Coherence of Gray categories via rewriting

Simon Forest and Samuel Mimram

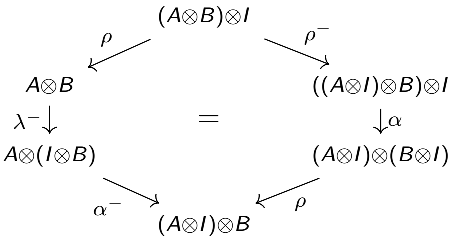
July 12th 2018

Coherence

We want to show coherence properties:

all the ways to prove that two objects are equivalent are equal

Think: MacLane's coherence theorem



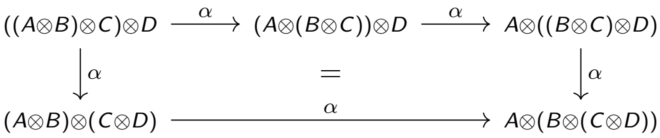
Coherence: all morphisms made of α, λ, ρ **and their inverses** between two objects are equal

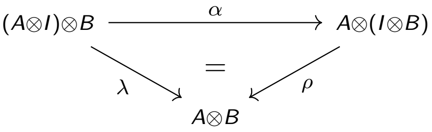
Coherence

- Structural isomorphisms of a monoidal category

$$
\begin{array}{rccc}
\alpha : & (A \otimes B) \otimes C & \xrightarrow{\sim} & A \otimes (B \otimes C) \\
\lambda : & (I \otimes A) & \xrightarrow{\sim} & A \\
\rho : & (A \otimes I) & \xrightarrow{\sim} & A
\end{array}
$$

- These isos satisfy axioms that imply coherence





Idea: such coherence conditions can be obtained by orienting the isos and considering the associated rewriting system

Coherence from rewriting

- **Rewriting system**
  Get a rewriting system: choose a "good" orientation for the isos of the considered structure

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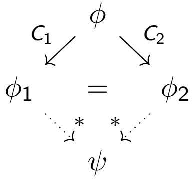
  In particular, we want $\to$ terminating

Coherence from rewriting

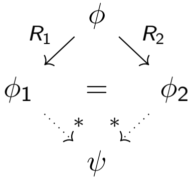
- **Rewriting system**
- **Critical pair lemma**: if critical branchings are confluent, then all local branchings are confluent

$\forall (C_1, C_2)$ critical



then

$\forall (R_1, R_2)$



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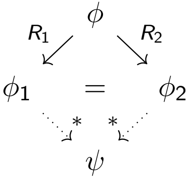

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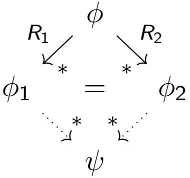

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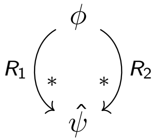

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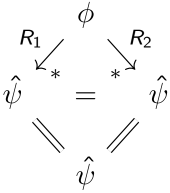

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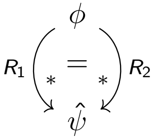

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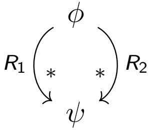

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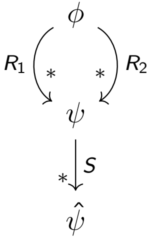

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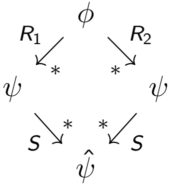

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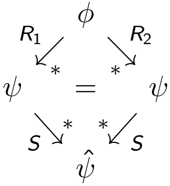

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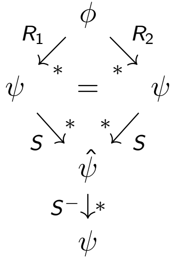

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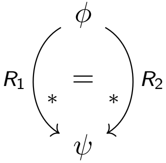

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  $\rightarrow$ Analogous to the proof of the Church-Rosser lemma

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Axioms for coherence:

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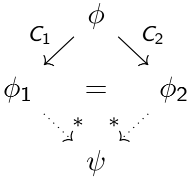

Algebraic structures in higher categories

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Algebraic structures in higher categories

- Coherence of monoidal categories is a special case of the coherence of monoids in a 2-category
- For *strict*-categories, it is well-known how to do rewriting using polygraphs
- What we would like: adapt these techniques and results to *weak*-categories
- In dimension $n \geq 3$, weak categories are **hard** !
- An easier step: semi-strict categories in dimension 3

Gray categories

Known results

- *A coherent approach to pseudomonads*, Lack, 2000
- *Coherence for Frobenius pseudomonoids and the geometry of linear proofs*,Dunn and Vicary, 2016
- *Coherence for braided and symmetric pseudomonoids*, Verdon, 2017
- ...

This work

Summary of the work:

- reflect the properties of Gray categories in a rewriting system
- adapt the usual tools of rewriting theory to show coherence
- give some automation to find the coherence conditions
- apply it on examples

Rewriting in Gray setting

Critical branchings

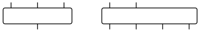
Examples

Rewriting in Gray setting

Gray categories

Elements of a Gray category:

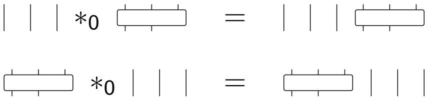
- 0-cells and 1-cells
- 2-cells:



- 3-cells:

Gray categories

- composition of 2-cells with identities on the left and the right



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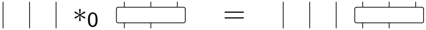
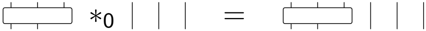
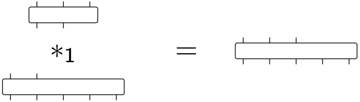




- composition: 2-cells can be composed vertically



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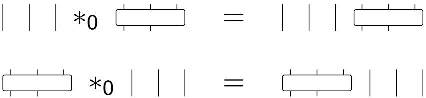

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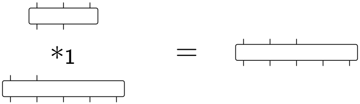
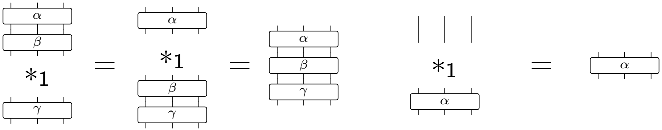


- 3-cells can be composed horizontally

Gray categories

- properties of associativity and unitality



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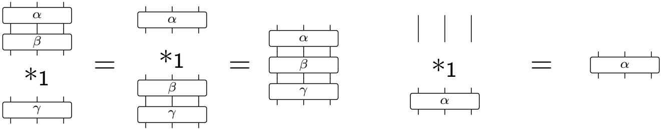

- … but no exchange law !

Gray categories

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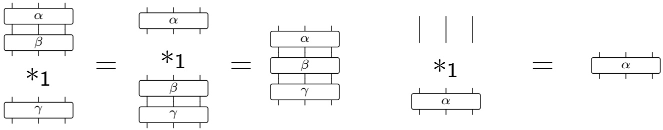
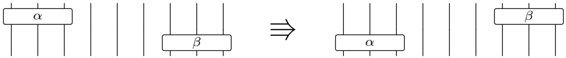


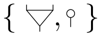
- … but no exchange law !
- instead, invertible 3-cell



Signatures

A signature S is given by:

- a set of elementary 2-dimensional diagrams called 2-generators

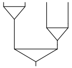


- some typing information about the source and target of these diagrams

Terms

- **slice**: a 2-generators with identities on the left and the right

- **terms** (or 2-**cells**): a sequence of composable slices

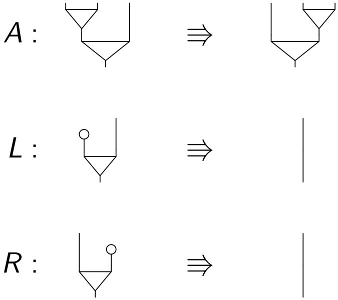


in particular, in this formalism, the following cell does not exist

because there is only one 2-generator per slice

Rewriting system

- A *rewriting system* is given by:
  - a signature S
  - a set P of rewriting rules (called 3-generators) on the terms of the signature

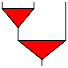
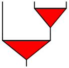


Rewriting step

Rewriting step: a rewriting rule in a context

- identities on the left and the right
- 2-cells above and below

Start from a rewriting rule, say:

$A :$  $\Rrightarrow$ 

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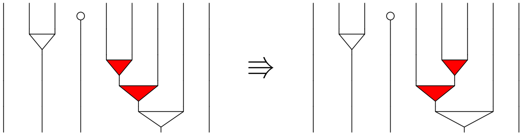
Put it inside a context:



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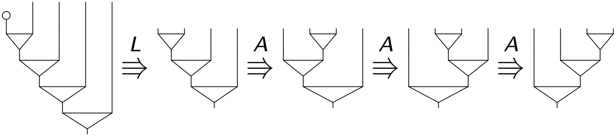
- *Rewriting path*: a sequence of rewriting steps



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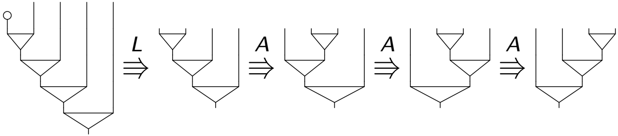

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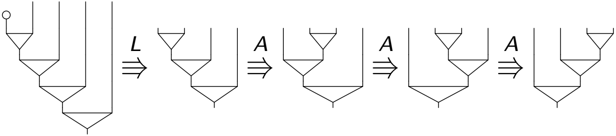

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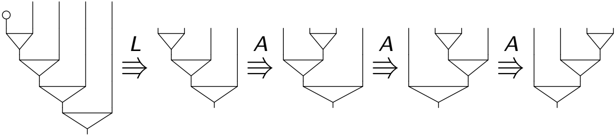
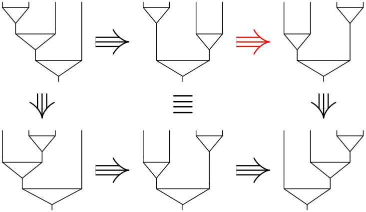


- *Rewriting zigzag*: a sequence of rewriting steps or inverse rewriting steps
- Let $\equiv$ a *congruence* on the zigzags
- **Coherence property**: between two 2-cells, at most one zigzag up to $\equiv$



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- More precisely: give P and $\equiv$ that will **present** a Gray category
- For this purpose:
  - interchangers
  - parallels paths
  - naturally equivalent paths
  - inverses

Interchangers

- Let S a **signature** and $\alpha, \beta \in$ S and *u* a sequence of identities

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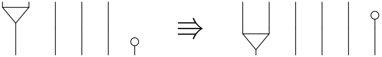
- Gray-cats induce **interchanger** $X_{\alpha,u,\beta}$: rewriting rule that exchanges α and β when separated by u

$X_{m,\bar{3},e}$:



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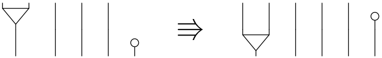
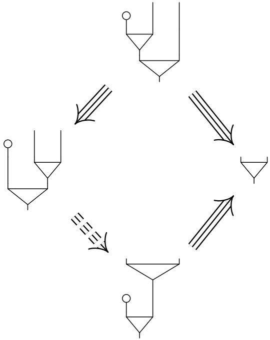


- Nice, because we had branchings that could not be closed



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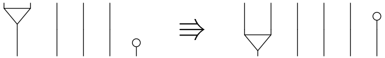

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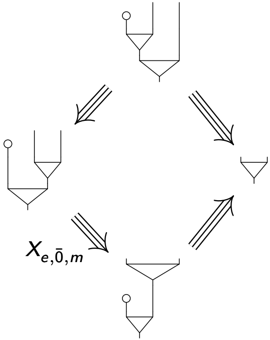
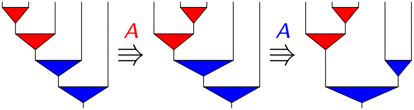


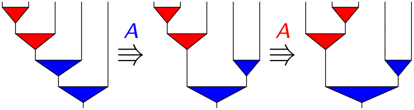
- From now on, interchangers are allowed rewriting steps

Parallel paths

- Consider the following two paths:



and



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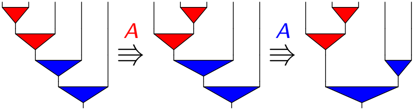

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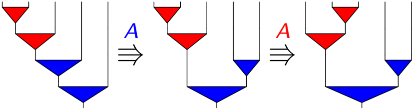

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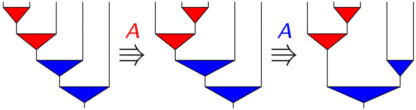

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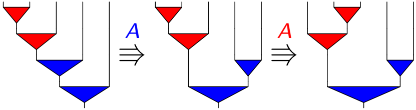

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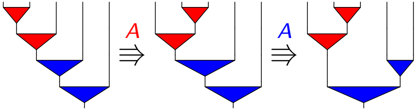

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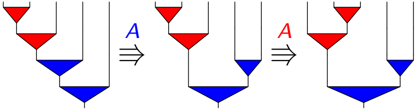

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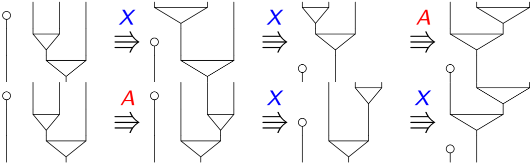

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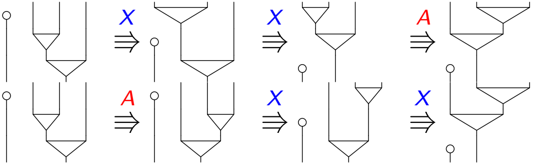

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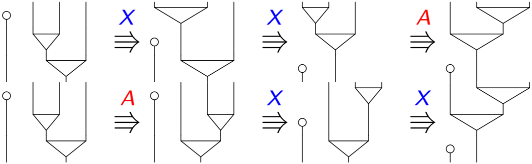

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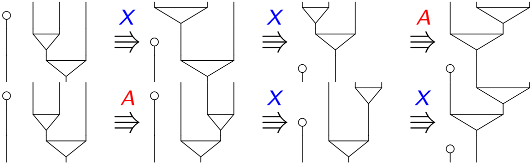

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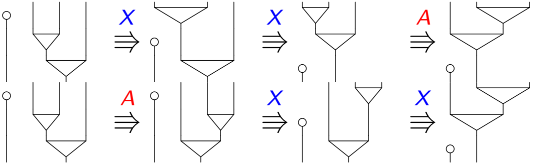

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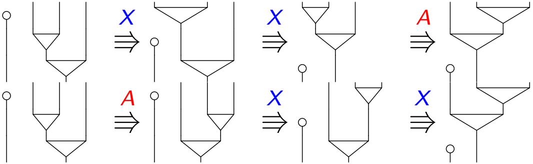

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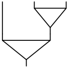
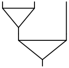
A^- :  $\Rrightarrow$ 

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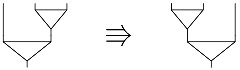
A^- : 

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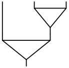
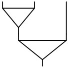
A^- :  $\Rrightarrow$ 

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- Solution: squares given by “critical branchings”

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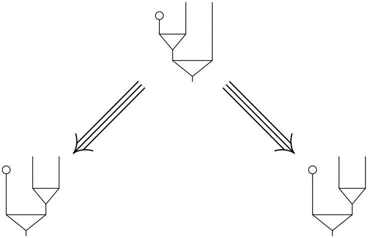

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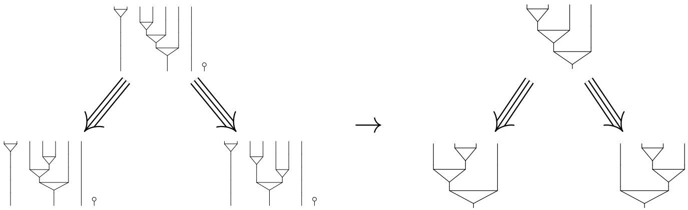

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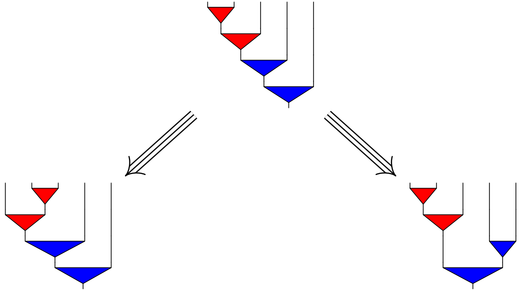

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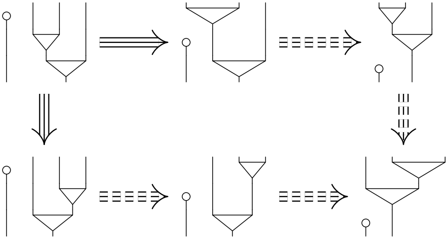

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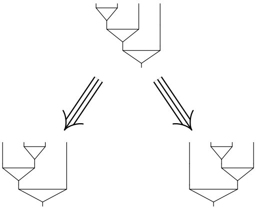

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Theorem(Critical pair lemma): if critical branchings are confluent then all local branchings are confluent

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- There is an infinite number of interchangers

$X_{m,\bar{3},e}$ $X_{m,\bar{4},e}$ …

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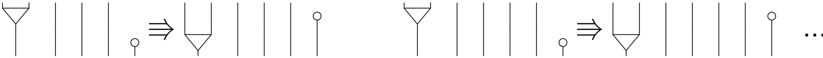

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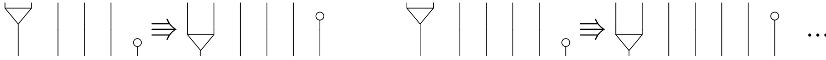

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- Concerning computability

  An algorithm exists to compute the critical branchings

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- between two operational rules
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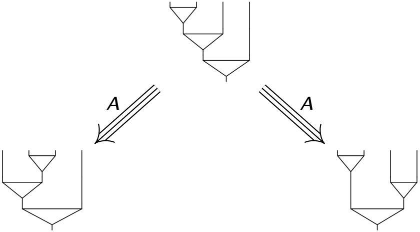

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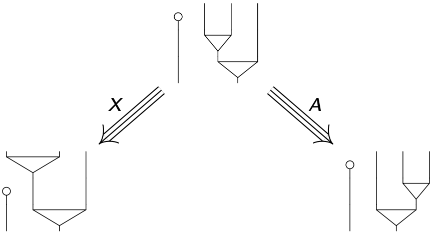

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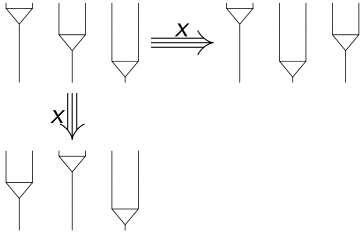

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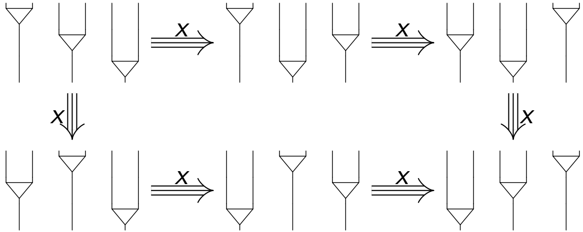

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Method to show coherence

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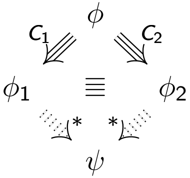
- Start from an algebraic structure
- Orient the isos to get a rewriting system
- Show that it is terminating
- Find the critical branchings (an algorithm exists)

Theorem: if the critical branchings are confluent, then the structure is coherent

$\forall(C_1, C_2)$ critical



Termination

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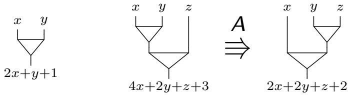

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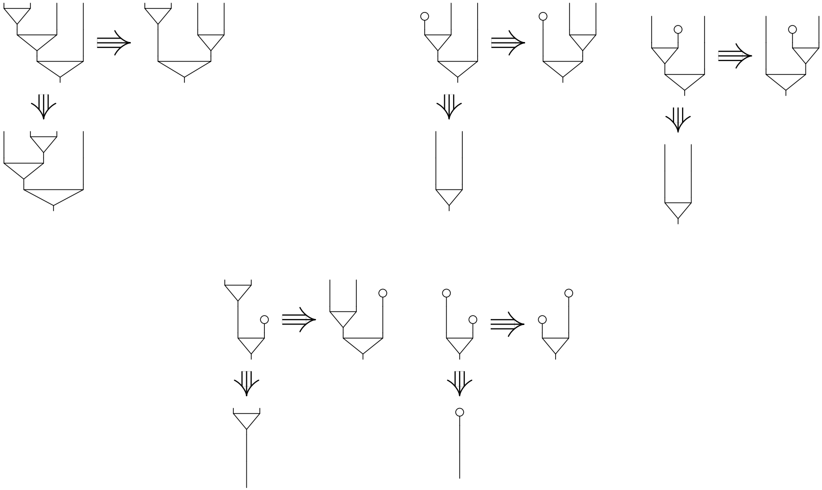
With monoids, we find five critical pairs



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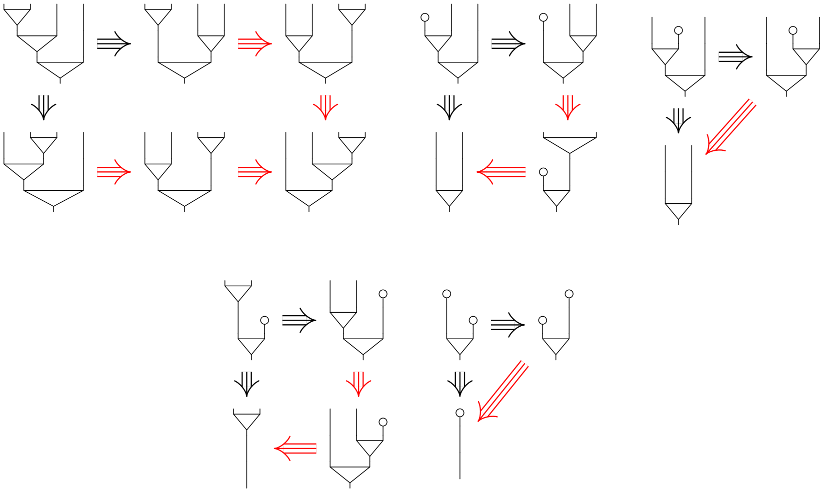

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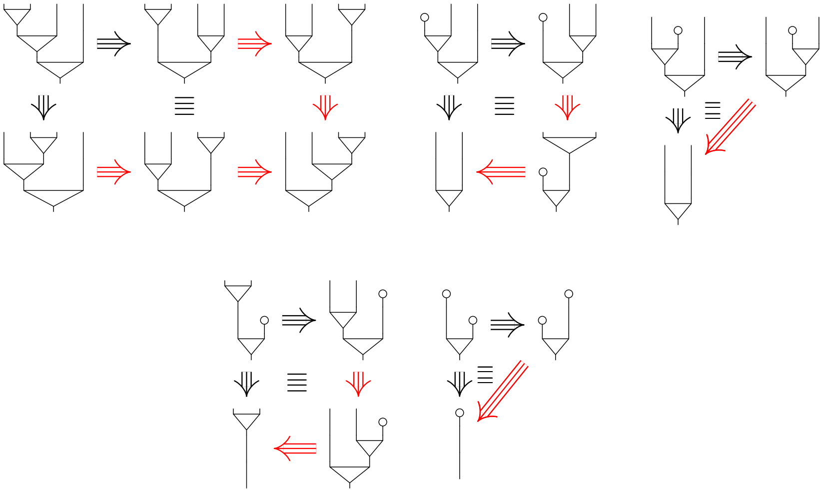


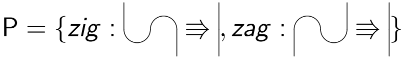
We deduce constraints on ≡ for coherence

Other examples

- Adjunctions
  - Signature

$$S = \{\cup, \cap\}$$

  - Rules



- Self-dualities
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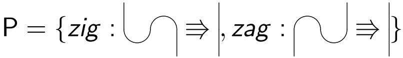
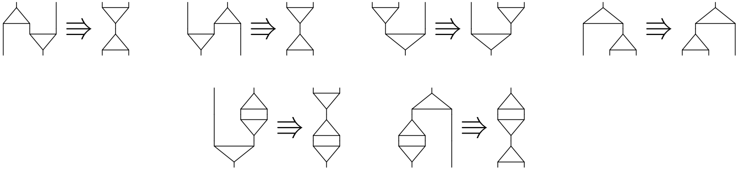


- Frobenius monoid

Frobenius monoid (without units)

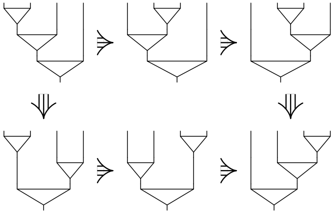
Signature

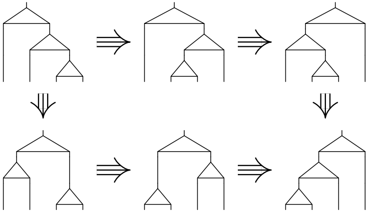
Rules

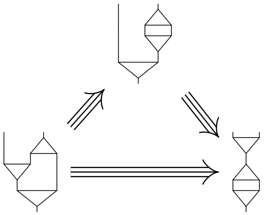


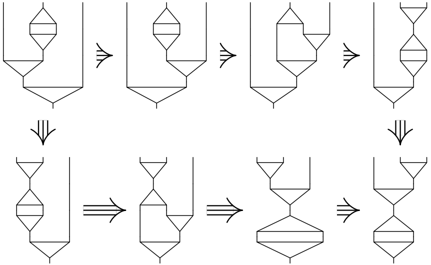
Coherence relations

19 relations found by the algorithm

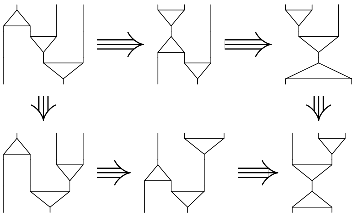


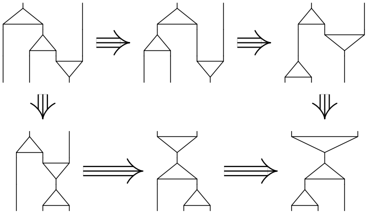


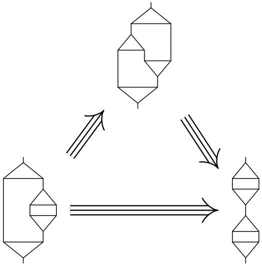




Coherence relations







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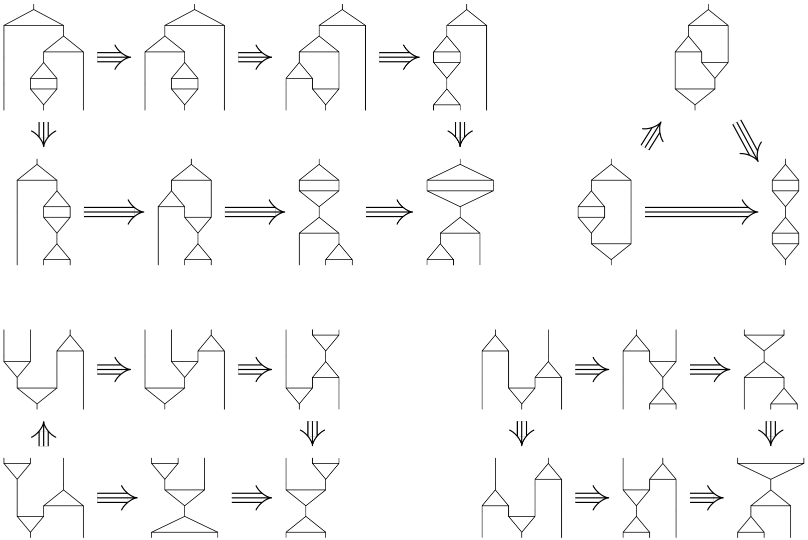

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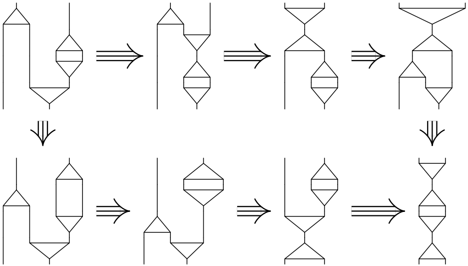

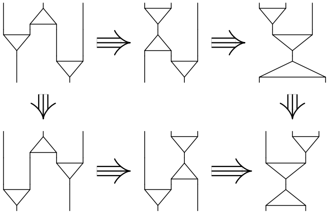

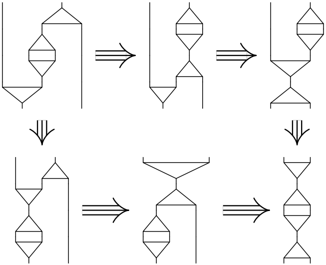

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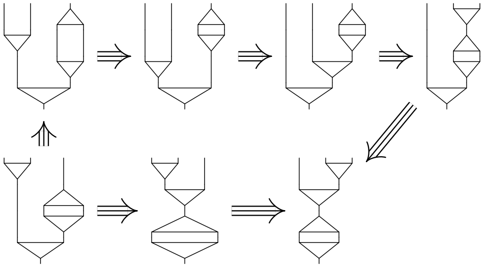

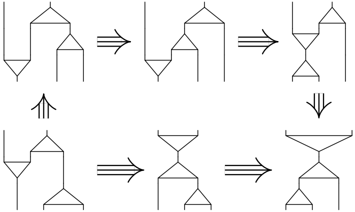

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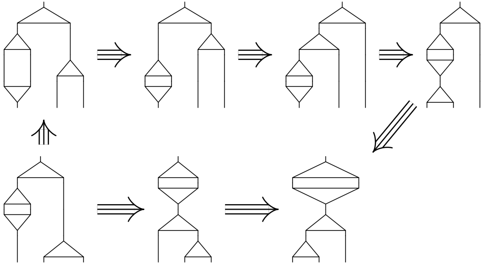

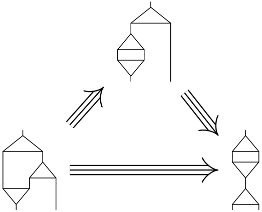

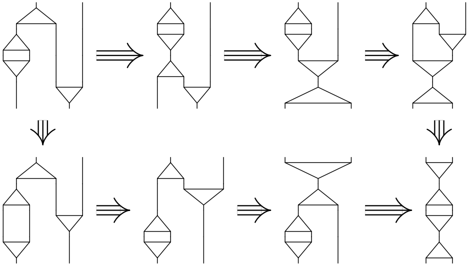

Conclusion

- A rewriting system that reflects the structure of Gray categories
- Adapted tools to show coherence in this setting
- More automated method for coherence
  - Algorithm to compute the coherence conditions
- Another proof of the coherence of monoids
- Coherence of other examples