to pseudomonads, Lack, 2000
- Coherence for Frobenius pseudomonoids and the geometry of linear proofs, Dunn and Vicary, 2016
- Coherence for braided and symmetric pseudomonoids, Verdon, 2017
- ...
Summary of the work:

- reflect the properties of Gray categories in a rewriting system
- adapt the usual tools of rewriting theory to show coherence
- give some automation to find the coherence conditions
- apply it on examples
Rewriting in Gray setting

Critical branchings

Examples
Rewriting in Gray setting
Elements of a Gray category:

- 0-cells and 1-cells
- 2-cells:
  
- 3-cells:
Gray categories

- composition of 2-cells with identities on the left and the right

\[
\begin{array}{c}
\text{\hspace{2cm}}
\end{array}
\]

\[
\begin{array}{c}
| \quad | \quad | \quad |\quad *0 \quad | \quad | \quad |\quad = \quad | \quad | \quad | \quad | \quad | \quad |
\end{array}
\]

\[
\begin{array}{c}
| \quad | \quad | \quad |\quad *0 \quad | \quad | \quad |\quad = \quad | \quad | \quad | \quad | \quad | \quad |
\end{array}
\]
Gray categories

- composition of 2-cells with identities on the left and the right

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics{image1.png}} \ast_0 \text{\includegraphics{image2.png}} = \\
\text{\includegraphics{image3.png}} \ast_0 \text{\includegraphics{image4.png}} =
\end{array}
\end{align*}
\]

- composition: 2-cells can be composed vertically

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics{image5.png}} \ast_1 \text{\includegraphics{image6.png}} = \\
\text{\includegraphics{image7.png}} \ast_1 
\end{array}
\end{align*}
\]
Gray categories

- composition of 2-cells with identities on the left and the right
  
  $\begin{array}{c}
  | | | *0 \quad \quad = \quad | | | \\
  \hline
  \end{array}$

  $\begin{array}{c}
  \hline
  | | | *0 \quad = \quad | | | \\
  \hline
  \end{array}$

- composition: 2-cells can be composed vertically

  $\begin{array}{c}
  \hline
  \quad *1 \quad = \quad \hline \\
  \end{array}$

- 3-cells can be composed horizontally

  $(\begin{array}{c}
  \hline
  \alpha \Rightarrow \beta \\
  \end{array}) *_2 (\begin{array}{c}
  \hline
  \beta \Rightarrow \gamma \\
  \end{array}) = (\begin{array}{c}
  \hline
  \alpha \Rightarrow \gamma \\
  \end{array})$
Gray categories

- properties of associativity and unitality

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \gamma
\end{align*}
\* 1

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \gamma
\end{align*}
\* 1

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \gamma
\end{align*}
\* 1

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \gamma
\end{align*}
\* 1

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \gamma
\end{align*}
\* 1

\[
\begin{align*}
\alpha & \\
\alpha
\end{align*}
\]

... but no exchange law!
Gray categories

- properties of associativity and unitality

\[ \alpha \beta \gamma = \alpha \beta \gamma = \alpha \beta \gamma \]

... but no exchange law!

\[ \alpha \beta \gamma \neq \beta \alpha \gamma \]
Gray categories

- properties of associativity and unitality

\[
\begin{align*}
\alpha & \quad \beta \\
\gamma & \\
\ast 1 & = \\
\beta & \quad \gamma \\
\gamma & \\
\ast 1 & = \\
\alpha & \quad \beta \\
\beta & \quad \gamma \\
\gamma & \\
\ast 1 & = \\
\alpha & \\
\end{align*}
\]

- ... but no exchange law!

- instead, invertible 3-cell
Signatures

A signature $S$ is given by:

- a set of elementary 2-dimensional diagrams called 2-generators
  \[
  \{ \begin{array}{c}
  \triangledown, \circ
  \end{array} \}
  \]

- some typing information about the source and target of these diagrams
Terms

- **slice**: a 2-generators with identities on the left and the right

  ![Diagram of a slice]

- **terms** (or **2-cells**): a sequence of composable slices

  ![Diagram of terms]

In particular, in this formalism, the following cell does not exist

![Diagram of a non-existing cell]

Because there is only one 2-generator per slice
A *rewriting system* is given by:

- a signature $S$
- a set $P$ of rewriting rules (called 3-generators) on the terms of the signature

\[
\begin{align*}
A : & \quad \quad \quad \quad \quad \quad \quad \quad \quad \equiv \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \\
L : & \quad \quad \quad \quad \quad \quad \quad \quad \quad \equiv \quad \quad \quad \quad \quad \quad \quad \quad \quad \\
R : & \quad \quad \quad \quad \quad \quad \quad \quad \quad \equiv \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{align*}
\]
Rewriting step: a rewriting rule in a context

- identities on the left and the right
- 2-cells above and below

Start from a rewriting rule, say:

\[ A : \quad \begin{array}{c} \downarrow \\downarrow \\
\end{array} \quad \Rightarrow \quad \begin{array}{c} \downarrow \\downarrow \\
\end{array} \]
**Rewriting step**: a rewriting rule in a context

- identities on the left and the right
- 2-cells above and below

Put it inside a context:
Coherence

- *Rewriting path*: a sequence of rewriting steps

- *Rewriting zigzag*: a sequence of rewriting steps or inverse rewriting steps

- *Coherence property*: between two 2-cells, at most one zigzag up to $\equiv$
Coherence

- **Rewriting path**: a sequence of rewriting steps

- **Rewriting zigzag**: a sequence of rewriting steps or inverse rewriting steps
Coherence

- **Rewriting path**: a sequence of rewriting steps

  ![Rewriting path diagram]

- **Rewriting zigzag**: a sequence of rewriting steps or inverse rewriting steps

- Let $\equiv$ a *congruence* on the zigzags
Coherence

- **Rewriting path**: a sequence of rewriting steps

  \[
  \begin{array}{c}
  \Rightarrow \\
  \Rightarrow \\
  \Rightarrow \\
  \Rightarrow \\
  \end{array}
  \]

- **Rewriting zigzag**: a sequence of rewriting steps or inverse rewriting steps

- Let $\equiv$ a *congruence* on the zigzags

- **Coherence property**: between two 2-cells, at most one zigzag up to $\equiv$
Gray rewriting

- Goal: reflect the structure of Gray category in rewriting
Gray rewriting

- **Goal:** reflect the structure of Gray category in rewriting
- More precisely: give $P$ and $\equiv$ that will **present** a Gray category
Gray rewriting

- **Goal:** reflect the structure of Gray category in rewriting
- More precisely: give $P$ and $\equiv$ that will **present** a Gray category
- For this purpose:
  - interchangers
  - parallels paths
  - naturally equivalent paths
  - inverses
Interchangers

Let $S$ a **signature** and $\alpha, \beta \in S$ and $u$ a sequence of identities.
Interchangers

Let $S$ a signature and $\alpha, \beta \in S$ and $u$ a sequence of identities.

Gray-cats induce interchanger $X_{\alpha,u,\beta}$: rewriting rule that exchanges $\alpha$ and $\beta$ when separated by $u$.

$X_{m,3,e} : \quad \Rightarrow \quad $
Interchangers

- Let $S$ a **signature** and $\alpha, \beta \in S$ and $u$ a sequence of identities
- Gray-cats induce **interchanger** $X_{\alpha, u, \beta}$: rewriting rule that exchanges $\alpha$ and $\beta$ when separated by $u$

\[
X_{m, \bar{3}, e} : \quad \Rightarrow
\]

- Nice, because we had branchings that could not be closed
Interchangers

- Let $S$ a **signature** and $\alpha, \beta \in S$ and $u$ a sequence of identities
- Gray-cats induce **interchanger** $X_{\alpha, u, \beta}$: rewriting rule that exchanges $\alpha$ and $\beta$ when separated by $u$

\[
X_{m, 3, e} : \quad \bullet \quad \Rightarrow \quad \bullet
\]

- Nice, because we had branchings that could not be closed

- From now on, interchangers are allowed rewriting steps
Parallel paths

Consider the following two paths:

\[ A \rightarrow A \]

and

\[ A \rightarrow A \]
Parallel paths

Consider the following two paths:

and

Parallel paths: the two paths obtained by applying two rules at independent positions
Parallel paths

Consider the following two paths:

\[ \begin{array}{c}
\text{A} \\
\Rightarrow \\
\text{A} \\
\Rightarrow \\
\end{array} \]

and

\[ \begin{array}{c}
\text{A} \\
\Rightarrow \\
\text{A} \\
\Rightarrow \\
\end{array} \]

Parallel paths: the two paths obtained by applying two rules at independent positions

In a Gray-cat, two parallel paths are equal
Parallel paths

- Consider the following two paths:

- Parallel paths: the two paths obtained by applying two rules at independent positions
- In a Gray-cat, two parallel paths are equal
- Nice because for coherence, these two paths need to be $\equiv$-equivalent
Consider the two following rewriting paths:

First path: "move down the unit" then A rule

Second path: A rule then "move down the unit"

These two paths are naturally-equivalent.

In a Gray-cat, two naturally-equivalent paths are equal.

Nice because for coherence, these two paths need to be equivalent.
Naturally-equivalent paths

Consider the two following rewriting paths:

First path: “move down the unit” then \( A \) rule
Naturally-equivalent paths

Consider the two following rewriting paths:

First path: “move down the unit” then $A$ rule
Second path: $A$ rule then “move down the unit”
Naturally-equivalent paths

Consider the two following rewriting paths:

First path: “move down the unit” then $A$ rule

Second path: $A$ rule then “move down the unit”

These two paths are naturally-equivalent
Consider the two following rewriting paths:

First path: “move down the unit” then A rule
Second path: A rule then “move down the unit”

These two paths are naturally-equivalent

In a Gray-cat, two naturally-equivalent paths are equal
Consider the two following rewriting paths:

- First path: “move down the unit” then $A$ rule
- Second path: $A$ rule then “move down the unit”

These two paths are naturally-equivalent

In a Gray-cat, two naturally-equivalent paths are equal

Nice because for coherence, these two paths need to be $\equiv$-equivalent
Recall that we want to consider structures with invertible 3-cells \((\alpha^-, \lambda^-, \ldots)\).
Inverses

- Recall that we want to consider structures with invertible 3-cells ($\alpha^-, \lambda^-, ...$)
- If $R : \alpha \Rightarrow \beta \in P$ a rewriting rule, denote $R^- : \beta \Rightarrow \alpha$ the **formal inverse**
Inverses

- Recall that we want to consider structures with invertible 3-cells \((\alpha^-, \lambda^-, \ldots)\)
- If \(R : \alpha \Rightarrow \beta \in P\) a rewriting rule, denote \(R^- : \beta \Rightarrow \alpha\) the **formal inverse**
- As an example, for monoids

\[
A^- : \begin{array}{c}
\text{Diagram 1} \\
\end{array} \Rightarrow \begin{array}{c}
\text{Diagram 2} \\
\end{array}
\]
Inverses

- Recall that we want to consider structures with invertible 3-cells \((\alpha^-, \lambda^-, \ldots)\)
- If \(R : \alpha \Rightarrow \beta \in \mathcal{P}\) a rewriting rule, denote \(R^- : \beta \Rightarrow \alpha\) the formal inverse
- As an example, for monoids
  \[
  A^- : \quad \Rightarrow \quad \\
  \]
- For coherence, equations like \(A \ast A^- \equiv 1_\alpha\) are needed
Inverses

- Recall that we want to consider structures with invertible 3-cells ($\alpha^-, \lambda^-, ...$)
- If $R : \alpha \Rightarrow \beta \in P$ a rewriting rule, denote $R^- : \beta \Rightarrow \alpha$ the **formal inverse**
- As an example, for monoids

\[
A^- : \begin{array}{c}
\text{Diagram 1}
\end{array} \Rightarrow \begin{array}{c}
\text{Diagram 2}
\end{array}
\]

- For coherence, equations like $A \ast A^- \equiv 1_\alpha$ are needed
- Nice: in a Gray-cat, these equations hold already
Gray presentation

- Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)

Coherence problem: what other axioms on $\equiv$ for the coherence property to hold?

Solution: squares given by "critical branchings"
Gray presentation

- Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)
- Let $\equiv$ a congruence on the zigzags (paths with inverses) such that

\[ R^{\ast} R^{-} = 1, \quad R^{-} R^{\ast} = 1 \]

Coherence problem: what other axioms on $\equiv$ for the coherence property to hold?

- Solution: squares given by "critical branchings"
Gray presentation

- Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)
- Let $\equiv$ a congruence on the zigzags (paths with inverses) such that
  - if $P_1, P_2$ parallel paths, then $P_1 \equiv P_2$
Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)

Let $\equiv$ a congruence on the zigzags (paths with inverses) such that

- if $P_1, P_2$ parallel paths, then $P_1 \equiv P_2$
- if $P_1, P_2$ naturally-equivalent, then $P_1 \equiv P_2$
Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)

Let $\equiv$ a congruence on the zigzags (paths with inverses) such that

- if $P_1, P_2$ parallel paths, then $P_1 \equiv P_2$
- if $P_1, P_2$ naturally-equivalent, then $P_1 \equiv P_2$
- $R \ast R^{-} \equiv 1$, $R^{-} \ast R \equiv 1$
Gray presentation

- Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)
- Let $\equiv$ a congruence on the zigzags (paths with inverses) such that
  - if $P_1, P_2$ parallel paths, then $P_1 \equiv P_2$
  - if $P_1, P_2$ naturally-equivalent, then $P_1 \equiv P_2$
  - $R \ast R^- \equiv 1$, $R^- \ast R \equiv 1$

**Theorem:** the set of zigzag quotiented by $\equiv$ induces canonically a Gray category.
Gray presentation

Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)

Let $\equiv$ a congruence on the zigzags (paths with inverses) such that

1. if $P_1, P_2$ parallel paths, then $P_1 \equiv P_2$
2. if $P_1, P_2$ naturally-equivalent, then $P_1 \equiv P_2$
3. $R \ast R^\perp \equiv 1$, $R^\perp \ast R \equiv 1$

Theorem: the set of zigzag quotiented by $\equiv$ induces canonically a Gray category.

Coherence problem: what other axioms on $\equiv$ for the coherence property

*If $Z_1, Z_2$ are rewriting zigzags between $\phi$ and $\psi$, then $Z_1 \equiv Z_2$ to hold?*
Gray presentation

▶ Let $S$ a signature and $P$ a set of rewriting rules (with interchangers)
▶ Let $\equiv$ a congruence on the zigzags (paths with inverses) such that
  ▶ if $P_1$, $P_2$ parallel paths, then $P_1 \equiv P_2$
  ▶ if $P_1$, $P_2$ naturally-equivalent, then $P_1 \equiv P_2$
  ▶ $R \ast R^- \equiv 1$, $R^- \ast R \equiv 1$

**Theorem:** the set of zigzag quotiented by $\equiv$ induces canonically a Gray category.

▶ **Coherence problem:** what other axioms on $\equiv$ for the coherence property

  *If $Z_1$, $Z_2$ are rewriting zigzags between $\phi$ and $\psi$, then $Z_1 \equiv Z_2$* to hold?

▶ Solution: squares given by “critical branchings”
Critical branchings
Critical branchings

Let $P_1 : \phi \Rightarrow \psi_1$, $P_2 : \phi \Rightarrow \psi_2$ a local branching:

- it is *trivial* when $P_1 = P_2$
Critical branchings

Let $P_1 : \phi \Rightarrow \psi_1$, $P_2 : \phi \Rightarrow \psi_2$ a local branching:

- it is \textit{trivial} when $P_1 = P_2$
- it is \textit{non-minimal} when a smaller context can be found

$\rightarrow$ it is \textit{independent} when $P_1$ and $P_2$ act on different parts of $\phi$

$\rightarrow$ it is \textit{natural} when $P_1$ and $P_2$ are the first steps of two naturally equivalent paths

$\rightarrow$ it is \textit{critical} when none of the above ones
Critical branchings

Let $P_1 : \phi \Rightarrow \psi_1$, $P_2 : \phi \Rightarrow \psi_2$ a local branching:

- it is \textit{trivial} when $P_1 = P_2$
- it is \textit{non-minimal} when a smaller context can be found
- it is \textit{independent} when $P_1$ and $P_2$ act on different parts of $\phi$
- it is \textit{natural} when $P_1$ and $P_2$ are the first steps of two naturally equivalent paths
- it is \textit{critical} when none of the above ones
Critical branchings

Let $P_1 : \phi \Rightarrow \psi_1$, $P_2 : \phi \Rightarrow \psi_2$ a local branching:

- it is *trivial* when $P_1 = P_2$
- it is *non-minimal* when a smaller context can be found
- it is *independent* when $P_1$ and $P_2$ act on different parts of $\phi$
- it is *natural* when $P_1$ and $P_2$ are the first steps of two naturally equivalent paths
Critical branchings

Let $P_1: \phi \Rightarrow \psi_1$, $P_2: \phi \Rightarrow \psi_2$ a local branching:

- it is *trivial* when $P_1 = P_2$
- it is *non-minimal* when a smaller context can be found
- it is *independent* when $P_1$ and $P_2$ act on different parts of $\phi$
- it is *natural* when $P_1$ and $P_2$ are the first steps of two naturally equivalent paths
- it is *critical* when none of the above ones
Critical branchings

Let $P_1 : \phi \Rightarrow \psi_1$, $P_2 : \phi \Rightarrow \psi_2$ a local branching:

- it is *trivial* when $P_1 = P_2$
- it is *non-minimal* when a smaller context can be found
- it is *independent* when $P_1$ and $P_2$ act on different parts of $\phi$
- it is *natural* when $P_1$ and $P_2$ are the first steps of two naturally equivalent paths
- it is *critical* when none of the above ones

**Theorem** (Critical pair lemma): if critical branchings are confluent then all local branchings are confluent
Finite number of critical pairs

- There is an infinite number of interchangers

\[ X_{m,3,e}, \quad X_{m,4,e}, \quad \ldots, \quad X_{m,n,e} \text{ for all } n \]
Finite number of critical pairs

► There is an infinite number of interchangers

\[
\begin{align*}
\triangledown & \quad \Rightarrow \quad \triangledown \\
X_{m,3,e} & \quad \Rightarrow \quad X_{m,4,e} \\
\end{align*}
\]

\[
X_{m,\bar{n},e} \text{ for all } n
\]

► So potentially an infinite number of critical branchings
Finite number of critical pairs

- There is an infinite number of interchangers

\[ X_{m,3,e} \quad X_{m,4,e} \quad X_{m,n,e} \text{ for all } n \]

- So potentially an infinite number of critical branchings

- In fact, no!

**Theorem:** A finite number of operational rules (and ...) gives a finite number of critical branchings.

(operational = that are not interchangers)
Finite number of critical pairs

- There is an infinite number of interchangers

\[ \text{...} \]

\[ \Rightarrow \]

\[ \text{...} \]

\[ X_{m, \bar{3}, e} \]

\[ \Rightarrow \]

\[ X_{m, \bar{4}, e} \]

\[ \text{...} \]

\[ X_{m, \bar{n}, e} \text{ for all } n \]

- So potentially an infinite number of critical branchings

- In fact, no!

**Theorem:** A finite number of operational rules (and ...) gives a finite number of critical branchings. (operational = that are not interchangers)

- Concerning computability

  An algorithm exists to compute the critical branchings
Why finiteness?

Three kinds of branchings:

- between two operational rules
  - finite number of operational rules implies finite number of critical branchings of this kind

- between an operational rule and an interchanger
  - for $n$ big enough, branchings with an operational rule and $\alpha, \beta, n$ can not be critical

- between two interchangers
  - they are never critical and are usually “natural branchings”
Why finiteness?

Three kinds of branchings:

- between two operational rules
  - finite number of operational rules implies finite number of critical branchings of this kind
- between an operational rule and an interchanger
  - for $n$ big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ can not be critical
- between two interchangers
  - they are never critical and are usually "natural branchings"
Why finiteness?

Three kinds of branchings:

- between two operational rules
  - finite number of operational rules implies finite number of critical branchings of this kind
- between an operational rule and an interchanger
  - for $n$ big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ cannot be critical
- between two interchangers
  - they are never critical and are usually “natural branchings”
Why finiteness?

Three kinds of branchings:

- between two operational rules
  - finite number of operational rules implies finite number of critical branchings of this kind
- between an operational rule and an interchanger
  - for $n$ big enough, branchings with an operational rule and $X_{\alpha,n,\beta}$ can not be critical
- between two interchangers
  - they are never critical and are usually “natural branchings”
Examples
**Summing up**

Method to show coherence

- Start from an algebraic structure
Summing up

Method to show coherence

- Start from an algebraic structure
- Orient the isos to get a rewriting system

Theorem: if the critical branchings are confluent, then the structure is coherent

∀ (∈, \in, =, ψ)

∀ (∈, \in, =, ψ)

∀ (∈, \in, =, ψ)
Summing up

Method to show coherence

- Start from an algebraic structure
- Orient the isos to get a rewriting system
- Show that it is terminating

Theorem: if the critical branchings are confluent, then the structure is coherent

∀ (C₁, C₂) critical
φ₁ = φ₂
ψ₁ = ψ₂
C₁ ∗ C₁ = C₂ ∗ C₂
Summing up

Method to show coherence

- Start from an algebraic structure
- Orient the isos to get a rewriting system
- Show that it is terminating
- Find the critical branchings (an algorithm exists)

**Theorem:** if the critical branchings are confluent, then the structure is coherent

\[ \forall (C_1, C_2) \text{ critical} \]

\[ \phi_1 \equiv \phi_2 \]

\[ \psi \]
Termination

Termination of $\Rightarrow$:  
- Taking into account operational rules and interchangers
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
- We can reduce the problem to operational rules

**Theorem**: (under reasonable conditions on the 2-generators) rewriting using only interchangers terminates.
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
- We can reduce the problem to operational rules

**Theorem**: (under reasonable conditions on the 2-generators) rewriting using only interchangers terminates.

- *Normal forms for planar connected string diagrams*, Delpeuch and Vicary, 2018
Termination

Termination of $\Rightarrow$:

- Taking into account operational rules and interchangers
- We can reduce the problem to operational rules

**Theorem**: (under reasonable conditions on the 2-generators) rewriting using only interchangers terminates.

- *Normal forms for planar connected string diagrams*, Delpeuch and Vicary, 2018

- Method for the operational rules:
  *Find a measure that is left unvariant by interchangers*
Example of monoids

With monoids, we find five critical pairs

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 1} \\
\Rightarrow \\
\Rightarrow \\
\Downarrow \\
\Rightarrow \\
\end{array}
\end{align*}
\]
Example of monoids

With monoids, we find five critical pairs and they are confluent.
Example of monoids

With monoids, we find five critical pairs and they are confluent

We deduce constraints on $\equiv$ for coherence
Other examples

- **Adjunctions**
  - **Signature**
    \[ S = \{ \cup, \cap \} \]
  - **Rules**
    \[ P = \{ zig : \cup \Rightarrow \cap, zag : \cap \Rightarrow \cup \} \]

- **Self-dualities**
  - **Signature**
    \[ S = \{ \cup, \cap \} \]
  - **Rules**
    \[ P = \{ zig : \cup \Rightarrow \cap, zag : \cap \Rightarrow \cup \} \]

- **Frobenius monoid**
Frobenius monoid (without units)

Signature

Rules
Coherence relations

19 relations found by the algorithm
Coherence relations

\[ \text{Diagram showing coherence relations.} \]
Coherence relations
Coherence relations
Coherence relations
Coherence relations
Conclusion

- A rewriting system that reflects the structure of Gray categories
- Adapted tools to show coherence in this setting
- More automated method for coherence
  - Algorithm to compute the coherence conditions
- Another proof of the coherence of monoids
- Coherence of other examples