Free precategories as presheaf categories

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Abstract

Precategories generalize both the notions of strict n-category and sesquicategory: their definition is essentially the same as the one of strict n-categories, excepting that we do not require the various interchange laws to hold. Those have been proposed as a framework in which one can express semi-strict definitions of weak higher categories: in dimension 3, Gray categories are an instance of them and have been shown to be equivalent to tricategories, and definitions of semi-strict tetracategories have been proposed, and used as the basis of proof assistants such as Globular. In this article, we are mostly interested in free precategories. Those can be presented by generators and relations, using an appropriate variation on the notion of polygraph (aka computad), and earlier works have shown that the theory of rewriting can be generalized to this setting, enjoying most of the fundamental constructions and properties which can be found in the traditional theory, contrarily to polygraphs for strict categories. We further study here why this is the case, by providing several results which show that precategories and their associated polygraphs bear properties which ensure that we have a good syntax for those. In particular, we show that the category of polygraphs for precategories form a presheaf category.

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Contents

Introduction

Strict polygraphs. The notion of *polygraph*, also known as *computad*, was introduced by Street [47] and Burroni [13] as a generalization of the notion of presentation for strict n-categories, thus extending the now classical notions of presentation for groups and monoids introduced by Dehn [19] and Thue [51]. From an algebraic point of view, they constitute the right notion of "free n-category", in the sense that they have been established as being the cofibrant objects in the folk model structure on the category of n-categories [43, 37]. They thus allow for computing various invariants of categories, as well as showing coherence theorems, based on the construction of resolutions (or cofibrant replacements) of categories of interest. For this reason, one is often interested in constructing coherent presentations of low-dimensional categories, which are polygraphs whose underlying free category is suitably equivalent to the original one.

In order to be able to perform practical computations, one is generally looking for polygraphs which are as small as possible. This task can be often be achieved by using techniques originating from rewriting theory [7, 50], suitably generalized to this setting, which exploits the orientation of relations in a presentation. Namely, when the presentation is terminating and confluent, generators corresponding to relations between relations can be found as confluence diagrams for critical branchings. This idea originates in the works of Squier on presented monoids [45, 46, 35] and has been the starting point of a series of works exploring higher dimensional rewriting [30, 31, 39, 24], which has since then been further generalized to various algebraic structures such as term rewriting systems [40], algebras [29] or operads [41]. While polygraphs have thus been proved to be quite a useful tool, they are still quite unsatisfactory on many aspects.

Limitations of strict polygraphs. From a categorical point of view, strict polygraphs are adapted to strict n-categories, but those are known not to be equivalent to weak n-categories, which are the real objects of interest. Namely, already starting from dimension 3, not every tricategory is equivalent to a 3-category: the best we can do is to strictify associativity and unitality, and show that every tricategory is equivalent to a Gray category [26] (we should underline here that this is not the only possible partial strictification [34]). Following our terminology, a Gray category is a 3-precategory equipped with interchange isomorphisms satisfying suitable axioms. Another categorical defect of polygraphs is the fact that they do not form a presheaf category. It is namely noted in [16] that this cannot be the case because of "the lack of an ordering" of 2-dimensional (and higher) cells, since composition is commutative for 2-cells with identity source and target. More formally, an abstract explanation of the fact that polygraphs do not form a presheaf category can be found in [38] and an elementary proof of this fact can be found in [17]. One route to solve this consists in restricting to polygraph where generators do not have identity sources (or targets), which has successfully been explored by Henry [33, 32]. Our exploration consists here in taking the other route and "add ordering" to morphisms.

From a rewriting point and computer science point of view, polygraphs, when considered as rewriting systems, lack a fundamental property found in most settings for rewriting: we expect that a finite rewriting system has a finite number of critical branchings. This was first observed by Lafont [36] and further studied by Guiraud and Malbos who showed that, because of this, there are finite convergent 3-polygraphs without finite derivation type [30]. From a practical perspective, this causes problems. Namely, representing the possibly infinite families of critical branchings is a difficult challenge, even in low dimensions [44]. But in fact, even providing a concrete representation of morphisms is a challenge, because there is no canonical representative of morphisms in free categories, up to the axioms of strict n-categories.

Polygraphs for precategories. For these reasons, it seems natural to investigate the framework of *n-precategories* whose definition is similar to the one of strict n-categories, excepting that we do not require the interchange laws to hold. In particular, in dimension 2, those correspond to Street's *sesquicategories* [48]. We have defined in [22, 23] an associated notion of polygraph and developed a theory of rewriting in this setting (interestingly, Araújo has recently independently come up with a very similar notion [5]). It seems that, in this setting, most of the limitations mentioned above vanish. First, we now have canonical representatives of morphisms in free n-precategories [23], a property which was first observed by Makkai while studying strict n-categories [38, Section 8], which makes them suitable for implementing software performing computation on morphisms. For this reason, they are also used internally in the *Globular* graphical proof assistant [52, 9]. Second, a finite rewriting system has a finite number of critical branchings, and those can be computed effectively. Third, we have a hope of being able to deal with weak higher-categories in this setting. Namely, we have already mentioned that Gray categories are equivalent to tricategories and are particular 3-precategories, and putative definitions of semistrict 4-categories based on 4-precategories have been proposed [9]. Note that the polygraph corresponding to a Gray category is almost never finite, but the infinite families of generators we add are regular enough to be dealt with in a uniform way [22, 23].

Properties of polygraphs. In this article, we further study of the category of n-polygraphs for n-precategories. Most importantly, we show that they form a presheaf category. Our proof is based on the characterization of concrete presheaf categories given by Makkai [38]. Simultaneously and independently, another proof of this result has been given by Araújo [5]. We should also mention that a notion of polygraph for weak categories has been developed and shown to be a presheaf category in [18]. Our approach gives rise to much smaller polygraphs and thus more amenable computations, although it is not entirely clear (yet) how to encode weak n-categories in our setting, excepting in low dimensions.

Plan of the paper. We begin by introducing precategories and associated polygraphs (Section 1) and show that functors between precategories induced by polygraphs have the important property of being Conduché (Section 2), which is used subsequently. Most of the remainder of the paper is devoted to showing that polygraphs form a presheaf category. Our proof is based on Makkai's theorem characterizing presheaf categories (recalled in Section 3). In order to make computations on cells in free precategories, it is useful to consider their support (Section 4). These allow defining and studying polyplexes (Section 5) which are shapes parametrizing compositions in precategories. This finally allows us to show that polygraphs form a presheaf category (Section 6). As a nice by-product, we derive a parametric adjunction together with an associated generic-free factorization for precategories, which gives a more conceptual view of the good syntactical properties of precategories (Section 7). Finally, we leave two open questions on homotopical aspects of polygraphs of precategories. First, whether polygraphs are the cofibrant objects for a reasonable model structure on precategories (we explain that the usual proof for strict categories does not immediately generalize to precategories), and second, whether the presheaf category of polygraphs is able to model homotopy types (we explain why the proof used by Henry for regular plexes [32] does not adapt here) (Section 8).

1 Precategories and their polygraphs

We recall here the definition of n-precategories as algebras over globular sets, as well as their elementary properties. We also recall the associated notion of polygraph, introduced in earlier works [22, 23], which is a particular instance of the very general notion of polygraph associated to a monad on globular sets introduced by Batanin [11].

The notion of precategory was first introduced by Street, in dimension 2, under the name of *sesquicategory*: this means a "1½-category", since sesquicategories have more structure than 1-categories, but less than 2-categories (they lack the interchange law). The general definition of precategories was (implicitly) given by Makkai in [38, Section 8], who used them to deal with the word problem for free strict categories. Later, they were used as data structures for the Globular proof assistant [8] and more recently for studying coherent presentations of Gray categories in [23] and coherence for adjunctions [6, 4].

In the following, given $n \in \mathbb{N}$, we write $\mathbb{N}_{<n}$ for the subset $\{0, \ldots, n-1\}$ of $\mathbb{N}$, and $\mathbb{N}_{\leq n}$ for $\mathbb{N}_{<n+1}$.

Globular sets. Given $n \in \mathbb{N} \cup \{\omega\}$, an *$n$-globular set* $(X, \partial^-, \partial^+)$ (often simply denoted X) is the data of sets X_k for $k \in \mathbb{N}_{\leq n}$ together with functions $\partial_i^-, \partial_i^+ \colon X_{i+1} \to X_i$ for $i \in \mathbb{N}_{<n}$ as in

$$X_0 \underset{\partial_0^+}{\overset{\partial_0^-}{\leftleftarrows}} X_1 \underset{\partial_1^+}{\overset{\partial_1^-}{\leftleftarrows}} X_2 \underset{\partial_2^+}{\overset{\partial_2^-}{\leftleftarrows}} \cdots \underset{\partial_{k-1}^+}{\overset{\partial_{k-1}^-}{\leftleftarrows}} X_k \underset{\partial_k^+}{\overset{\partial_k^-}{\leftleftarrows}} X_{k+1} \underset{\partial_{k+1}^+}{\overset{\partial_{k+1}^-}{\leftleftarrows}} \cdots$$

such that

$$\partial_i^- \circ \partial_{i+1}^- = \partial_i^- \circ \partial_{i+1}^+ \qquad\qquad \partial_i^+ \circ \partial_{i+1}^- = \partial_i^+ \circ \partial_{i+1}^+$$

for $i \in \mathbb{N}_{<n}$. When there is no ambiguity on i, we often write ∂^- and ∂^+ for ∂_i^- and ∂_i^+ respectively. An element u of X_i is called an *i-globe* of X and, for $i > 0$, the globes $\partial_{i-1}^-(u)$ and $\partial_{i-1}^+(u)$ are respectively called the *source* and *target* and u. Given n-globular sets X and Y, a *morphism* of n-globular sets between X and Y is a family of functions $F = (F_k \colon X_k \to Y_k)_{k \in \mathbb{N}_{\leq n}}$, such that

$$\partial_i^- \circ F_{i+1} = F_i \circ \partial_i^-$$

for $i \in \mathbb{N}_{<n}$. We write $\mathbf{Glob}_n$ for the category of n-globular sets. We have canonical truncation and inclusion functors

$$\mathcal{T}_n^{\mathrm{G}} \colon \mathbf{Glob}_{n+1} \to \mathbf{Glob}_n \qquad \text{and} \qquad \mathcal{I}_n^{\mathrm{G}} \colon \mathbf{Glob}_n \to \mathbf{Glob}_{n+1}$$

which respectively forget the $(n{+}1)$-globes and add an empty set of $(n{+}1)$-globes. They organize into an adjunction $\mathcal{I}_{n+1}^{\mathrm{G}} \dashv \mathcal{T}_n^{\mathrm{G}}$. It is direct from definition that globular sets are the models of an (essentially) algebraic theory, so that the category $\mathbf{Glob}_n$ is essentially algebraic. In particular, it implies that it is locally finitely presentable, complete and cocomplete [1].

For $\epsilon \in \{-, +\}$ and $j \geq 0$ with $j \leq n - i$, we define a morphism $\partial_{i,j}^\epsilon : X_{i+j} \to X_i$ by

$$\partial_{i,j}^\epsilon = \partial_i^\epsilon \circ \partial_{i+1}^\epsilon \circ \cdots \circ \partial_{i+j-1}^\epsilon$$

called the *iterated source* (resp. *target*) operation when $\epsilon = -$ (resp. $\epsilon = +$). We generally omit the index j when there is no ambiguity and simply write $\partial_i^\epsilon(u)$ for $\partial_{i,j}^\epsilon(u)$. Given $i, k, l \in \mathbb{N}_{\leq n}$

with $i < \min(k,l)$, we write $X_k \times_i X_l$ for the pullback

$$\begin{tikzcd} X_k \times_i X_l \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & X_l \arrow[d, "\partial_i^-"] \\ X_k \arrow[r, "\partial_i^+"'] & X_i \end{tikzcd} .$$

Given $p \in \mathbb{N}$ and $k_0, \ldots, k_p \in \mathbb{N}_{\le n}$, a sequence of globes $u_0 \in X_{k_0}, \ldots, u_p \in X_{k_p}$ is said *i-composable* for some $i < \min(k_0, \ldots, k_p)$, when $\partial_i^+(u_j) = \partial_i^-(u_{j+1})$ for $j \in \mathbb{N}_{<p}$. Given $k \in \mathbb{N}_{\le n}$ and $u, v \in X_k$, u and v are said *parallel* when $k = 0$ or $\partial_{k-1}^\epsilon(u) = \partial_{k-1}^\epsilon(v)$ for $\epsilon \in \{-,+\}$. In order to avoid dealing with the side condition $k = 0$, we use the convention that X_{-1} is the terminal set $\{*\}$ and that $\partial_{-1}^-, \partial_{-1}^+$ are the unique function $X_0 \to X_{-1}$.

Precategories. Given $n \in \mathbb{N} \cup \{\omega\}$, an *$n$-precategory* C is an n-globular set (whose k-globes are called *k-cells* in this context) together with, for $k \in \mathbb{N}_{<n}$, identity operations

$$\mathrm{id}^{k+1} \colon C_k \to C_{k+1}$$

for which we use the same notation conventions than the identity operations on strict categories, and, for $k, l \in \mathbb{N}_n^*$, composition operations

$$*_{k,l} \colon C_k \times_{\min(k,l)-1} C_l \to C_{\max(k,l)}$$

which satisfy the axioms below. Given $i, k, l \in \mathbb{N}_{\le n}$ with $i = \min(k,l)$, since the dimensions of the cells determine the indices of the composition to be used, we often write $*_i$ for $*_{k,l}$. In this way, we still make explicit the most important information which is the dimension i of composition. The axioms of n-precategories are the following:

(P-i) for $k \in \mathbb{N}_{<n}$ and $u \in C_k$,
$$\partial_k^-(\mathrm{id}_u^{k+1}) = u = \partial_k^+(\mathrm{id}_u^{k+1}),$$

(P-ii) for $i, k, l \in \mathbb{N}_{\le n}$ such that $i = \min(k,l) - 1$, $(u,v) \in C_k \times_i C_l$, and $\epsilon \in \{-,+\}$,

$$\partial^\epsilon(u *_i v) = \begin{cases} u *_i \partial^\epsilon(v) & \text{if } k < l, \\ \partial^-(u) & \text{if } k = l \text{ and } \epsilon = -, \\ \partial^+(v) & \text{if } k = l \text{ and } \epsilon = +, \\ \partial^\epsilon(u) *_i v & \text{if } k > l, \end{cases}$$

(P-iii) for $i, k, l \in \mathbb{N}_{\le n}$ with $i = \min(k,l) - 1$, given $(u,v) \in C_{k-1} \times_i C_l$,

$$\mathrm{id}_u *_i v = \begin{cases} v & \text{if } k \le l, \\ \mathrm{id}_{u *_i v} & \text{if } k > l, \end{cases}$$

and, given $(u,v) \in C_k \times_i C_{l-1}$,

$$u *_i \mathrm{id}_v = \begin{cases} u & \text{if } l \le k, \\ \mathrm{id}_{u *_i v} & \text{if } l > k, \end{cases}$$

(P-iv) for $i,k,l,m \in \mathbb{N}_{\le n}$ with $i = \min(k,l)-1 = \min(l,m)-1$, and $u \in C_k$, $v \in C_l$ and $w \in C_w$ such that u,v,w are i-composable,

$$(u *_i v) *_i w = u *_i (v *_i w),$$

(P-v) for $i,j,k,l,l' \in \mathbb{N}_{\le n}$ such that

$$i = \min(k, \max(l,l')) - 1, \qquad j = \min(l,l') - 1 \qquad \text{and} \qquad i < j,$$

given $u \in C_k$ and $(v,v') \in C_l \times_j C_{l'}$ such that u,v are i-composable,

$$u *_i (v *_j v') = (u *_i v) *_j (u *_i v')$$

and, given $(u,u') \in C_l \times_j C_{l'}$ and $v \in C_k$ such that u,v are i-composable,

$$(u *_j u') *_i v = (u *_i v) *_j (u' *_i v).$$

Note that, provided that the Axioms (P-i) to (P-iv) are satisfied, Axiom (P-v) can be shown equivalent to the more symmetrical axiom

(P-v)' for every $i,j,k \in \mathbb{N}_{\le n}$ satisfying $i < j < k$, and cells $u_1,u_2 \in C_{i+1}$, $v_1,v_2 \in C_{j+1}$ and $w \in C_k$ such that u_1,w,u_2 are i-composable and v_1,w,v_2 are j-composable, we have

$$u_1 *_i (v_1 *_j w *_j v_2) *_i u_2 = (u_1 *_i v_1 *_i u_2) *_j (u_1 *_i w *_i u_2) *_j (u_1 *_i v_2 *_i u_2).$$

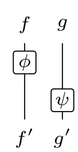
Example 1. Given a 2-precategory C with two 2-cells ϕ and ψ as in

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\phi"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\psi"] & z \end{tikzcd}$$

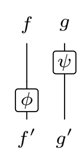
there are two ways to compose ϕ and ψ together, given by

$$(\phi *_0 g) *_1 (f' *_0 \psi) \quad \text{and} \quad (f *_0 \psi) *_1 (\phi *_0 g')$$

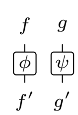
that can be represented using string diagrams by



and



and these two composites are not expected to be equal in C. Moreover, by our definition of precategories, there is no such thing as a valid cell $\phi *_0 \psi$, and the string diagram



makes no sense in this setting.

Given two n-precategories C and D, a *morphism* of n-precategories (or n-*prefunctor*) between C and D is a morphism of n-globular sets $F\colon C \to D$ such that

– $F(\mathrm{id}^{k+1}_u) = \mathrm{id}^{k+1}_{F(u)}$ for $k \in \mathbb{N}_{<n}$ and $u \in C_k$,

– $F(u *_i v) = F(u) *_i F(v)$ for $i, k, l \in \mathbb{N}_{\le n}$ with $i = \min(k, l) - 1$ and $(u, v) \in C_k \times_i C_l$.

We write $\mathbf{PCat}_n$ for the category of n-precategories thus defined. We have canonical truncation and inclusion functors

$$\mathcal{T}^{\mathrm{C}}_n\colon \mathbf{PCat}_{n+1} \to \mathbf{PCat}_n \qquad \text{and} \qquad \mathcal{I}^{\mathrm{C}}_n\colon \mathbf{PCat}_n \to \mathbf{PCat}_{n+1}$$

which respectively forget the $(n+1)$-cells and add a set of $(n+1)$-cells consisting of formal identities of n-cells. They organize into an adjunction $\mathcal{I}^{\mathrm{C}}_{n+1} \dashv \mathcal{T}^{\mathrm{C}}_n$.

The globular monad of n-precategories. The above definition of n-precategories directly translates into an essentially algebraic theory so that the category $\mathbf{PCat}_n$ is locally finitely presentable [1]. There is a forgetful functor

$$\mathcal{U}_n\colon \mathbf{PCat}_n \to \mathbf{Glob}_n$$

which maps an n-precategory to its underlying n-globular set, and this functor is induced by the inclusion of the essentially algebraic theory of n-globular sets into the one of n-precategories. We thus have the following [20, Proposition 1.4.2.4]:

Proposition 2. *The category $\mathbf{PCat}_n$ is locally finitely presentable, complete and cocomplete. Moreover, the functor $\mathcal{U}_n$ is a right adjoint which preserves directed colimits.*

The above proposition states the existence of a functor

$$\mathcal{F}_n\colon \mathbf{Glob}_n \to \mathbf{PCat}_n$$

which is left adjoint to $\mathcal{U}_n$, sending an n-globular set to the n-precategory it freely generates. Moreover, the functor $\mathcal{U}_n$ can be shown monadic using Beck's monadicity theorem [20, Proposition 1.4.2.5]:

Proposition 3. *For every $n \in \mathbb{N} \cup \{\omega\}$, the functor $\mathcal{U}_n$ is monadic.*

This shows that, for $n \in \mathbb{N}\cup\{\omega\}$, $\mathbf{PCat}_n$ is the category of algebras for a monad $T^n\colon \mathbf{Glob}_n \to \mathbf{Glob}_n$ on n-globular sets (the monad induced by the above adjunction).

Polygraphs of precategories. In fact, for $n \in \mathbb{N}$, the monads T^n is adequately derived by truncation from T^ω [20, Theorem 1.4.2.8], the latter being *truncable* in the sense of Batatnin [11]. By general arguments on globular algebras, this allows the definition of *polygraphs* for the theory of precategories.

The category of n-*polygraphs* $\mathbf{Pol}_n$ (for n-precategories) is defined by induction on n, together with a functor

$$(-)^{*,n}\colon \mathbf{Pol}_n \to \mathbf{PCat}_n$$

often written $(-)^*$, which associates to an n-polygraph the n-precategory it freely generates, as follows. We first define $\mathbf{Pol}_0 = \mathbf{Glob}_0$ (which is isomorphic to $\mathbf{Set}$) and $(-)^{*,0} = \mathcal{F}_0$ (which is the identity functor on **Set**). Now, given $n \in \mathbb{N}$, assuming $\mathbf{Pol}_n$ and $(-)^{*,n}$ defined in dimension n, we define $\mathbf{Pol}_{n+1}$ as the pullback

$$
\begin{tikzcd}
\mathbf{Pol}_{n+1} \arrow[r, dashed, "\mathcal{G}_{n+1}"] \arrow[d, dashed, "\mathcal{T}^{\mathrm{P}}_n"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{Glob}_{n+1} \arrow[d, "\mathcal{T}^{\mathrm{G}}_n"] \\
\mathbf{Pol}_n \arrow[r, "\mathcal{U}_n(-)^{*,n}"'] & \mathbf{Glob}_n
\end{tikzcd}
$$

The functor $\mathcal{T}^{\mathrm{P}}_n : \mathbf{Pol}_{n+1} \to \mathbf{Pol}_n$, called the *$n$-truncation* functor for polygraphs, admits a left adjoint $\mathcal{I}^{\mathrm{P}}_{n+1} : \mathbf{Pol}_n \to \mathbf{Pol}_{n+1}$, which extends an n-polygraph P as an $(n+1)$-polygraph with an empty set of $(n+1)$-generators (using the description of polygraphs given just below). The image P^* under $(-)^{*,n+1}$ of an $(n+1)$-polygraph P is defined as the pushout

$$
\begin{tikzcd}
\mathcal{F}_{n+1}\,\mathcal{I}^{\mathrm{G}}_{n+1}\,\mathcal{T}^{\mathrm{G}}_n\,\mathcal{G}_{n+1}\,\mathsf{P} \arrow[r, "(\mathcal{F}_{n+1}\,\mathrm{i}^{\mathrm{G}}_n\,\mathcal{G}_{n+1})_{\mathsf{P}}"] \arrow[d, "\alpha_{\mathsf{P}}"'] & \mathcal{F}_{n+1}\,\mathcal{G}_{n+1}\,\mathsf{P} \arrow[d, dashed] \\
\mathcal{I}^{\mathrm{C}}_{n+1}(\mathcal{T}^{\mathrm{P}}_n\,\mathsf{P})^* \arrow[r, dashed] \arrow[ur, phantom, very near end, "\ulcorner"] & \mathsf{P}^*
\end{tikzcd}
$$

where $\mathrm{i}^{\mathrm{G}}_n$ is the counit of the adjunction $\mathcal{I}^{\mathrm{G}}_{n+1} \dashv \mathcal{T}^{\mathrm{G}}_n$ and α_{P} is the composite

$$
\alpha_{\mathsf{P}} \quad = \quad \mathcal{F}_{n+1}\,\mathcal{I}^{\mathrm{G}}_{n+1}\,\mathcal{T}^{\mathrm{G}}_n\,\mathcal{G}_{n+1}\,\mathsf{P} \xrightarrow{\;\sim\;} \mathcal{I}^{\mathrm{C}}_{n+1}\,\mathcal{F}_n\,\mathcal{U}_n(-)^{*,n}\,\mathcal{T}^{\mathrm{P}}_n\,\mathsf{P} \xrightarrow{(\mathcal{I}^{\mathrm{C}}_{n+1}\,\varepsilon_n(-)^{*,n}\,\mathcal{T}^{\mathrm{P}}_n)_{\mathsf{P}}} \mathcal{I}^{\mathrm{C}}_{n+1}(\mathcal{T}^{\mathrm{P}}_n\,\mathsf{P})^*
$$

where ε_n is the counit of the adjunction $\mathcal{F}_n \dashv \mathcal{U}_n$. Intuitively, P^* is obtained by freely generating an $(n+1)$-precategory from $(\mathcal{T}^{\mathrm{P}}_n\,\mathsf{P})^*$ by attaching the $(n+1)$-generators described by $\mathcal{G}_{n+1}\,\mathsf{P}$. The mapping $\mathsf{P} \mapsto \mathsf{P}^*$ then naturally extends to a functor $(-)^{*,n} : \mathbf{Pol}_{n+1} \to \mathbf{PCat}_n$, which concludes the inductive definition of polygraphs of precategories. More details on this construction can be found in [20, 23].

Since the monad of the theory of precategory is truncable, given $n \in \mathbb{N}$, an n-polygraph P can be alternatively described as a diagram in **Set** of the form

$$
\begin{tikzcd}
\mathsf{P}_0 \arrow[d, "\mathrm{e}_0"] & \mathsf{P}_1 \arrow[dl, shift right, "\mathrm{d}^-_0"'] \arrow[dl, shift left, "\mathrm{d}^-_0"] \arrow[d, "\mathrm{e}_1"] & \mathsf{P}_2 \arrow[dl, shift right, "\mathrm{d}^-_1"'] \arrow[dl, shift left, "\mathrm{d}^-_1"] \arrow[d, "\mathrm{e}_2"] & \cdots & \mathsf{P}_{n-1} \arrow[dl, shift right, "\mathrm{d}^-_{n-2}"'] \arrow[dl, shift left, "\mathrm{d}^-_{n-2}"] \arrow[d, "\mathrm{e}_n"] & \mathsf{P}_n \arrow[dl, shift right, "\mathrm{d}^-_{n-1}"'] \arrow[dl, shift left, "\mathrm{d}^-_{n-1}"] \\
\mathsf{P}^*_0 & \mathsf{P}^*_1 \arrow[l, shift right, "\partial^-_0"'] \arrow[l, shift left, "\partial^+_0"] & \cdots \arrow[l, shift right, "\partial^-_1"'] \arrow[l, shift left, "\partial^+_1"] & \mathsf{P}^*_{n-2} & \mathsf{P}^*_{n-1} \arrow[l, shift right, "\partial^-_{n-2}"'] \arrow[l, shift left, "\partial^+_{n-2}"] &
\end{tikzcd}
$$

where, for $i \in \mathbb{N}_{<n}$, e_i is the embedding of the i-generators P_i into the set P^*_i of freely generated i-cells, such that

$$
\partial^-_i \circ \mathrm{d}^-_{i+1} = \partial^-_i \circ \mathrm{d}^+_{i+1} \qquad \text{and} \qquad \partial^+_i \circ \mathrm{d}^-_{i+1} = \partial^+_i \circ \mathrm{d}^+_{i+1}
$$

for $i \in \mathbb{N}_{<n}$. Note that the above description is the same as the original definition of polygraphs by Burroni [13], excepting that the sets P^*_i of i cells are freely generated as i-precategories instead of strict i-categories.

By general properties on locally presentable categories, we have:

Proposition 4. *Given $n \in \mathbb{N} \cup \{\omega\}$, $\mathbf{Pol}_n$ is a locally finitely presentable category. In particular, it is complete and cocomplete.*

Proof. The 2-category of locally presentable categories, right adjoints (resp. left adjoints) and natural transformations is closed under bipullbacks (see [12, Theorem 2.18, Theorem 3.15]). A pullback along an isofibration happens to be a bipullback and the pullback of $\mathcal{T}_n^{\mathrm{G}}$ along $\mathcal{U}_n$ can be shown to be a left adjoint and again an isofibration. Then, its pullback by $(-)^{*,n}$, which is known (see [11, Proposition 3.1]) to be a left adjoint, is again a left adjoint whose domain $\mathbf{Pol}_n$ is a locally presentable category. A more detailed study shows that $\mathbf{Pol}_n$ is locally *finitely* presentable with finite polygraphs as finitely presentable objects. See [21, Proposition 3.3.3] for the local presentability and [20, Theorem 1.3.3.19] for the local *finite* presentability. □

In the following, we will write $\mathbf{1}$ for the terminal object of $\mathbf{Pol}_n$, for $n \in \mathbb{N} \cup \{\omega\}$.

2 Free functors are Conduché

Free precategories on polygraphs enjoy useful properties, thanks to which we have a nice syntax for morphisms in those, as we now show. It should be noted that many those are not valid in the usual setting of polygraph for strict categories (as opposed to precategories). One remarkable such property of free precategories is that their cells can be described as canonical compositions of generators, which happen to be unique for a given cell, so that we prefer to call them *normal forms.* These normal forms are adequately reflected by free functors, since the latter reflect elementary compositions: in other words, they satisfy the analogue of the Conduché property for strict categories [28]. In addition to providing convenient tools in the proofs, we will see in subsequent sections that these properties entail the existence universal shapes of compositions.

Types and contexts. Given $m \leq n \in \mathbb{N}$, an n-precategory C, an *m-type* is a pair of parallel $(m{-}1)$-cells of C. We use the convention that there is a unique 0-type, and all pairs of 0-cells of a precategory are parallel. Given a k-cell $u \in C$ for some $k \geq m$, u has a canonical associated m-type: $(\partial^-_{m-1}(u), \partial^+_{m-1)}(u))$. In the following, an m-type is thought of as the type for a formal variable, which suggests defining the notion of context (a morphism in which the variable occurs exactly once) and of substitution (replacing the variable by a morphism).

An *m-context* E for an m-type (s,t) is defined by induction on m, together with the *evaluation* $E[u]$ of E at a cell of m-type (s,t):

- there is a unique 0-context of the unique 0-type, denoted $[-]$, and the evaluation of it at a cell $u \in C$ is u,
- an $(m{+}1)$-context of type (s,t) is a triple $E = (l, E', r)$ with $l, r \in C_m$, and E' an m-context of type $(\partial^-_{m-1}(s), \partial^+_{m-1}(t))$ such that $\partial^+_m(l) = E'[s]$ and $E'[t] = \partial^-_m(r)$, and the evaluation $E[u]$ of E at a cell u is defined by $E[u] = l *_m E'[u] *_m r$.

Alternatively, an m-context E can be thought of as an expression of the form

$$l_m *_{m-1} (\cdots *_1 (l_1 *_0 [-] *_0 r_1) *_1 \cdots) *_{m-1} r_m$$

where the $l_i, r_i \in C_i$ are the i-cells occurring in the definition of E for $i \in \mathbb{N}_{\leq m}$, and its evaluation at a cell u as the cell obtained by replacing $[-]$ by u in the above expression.

Normal forms. We have the following normal form for the cells of free precategories:

Theorem 5. *Given $m \in \mathbb{N}$ and a polygraph $\mathsf{P} \in \mathbf{Pol}_\omega$, every m-cell of P^* can be written uniquely as*

$$E_1[g_1] *_{m-1} \cdots *_{m-1} E_k[g_k]$$

for some unique $g_1, \cdots, g_k \in \mathsf{P}_m$ and $(m-1)$-contexts $E_1, \ldots, E_k$ of the corresponding types.

Proof. We only sketch the proof, which is detailed in [23, Theorem 1.8.3]. One can adequately orient the axioms (P-i)–(P-v) of precategories in order to obtain a terminating and locally confluent rewriting system on the formal expressions of cells of free precategories. By standard arguments of rewriting theory [7], this gives the existence and unicity of normal forms. □

Remark 6. A consequence of the above theorem is that the embeddings $\mathrm{e}_i \colon \mathsf{P}_i \to \mathsf{P}_i^*$ introduced earlier are injective. Thus, given $g \in \mathsf{P}_i$, we will often omit e_i and write g for both the element of P_i and the cell of P_i^*.

The unicity of normal forms directly entails the that the image under a free functor of an identity is an identity (resp. of a generator is a generator):

Proposition 7. *Let $F \colon \mathsf{P} \to \mathsf{Q} \in \mathbf{Pol}_\omega$ be a morphism of polygraphs, $k \in \mathbb{N}_{\leq n}$ and $u \in \mathsf{P}_k^*$. The following hold:*

- *(i) when $k > 0$, there exists a cell $u' \in \mathsf{P}_{k-1}^*$ such that $u = \mathrm{id}_{u'}^k$ if and only if there exists a cell $\tilde{u}' \in \mathsf{Q}_{k-1}^*$ such that $F^*(u) = \mathrm{id}_{\tilde{u}'}^k$,*
- *(ii) there exists a generator $g \in \mathsf{P}_k$ such that $u = g$ if and only if there exists a generator $\tilde{g} \in \mathsf{Q}_k$ such that $F^*(u) = \tilde{g}$.*

We should also mention now that composition in free precategories is cancellative. This does not seem to be deducible from the more general properties developed in the next sections.

Proposition 8. *Given $\mathsf{P} \in \mathbf{Pol}_\omega$ and $u, v_1, v_2 \in \mathsf{P}^*$ such that $u *_i v_1 = u *_i v_2$ for some i, then $v_1 = v_2$.*

Proof. Note that, by the input and output dimension conditions of $*_i$, we necessarily have that the dimension of v_1 is the one of v_2. We do an induction on the dimension of the resulting cell $u *_i v_1$ and distinguish three cases depending on the relative dimensions of u, v_1 and v_2.

- Suppose that $u, v_1, v_2 \in \mathsf{P}_{i+1}^*$. By unicity of the decomposition of $(i+1)$-cells of free precategories (Theorem 5) and its compatibility with i-composition as concatenation, we have $v_1 = v_2$.
- Suppose $u \in \mathsf{P}_{i+1}^*$ and $v_1, v_2 \in \mathsf{P}_n^*$ with $n > i + 1$. We reason by induction on v_1.
  - Suppose that $v_1 = \alpha$ for some generator $\alpha \in \mathsf{P}_n$. Then, by the definition of composition and the normal forms, we have that $v_2 = \alpha$.
  - Suppose that $v_1 = E_1[\alpha]$ for some generator $\alpha \in \mathsf{P}_n$ and m-context E_1 with $0 < m < n$. By the definition of composition and the unicity of normal forms, we have $v_2 = E_2[\alpha]$. Let $(l_j, E'_j, r_j) = E_j$ for $j \in \{1, 2\}$. If $m = i + 1$, then, by unicity of normal forms, we have $u *_i l_1 = u *_i l_2$, $E'_1[\alpha] = E'_2[\alpha]$ and $r_1 = r_2$. By the beginning of the proof, we have $l_1 = l_2$, so that $v_1 = v_2$. Otherwise, if $m > i + 1$, then $u *_i l_1 = u *_i l_2$, $u *_i E'_1[\alpha] = u *_i E'_2[\alpha]$ and $u *_i r_1 = u *_i r_2$. By the different induction hypotheses, we have $l_1 = l_2$, $E'_1[\alpha] = E'_2[\alpha]$ and $r_1 = r_2$, so that $v_1 = v_2$.

  - If $v_1 = E_1^1[\alpha_1] *_{n-1} \cdots *_{n-1} E_1^k[\alpha_k]$, then we necessarily have $v_2 = E_2^k[\alpha_1] *_{n-1} \cdots *_{n-1} E_2^k[\alpha_k]$ such that $u *_i E_1^j[\alpha_j] = u *_i E_2^j[\alpha_j]$. By the previous argument, we have $E_1^j[\alpha_j] = E_2^j[\alpha_j]$, so that $v_1 = v_2$.

- Suppose that $u \in \mathsf{P}_n^*$, $v_1, v_2 \in \mathsf{P}_{i+1}^*$ with $n > i+1$. We reason by induction on u.
  - If $u = \mathrm{id}_{u'}$, then we have $u' *_i v_1 = u' *_i v_2$, so that $v_1 = v_2$ by induction.
  - If $u = E[\alpha]$ for some $(n-1)$-context E, then let $(l, E', r) = E$. We then have $r *_i v_1 = r *_i v_2$ so that, by induction hypothesis, $v_1 = v_2$.
  - If $u = E_1[\alpha_1] *_{n-1} \cdots *_{n-1} E_k[\alpha_k]$ for some $k \geq 1$, $\alpha_1, \ldots, \alpha_k \in \mathsf{P}_n$ and contexts $E_1, \ldots, E_k$, then we have in particular $E_1[\alpha_1] *_i v_1 = E_1[\alpha_1] *_i v_2$ so that we can conclude $v_1 = v_2$ by the previous case.

- Suppose that $u, v_1, v_2 \in \mathsf{P}_{i+1}^*$, $v_1, v_2 \in \mathsf{P}_{i+1}^*$ with $n > 0$. By the unicity of normal forms, we can uniquely write u as $E^1[\alpha^1] *_i \cdots *_i E^k[\alpha^k]$ and v_j as $E_j^1[\beta_j^1] *_i \cdots *_i E_j^{l_j}[\beta_j^{l_j}]$ for $j \in \{1,2\}$ for some adequate $k, l_1, l_2 \in \mathbb{N}$, i-contexts $E^{\cdot}$, $E_1^{\cdot}$, $E_2^{\cdot}$ and $(i+1)$-generators $\alpha^{\cdot}$, $\beta_1^{\cdot}$ and $\beta_2^{\cdot}$. By considering the induced normal forms on $u *_i v_1$ and $u *_i v_2$ by concatenation, we deduce by unicity of normal forms that $l_1 = l_2$ and $E_1^{\cdot} = E_2^{\cdot}$ and $\beta_1^{\cdot} = \beta_2^{\cdot}$, so that $v_1 = v_2$. □

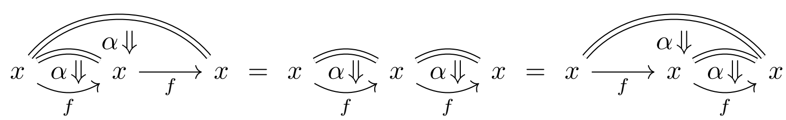
Remark 9. Note that such a property does not hold for polygraphs of strict categories. Indeed, considering the 2-polygraph of strict precategories P defined by

$$\mathsf{P}_0 = \{x\} \qquad \mathsf{P}_1 = \{f\colon x \to x\} \qquad \mathsf{P}_2 = \{\alpha\colon \mathrm{id}_x \Rightarrow f\},$$

we have $\alpha *_0 \mathrm{id}_f \neq \mathrm{id}_f *_0 \alpha$ while

$$\alpha *_1 (\alpha *_0 \mathrm{id}_f) = \alpha *_0 \alpha = \alpha *_1 (\mathrm{id}_f *_0 \alpha)$$

in the free strict 2-category P^*. Graphically,



Conduché functors. We now introduce the notion of (strict) Conduché functor for precategories, following the work of Guetta in the case of strict categories [28]. Informally, these functors have a "co-functoriality" property, in the sense that cells mapped to composites are themselves composites. The notion of weak Conduché functor was introduced by Guiraud in a seemingly unrelated context [25] as a necessary and sufficient condition for a functor $F\colon C \to D$ between strict n-categories to be *exponentiable*, i.e., for the pullback functor $F^{\leftarrow}\colon \mathbf{Cat}/D \to \mathbf{Cat}/C$ to have a right adjoint.

Let $n \in \mathbb{N} \cup \{\omega\}$, $C, D \in \mathbf{PCat}_n$ and $F\colon C \to D$ be an n-prefunctor. We say that F is *n-Conduché* when it satisfies that, for all $i, k_1, k_2, k \in \mathbb{N}_n^*$ with $i = \min(k_1, k_2) - 1$ and $k = \max(k_1, k_2)$, $u \in C_k$, i-composable $v_1 \in D_{k_1}$ and $v_2 \in D_{k_2}$ such that

$$F(u) = v_1 *_i v_2,$$

there exist unique i-composable $u_1 \in C_{k_1}$ and $u_2 \in C_{k_2}$ such that

$$F(u_1) = v_1 \qquad \text{and} \qquad F(u_2) = v_2 \qquad \text{and} \qquad u_1 *_i u_2 = u.$$

As in the case of strict categories, the Conduché property implies a unique lifting of identities:

Proposition 10. *Given $n \in \mathbb{N} \cup \{\omega\}$ and an n-Conduché prefunctor $F \colon C \to D \in \mathbf{PCat}_n$, if*

$$F(u) = \mathrm{id}_v$$

for some $k \in \mathbb{N}_{<n}$, $u \in C_{k+1}$, and $v \in D_k$, then there exists a unique $u' \in C_k$ such that

$$F(u') = v \qquad \text{and} \qquad u = \mathrm{id}_{u'}.$$

Proof. Since $\mathrm{id}_v = \mathrm{id}_v *_k \mathrm{id}_v$, by the Conduché property, there exist unique $u_1, u_2 \in C_{k+1}$ such that $F(u_1) = \mathrm{id}_v$, $F(u_2) = \mathrm{id}_v$ and $u = u_1 *_k u_2$. Moreover, we have that $F(\mathrm{id}_{\partial_k^-(u)}) = v$ and $u = \mathrm{id}_{\partial_k^-(u)} *_k u$ so that $u_1 = \mathrm{id}_{\partial_k^-(u)}$ and $u_2 = u$. Symmetrically, we have that $u_1 = u$ and $u_2 = \mathrm{id}_{\partial_k^+(u)}$. Thus, $u = \mathrm{id}_{\partial_k^-(u)} *_k \mathrm{id}_{\partial_k^+(u)}$, so that $\partial_k^-(u) = \partial_k^+(u)$ and $u = \mathrm{id}_{u'}$ with $u' = \partial_k^-(u)$. Uniqueness is immediate. □

Unlike for strict categories, we have the remarkable property that all free functors of precategories are Conduché:

Proposition 11. *Given $m \in \mathbb{N} \cup \{\omega\}$ and a morphism $F \colon \mathsf{P} \to \mathsf{Q} \in \mathbf{Pol}_m$, the prefunctor $F^* \colon \mathsf{P}^* \to \mathsf{Q}^*$ is Conduché.*

Proof. For the sake of simplicity, we only handle the case $m = \omega$. Suppose given $n \in \mathbb{N}$ and an n-cell $u \in \mathsf{P}_n^*$ such that $F(u) = \bar{u}_1 *_i \bar{u}_2$. We reason by case analysis on the relative dimensions of $\bar{u}_1$ and $\bar{u}_2$.

- If $\bar{u}_1, \bar{u}_2 \in \mathsf{Q}_n^*$ then $i = n-1$. By the unicity of normal forms and its compatibility with $*_{n-1}$, there are unique u_1, u_2 such that $F^*(u_k) = \bar{u}_k$ for $k \in \{1, 2\}$ and $u = u_1 *_{n-1} u_2$.
- Suppose $\bar{u}_1 \in \mathsf{Q}_{i+1}^*$ and $\bar{u}_2 \in \mathsf{Q}_n^*$. If there are u_1, u_2 such that $F^*(u_k) = \bar{u}_k$ for $k \in \{1, 2\}$ and $u = u_1 *_i u_u$, then they are unique since, by the previous point, u_1 and $\partial_{i+1}^-(u_2)$ are uniquely determined by $\partial_{i+1}^-(u) = u_1 *_i u_2^-$ and $F^*(u_1) = \bar{u}_1$ and $F^*(u_2^-) = \partial_{i+1}^-(u_2)$. Moreover, since $u = u_1 *_i u_2$, we have that u_2 is unique by Proposition 8. So unicity holds. For existence, we reason by induction on n and $\bar{u}_2$.
  - If $\bar{u}_2 = \bar{E}[\bar{\alpha}]$ for some $(i{+}1)$-context $\bar{E} = (\bar{l}, \bar{E}', \bar{r})$, then, by unicity of normal forms, $u = E[\alpha]$ for some $\alpha \in \mathsf{P}$, and $(i{+}1)$-context $E = (l, E', r)$, and we moreover have $F^*(l) = \bar{u}_1 *_i \bar{l}$, $F^*(E[\alpha]) = \bar{E}[\bar{\alpha}]$ and $F^*(r) = \bar{r}$. By the first part, there are u_1 and $\tilde{l}$ such that $l = u_1 *_i \tilde{l}$, so that $\tilde{E} = (\tilde{l}, E', r)$ satisfies that $u = u_1 *_i \tilde{E}[\alpha]$, $F^*(u_1) = \bar{u}_1$ and $F^*(\tilde{E}[\alpha]) = \bar{E}[\bar{\alpha}]$.
  - If $\bar{u}_2 = \bar{E}[\bar{\alpha}]$ for some $(j{+}1)$-context $\bar{E} = (\bar{l}, \bar{E}', \bar{r})$ with $j > i$, then, by unicity of normal forms, $u = E[\alpha]$ for some $\alpha \in \mathsf{P}$, and $(j{+}1)$-context $E = (l, E', r)$, and we moreover have $F^*(l) = \bar{u}_1 *_i \bar{l}$, $F^*(E'[\alpha]) = \bar{u}_1 *_i \bar{E}'[\bar{\alpha}]$ and $F^*(r) = \bar{u}_1 *_i \bar{r}$. By the other induction hypothesis, there are $u_1^1, u_1^2, u_1^3, \tilde{l}, \tilde{u}', \tilde{r}$ such that $l = u_1^1 *_i \tilde{l}$, $E'[\alpha] = u_1^2 *_i \tilde{u}'$ and $r = u_1^3 *_i \tilde{r}$. Since $u_1^1 *_i \partial_j^+(\tilde{l}) = \partial_j^+(l) = \partial_j^-(\tilde{u}') = u_1^2 *_i \partial_j^-(\bar{E}'[\bar{\alpha}])$ and $F^*(u_1^1) = F^*(u_1^2) = \bar{u}_1$ and $F^*(\partial_j^+(\tilde{l})) = F^*(\partial_j^-(\tilde{u}')) = \partial_j^+(\bar{l})$, by unicity, we have $u_1^1 = u_1^2$ and $\partial_j^+(\tilde{l}) = \partial_j^-(\tilde{u}')$. Similarly, $u_1^2 = u_1^3$ and $\partial_j^+(\tilde{u}') = \partial_j^-(\tilde{r})$. Thus, writing u_1 for u_1^1 and u_2 for $\tilde{l} *_j \tilde{u}' *_j \tilde{r}$, we have that $u = u_1 *_i u_2$ is the wanted decomposition for u.

- If $\bar{u}_2 = \bar{E}_1[\bar{\alpha}_1] *_{n-1} \cdots *_{n-1} \bar{E}_k[\bar{\alpha}_k]$, then, by unicity of normal forms, we have that $u = E_1[\alpha_1] *_{n-1} \cdots *_{n-1} E_k[\alpha_k]$ such that $F^*(E_l[\alpha_l]) = \bar{u}_1 *_i \bar{E}_l[\bar{\alpha}_l]$. By induction hypothesis, we get u_1^l and u_2^l such that $E_l[\alpha_l] = u_1^l *_i u_2^l$, $F^*(u_1^l) = \bar{u}_1$ and $F^*(u_2^l) = \bar{E}_l[\bar{\alpha}_l]$. Using the same argument as earlier, we get that $u_1^1 = \cdots = u_1^k$ and $u_2^1, \ldots, u_2^k$ are $(n-1)$-composable so that, writing u_1 for u_1^1 and u_2 for $u_2^1 *_{n-1} \cdots *_{n-1} u_2^k$, we have a decomposition $u = u_1 *_i u_2$ satisfying the wanted properties.

- Suppose $\bar{u}_1 \in \mathsf{Q}_n^*$ and $\bar{u}_2 \in \mathsf{Q}_{i+1}^*$. This case is similar to the previous one. ☐

Remark 12. As a counter-example for the above property in the context of strict categories, consider the polygraphs P and Q defined by

$$\begin{array}{lll} \mathsf{P}_0 = \{x\} & \mathsf{P}_1 = \emptyset & \mathsf{P}_2 = \{\alpha\colon \mathrm{id}_x \Rightarrow \mathrm{id}_x, \beta\colon \mathrm{id}_x \Rightarrow \mathrm{id}_x\} \\ \mathsf{Q}_0 = \{y\} & \mathsf{Q}_1 = \emptyset & \mathsf{Q}_2 = \{\gamma\colon \mathrm{id}_y \Rightarrow \mathrm{id}_y\}. \end{array}$$

We then have a morphism $F\colon \mathsf{P} \to \mathsf{Q}$ sending α and β to γ, and the associated prefunctor F^* sends both $\alpha *_0 \beta$ and $\beta *_0 \alpha$ to $\gamma *_0 \gamma$.

A nice application of the above Conduché properties is the characterization of monomorphisms of polygraphs. First, we briefly observe the equivalence between monomorphisms of precategories and dimensionwise injections.

Proposition 13. *Given $n \in \mathbb{N} \cup \{\omega\}$ and $F\colon C \to D \in \mathbf{PCat}_n$, the following are equivalent:*

(i) F is a monomorphism,

(ii) F_k is a monomorphism for every $k \leq n$.

Proof. The theory of n-precategories is sketchable and the functor $(-)_k\colon \mathbf{PCat}_n \to \mathbf{Set}$, which to a precategory associates its set of k-cells is induced by a sketch morphism. It is thus a right adjoint [10, Section 4, Theorem 4.1]. In particular, it preserves monomorphisms. Thus, (i) implies (ii). Moreover, since the functors $(-)_k$ for $k < n+1$ are jointly faithful, we have that (ii) implies (i). ☐

We then have the following characterization property for monomorphisms of polygraphs, which are in particular preserved by the functor $(-)^*$:

Proposition 14. *Given $n \in \mathbb{N} \cup \{\omega\}$ and $F\colon \mathsf{P} \to \mathsf{Q} \in \mathbf{Pol}_n$, the following are equivalent:*

(i) F is a monomorphism,

(ii) F_k is a monomorphism for every $k \leq n$,

(iii) F^* is a monorphism in $\mathbf{PCat}_n$.

Proof. We show this property by induction on n. (ii) clearly implies (i).

Conversely, assuming (i), by induction hypothesis, we have that F_k and F_k^* are monomorphisms for $k < n$. Now, let $x, y \in \mathsf{P}_n$ such that $F_n(x) = F_n(y)$. In particular, we have $F_{n-1}^*(\partial^\epsilon(x)) = F_{n-1}^*(\partial^\epsilon(y))$ for $\epsilon \in \{-,+\}$, so that $\partial^\epsilon(x) = \partial^\epsilon(y)$ for $\epsilon \in \{-,+\}$, by injectivity of F_{n-1}^*. Consider the n-polygraph R such that $\mathcal{T}_{n-1}^{\mathrm{P}} \mathsf{R} = \mathcal{T}_{n-1}^{\mathrm{P}} \mathsf{P}$ and $\mathsf{R}_n = z$ with $\mathrm{d}_{n-1}^\epsilon(z) = \partial^\epsilon(x)$ for $\epsilon \in \{-,+\}$. Then, we have two canonical morphisms $G^x, G^y\colon \mathsf{R} \to \mathsf{P}$, verifying $G^x(z) = x$ and $G^y(z) = y$. We then have $F \circ G^x = F \circ G^y$, so that $G^x = G^y$ since F is a monomorphism. In particular, we have $x = y$. Thus, F_n is injective, so (ii) holds.

By Theorem 5, the embedding $\mathrm{e}_k^{\mathsf{P}} : \mathsf{P}_k \to \mathsf{P}_k^*$ (resp. $\mathrm{e}_k^{\mathsf{Q}} : \mathsf{Q}_k \to \mathsf{Q}_k^*$) is a monomorphism. Thus, (iii) implies (ii), since $\mathrm{e}_k^{\mathsf{Q}} \circ F_k = F_k^* \circ \mathrm{e}_k^{\mathsf{P}}$ and the right-hand side of the latter equation is a monomorphism by Proposition 13.

Conversely, assume (ii). Let $u, v \in \mathsf{P}_n^*$ such that $F^*(u) = F^*(v)$. We show that $u = v$ by induction on an expression defining u. If $u = \mathrm{id}_{u'}$ for some $u' \in \mathsf{P}_{n-1}^*$, by Propositions 10 and 11, there exists $v' \in \mathsf{P}_{n-1}^*$ such that $v = \mathrm{id}_{v'}$. We thus have $F^*(u') = F^*(v')$ and $u' = v'$ by induction hypothesis. If $u = u_1 *_i u_2$ for some $i < n$ and i-composable $u_1, u_2 \in \mathsf{P}^*$, then by Proposition 11, there exists i-composable $v_1, v_2 \in \mathsf{P}^*$ such that $v = v_1 *_i v_2$ and $F^*(u_k) = F^*(v_k)$ for $k \in \{1, 2\}$, so that $u_k = v_k$ by induction hypothesis, and $u = v$. Finally, if $u = \mathsf{e}_{\mathsf{P},n}(g)$ for some $g \in \mathsf{P}_n$ then $v = \mathrm{e}_n^{\mathsf{P}}(h)$ for some $h \in \mathsf{P}_n$ by Proposition 7. But then, we have

$$\mathrm{e}_n^{\mathsf{Q}}(F_n(g)) = F^*(\mathrm{e}_n^{\mathsf{P}}(g)) = F^*(\mathrm{e}_n^{\mathsf{P}}(h)) = \mathrm{e}_n^{\mathsf{Q}}(F_n(h))$$

where $\mathrm{e}_n^{\mathsf{Q}} \circ F_n$ is a monomorphism by hypothesis and Remark 6. Thus, $g = h$ and $u = v$. Hence, (iii) holds. □

3 Makkai's criterion for presheaf categories

We now recall the criterion given by Makkai [38] to detect whether a category $\mathcal{C}$ is a presheaf category in the expected way, i.e., relatively to a concretization functor $\mathcal{C} \to \mathbf{Set}$. In the case of a presheaf category, the objects of the base category are recognized as the "suitably initial" elements of the concretization. Makkai used this criterion to show that polygraphs for strict categories do not form a presheaf categories in the expected way, where the concretization functor maps a polygraph to the set of all generators. We will use this criterion in Section 6 to prove that, in the case of precategories, we do get a presheaf category.

A *concrete category* is a category $\mathcal{C}$ endowed with a functor

$$|-|^{\mathcal{C}} \colon \mathcal{C} \to \mathbf{Set}.$$

The above *concretization functor* should be understood as a candidate set-theoretic representation of $\mathcal{C}$: for c an object of C, the set $|c|$ describes the candidate elements of the associated presheaf. The following canonical example should provide a good illustration of this intuition.

Example 15. Let C be a small category. $\hat{C}$ has a canonical structure of concrete category, where $|-|^{\hat{C}}$ is defined on preasheaves $P \in \hat{C}$ by

$$|P|^{\hat{C}} = \bigsqcup_{c \in C_0} P(c)$$

and extended naturally to morphisms between presheaves.

In the following, we will be interested in the concretization functor given by the following example:

Example 16. The functor $|-| \colon \mathbf{Pol}_\omega \to \mathbf{Set}$ which maps $\mathsf{P} \in \mathbf{Pol}_\omega$ to

$$|\mathsf{P}| = \bigsqcup_{k \in \mathbb{N}} \mathsf{P}_k$$

equips $\mathbf{Pol}_\omega$ with a structure of concrete category.

Later, we will study the properties of $\mathbf{Pol}_\omega$ equipped with the above concretization functor. Another concretization functor on $\mathbf{Pol}_\omega$ that will be of interest for us is given by the example below:

Example 17. There is a functor $|-|\colon \mathbf{Cat}_\omega \to \mathbf{Set}$ which maps $C \in \mathbf{Cat}_\omega$ to

$$|C| = \bigsqcup_{k\in\mathbb{N}} C_k$$

By precomposition with the functor $(-)^* : \mathbf{Pol}_\omega \to \mathbf{Cat}_\omega$, we obtain a functor $|(-)^*|\colon \mathbf{Pol}_\omega \to \mathbf{Set}$ which maps $\mathsf{P} \in \mathbf{Pol}_\omega$ to

$$|\mathsf{P}^*| = \bigsqcup_{k\in\mathbb{N}} \mathsf{P}^*_k$$

and also equips $\mathbf{Pol}_\omega$ with a structure of concrete category.

In order to distinguish between the two preceding structures of concrete category on $\mathbf{Pol}_\omega$, we use the convention that we write $\mathbf{Pol}_\omega$ when considering the concrete category structure on $\mathbf{Pol}_\omega$ given by $|-|$ and $\mathbf{Pol}^*_\omega$ when considering the concrete category structure on $\mathbf{Pol}_\omega$ given by $|(-)^*|$.

An *equivalence of concrete categories* between concrete categories $(\mathcal{C}, |-|^{\mathcal{C}})$ and $(\mathcal{D}, |-|^{\mathcal{D}})$ is the data of an equivalence of categories $\mathcal{E}\colon \mathcal{C} \to \mathcal{D}$ and a natural isomorphism

$$\Phi\colon |-|^{\mathcal{D}} \circ \mathcal{E} \Rightarrow |-|^{\mathcal{C}}.$$

When such an equivalence exists, $(\mathcal{C}, |-|^{\mathcal{C}})$ and $(\mathcal{D}, |-|^{\mathcal{D}})$ are said *concretely equivalent*. One might then consider the following natural question:

When is some concrete category $(\mathcal{C}, |-|^{\mathcal{C}})$ concretely equivalent
to a presheaf category $(\hat{C}, |-|^{\hat{C}})$ for some small category C?

When it is the case, we say that $(\mathcal{C}, |-|^{\mathcal{C}})$ is a *concrete presheaf category.*

Given a concrete category $(\mathcal{C}, |-|^{\mathcal{C}})$, the *category of elements* $\mathrm{Elt}(\mathcal{C})$ of $\mathcal{C}$ is the category

- whose objects are the pairs (X, x) where $X \in \mathcal{C}_0$ and $x \in |X|^{\mathcal{C}}$, and
- whose morphisms from (X, x) to (Y, y) are the morphisms $f\colon X \to Y \in \mathcal{C}$ such that $|f|^{\mathcal{C}}(x) = y$.

Given a morphism $f\colon (X, x) \to (Y, y)$ as above, we say that y is a *specialization* of x. An object $(X, x) \in \mathrm{Elt}(\mathcal{C})$ is *principal* when, for every morphism $f\colon (Y, y) \to (X, x) \in \mathrm{Elt}(\mathcal{C})$ such that f is a monomorphism in $\mathcal{C}$, we have that f is an isomorphism; it is *primitive* when it is principal and, for all $f\colon (Y, y) \to (X, x) \in \mathrm{Elt}(\mathcal{C})$ where (Y, y) is principal, f is an isomorphism.

Example 18. Let C be a small category and consider the canonical concrete category structure on $\hat{C}$ given by Example 15. Given $P \in \hat{C}$ and $c \in C$, we write $\iota_c : P(c) \to \bigsqcup_{c\in C} P(c)$ for the canonical injection. The category $\mathrm{Elt}(\hat{C})$ has

- as objects the pairs $(P, \iota_c(x))$ where $P \in \hat{C}$ and $x \in P(c)$, and
- as morphisms from $(P, \iota_c(x))$ to $(Q, \iota_d(y))$ the natural transformations $\alpha\colon P \Rightarrow Q$ such that $c = d$ and $\alpha_c(x) = y$.

Given $(P, \iota_c(x)) \in \mathrm{Elt}(\hat{C})$, we have the following.

- $(P, \iota_c(x))$ is principal when P is the smallest subpresheaf P' of P such that $x \in P'(c)$. In particular, for all $c \in C$, $(C(-,c), \iota_c(\mathrm{id}_c)) \in \mathrm{Elt}(\hat{C})$ is principal.
- $(P, \iota_c(x))$ is primitive when the natural transformation $\theta\colon C(-,c) \to P$ which maps id_c to x is an isomorphism.

The characterization of concrete presheaf categories given by Makkai is the following [38, Theorem 4]:

Theorem 19. *Let $(\mathcal{C}, |-|^{\mathcal{C}})$ be a concrete category. $\mathcal{C}$ is concretely equivalent to a presheaf category if and only if the following conditions are all satisfied:*

(a) $|-|^{\mathcal{C}}$ *reflects isomorphisms,*

(b) $\mathcal{C}$ *is cocomplete and* $|-|^{\mathcal{C}}$ *preserves all small colimits,*

(c) the collection of isomorphism classes of primitive elements of $\mathrm{Elt}(\mathcal{C})$ *is small,*

(d) for every element $(X, x) \in \mathrm{Elt}(C)$*, there is a morphism* $(U, u) \to (X, x)$ *for some primitive element* (U, u)*,*

(e) given two morphisms $f, g\colon (U, u) \to (X, x) \in \mathrm{Elt}(\mathcal{C})$ *where* (U, u) *is primitive, we have* $f = g$*,*

(f) given two morphisms $f\colon (U, u) \to (X, x)$ *and* $g\colon (V, v) \to (X, x)$ *of* $\mathrm{Elt}(\mathcal{C})$ *where both* (U, u) *and* (V, v) *are primitive, there is an isomorphism* $\theta\colon (U, u) \to (V, v)$ *such that* $g \circ \theta = f$*.*

4 The support function

It is often useful to consider the support of a cell in a precategory, which informally consists in the set of generators occurring in this cell. In particular, the support will allow us to retrieve some properties of a morphism of polygraphs F from the associated free functor F^*, which will turn out to be useful when studying polyplexes. A support function for free strict categories was already introduced by Makkai for his study of the word problem on these categories [38].

Given $n \in \mathbb{N} \cup \{\omega\}$ and an n-polygraph P, we define the *support function*

$$\mathrm{supp}^{\mathsf{P}}\colon |\mathsf{P}^*| \to \mathcal{P}(|\mathsf{P}|)$$

which to any cell in P^* associates a set of generators of P, by induction on $u \in \mathsf{P}^*$ as follows:

- if $u = g \in \mathsf{P}_0$, then $\mathrm{supp}(u) = \{g\}$,
- if $u = g \in \mathsf{P}_{k+1}$ for some $k < n$, then $\mathrm{supp}(u) = \{g\} \cup \mathrm{supp}(\mathrm{d}^-(g)) \cup \mathrm{supp}(\mathrm{d}^+(g))$,
- if $u = \mathrm{id}_{u'}$ for some $k < n$ and $u' \in \mathsf{P}^*_k$, then $\mathrm{supp}(u) = \mathrm{supp}(u')$,
- if $u = u_1 *_i u_2$ for some $0 < k_1, k_2 < n+1$, $i = \min(k_1, k_2) - 1$ and i-composable $u_1 \in \mathsf{P}^*_{k_1}$ and $u_2 \in \mathsf{P}^*_{k_2}$, then $\mathrm{supp}(u) = \mathrm{supp}(u_1) \cup \mathrm{supp}(u_2)$.

One can easily verify that supp respects the axioms of precategories, so that:

Lemma 20. *The function* supp *is well-defined.*

The function supp is moreover natural:

Lemma 21. *Let $n \in \mathbb{N} \cup \{\omega\}$ and $F\colon \mathsf{P} \to \mathsf{Q} \in \mathbf{Pol}_n$. Then, we have that* $\operatorname{supp}^{\mathsf{Q}} \circ |F^*| = |F| \circ \operatorname{supp}^{\mathsf{P}}$.

Proof. By induction on $u \in \mathsf{P}^*$. □

Given a polygraph P and a cell $u \in \mathsf{P}^*$, the support of u is always finite. By restricting P to the generators occurring in this support, on can show the following:

Proposition 22. *Given $n \in \mathbb{N} \cup \{\omega\}$, an n-polygraph P and $u \in \mathsf{P}^*$, there exist a finite n-polygraph $\tilde{P}$, a monomorphism $F\colon \tilde{P} \to \mathsf{P}$ and $\tilde{u} \in \tilde{P}^*$ such that $F^*(\tilde{u}) = u$ and* $\operatorname{supp}(\tilde{u}) = |\tilde{\mathsf{P}}|$.

Given $F\colon \mathsf{P} \to \mathsf{Q}$ and $u \in \mathsf{P}^*$, we write $F/u\colon \operatorname{supp}(u) \to \operatorname{supp}(F^*(u))$ for the restriction of F to the support and the image of the support of u.

Lemma 23. *Given a pair of parallel morphisms*

$$\mathsf{P} \underset{G}{\overset{F}{\rightrightarrows}} \mathsf{Q}$$

of $\mathbf{Pol}_\omega$ such that $F^*(u) = G^*(u)$ for some $u \in \mathsf{P}^*$, we have $F/u = G/u$.

Proof. By induction on n and a formula defining u.

- If $u = \alpha$ for some $\alpha \in \mathsf{P}$, then $F(\alpha) = G(\alpha)$. We then also have that $F^*(\partial^\epsilon(\alpha)) = G^*(\partial^\epsilon(\alpha))$ for $\epsilon \in \{-,+\}$, so that $F/\partial^\epsilon(u) = G/\partial^\epsilon(u)$ by induction. Thus, $F/\alpha = G/\alpha$.

- If $u = \mathrm{id}_{u'}$, then the property follows by induction hypothesis.

- If $u = u_1 *_i u_2$. Then, we have $F^*(u_1) *_i F^*(u_2) = G^*(u_1) *_i G^*(u_2)$. Writing $!_{\mathsf{P}}\colon \mathsf{P} \to \mathbf{1}$ for the terminal morphism in $\mathbf{Pol}_\omega$, we have $!^*_{\mathsf{Q}}(F^*(u_j)) = !^*_{\mathsf{P}}(u_j) = !^*_{\mathsf{Q}}(G^*(u_j))$ for $j \in \{1,2\}$. Since $!^*_{\mathsf{P}}$ is Conduché by Proposition 11, we have $F^*(u_j) = G^*(u_j)$ for $j \in \{1,2\}$. Thus, $F/u_j = G/u_j$ for $j \in \{1,2\}$ so that $F/u = G/u$. □

We have the following nice description of principal elements of $\operatorname{Elt}(\mathbf{Pol}_\omega)$ and $\operatorname{Elt}(\mathbf{Pol}^*_\omega)$ using support:

Lemma 24. *An element (P, u) of $\operatorname{Elt}(\mathbf{Pol}_\omega)$ (resp. $\operatorname{Elt}(\mathbf{Pol}^*_\omega)$) is principal if and only if* $\operatorname{supp}(u) = |\mathsf{P}|$.

Proof. Assume that (P, u) is principal. Then, by Proposition 22, there exist an element $(\tilde{\mathsf{P}}, \tilde{u})$ and $F\colon (\tilde{\mathsf{P}}, \tilde{u}) \to (\mathsf{P}, u)$ such that F is a monomorphism of $\mathbf{Pol}_\omega$ and $\operatorname{supp}(\tilde{u}) = |\tilde{\mathsf{P}}|$. Since (P, u) is principal, we have that F is an isomorphism. Thus, by Lemma 21, we have that $\operatorname{supp}(u) = |\mathsf{P}|$.

Conversely, assume that $\operatorname{supp}(u) = |\mathsf{P}|$. Let (Q, v) be an element and $F\colon (\mathsf{Q}, v) \to (\mathsf{P}, u)$ be a morphism where F is a monomorphism in $\mathbf{Pol}_\omega$. By Lemma 21, we have that $|F(\operatorname{supp}(v))| = \operatorname{supp}(u) = |\mathsf{P}|$. Thus, F_k is surjective for every $k \in \mathbb{N}$. Moreover, F_k is injective by Proposition 14, so that F_k is an isomorphism for every k. Since $|-|$ reflects isomorphisms (exercise to the reader), we have that F is an isomorphism. Thus, (P, u) is principal. □

Finally, as a consequence of Lemmas 23 and 24, we have:

Lemma 25. *Given a pair of parallel morphisms*

$$(\mathsf{P}, u) \underset{G}{\overset{F}{\rightrightarrows}} (\mathsf{Q}, v)$$

of $\operatorname{Elt}(\mathbf{Pol}^*_\omega)$ where (P, u) is principal, then $F = G$.

5 Polyplexes

We now introduce the construction of polyplexes for the cells of free precategories. Those are polygraphs representing composition shapes such that every such cell in a polygraph is the composite of a polyplex in a unique way. Polyplexes are themselves composed of plexes (see next section) which are polygraphs representing generators in a polygraph. These notions are due to Burroni [14], and were further developed by Henry [33].

Formally, a *polyplex* is an element $(\mathsf{P}, u) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$ which is primitive (for the concrete structure introduced in Example 17). Given an element (Q, v) in $\mathrm{Elt}(\mathbf{Pol}^*_\omega)$, a *polyplex lifting* is the data of a polyplex (P, u) and a morphism of elements $F\colon (\mathsf{P}, u) \to (\mathsf{Q}, v) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$.

The construction of polyplexes will be carried out by induction on a formula defining a cell. The inductive case of identities is handled by the following lemma:

Lemma 26. *Given an element $(\mathsf{P}, u) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$, (P, u) is a polyplex if and only if $(\mathsf{P}, \mathrm{id}_u)$ is a polyplex.*

Proof. By Lemma 24, (P, u) is principal if and only if $(\mathsf{P}, \mathrm{id}_u)$ is principal. So we can assume that both are principal.

Suppose that (P, u) is primitive. Let $F\colon (\mathsf{Q}, v) \to (\mathsf{P}, \mathrm{id}_u)$ be a morphism of elements where Q is principal. Then, by Proposition 7, we have that $v = \mathrm{id}_{v'}$ for some $v' \in \mathsf{Q}^*$, and, by compatibility of F^* with ∂^-, we moreover have $F^*(v') = u$. Since $\mathrm{supp}(v) = \mathrm{supp}(v')$, (Q, v') is still a principal element. Thus, F is an isomorphism since (P, u) is primitive. Hence, $(\mathsf{P}, \mathrm{id}_u)$ is primitive. The converse is similar. □

The lemmas and propositions until the end of this section, describing the remaining cases characterizing polyplexes for composites and generators together with global existence and unicity properties, are proved by <u>mutual induction</u> on a formula defining the cell u appearing in the statements. First, the case of generators:

Lemma 27. *Let $(U, u) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$. Then, the following are equivalent:*

(i) *(U, u) is a polyplex and there exist $\alpha \in U$ such that $u = \alpha$,*

(ii) *there exist polyplex liftings*
$$G^\epsilon\colon (U^\epsilon, u^\epsilon) \to (U, \partial^\epsilon(u))$$
for $\epsilon \in \{-, +\}$, principal elements (S, s) and (T, t), and morphisms
$$F^{\epsilon-}\colon (S, s) \to (U^\epsilon, \partial^-(u^\epsilon)) \qquad\qquad F^{\epsilon+}\colon (T, t) \to (U^\epsilon, \partial^+(u^\epsilon))$$
for $\epsilon \in \{-, +\}$, such that, considering the pushout
$$\begin{tikzcd} S \sqcup T \arrow[r, "{[F^{+-}, F^{++}]}"] \arrow[d, "{[F^{--}, F^{-+}]}"'] & U^+ \arrow[d, dotted, "\bar{G}^+"] \\ U^- \arrow[r, dotted, "\bar{G}^-"'] & \partial U \end{tikzcd}$$
(U, u) is isomorphic to $(\bar{U}, \bar{\alpha})$, where $\bar{U}$ is obtained from ∂U by adding a generator
$$\bar{\alpha} : \bar{G}^-(u^-) \to \bar{G}^+(u^+).$$

Proof. Suppose that (ii) holds. By the unicity of normal forms (Theorem 5), it is enough to show that $(\bar{U}, \bar{\alpha})$ is primitive. First, it is principal by Lemma 24 since

$$\mathrm{supp}(\bar{\alpha}) = \{\alpha\} \cup \mathrm{supp}(\bar{G}^-(u^-)) \cup \mathrm{supp}(\bar{G}^+(u^+)) = |\bar{U}|.$$

Second, consider a morphism $H \colon (V, v) \to (\bar{U}, \alpha)$ with (V, v) principal. By induction hypothesis on Proposition 29, we have polyplex liftings

$$H^\epsilon \colon (\tilde{U}^\epsilon, \tilde{u}^\epsilon) \to (V, \partial^\epsilon(v))$$

for $\epsilon \in \{-, +\}$. Since $H^*(\partial^\epsilon(v)) = \partial^\epsilon(\bar{u})$, by Proposition 30, we can assume that $(\tilde{U}^\epsilon, \tilde{u}^\epsilon) = (U^\epsilon, u^\epsilon)$ for $\epsilon \in \{-, +\}$. Since (S, s) is principal, we have, by Lemma 25

$$H^- \circ F^{--} = H^+ \circ F^{+-} \qquad \text{and} \qquad H^- \circ F^{-+} = H^+ \circ F^{++}.$$

Thus, we derive a morphism $\partial H' \colon \partial U \to V$ from the pushout. By unicity of normal forms, $v = \beta$ for some $\beta \in V$. Thus, $\partial H'$ can be extended to $H' \colon \bar{U} \to V$ by putting $H'(\alpha) = \beta$. Using Lemma 25, we can easily verify that H' is the inverse of H. Hence, $(\bar{U}, \bar{u})$ is a polyplex.

Now, assume that (i) holds. By induction hypothesis, there are polyplex liftings

$$G^\epsilon \colon (U^\epsilon, u^\epsilon) \to (\mathsf{P}, \partial^\epsilon(u))$$

for $\epsilon \in \{-, +\}$. By induction hypothesis on Proposition 29, there exists a polyplex lifting $F^{--} \colon (S, s) \to (U^-, \partial^-(u^-))$. Similarly, there is a polyplex lifting of $(U^+, \partial^-(u^+))$ and, since $(G^-)^*(\partial^-(u^-)) = (G^+)^*(\partial^-(u^+))$, by Proposition 30, it can be chosen to be of the form

$$F^{+-} \colon (S, s) \to (U^+, \partial^-(u^+)).$$

Similarly, there are polyplex liftings

$$F^{-+} \colon (T, t) \to (U^-, \partial^+(u^-)) \qquad \text{and} \qquad F^{++} \colon (T, t) \to (U^+, \partial^+(u^+)).$$

Writing F^ϵ for $[F^{\epsilon-}, F^{\epsilon+}]$ for $\epsilon \in \{-, +\}$, consider the pushout

$$\begin{tikzcd} S \sqcup T \arrow[r, "F^+"] \arrow[d, "F^+"'] & U^+ \arrow[d, dotted, "\bar{G}^+"] \\ U^- \arrow[r, dotted, "\bar{G}^-"'] & \partial U \end{tikzcd}$$

and write $\bar{U}$ for the ω-polygraph obtained from ∂U by adding a generator $\bar{\alpha} \colon \bar{G}^-(u^-) \to \bar{G}^+(u^+)$ (this is well-defined, since the definition of ∂U ensures that $\partial^\epsilon(\bar{G}^-(u^-)) = \partial^\epsilon(\bar{G}^+(u^+))$ for $\epsilon \in \{-, +\}$). By the first part, $(\bar{U}, \bar{\alpha})$ is a polyplex, and we easily deduce a polyplex lifting $H \colon (\bar{U}, \bar{\alpha}) \to (U, \alpha)$ from the above pushout. Since (U, α) is primitive, H is an isomorphism. Thus, (ii) holds. □

The next lemma deals with the case of composites of the polyplex construction:

Lemma 28. *Let $(U, u) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$, $u_1 \in U^*_k$, $u_2 \in U^*_l$ for some $k, l \in \mathbb{N}$, with u_1 and u_2 are i-composable for $i = \min(k, l)$. Then, the following are equivalent:*

(i) (U, u) is a polyplex and $u = u_1 *_i u_2$,

(ii) there exist a principal element (U', u') and polyplexes (U_1, u_1) and (U_2, u_2), and morphisms $F_j : U' \to U_j \in \mathbf{Pol}_\omega$ and $G_j : U_j \to U$ for $j \in \{1, 2\}$, such that

$$\begin{tikzcd} U' \arrow[r, "F_2"] \arrow[d, "F_1"'] & U_2 \arrow[d, "G_2"] \\ U_1 \arrow[r, "G_1"'] & U \end{tikzcd}$$

is a pushout diagram in $\mathbf{Pol}_\omega$, $F_1^*(u') = \partial_i^+(u_1)$, $F_2^*(u') = \partial_i^-(u_2)$ and $u = G_1(u_1) *_i G_2(u_2)$.

Proof. Suppose that (ii) holds. We have

$$\begin{aligned} \operatorname{supp}(G_1^*(u_1) *_i G_2^*(u_2)) &= \operatorname{supp}(G_1^*(u_1)) \cup \operatorname{supp}(G_2^*(u_2)) \\ &= G_1(\operatorname{supp}(u_1)) \cup G_2(\operatorname{supp}(u_2)) \\ &= G_1(|U_1|) \cup G_2(|U_2|) \\ &= |U| \end{aligned}$$

thus (U, u) is principal by Lemma 24. Now, consider $H : (\mathsf{R}, w) \to (U, u) \in \operatorname{Elt}(\mathbf{Pol}_\omega^*)$ with (R, w) principal. We have

$$H(|\mathsf{R}|) = H(\operatorname{supp}(w)) = \operatorname{supp}(H^*(w)) = \operatorname{supp}(u) = |U|$$

so that the functions $H_j : \mathsf{R}_j \to U_j$ are surjective for every j. Thus, H is an epimorphism. Since H^* is Conduché by Proposition 11 and $H^*(w) = G_1^*(u_1) *_i G_2^*(u_2)$, there exist unique w_1, w_2 such that $H^*(w_j) = G_j^*(u_j)$ for $j \in \{1, 2\}$ and $w = w_1 *_i w_2$. By induction hypothesis on Proposition 29, there exist polyplex liftings $H'_j : (\tilde{U}_j, \tilde{u}_j) \to (\mathsf{R}, w_j)$ for $j \in \{1, 2\}$. By induction hypothesis on Proposition 30, since both $(\tilde{U}_j, \tilde{u}_j)$ and (U_j, u_j) are polyplex liftings of $(U, G_j^*(u_j))$, we may assume that $(\tilde{U}_j, \tilde{u}_j) = (U_j, u_j)$ for $j \in \{1, 2\}$. By Lemma 25, we have $H'_1 \circ F_1 = H'_2 \circ F_2$, so that we obtain $H' : U \to \mathsf{R}$ from the pushout. We compute that

$$H'(u) = H'(G_1^*(u_1) *_i G_2^*(u_2)) = H'_1(u_1) *_i H'_2(u_2) = w_1 *_i w_2 = w.$$

Thus, using Lemma 25, we easily have that $H' \circ H = \mathrm{id}_{\mathsf{R}}$ and $H \circ H' = \mathrm{id}_U$. Hence, (U, u) is primitive.

Conversely, suppose that (i) holds. Then, by induction hypothesis on Proposition 29, there exist $G_k : (U_k, \bar{u}_k) \to (\mathsf{P}, u_k)$ with $(U_k, \bar{u}_k)$ primitive for $k \in \{1, 2\}$. By induction hypothesis on Proposition 29, there exist $\tilde{F}_k : (\tilde{U}_k, \tilde{u}_k) \to (U_k, \partial_i^{\epsilon(k)}(u_k))$ with $(\tilde{U}_k, \tilde{u}_k)$ primitive for $i \in \{1, 2\}$ and $\epsilon(1) = +$ and $\epsilon(2) = -$. In particular, $(\tilde{U}_1, \tilde{u}_1)$ and $(\tilde{U}_2, \tilde{u}_2)$ are both polyplex liftings of $(U, \partial_i^+(u_1))$. By induction hypothesis on Proposition 30, we can assume that $(\tilde{U}_1, \tilde{u}_1) = (\tilde{U}_2, \tilde{u}_2)$ and write $(\tilde{U}, \tilde{u})$ for this element. By Lemma 25, we have $G_1 \circ F_1 = G_2 \circ F_2$. Consider the pushout

$$\begin{tikzcd} \tilde{U} \arrow[r, "F_2"] \arrow[d, "F_1"'] & U_2 \arrow[d, dotted, "\bar{G}_2"] \\ U_1 \arrow[r, dotted, "\bar{G}_1"'] & \bar{U} \end{tikzcd}$$

By its universal property, we get a morphism $H\colon \bar{U} \to U$ from G_1 and G_2. By the first implication, $(\bar{U}, G_1^*(\bar{u}_1) *_i G_2^*(\bar{u}_2))$ is a primitive element. Moreover, H induces a morphism

$$H\colon (\bar{U}, G_1^*(\bar{u}_1) *_i G_2^*(\bar{u}_2)) \to (U, u_1 *_i u_2)$$

of $\mathrm{Elt}(\mathbf{Pol}^*_\omega)$. Thus, since $(U, u_1 *_i u_2)$ is primitive, H is an isomorphism. □

The previous lemmas lead to the following polyplex lifting existence property:

Proposition 29. *Given an element* $(\mathsf{P}, u) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$*, there exists a polyplex lifting*

$$F\colon (U, \bar{u}) \to (\mathsf{P}, u)$$

where $(U, \bar{u})$ *is primitive.*

Proof. We reason by case analysis on a formula for u.

- If $u = \mathrm{id}_{u'}$, then, by Lemma 26, the conclusion follows from induction hypothesis.
- If $u = u_1 *_i u_2$, then, by induction hypothesis, there are morphisms

  $$G^k\colon (U^k, \bar{u}_k) \to (\mathsf{P}, u)$$

  with $(U^k, \bar{u}_k)$ primitive for $k \in \{1, 2\}$. By induction hypothesis, there are polyplex liftings

  $$F^k\colon (\tilde{U}^k, \tilde{u}_k) \to (U^k, \partial_i^{\epsilon(k)}(\bar{u}_k))$$

  with $\epsilon(1) = +$ and $\epsilon(2) = -$. By induction hypothesis on Proposition 30, we can assume that $(\tilde{U}^1, \tilde{u}_1) = (\tilde{U}^2, \tilde{u}_2)$ and write $(\tilde{U}, \tilde{u})$ for this element. Since $(\tilde{U}, \tilde{u})$ is principal, we have $G^1 \circ F^1 = G^2 \circ F^2$. Consider the pushout

  $$\begin{tikzcd} \tilde{U} \arrow[r, "F^2"] \arrow[d, "F^1"'] & U^2 \arrow[d, dotted, "\bar{G}^2"] \\ U^1 \arrow[r, dotted, "\bar{G}^1"'] & \bar{U} \end{tikzcd}$$

  Then, by Lemma 28, $(\bar{U}, \bar{G}^1(\bar{u}_1) *_i \bar{G}^2(\bar{u}_2))$ is a polyplex, and the universal property of pushouts gives a polyplex lifting

  $$H\colon (\bar{U}, \bar{G}^1(\bar{u}_1) *_i \bar{G}^2(\bar{u}_2)) \to (\mathsf{P}, u).$$

- If $u = \alpha$ for some generator $\alpha \in \mathsf{P}$, by induction, there are polyplex liftings

  $$G^\epsilon\colon (U^\epsilon, u^\epsilon) \to (\mathsf{P}, \partial^\epsilon(u))$$

  for $\epsilon \in \{-, +\}$. By induction on Proposition 29, there exists a polyplex lifting

  $$F^{--}\colon (S, s) \to (U^-, \partial^-(u^-)).$$

  Similarly, there is a polyplex lifting of $(U^+, \partial^-(u^+))$ and, since

  $$(G^-)^*(\partial^-(u^-)) = (G^+)^*(\partial^-(u^+))$$

by Proposition 30, it can be chosen to be of the form

$$F^{+-}\colon (S,s) \to (U^+, \partial^-(u^+)).$$

Similarly, there are polyplex liftings

$$F^{-+}\colon (T,t) \to (U^-, \partial^+(u^-)) \qquad \text{and} \qquad F^{++}\colon (T,t) \to (U^+, \partial^+(u^+)).$$

Writing F^ϵ for $[F^{\epsilon-}, F^{\epsilon+}]$ for $\epsilon \in \{-,+\}$, consider the pushout

$$\begin{tikzcd} S \sqcup T \arrow[r, "F^+"] \arrow[d, "F^+"'] & U^+ \arrow[d, dotted, "\bar{G}^+"] \\ U^- \arrow[r, dotted, "\bar{G}^-"'] & \partial U \end{tikzcd}$$

and write U for the ω-polygraph obtained from ∂U by adding a generator

$$\bar{\alpha} : \bar{G}^-(u^-) \to \bar{G}^+(u^+)$$

(this is well-defined, since the definition of ∂U ensures that $\partial^\epsilon(\bar{G}^-(u^-)) = \partial^\epsilon(\bar{G}^+(u^+))$ for $\epsilon \in \{-,+\}$). By Lemma 27, $(U, \bar{\alpha})$ is a polyplex, and we easily deduce a polyplex lifting $H\colon (U, \bar{\alpha}) \to (\mathsf{P}, \alpha)$. □

Finally, we have the following uniqueness property of polyplex liftings:

Proposition 30. *Given two morphisms $L^1\colon (U^1, u_1) \to (\mathsf{P}, u)$ and $L^2\colon (U^2, u_2) \to (\mathsf{P}, u)$ of $\mathrm{Elt}(\mathbf{Pol}^*_\omega)$ where both (U^1, u_1) and (U^2, u_2) are primitive, there is an isomorphism $\Theta\colon (U^1, u_1) \to (U^2, u_2)$ such that $L^2 \circ \Theta = L^1$.*

Proof. We reason by case analysis on a formula for u.

- If $u = \mathrm{id}_{u'}$, then the conclusion follows from induction hypothesis on u.
- If $u = \alpha$ for some $\alpha \in \mathsf{P}$, then, by Lemma 27, U^1 and U^2 are obtained by adding respective top-level generators α^1 and α^2 to polygraphs ∂U^1 and ∂U^2, the latter being expressed as pushouts

$$\begin{tikzcd} S^i \sqcup T^i \arrow[r, "{[F^{i,+-}, F^{i,++}]}"] \arrow[d, "{[F^{i,--}, F^{i,-+}]}"'] & U^{i,+} \arrow[d, dotted, "\bar{G}^{i,+}"] \\ U^{i,-} \arrow[r, dotted, "\bar{G}^{i,-}"'] & \partial U^i \end{tikzcd}$$

for some principal (S^i, s^i), (T^i, t^i) and some primitive $(U^{i,-}, u^{i,-})$, $(U^{i,+}, u^{i,+})$ for $i \in \{1,2\}$ as in the statement of that lemma. In particular, $(U^{i,\epsilon}, u^{i,\epsilon})$ are polyplex liftings of $\partial^\epsilon(\alpha)$ for $i \in \{1,2\}$ and $\epsilon \in \{-,+\}$. By induction hypothesis, for $\epsilon \in \{-,+\}$, there are isomorphisms $\Theta^\epsilon\colon (U^{1,\epsilon}, u^{1,\epsilon}) \to (U^{2,\epsilon}, u^{2,\epsilon})$. Since (S^1, s^1) and (T^1, t^1) are principal, we can easily verify with Lemma 25 that

$$G^{2,-} \circ \Theta^- \circ [F^{1,--}, F^{1,-+}] = G^{2,+} \circ \Theta^+ \circ [F^{1,+-}, F^{1,++}]$$

so that we get a morphism $\partial\Theta\colon \partial U^1 \to \partial U^2$, which extends to a morphism $\Theta\colon U^1 \to U^2$ such that $\Theta(\alpha^1) = \alpha^2$. Symmetrically, a morphism $\Theta'\colon (U^2, \alpha^2) \to (U^1, \alpha^1)$ can be built. Using Lemma 25, we easily verify that Θ and Θ' are inverse of each other.

– If $u = u_1 *_i u_2$, we use the pushout description from Lemma 27 and this case is then handled just like the previous one. $\square$

Remark 31. A consequence of the existence and unicity properties above, together with Lemma 25, is that the functor $|(-)^*|\colon \mathbf{Pol}_\omega \to \mathbf{Set}$ of Example 17 is *familially representable* [15], i.e., can be expressed as a functor of the form

$$\bigsqcup_{i \in I} \operatorname{Hom}(U^i, -)\colon \mathbf{Pol}_\omega \to \mathbf{Set}.$$

Here, I is a set of representatives (U^i, u^i) of all polyplexes (considered up to isomorphism of elements in $\operatorname{Elt}(\mathbf{Pol}^*_\omega)$) of any dimension. Those can for instance be enumerated by constructing one polyplex liftings for each cell of the free precategory on the terminal polygraph. A similar description holds for the functor $(-)^*_k$, mapping a polygraph to the set of k-cells of the associated free precategory: the family I is now a set of representatives for the polyplexes of dimension k up to isomorphism.

Remark 32. A consequence of the canonicity of a polyplex liftings given by the above properties is that one can define a "polyplex measure" on the cells of free precategories. Let $\mathsf{P} \in \mathbf{Pol}_\omega$, and write $\mathbb{Z}\mathsf{P}$ for the free $\mathbb{Z}$-module on $|\mathsf{P}|$. Given $u \in \mathsf{P}^*$, one can define $\delta^{\mathsf{P}}(u)$ as follows. Consider a polyplex lifting $F\colon (V, v) \to (\mathsf{P}, u)$ and define $S_V \in \mathbb{Z}V$ by $S_V = \sum_{g \in V} g$. Then, one defines $\delta^{\mathsf{P}}(u)$ as $\mathbb{Z}F(S_V)$. The definition of $\delta^{\mathsf{P}}(u)$ does not depend on the choice of (V, v) by Lemma 25 and proposition 30. The question of the existence of a similar measure for free strict categories was raised by Makkai in [38]. Later, using the standard Eckmann–Hilton for strict categories, the non-existence of such a measure was proven [20, Proposition 2.5.2.13].

6 Polygraphs as a presheaf category

We can now use the results of the previous section in order to conclude that $\mathbf{Pol}_\omega$ is a (concrete) presheaf category on the base category (also called shape category) of *plexes*, which are the elementary shapes polygraphs are made of. In addition to the works of Burroni [14] and Henry [33], this notion was also studied by Makkai [38] under the name "computopes".

Formally, a *plex* is an element $(\mathsf{P}, u) \in \operatorname{Elt}(\mathbf{Pol}_\omega)$ which is primitive (for the concrete structure introduced in Example 16). Given an element (Q, v) in $\operatorname{Elt}(\mathbf{Pol}_\omega)$, a *plex lifting* is the data of a plex (P, u) and a morphism of elements $F\colon (\mathsf{P}, u) \to (\mathsf{Q}, v) \in \operatorname{Elt}(\mathbf{Pol}_\omega)$.

In order to relate the properties of plexes to the ones of polyplexes proved in the previous section, we first need to briefly discuss the link between $\operatorname{Elt}(\mathbf{Pol}_\omega)$ and $\operatorname{Elt}(\mathbf{Pol}^*_\omega)$. We write $\mathcal{U}\colon \operatorname{Elt}(\mathbf{Pol}_\omega) \to \operatorname{Elt}(\mathbf{Pol}^*_\omega)$ for the canonical embedding. First note that, as a consequence of Proposition 7(ii), that

Lemma 33. *The functor $\mathcal{U}$ is fully faithful.*

We then have the following.

Proposition 34. *Let $(\mathsf{P}, g) \in \operatorname{Elt}(\mathbf{Pol}_\omega)$. Then*

(1) (P, g) is principal if and only if $\mathcal{U}(\mathsf{P}, g)$ is principal,

(2) (P, g) is a plex if and only if $\mathcal{U}(\mathsf{P}, g)$ is a polyplex.

Proof. By Lemma 24, (1) holds. Suppose now that both (P, g) and $\mathcal{U}(\mathsf{P}, g)$ are principal. By Proposition 7(ii), $\mathcal{U}$ is fully faithful, so that it reflects isomorphisms. Thus, if $\mathcal{U}(\mathsf{P}, g)$ is a polyplex, then (P, g) is a plex. For the converse, note that if $f\colon (\mathsf{Q}, v) \to \mathcal{U}(\mathsf{P}, g)$ is a morphism of $\mathrm{Elt}(\mathbf{Pol}^*_\omega)$, then, by Proposition 7(ii), $v \in \mathsf{Q}$, so that

$$(\mathsf{Q}, v) = \mathcal{U}(\mathsf{Q}, v) \quad \text{and} \quad f = \mathcal{U}(f).$$

Hence, if (P, g) is a plex, then $\mathcal{U}(\mathsf{P}, g)$ is a polyplex. □

Theorem 35. *The category $\mathbf{Pol}_\omega$ is a concrete presheaf category.*

Proof. We verify that the various conditions of Makkai's criterion (Theorem 19) are satisfied.

(a) Clear from the definition of $\mathbf{Pol}_\omega$.

(b) A consequence of general properties satisfied by categories of polygraphs derived from a globular monad (see Propositions 1.3.3.7 and 1.3.3.15 of [20]).

(c) Since a primitive element (P, g) is principal, the polygraph P is finite. Thus, up to isomorphism, the sets P_i can be assumed to be subsets of $\mathbb{N}$. So that (c) holds.

(d) Given an element $(\mathsf{P}, g) \in \mathrm{Elt}(\mathbf{Pol}_\omega)$, by Proposition 29, there exists a polyplex lifting $F\colon (U, u) \to (\mathsf{P}, g)$ of $\mathcal{U}(\mathsf{P}, g)$. By Proposition 7(ii), we have that $u \in U$. Moreover, by Proposition 34(2), we have that (U, u) is a plex, so (d) holds.

(e) Given $f, g\colon (U, u) \to (X, x) \in \mathrm{Elt}(\mathbf{Pol}_\omega)$ with (U, u) a primitive plex, then we have that $\mathcal{U}(f), \mathcal{U}(g)\colon (U, u) \to (X, x) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$, so that $\mathcal{U}(f), \mathcal{U}(g)$ by Lemma 25, and $f = g$ by faithfulness.

(f) Given two morphisms $f\colon (U, u) \to (X, x)$ and $g\colon (V, v) \to (X, x)$ of $\mathrm{Elt}(\mathbf{Pol}_\omega)$ where both (U, u) and (V, v) are primitive, we have by Proposition 30 that there is an isomorphism

$$\theta\colon \mathcal{U}(U, u) \to \mathcal{U}(V, v) \in \mathrm{Elt}(\mathbf{Pol}^*_\omega)$$

such that $\mathcal{U}(g) \circ \theta = \mathcal{U}(f)$. We conclude by the full faithfulness of $\mathcal{U}$. □

Remark 36. Following Makkai's proof of [38, Theorem 4], the base category of the presheaf category given by the above theorem is a small full subcategory of $\mathbf{Pol}_\omega$, whose objects are (the underlying polygraphs of) plexes, and such that every (underlying polygraph of a) plex is isomorphic to exactly one object of this subcategory. The objects of the latter can thus be easily enumerated, since they are in correspondence with the generators of the terminal polygraph $\mathbf{1}$, as plex liftings.

Remark 37. Like the familial representability observed in Remark 31, the conditions (d) to (f) proved above entails a familial representability for the functor $|-|$ of Example 16, which can be expressed as

$$\coprod_{i \in I} \mathrm{Hom}(U^i, -)\colon \mathbf{Pol}_\omega \to \mathbf{Set}.$$

Here, I is a set of representatives (U^i, g^i) of all plexes (considered up to isomorphism of $\mathrm{Elt}(\mathbf{Pol}_\omega)$). By taking I to be a set of representatives of all plexes of dimension k for some $k \in \mathbb{N}$, one get a familial representability of the functor $(-)_k\colon \mathbf{Pol}_\omega \to \mathbf{Set}$.

Remark 38. In [5], Araújo relies on [18, Proposition 5.14], which gives sufficient conditions for a category to be a presheaf category on a given full subcategory. The difference with [38, Theorem 4] is that the latter is relative to a concrete presheaf structure, and is able to characterize the shape category as a full subcategory of primitive elements.

7 Parametric adjunction and genericic factorization

While Remark 31 asserts that the cells of free precategories on polygraphs are instances of "universal shapes" (i.e., polyplexes), a more conceptual and general syntactical result can be given, which encompasses both the existence of those universal shapes and the Conduché property of free functors. This result relies on the existence of a parametric adjunction and an associated generic factorization for the free functor $(-)^*$. Parametric adjunctions and generic factorizations appear frequently in the context of algebraic higher category [49, 53, 32]: for example, the free ω-category monad functor on globular sets is parametric right adjoint, and has an associated generic factorization. While the classical parametric right adjoints are monad functors on presheaf categories (for which characterization criteria have been developed, for example [54, Theorem 2.13]), the unusual fact here is that the parametric right adjoint $(-)^*$ is a left adjoint, whose codomain is not a presheaf category, but the category of n-precategories: for us, this fact reflects and summarize the good syntactical properties of the theory of precategories.

While parametric adjunctions can easily be deduced from familial representability properties (like Remark 31) in a presheaf setting (see [54, Proposition 2.10]), there is no direct criterion in our setting, so that we have to show the parametric representability "by hand": we need to show that the functor $(-)^*_{\mathbf{1}} \colon \mathbf{Pol}_\omega \to \mathbf{PCat}_\omega/\mathbf{1}^*$ is a right adjoint, where $\mathbf{PCat}_\omega/\mathbf{1}^*$ is the slice category of $\mathbf{PCat}_\omega$ over the free precategory on the terminal polygraph $\mathbf{1}$, and $(-)^*_{\mathbf{1}}$ the functor induced by $(-)^*$. Since both $\mathbf{Pol}_\omega$ and $\mathbf{PCat}_\omega/\mathbf{1}^*$ are locally presentable categories, and that $(-)^*$ is a left adjoint, we are only required to show that $(-)^*_{\mathbf{1}}$ preserves limits (see [1, Theorem 1.66]). Since $\mathbf{Pol}_\omega$ has a terminal object and the computation of limits in $\mathbf{PCat}_\omega/\mathbf{1}^*$ amounts to the computation of a connected limit in $\mathbf{PCat}_\omega$, we simply need to show that connected limits are preserved, recovering [54, Theorem 2.13] in our context. In the following, given $k, n \in \mathbb{N}$, we write $D^{k,n}$, or simply D^k for the free n-precategory with one non-identity k-cell.

Proposition 39. *Given $n \in \mathbb{N} \cup \{\omega\}$, the functor $(-)^* \colon \mathbf{Pol}_n \to \mathbf{PCat}_n$ preserves connected limits.*

Proof. First note that the functor $|-| \colon \mathbf{PCat}_n \to \mathbf{Set}$ is conservative; it is moreover familially representable by the D^k's for $k < n+1$ (a k-cell of an n-precategory C is the same thing as a functor $D^k \to C$) and thus preserves connected limits by [15, Theorem 2.5] and [1, Corollary 2.45]. Since $\mathbf{PCat}_n$ is complete, it is sufficient to show that the functor $|(-)^*| \colon \mathbf{Pol}_n \to \mathbf{Set}$ preserves connected limits. But this functor is familially representable by Remark 31, so that it preserves connected limits by [15, Theorem 2.5]. $\square$

By the argument exposed earlier, we can conclude that:

Theorem 40. *Given $n \in \mathbb{N} \cup \{\omega\}$, the functor $(-)^*_{\mathbf{1}} \colon \mathbf{Pol}_n \to \mathbf{PCat}_n/\mathbf{1}^*$ is a right adjoint. In other words, $(-)^*$ is a parametric right adjoint.*

As a consequence, we have a generic factorization for the functor $(-)^*$. We recall from [53] the notion of generic morphism in the present case: given $C \in \mathbf{PCat}_n$ and $\mathsf{P} \in \mathbf{Pol}_n$, a morphism $F \colon C \to \mathsf{P}^*$ is *generic* when, for any commutative square of the form

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "F"'] & \mathsf{Q}^* \arrow[d, "H^*"] \\ \mathsf{P}^* \arrow[r, "K^*"'] \arrow[ur, dotted, "L^*"] & \mathsf{R}^* \end{tikzcd}$$

for some $\mathsf{Q}, \mathsf{R} \in \mathbf{Pol}_n$, $G\colon C \to \mathsf{Q}^*$ in $\mathbf{PCat}_n$, $H\colon \mathsf{Q} \to \mathsf{R}$ and $K\colon \mathsf{P} \to \mathsf{R}$ in $\mathbf{Pol}_n$, there exists a unique $L\colon \mathsf{P} \to \mathsf{Q}$ such that $G = L^* \circ F$ and $K = H \circ L$. Now, given a morphism $F\colon C \to \mathsf{P}^*$, a *generic factorization* is a decomposition of F as $H^* \circ G$ for some $\mathsf{Q} \in \mathbf{Pol}_n$, some generic $G\colon C \to \mathsf{Q}^*$ and $H\colon \mathsf{Q} \to \mathsf{P} \in \mathbf{Pol}_n$. By the universal property of generic morphisms, such a decomposition is unique up to an isomorphism $\mathsf{Q} \to \mathsf{Q}'$.

Corollary 41. *Given $n \in \mathbb{N} \cup \{\omega\}$, $C \in \mathbf{PCat}_n$ and $\mathsf{P} \in \mathbf{Pol}_n$, every $F\colon C \to \mathsf{P}^* \in \mathbf{PCat}_n$ admits a generic factorization.*

Proof. By [54, Proposition 2.6], the existence of generic factorizations follows from the fact that $(-)^*_{\mathbf{1}}$ is a parametric right adjoint. ☐

Some generic morphisms are easy to identify:

Proposition 42. *Given $n \in \mathbb{N}$ and $\mathsf{P} \in \mathbf{Pol}_\omega$, writing u for the non-identity n-cell of D^n, a functor $F\colon D^n \to \mathsf{P}^*$ is generic if and only if $(\mathsf{P}, F(u))$ is a polyplex.*

Proof. We start with the first implication. Let $H\colon (\mathsf{Q}, v) \to (\mathsf{P}, F(u))$ be a polyplex lifting of $(\mathsf{P}, F(u))$. Then, writting $G\colon D^n \to \mathsf{Q}^*$ sending u to v, we have $H^* \circ G = (\mathrm{id}_{\mathsf{P}})^* \circ F$. Thus, there exists a unique lifting $L\colon \mathsf{P} \to \mathsf{Q}$ such that $L^* \circ F = G$ and $H \circ L = \mathrm{id}_{\mathsf{P}}$. In particular, we have that $L^*(F(u)) = v$ and L is a monomorphism. Thus, since (Q, v) is principal, $L\colon (\mathsf{P}, F(u)) \to (\mathsf{Q}, v)$ is an isomorphism.

Conversely, let

$$\begin{tikzcd} D^n \arrow[r, "G"] \arrow[d, "F"'] & \mathsf{Q}^* \arrow[d, "H^*"] \\ \mathsf{P}^* \arrow[r, "K^*"'] & \mathsf{R}^* \end{tikzcd}$$

be a commutative square where $(\mathsf{P}, F(u))$ is assumed to be a polyplex. Consider a polyplex lifting $L\colon (\bar{\mathsf{P}}, \bar{u}) \to (\mathsf{Q}, G(u))$. By applying H^*, $(\bar{\mathsf{P}}, \bar{u})$ is a polyplex lifting of $H^*(G(u)) = K^*(F(u))$ and so is $(\mathsf{P}, F(u))$. By Proposition 30, we may assume $(\bar{\mathsf{P}}, \bar{u}) = (\mathsf{P}, F(u))$ with $H \circ L = K$. Moreover, since $L^*(F(u)) = G(u)$, we have $L \circ F = G$ by freeness of D^n. Finally, the unicity of the lifting L of the above square is a consequence of Lemma 25. ☐

Remark 43. In a related manner, given $n \in \mathbb{N}$ and $v \in \mathbf{1}^*_n$, the image of $D^n \xrightarrow{v} \mathbf{1}^*$, seen as an object of $\mathbf{PCat}_\omega/\mathbf{1}^*$, by a left adjoint to $(-)^*_{\mathbf{1}}$ is the underlying polygraph of a polyplex lifting of v.

Remark 44. The above generic factorization can be seen as a stronger version of Proposition 11. Indeed, given $k, l > 0$ and $i = \min(k, l) - 1$, there exists a polygraph $\mathsf{D}^{k,l}$ such that $(\mathsf{D}^{k,l})^*$ is the free ω-precategory with one k-cell u_1 and one l-cell u_2, such that $\partial^+_i(u_1) = \partial^-_i(u_2)$. The construction of $\mathsf{D}^{k,l}$ can be seen to induce a polyplex $(\mathsf{D}^{k,l}, u_1 *_i u_2)$ by Lemma 28. Writting n for $\max(k, l)$ and $F^{k,l}\colon D^n \to (\mathsf{D}^{k,l})^*$ for the functor sending the non-trivial n-cell of D^n to $u_1 *_i u_2$, Proposition 11 amounts to observe that the $F^{k,l}$'s are generic by Proposition 42.

8 Toward homotopical properties of precategories

In this section, we report on failed attempts to study homotopical properties of categories, leaving open questions for future works.

A folk model structure on precategories? In the setting of strict n-categories, the usefulness of polygraphs can be explained by the facts that they are free objects such that every category admits a description by such an object, and any two descriptions are suitably equivalent. In more precise and modern terms, this was formalized by Lafont, Métayer and Worytkiewicz [37], who constructed a structure of model category on the category $\mathbf{Cat}_\omega$ of strict ω-categories, in which weak equivalences are the expected equivalences of ω-categories and cofibrant objects are ω-categories freely generated by polygraphs. One could expect that we could perform a similar construction on precategories, and construct a model structure where weak equivalences are the expected ones and cofibrant objects are polygraphs in the sense of this article. Whether this is possible or not is left as an open question, but explain here that a direct adaptation of the proof of [37] does not go through easily.

Let us first introduce some terminology. Given an ω-precategory C, we make the following coinductive mutual definitions:

- two cells $x, y \in C$ of the same dimension are *equivalent*, denoted $x \sim y$, when there exists an equivalence $u : x \to y$;
- a cell $u : x \to y$ is an *equivalence* when there exists $\bar{u} : y \to x$ such that $u * \bar{u} \sim \mathrm{id}_x$ and $\bar{u} * u \sim \mathrm{id}_y$.

We could then have hope for the following definition of weak equivalences. Given an ω-prefunctor $F : C \to D$, F is a *weak equivalence* when it is "essentially surjective in every dimension", i.e.,

- for every 0-cell $y \in D_0$, there exists $x \in C_0$ such that $Fx \sim y$,
- for every pair of parallel cells $u, u' \in C$ and cell $\bar{v} : F(u) \to F(u')$, there exists $v \in C$ such that $F(v) \sim \bar{v}$.

The above definitions directly generalize the ones for strict categories. The construction of the folk model structure on strict ω-categories then requires a *weak division property* [37, Lemma 5], which the authors present as being "crucial". The direct generalization of it in the setting of precategories is as follows:

Property 45 (Weak division)**.** *Given an ω-precategory C and an equivalence $u : x \to y \in C_1$, for any 1-cells $s, t : y \to z$ and for any 2-cell $w : u *_0 s \Rightarrow u *_0 t$,*

(a) there is a 2-cell $v : s \Rightarrow t$ such that $u *_0 v \sim w$,

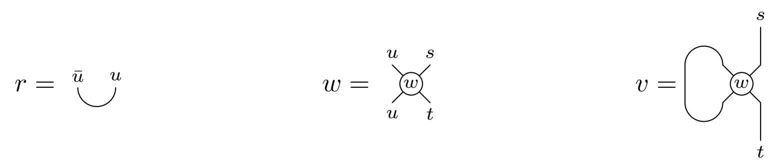
$$
\begin{tikzcd}
 & y \arrow[dr, bend left, "s"] & \\
x \arrow[ur, bend left, "u"] \arrow[dr, bend right, "u"'] & w \Downarrow & z \\
 & y \arrow[ur, bend right, "t"'] &
\end{tikzcd}
\qquad \sim \qquad
\begin{tikzcd}
x \arrow[r, "u"] & y \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "v \Downarrow"] & z
\end{tikzcd}
$$

(b) for any 2-cells $v, v' : s \Rightarrow t$ such that $u *_0 v \sim w \sim u *_0 v'$ we have $v \sim v'$.

We would also need a generalization of the above property for n-cells, but we will see that the proof of the stated property in dimension 1 already fails to generalize from strict categories to precategories. Consider cells u and w as in the above property, with u reversible, and let us try to define the cell v. Writing $r : \bar{u} *_0 u \to 1_x$ for the 2-cell witnessing that u is reversible, following [37], we are tempted to define v as

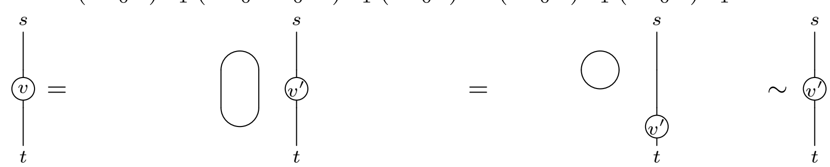
$$v = \overline{(r *_0 s)} *_1 (\bar{u} *_0 w) *_1 (r *_0 t)$$

If we picture r and w as on the left, v can be pictured as on the right:



In particular, in the case where w is of the form $w = u *_0 v'$ for some 2-cell $v' : s \Rightarrow t$, we should have $v \sim v'$ by (b). In the case of strict categories, this holds thanks to the interchange law:

$$v = \overline{(r *_0 s)} *_1 (\bar{u} *_0 u *_0 v') *_1 (r *_0 t) = \overline{(r *_0 s)} *_1 (r *_0 s) *_1 v' \sim v'$$



However, in the case of precategories there is no reason why this should hold. Of course, this does not directly imply that Property 45 does not hold or that there is no suitable model structure on precategories, but more work is required than a mere adaptation of [37]. The above also suggests that it could be interesting to investigate structures "in between" precategories and strict categories, where the interchange law is only required to hold for some morphisms (such as r in the above example).

A cone construction? Another homotopy-related question one might ask is whether the underlying shape category of the presheaf category of polygraphs of precategories is able to model homotopy types. A now standard approach to get a positive answer is to show that this shape category is a *weak test category* [27, 42], i.e., a category C whose presheaf category $\hat{C}$ can be equipped with a canonical class of weak equivalences $\mathcal{W}$, such that the induced localization $\hat{C}[\mathcal{W}^{-1}]$ is canonically isomorphic to the homotopy category **Hot**, so that, in particular, $\hat{C}$ models all homotopy types.

A common way to show that a category is a weak test category is to exhibit a *separating décalage* [42] on this category. Formally, a *décalage* on a catégorie C is given by a functor $D : C \to C$ together with natural transformations

$$1_C \xrightarrow{\alpha} D \xleftarrow{\beta} \top$$

where $\top$ is an object of C seen as a constant functor. Such a décalage is *separating* when we moreover have that

(a) for every $c \in C_0$, the arrow $\alpha_c : c \to D(c)$ is a monomorphism,

(b) α is *cartesian*: for every morphism $f : c \to c' \in C$, the diagram

$$\begin{tikzcd} c \arrow[r, "\alpha_c"] \arrow[d, "f"'] & D(c) \arrow[d, "D(f)"] \\ c' \arrow[r, "\alpha_{c'}"'] & D(c') \end{tikzcd}$$

is a pullback,

(c) for every $c \in C_0$, there is no commutative diagram of the form

$$\begin{tikzcd} c' \arrow[r, "g"] \arrow[d, "f"'] & \top \arrow[d, "\beta_c"] \\ c \arrow[r, "\alpha_c"'] & D(c) \end{tikzcd}$$

for some $c' \in C_0$ and $f\colon c' \to c$ and $g\colon c' \to \top$ in C_1.

Following Henry's line of proof for the case of regular plexes [32], a promising choice of décalage in a polygraphic setting is the one where D is "cone construction" functor, also called *expansion functor*: starting from a polygraph P, this functor adds to P a 0-generator o, a 1-generator $x \to o$ for each $x \in \mathsf{P}_0$, and more generally an $(i{+}1)$-generator for each i-generator of P, so that $D\mathsf{P}$ appears as a combinatorial description of a cone over the "space" defined by P. Then, continuing the definition of a décalage, one can take α to be the canonical embedding of a polygraph into the base of its cone, $\top$ to be the polygraph with only one 0-generator o, and β to be the marking of o as the top of each constructed cone.

While Henry [32] used the join of strict categories [3] to define the expansion functor on regular plexes, a more direct description of this construction was used by Ara et al. [2] in the case of strict categories that we unsuccessfully tried to adapt to precategories. In the following, we describe this attempt, hoping it can still benefit other settings. Write $\mathbf{PCat}_\omega^\bullet$ for the category of *pointed ω-precategories*, that is, the category whose objects are the pairs (C, o) where $o \in C_0$ and the morphisms $(C, o_C) \to (D, o_D)$ are the functor $F\colon C \to D$ such that $F(o_C) = o_D$. We have an evident adjunction

$$\begin{tikzcd} \mathbf{PCat}_\omega \arrow[r, bend left, "(-)\sqcup\{o\}"] \arrow[r, phantom, "\perp"] & \mathbf{PCat}_\omega^\bullet \arrow[l, bend left, "\mathrm{U}"] \end{tikzcd} \tag{1}$$

where U simply forgets the pointed 0-cell o. In order to define an expansion functor on precategories, one wants to introduce a functor

$$\Lambda\colon \mathbf{PCat}_\omega^\bullet \to \mathbf{PCat}_\omega^\bullet$$

such that $\Lambda(C, o)$ is the ω-precategory of *i-cones* on (C, o) for $i \in \mathbb{N}$: a 0-cone is some "base" 0-cell $x_{\mathrm{b}} \in C$ together with some 1-cell $x_{\mathrm{c}}\colon x_{\mathrm{b}} \to o$ of C, a 1-cone between $(x_{\mathrm{b}}, x_{\mathrm{c}})$ and $(y_{\mathrm{b}}, y_{\mathrm{c}})$ is a "base" 1-cell $f_{\mathrm{b}}\colon x \to y$ and $f_{\mathrm{c}}\colon f_{\mathrm{b}} *_0 y_{\mathrm{c}} \Rightarrow x_{\mathrm{c}}$, and so on. There is then a natural embedding $\gamma_{(C,o)}\colon \Lambda(C, o) \to (C, o)$, mapping every i-cone to its "base" i-cell. If such a functor exists, one could then define the category of *conic precategories* $\mathbf{PCat}_\omega^{\mathrm{C}}$ whose objects are the triples (C, o, σ), where (C, o) is a pointed ω-precategory and σ is a section of $\gamma_{(C,o)}$ satisfying adequate degeneracy conditions (see [2, Definition 2.2.1]), and whose morphisms are the ones of $\mathbf{PCat}^{\bullet}_{\omega}$ which adequately commute with the sections. In other words, an object of $\mathbf{PCat}^{\mathrm{C}}_{\omega}$ is a pointed ω-precategory with the data of a compatible i-cone for every i-cell, satisfying degeneracy conditions. The forgetful functor $\mathrm{V}\colon \mathbf{PCat}^{\mathrm{C}}_{\omega} \to \mathbf{PCat}^{\bullet}_{\omega}$ should then admit a left adjoint, so that we get an adjunction

$$\begin{tikzcd} \mathbf{PCat}^{\bullet}_{\omega} \arrow[r, bend left, "(-)^{\mathrm{C}*}"] \arrow[r, phantom, "\perp"] & \mathbf{PCat}^{\mathrm{C}}_{\omega} \arrow[l, bend left, "\mathrm{V}"] \end{tikzcd} . \tag{2}$$

The expansion functor is then the functor

$$\tilde{D} = \mathrm{U} \circ \mathrm{V} \circ (-)^{\mathrm{C}*} \circ ((-) \sqcup \{o\}) \colon \mathbf{PCat}_{\omega} \to \mathbf{PCat}_{\omega}$$

which is the underlying functor of the monad of the composition of the two adjunctions (1) and (2). Then, one could show that this functor restricts well to polygraphs (just like for the case of strict categories [2]), so that we get $D\colon \mathbf{Pol}_{\omega} \to \mathbf{Pol}_{\omega}$, and then show that D is the underlying functor of a separating décalage.

Sadly, the definition of Λ does not go through for precategories. Given a pointed ω-precategory (C, o), even though one can follow the concrete definition of [2] to get a globular set $\Lambda(C, o)$ equipped with precategorical compositions operations, one can show that the latter do not satisfy axiom (P-v) of precategories in general: the lack of interchange law for precategories is to blame here.

While it is not formally excluded that the shape category of plexes is a weak test category, the fact that it does not admit an expansion functor while the one of regular plexes does is already a bad sign which suggests, in addition to the difficulty to define a notion of weak equivalences with good properties (as discussed at the beginning of this section), that “bare” precategories are not an adequate tool for homotopical purposes (but that does not prevent them to be used to define other adequate tools, like Gray categories [23]). Maybe this could be linked to the fact that the underlying operad of precategories is not contractile, and should be better understood in future work.

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