The Cartesian Closed Bicategory of Thin Spans of Groupoids

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***Abstract*—Recently, there has been growing interest in bicategorical models of programming languages, which are "proof-relevant" in the sense that they keep distinct account of execution traces leading to the same observable outcomes, while assigning a formal meaning to reduction paths as isomorphisms.**

In this paper we introduce a new model, a bicategory called *thin spans of groupoids*. Conceptually it is close to Fiore et al.'s *generalized species of structures* and to Melliès' *homotopy template games*, but fundamentally differs as to how replication of resources and the resulting symmetries are treated. Where those models are *saturated* – the interpretation is inflated by the fact that semantic individuals may carry arbitrary symmetries – our model is *thin*, drawing inspiration from *thin concurrent games*: the interpretation of terms carries no symmetries, but semantic individuals satisfy a subtle invariant defined via biorthogonality, which guarantees their invariance under symmetry.

We first build the bicategory $\mathbf{Thin}$ of thin spans of groupoids. Its objects are certain groupoids with additional structure, its morphisms are spans composed via plain pullback with identities the identity spans, and its 2-cells are span morphisms making the induced triangles commute only up to natural isomorphism. We then equip $\mathbf{Thin}$ with a pseudocomonad $!$, and finally show that the Kleisli bicategory $\mathbf{Thin}_!$ is cartesian closed.

I. Introduction

The *relational model* [1] is one of the most basic and elementary denotational models for linear logic or the λ-calculus. At its heart, it is simply an interpretation of formulas / types as *sets* and proofs / programs as *relations*, *i.e.* in the category $\mathbf{Rel}$. Despite its simplicity the relational model is ubiquitous: it is the basic substrate for the spectrum of so-called *web-based* models of linear logic, including coherence or finiteness spaces [2]. It faithfully predicts reduction time [3]. It supports quantitative extensions such as in probabilistic coherence spaces [4], the weighted relational model [5], and even up to quantum computation [6] – quantitative extensions which enjoy powerful full abstraction results [7], [8]. Presented syntactically, the relational model exactly corresponds to *non-idempotent intersection types* [9], a currently active research topic in its own right (see *e.g.* [10], [11]) which enables a syntactic methodology to addressing semantic questions. Finally, it has a tight connection with *game semantics* [12], [13], of which it appears as a desequentialization (see *e.g.* [8], [14]–[16]). In short, it is at the crossroads of multiple topics, past and current, of the denotational semantics universe.

Another recent trend in denotational semantics is the adoption of *bicategorical models* [17] where the familiar categorical laws hold only up to certain 2-*cells* satisfying coherence conditions – in particular, Fiore and Saville have recently thoroughly explored *cartesian closed bicategories* [18]. In such models, the denotation is no longer an invariant of reduction: two convertible terms yield merely *isomorphic* objects, and *reduction paths* have a genuine interpretation as specific *isomorphisms* [19] – thus bringing reduction into the categorical model. There are still not many concrete bicategorical models, and we are aware of only three (families of) such models that can deal with non-linear computation, in chronological order: firstly, Fiore, Gambino, Hyland and Winskel's cartesian closed bicategory of *generalized species of structure* [20]; secondly, Castellan, Clairambault and Winskel's *thin concurrent games* [21] (as established by Paquet [22]); thirdly, Melliès' *homotopy template games* [23]. Of these three, the first is by far the most studied with various works including generalizations and application to semantics [24]–[26], links with intersection types and Taylor expansion [27], [28], or applications to the pure λ-calculus [29]. Beyond giving a non-degenerated interpretation to reduction paths, those concrete bicategorical models are "proof-relevant", in the sense that they keep distinct semantic witnesses for the possibly multiple evaluation traces with the same observable behaviour and thus keep a clear, branching account of non-determinism.

These models have something else in common: in their construction, the main subtlety has to do with replication, *i.e.* the modality $!$ of linear logic. In the relational model, $!A$ is the set $\mathcal{M}(A)$ of finite multisets of elements of A, or alternatively, the free monoid A^* quotiented by permutations. In bicategorical models, this is replaced by a *categorification* of $\mathcal{M}(A)$: a category (or groupoid) whose objects keep separate individual resource usages (*e.g.* A^*). Its morphisms are explicit permutations, often called *symmetries* in this paper. Individuals in the model must refer to specific resources (*e.g.* a_i in $a_1 \dots a_n \in A^*$), but the categorical laws expected for models of programming languages requires that their behaviour should still be invariant under symmetry. In both generalized species of structure and template games, this is done by *saturating* the set of witnesses with respect to symmetries: intuitively, the behaviour of an individual cannot depend on the specific identity of resources, because those resources are seen through the "noise" of all possible symmetries –

this shall be reviewed gently in Section II. This saturation complicates models and their construction, though for good reasons. But this contrasts with *thin concurrent games*, which handles symmetry with a mechanism inspired by Abramsky-Jagadeesan-Malacaria games [12] and Melliès' *orbital games* [15]: strategies are not saturated, but their invariance under "Opponent's symmetries" is ensured by a subtle bisimulation-like structure – we call this the *thin* approach.

We believe that the thin approach is helpful at least for applications to semantics: the absence of symmetries on witnesses allow a more concrete flavour which may help when ordering individuals allowing continuous reasoning[1], or simplify quantitative extensions such as [24]. But more fundamentally, there is a clear tension between these two worlds that deserves investigation. Are proof-relevant relational models inherently saturated? Is the thin approach only possible in games thanks to the presence of time and causality? These fundamental questions may be of interest beyond denotational semantics, as the handling of symmetry in such models is deeply connected to algebraic combinatorics [20] and homotopy theory [23].

a) Contributions: We introduce the bicategory $\mathbf{Thin}$ of *thin spans of groupoids*: its objects are certain groupoids with additional structure, its morphisms certain spans, and its 2-cells certain *weak* span morphisms, *i.e.* making the induced triangles commute up to chosen natural isos. Identities are identity spans, and composition of spans is by plain pullback.

Of course, plain pullbacks are too weak to support the horizontal composition of weak span morphisms. To address this, we first define *uniform spans* via a biorthogonality construction, ensuring that the composition pullbacks also satisfy the *bipullback* universal property. This allows us to compose 2-cells horizontally, but that horizontal composition is still not canonically defined and fails to give a bicategory.

For the next step, we import from thin concurrent games and from Melliès' orbital games a decomposition of symmetries into *positive* symmetries (due to the program), and *negative* symmetries (due to the environment). We then define *thin spans* via a second biorthogonality construction, which ensures that the horizontal composition of weak span morphisms are canonically defined as long as we consider *positive* weak span morphisms, where the chosen iso only involves positive symmetries. We show this results in a bicategory $\mathbf{Thin}$. Furthermore, we equip $\mathbf{Thin}$ with a pseudocomonad !, and show that the Kleisli bicategory $\mathbf{Thin}_!$ is cartesian closed.

b) Outline: In Section II we start with a gentle introduction to the relational model and its proof-relevant extensions. In Section III we introduce the bicategory $\mathbf{Thin}$, deploying first the uniform orthogonality and then the thin orthogonality. In Section IV we introduce the pseudocomonad !, and show that the Kleisli bicategory $\mathbf{Thin}_!$ is cartesian closed.

[1]For instance, in [29], the generalization from finite to infinite computation is not simply by continuity as per usual in denotational semantics, because of the quotient involved in the management of saturation.

II. Relational Models, Spans, Species

A. The Relational Model

The *relational model* is one of the simplest denotational models of the λ-calculus, linear logic, or simple programming languages such as PCF. It consists in simply interpreting every type A as a set $[\![A]\!]$, and a program $\vdash M : A$ as a subset of $[\![A]\!]$. This set $[\![A]\!]$ is often called the *web* seeing that it is the first component of the so-called web-based models of linear logic such as coherence spaces and their extensions. One may think of elements of $[\![A]\!]$ as completed executions (which is straightforward enough for ground types such as booleans or natural numbers but may be more complex for higher-order types), and of $[\![M]\!] \subseteq [\![A]\!]$ as simply the collection of all the completed executions that M may achieve.

Example 1. *The ground type for booleans is interpreted as* $[\![\mathbb{B}]\!] = \{\mathbf{tt}, \mathbf{ff}\}$, *and the constant* $\vdash \mathbf{tt} : \mathbb{B}$ *as* $[\![\mathbf{tt}]\!] = \{\mathbf{tt}\}$.

The interpretation of a program M is computed compositionally, following the methodology of denotational semantics, organized by the categorical structure of sets and relations.

1) Basic categorical structure: There is a category $\mathbf{Rel}$ with sets as objects, and as morphisms the *relations* from A to B, *i.e.* subsets $R \subseteq A \times B$. The identity on A is the diagonal relation $\{(a,a) \mid a \in A\} \subseteq A \times A$, and the composition of $R \subseteq A \times B$ and $S \subseteq B \times C$ consists in all pairs $(a,c) \in A \times B$ such that $(a,b) \in R$ and $(b,c) \in S$ for some $b \in B$.

Besides, $\mathbf{Rel}$ has a monoidal structure given by the cartesian product on objects, and for $R_i \in \mathbf{Rel}(A_i, B_i)$, $R_1 \times R_2 \in \mathbf{Rel}(A_1 \times A_2, B_1 \times B_2)$ set as comprising all $((a_1,a_2),(b_1,b_2))$ when $(a_i,b_i) \in R_i$ – the unit I is a fixed singleton set, say $\{*\}$. Additionally, $\mathbf{Rel}$ is *compact closed*: each set A has a dual A^* defined simply as A itself, and there are relations $\eta_A \in \mathbf{Rel}(I, A \times A)$ and $\epsilon_A \in \mathbf{Rel}(A \times A, I)$, both diagonal relations, satisfying coherence conditions [30]. In particular, $\mathbf{Rel}$ is $\star$-*autonomous* and as such a model of multiplicative linear logic, and the linear λ-calculus: the linear arrow type is interpreted as $[\![A \multimap B]\!] = [\![A]\!] \times [\![B]\!]$. Finally, $\mathbf{Rel}$ has finite products, with the binary product of sets A and B given by the disjoint union $A + B = \{1\} \times A \uplus \{2\} \times B$.

2) The exponential modality: The exponential modality of $\mathbf{Rel}$ is based on *finite multisets*. If A is a set, we write $\mathcal{M}(A)$ for the set of finite multisets on A. To denote specific multisets we use a list-like notation, as in *e.g.* $[0,1,1] \in \mathcal{M}(\mathbb{N})$ – we write $[] \in \mathcal{M}(A)$ for the empty multiset.

For A a set, its **bang** $!A$ is simply the set $\mathcal{M}(A)$. This extends to a comonad on $\mathbf{Rel}$, satisfying the required conditions to form a so-called **Seely category** – in particular, there is

$$\mathcal{M}(A+B) \cong \mathcal{M}(A) \times \mathcal{M}(B)$$

a bijection providing the *Seely isomorphism*. Altogether, this makes $\mathbf{Rel}$ a model of intuitionistic linear logic; and this makes the Kleisli category $\mathbf{Rel}_!$ cartesian closed so that we may interpret (among others) the simply-typed λ-calculus.

Example 2. *Considering the term* $\vdash M : \mathbb{B} \to \mathbb{B}$ *of* PCF

$$\vdash \lambda x^{\mathbb{B}}.\, \texttt{if}\, x\, \texttt{then}\, x \quad \texttt{else}\, \texttt{if}\, x\, \texttt{then}\, \mathbf{ff}\, \texttt{else}\, \mathbf{tt} : \mathbb{B} \to \mathbb{B}\,,$$

we have $[\![M]\!] = \{([\mathbf{tt},\mathbf{tt}],\mathbf{tt}), ([\mathbf{tt},\mathbf{ff}],\mathbf{ff}), ([\mathbf{ff},\mathbf{ff}],\mathbf{tt})\}$.

Here we can observe that the model is quantitative*, in that it records how many resources each execution consumes: one may observe output* $\mathbf{tt}$ *either with two evaluations of* x *to* $\mathbf{tt}$*, or with two evaluations of* x *to* $\mathbf{ff}$*. One may observe output* $\mathbf{ff}$ *with two evaluations of* x*, one to* $\mathbf{tt}$ *and one to* $\mathbf{ff}$*. Recall that in* $[\mathbf{tt},\mathbf{ff}] = [\mathbf{ff},\mathbf{tt}]$*, the order is irrelevant.*

The relational model also supports the interpretation of non-determinism: if $\vdash$ **choice** $: \mathbb{B}$ is a new primitive evaluating non-deterministically to $\mathbf{tt}$ or $\mathbf{ff}$, then we may simply set

$$[\![\mathbf{choice}]\!] = \{\mathbf{tt},\mathbf{ff}\}\,.$$

3) Extensions of the relational model: The relational model is extremely flexible, and can be extended in multiple different ways. In one direction one may add to the objects a *coherence relation* and restrict to compatible morphisms – we obtain in this way (multiset-based) *coherence semantics*.

Another extension is the *weighted relational model* [5], [31] where a term $\vdash M : A$, instead of denoting a subset of $[\![A]\!]$ – *i.e.* a function $[\![M]\!] : [\![A]\!] \to \{0,1\}$ – denotes a function

$$[\![M]\!] : [\![A]\!] \to \mathcal{R}$$

assigning to each point of the web $a \in [\![A]\!]$ a *weight* $[\![M]\!]_a \in \mathcal{R}$. The weight may be used to record additional information about executions. One may record the number of distinct non-deterministic branches leading to a certain result: for instance, if $\mathcal{R} = \mathbb{N} \cup \{+\infty\}$, then $[\![\texttt{if}\,\mathbf{choice}\,\texttt{then}\,\mathbf{tt}\,\texttt{else}\,\mathbf{tt}]\!]_{\mathbf{tt}} = 2$. With $\mathcal{R} = \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, we may track the *probability* with which a certain result occurs, obtaining a model fully abstract for probabilistic PCF [7]. The paper [5] contains other examples: resource consumption, must convergence, *etc*.

It is natural to go one step further and make the relational model "proof-relevant". This means not merely recording a weight or counting non-deterministic branches, but keeping track of a set $[\![M]\!]_a \in \mathbf{Set}$ of *witnesses* of the execution of M to a, for each $\vdash M : A$ and $a \in [\![A]\!]$. There are well-documented ways to do that which we shall review later on, but for now let us attempt this naively.

B. The Bicategory of Spans

A first idea is to simply replace *relations* with *spans*.

1) Spans: Recall that if $\mathcal{C}$ is a category with pullbacks, then we form $\mathbf{Span}(\mathcal{C})$ has having as objects those of $\mathcal{C}$, and as morphisms from A to B triples $(S, \partial_A^S, \partial_B^S)$ forming a diagram

$$A \quad \overset{\partial_A^S}{\leftarrow} \quad S \quad \overset{\partial_B^S}{\rightarrow} \quad B\,,$$

where intuitively S is a set of *internal witnesses*, projected to A and B via the maps ∂_A^S and ∂_B^S. For $\mathcal{C} = \mathbf{Set}$ one obtains a relation by collecting the pairs $(\partial_A^S(s), \partial_B^S(s))$ for $s \in S$, but we have more: for each pair $(a,b) \in A \times B$ we have

$$\mathsf{wit}^S(a,b) = \{s \in S \mid \partial_A^S(s) = a \;\&\; \partial_B^S(s) = b\}\,,$$

a set of **witnesses** that a and b are related – hence this indeed provides a notion of a *proof-relevant* relational model.

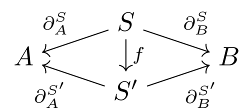
Example 3. *Writing* $\mathbb{B} = \{\mathbf{tt},\mathbf{ff}\}$ *and* $1 = \{*\}$*, we may represent the program* $\vdash \texttt{if}\,\mathbf{choice}\,\texttt{then}\,\mathbf{tt}\,\texttt{else}\,\mathbf{tt}$ *as*

$$1 \quad \overset{\partial_l}{\leftarrow} \quad \{a,b\} \quad \overset{\partial_r}{\rightarrow} \quad \mathbb{B}$$

a span*, where* $\partial_l(a) = \partial_l(b) = *$*,* $\partial_r(a) = \partial_r(b) = \mathbf{tt}$*.*

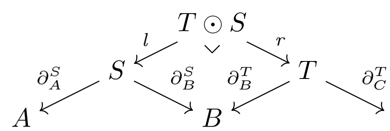
Thus, the evaluation of the program to $\mathbf{tt}$ has two witnesses.

2) A bicategory: The exact identity of S does not matter – the same span above with $S' = \{a', b'\}$ should not be treated distinctly. A **morphism** between spans is $f : S \to S'$ making



commute; an **isomorphism** of span is an invertible morphism.

The **identity span** on A is simply $A \leftarrow A \to A$ with two identity maps. The **composition** of $A \leftarrow S \to B$ and $B \leftarrow T \to C$ is obtained by first forming the pullback



(1)

and setting $\partial_A^{T\odot S} = \partial_A^S \circ l$ and $\partial_C^{T\odot S} = \partial_C^T \circ r$ – for $\mathbf{Span}(\mathbf{Set})$, this means that $T \odot S$ has elements all pairs (s,t) such that $\partial_B^S(s) = \partial_B^T(t)$, projected to A and C via $\partial_A^{T\odot S}((s,t)) = \partial_A^S(s)$ and $\partial_C^{T\odot S}((s,t)) = \partial_C^T(t)$.

This composition need not be associative on the nose, but the universal property of pullbacks entails that it is associative up to canonical isomorphism – forming a *bicategory*:

Theorem 1. *If* $\mathcal{C}$ *has pullbacks, then* $\mathbf{Span}(\mathcal{C})$ *defined with*

objects:	*objects of* $\mathcal{C}$*,*
morphisms:	*spans* $A \leftarrow S \to B$*,*
2-cells:	*morphisms of spans,*

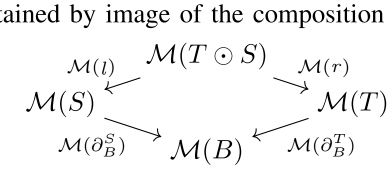
forms a bicategory, denoted $\mathbf{Span}(\mathcal{C})$.

In fact, $\mathbf{Span}(\mathcal{C})$ is a compact closed bicategory [32], and thus a model of the linear λ-calculus. In particular, $\mathbf{Span}(\mathbf{Set})$ shares much structure with $\mathbf{Rel}$: it has the same objects and the operation sending a span $A \leftarrow S \to B$ to the pairs $(\partial_A^S(s), \partial_B^S(s))$ for $s \in S$ is a functor, establishing $\mathbf{Span}(\mathbf{Set})$ as a natural candidate for a proof-relevant relational model.

3) The exponential: However, the exponential of $\mathbf{Rel}$ does not directly transport to $\mathbf{Span}$. The operation $\mathcal{M}(-)$ does yield a functor on $\mathbf{Set}$ obtained by setting, for $f : A \to B$,

$$\mathcal{M}(f)([a_1,\dots,a_n]) = [f(a_1),\dots,f(a_n)]$$

defining $\mathcal{M}(f) : \mathcal{M}(A) \to \mathcal{M}(B)$. But $\mathcal{M}(f)$ does not lift to $\mathbf{Span}(\mathbf{Set})$ as it does not preserve pullbacks. Indeed, the diagram obtained by image of the composition pullback



is no pullback: this would need a bijection of $\mathcal{M}(T \odot S)$ with

$$\{(\mu, \nu) \in \mathcal{M}(S) \times \mathcal{M}(T) \mid \mathcal{M}(\partial_B^S)(\mu) = \mathcal{M}(\partial_B^T)(\nu)\}\,,$$

which fails in general. If $S = T = \mathbb{B}$ and $B = 1$, the pair of multisets $([\mathbf{tt}, \mathbf{ff}], [\mathbf{tt}, \mathbf{ff}])$ does not uniquely specify who is synchronized with whom: it may correspond to both multisets $[(\mathbf{tt}, \mathbf{tt}), (\mathbf{ff}, \mathbf{ff})]$ and $[(\mathbf{tt}, \mathbf{ff}), (\mathbf{ff}, \mathbf{tt})]$ in $\mathcal{M}(T \odot S)$.

This might be expected: a finite multiset only remembers the multiplicity of elements, but does not track distinct individual occurrences. This is in tension with the goal of a proof-relevant relational semantics, for which specific witnesses are naturally associated with individual resource occurrences.

4) Categorifying objects: If the exponential is to track individual resource occurrences, that means avoiding the quotient of finite multisets: an element of $!A$ may for instance be a *list*, or a *word* $a_1 \dots a_n \in A^*$ of elements of A. We must of course still account for reorderings, which turn A^* into a *groupoid* – in fact, it is an instance of the construction of the *free symmetric monoidal category* $\mathbf{Sym}(A)$ over a category A: its objects are finite words $a_1 \dots a_n$ of objects of A, and a morphism from $a_1 \dots a_n$ to $a'_1 \dots a'_n$ consists of a permutation $\pi \in \mathcal{S}_n$, and a family $(f_i \in A(a_i, a_{\pi(i)}))_{1 \le i \le n}$.

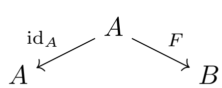
Thus, objects are not mere sets but categories, which means that we move from $\mathbf{Span}(\mathbf{Set})$ to $\mathbf{Span}(\mathbf{Cat})$. Indeed, $\mathbf{Cat}$ also has pullbacks, and so the exact same construction as above yields a bicategory $\mathbf{Span}(\mathbf{Cat})$ – except that now the functor $\mathbf{Sym} : \mathbf{Cat} \to \mathbf{Cat}$ preserves pullbacks and thus lifts to

$$\mathbf{Sym} : \mathbf{Span}(\mathbf{Cat}) \to \mathbf{Span}(\mathbf{Cat})\,.$$

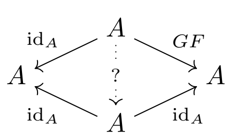
However, in this categorification, the Seely isomorphism $\mathcal{M}(A + B) \cong \mathcal{M}(A) \times \mathcal{M}(B)$ is lost. Instead, we only get

$$\mathbf{Sym}(A + B) \simeq \mathbf{Sym}(A) \times \mathbf{Sym}(B)$$

an *equivalence* of categories. In order to lift it to spans, we observe that given a functor $F : A \to B$ we get a span

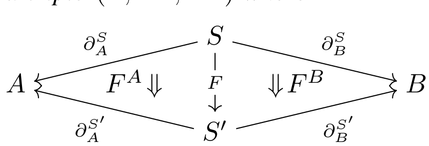
$\hat{F} \quad = \quad$  $\quad \in \quad \mathbf{Span}(\mathbf{Cat})(A, B)$

so that lifting an equivalence $F : A \simeq B : G$ to spans requires us to provide a family of 2-cells, *i.e.* for each category A:



however whatever our choice for the mediating map is, one of the triangles fails to commute on the nose but only up to isomorphism, which the 2-cells of $\mathbf{Span}(\mathbf{Cat})$ are too strict to accommodate. This invites weakening the 2-cells to:

Definition 1. *A **weak morphism** from $A \leftarrow S \to B$ to $A \leftarrow S' \to B$ is a triple (F, F^A, F^B) where*



*with $F^A : \partial_A^S \Rightarrow \partial_A^{S'} \circ F$ and $F^B : \partial_B^S \Rightarrow \partial_B^{S'} \circ F$ natural isos. We call this a **strong morphism** if F^A and F^B are identities.*

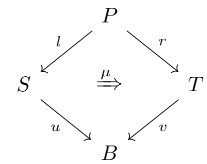


Fig. 1. A bipullback

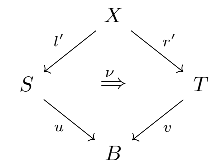
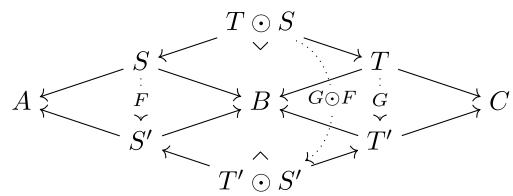


Fig. 2. Alternative square

Adopting *weak morphisms* seems to solve the problem above, but only to run into a much more subtle one: in $\mathbf{Span}(\mathbf{Cat})$, the horizontal composition of 2-cells $F : S \Rightarrow S'$ and $G : T \Rightarrow T'$ as required by the bicategorical structure follows from the universal property of the pullback $T' \odot S'$:



(2)

but this universal property is powerless to compose horizontally weak morphisms. We cannot have the cake and eat it too: if our method to compose spans ignores the 2-categorical nature of $\mathbf{Cat}$, then we cannot hope composition to preserve an equivalence between spans that relies on it, as required for a model of linear logic. So it seems that this road to a proof-relevant relational model is doomed – except that this is exactly what we shall do in this paper!

Before we delve into that, we review existing solutions.

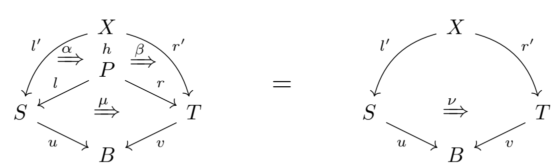
C. Proof-Relevant Relational Models, and Other Related Work

As plain pullbacks are "too 1-dimensional", it is natural to compose spans with a 2-dimensional version.

1) Bipullbacks: There are multiple variants for weakened versions of pullbacks in a 2-category. In this paper, a central notion will be that of a *bipullback*:[2]

Definition 2. *In a 2-category $\mathcal{C}$, a **bipullback** of the cospan $S \xrightarrow{u} B \xleftarrow{v} T$ is a square commuting up to an invertible 2-cell as in Figure 1, such that for any square as in Figure 2:*

(a) There is a morphism $h : X \to P$ and 2-cells α and β s.t.:



*(b) h, α, β are unique up to unique 2-cell – see **??**.*

The important observation is that this alternative universal property is sufficient to extend the definition of the horizontal composition in (2) to *weak morphisms* – with the proviso that

[2]According to the nlab, its proper name is a *bi-iso-comma-object*.

this defines horizontal composition only up to iso; as *(b)* does not guarantee uniqueness of h on the nose.

2) Hoffnung's monoidal tricategory: Hoffnung [33] constructs a categorification of $\mathbf{Span}(\mathbf{Cat})$ following this idea. He exploits that $\mathbf{Cat}$ actually has *pseudo-pullbacks*[3], which are a special case of the definition above where α and β are required to be identities and h is unique on the nose – making horizontal composition of weak morphisms of spans a well-defined function once a choice of pseudo-pullbacks is fixed.

Concretely, a pseudo-pullback of a cospan $S \xrightarrow{u} B \xleftarrow{v} T$ may be constructed as a category with objects triples (s, θ, t) where $\theta \in B(u(s), v(t))$. So for instance, if $S = T = \mathbf{Sym}(\mathbb{B})$ and $B = \mathbf{Sym}(1)$, the pseudo-pullback would have two objects synchronizing $[\mathbf{tt}, \mathbf{ff}] \in S$ and $[\mathbf{tt}, \mathbf{ff}] \in T$: $([\mathbf{tt}, \mathbf{ff}], \mathrm{id}, [\mathbf{tt}, \mathbf{ff}])$ and $([\mathbf{tt}, \mathbf{ff}, \mathrm{swap}, [\mathbf{tt}, \mathbf{ff}])$. The issue of Section II-B3 is avoided by adding new witnesses carrying all possible symmetries. This is a fundamental phenomenon in models of linear logic, which we refer to as *saturation*.

Because saturation *inflates* the number of witnesses at each composition, spans composed by pseudo-pullbacks no longer form a bicategory. In particular, the post-composition of a span $A \leftarrow S \rightarrow B$ with the identity span $B \leftarrow B \rightarrow B$ yields an inflated S' much bigger than S. So neutrality of identity no longer holds up to isomorphism, but only up to *equivalence* factoring in *maps between maps of spans*. Accordingly, Hoffnung actually constructs a *monoidal tricategory* of categorical spans with weak morphisms, *i.e.* a one-object *tetracategory*!

3) Melliès' template games: Recently, Melliès introduced *template games* [34], in an attempt to unify various games models. This is essentially a model of categorical spans where categories are regarded as games and structured by a projection to a category called the *template*, capturing the mechanisms of synchronization and scheduling between players. Though [34] was developped in a purely linear setting with spans related by strong morphisms, Melliès proposed a non-linear extension, forming a model of differential linear logic [23].

Melliès' contribution puts into play notions from *homotopy theory*: he starts not with $\mathbf{Cat}$, but from any 2-category $\mathbb{S}$ equipped with a Quillen model structure (with additional conditions). Spans are composed by mere pullbacks, but a span

$$A \xleftarrow{u} S \xrightarrow{v} B$$

must satisfy a fibration property to the effect that symmetries in A and B can be lifted uniquely in S – in our terminology, S is *saturated*. Saturation ensures that pullbacks between those spans are actually *homotopy pullbacks*, and thus that they may be used for the horizontal composition of weak morphisms. The higher dimensional structure seen in Hoffnung [33] is then managed by the homotopy-theoretic operation of *localization*, formally inverting weak equivalences. This yields an actual *bicategory* of objects of $\mathbb{S}$ related by *homotopy spans*.

This elegant construction gives a model of differential linear logic, showing that the symmetries implicit in linear logic may be naturally managed via the tools of homotopy theory.

[3] According to the nlab, its proper name is an *iso-comma-object*.

4) Generalized Species of Structures: Last but not least, the most well-studied proof-relevant extension of $\mathbf{Rel}$ is definitely Fiore, Gambino, Hyland and Winskel's cartesian closed bicategory of generalized species of structure [20]. Relations from A to B are replaced with *distributors* or *profunctors*:

$$F : A^{\mathrm{op}} \times B \to \mathbf{Set}$$

for A and B categories. This forms a (compact closed) bicategory $\mathbf{Dist}$ of (small) categories, distributors and natural transformations between them. The free symmetric monoidal construction $\mathbf{Sym}(-)$ yields a pseudocomonad on $\mathbf{Dist}$, whose Kleisli bicategory $\mathbf{Esp}$ is cartesian closed.

As for the span-based approaches above, the way in which $\mathbf{Dist}$ and $\mathbf{Esp}$ handle symmetries is saturated. This may first be seen in the identity distributor which is defined to be

$$A[-,-] : A^{\mathrm{op}} \times A \to \mathbf{Set}$$

the Yoneda embedding, which associates as witnesses over a pair (a, a) the homset $A[a, a]$, including all symmetries on a.

Composition of distributors is via the coend formula

$$G \odot F = \int^{b \in B} F(-, b) \times G(b, -)$$

which sets witnesses in $(G \odot F)(a, c)$ to be pairs $(s, t) \in F(a, b) \times G(b, C)$ quotiented by a relation identifying the action of a morphism in B on s or on t.

Accordingly, when computing the interpretation of a program $\vdash M : A$ in $\mathbf{Esp}$, for every $a \in [\![A]\!]$ we get $[\![M]\!](a)$ a set of witnesses carrying around explicit symmetries, quotiented by an equivalence relation letting symmetries flow around – this is described syntactically elegantly by Olimpieri [28]. The treatment of symmetry in $\mathbf{Esp}$ is, again, saturated.

5) Game semantics: To our knowledge, this saturation phenomenon in models of linear logic first appears in Baillot, Danos, Ehrhard and Regnier's (BDER) version [35] of Abramsky-Jagadeesan-Malacaria (AJM) games [12].

In AJM games, the *moves* of a game $!A$ are defined as pairs (i, a) of $i \in \mathbb{N}$ a **copy index**, and $a \in A$ a move in A – a fundamental difficulty in setting up the games model, is that of *uniformity*: ensuring that the behaviour of strategies does not depend on the specific choice of copy indices (which is the game semantics analogue of composition preserving weak morphisms). In BDER, uniformity is guaranteed by requiring strategies to be *saturated*: they are morally wrapped by copycat processes exchanging non-deterministically all copy indices. This "noise" prevents strategies from seeing specific copy indices, let alone depending on them – this is analogous to the saturation phenomenon above.

But in AJM games there is another choice: in the original AJM setting [12], strategies carry a deterministic choice of copy indices. Instead of saturation, uniformity is guaranteed by requiring that strategies satisfy a bisimulation-like property, which ensures that whenever Opponent swaps their copy indices, Player can swap theirs accordingly, leaving the behaviour "up to copy indices" invariant. In contrast to the "saturated" approach to uniformity, we refer to this as the "thin" approach. Similar ideas are at play in other game models based on copy indices: in Melliès' orbital games [15], and more recently in *thin concurrent games*[4] [21].

Thin concurrent games are a particularly striking related work, because just as $\mathbf{Esp}$, they *also* form a cartesian closed bicategory as proved by Paquet [22], and also generalize the relational model [37]. In thin concurrent games, strategies are composed by pullback. But it is a theorem that this pullback is also a bipullback, which can be used to compose horizontally weak morphisms even though strategies are not saturated. But this bipullback property follows from a subtle interactive reindexing mechanism between strategies, relying crucially on the fact that we have access to time – it seems hard to replicate it purely statically as in a relational model.

III. The Bicategory of Thin Spans

This long discussion lets us state the main question in this paper: can we construct a *thin* version of categorical spans?

A. Pullbacks and Bipullbacks in Groupoids

For simplicity, we focus on spans of *groupoids* rather than categories, which are sufficient for the interpretation of types – we write $\mathbf{Gpd}$ for the small 2-category of groupoids. So we aim to construct a bicategory whose objects are small groupoids, whose morphisms are spans $A \leftarrow S \rightarrow B$ with identity the identity span $A \leftarrow A \rightarrow A$, whose composition is plain pullback and yet, whose 2-cells are *weak* morphisms.

1) Notations and terminology: A span $A \leftarrow S \rightarrow B$ may be presented as a functor $S \rightarrow A \times B$, so it is convenient not to focus on spans, but on functors $S \rightarrow A$ over a groupoid A. We refer to those with terminology inspired from game semantics. A **prestrategy** on groupoid A is a pair (S, ∂^F) where $\partial^F : S \rightarrow A$ is called the **display map**. We often refer to the prestrategy only with S, and write $\mathsf{PreStrat}(A)$ for the set of prestrategies on A. A **prestrategy from** A **to** B is a prestrategy on $A \times B$ – then, we write $\partial^F_A : S \rightarrow A$ and $\partial^F_B : S \rightarrow B$ for the two display maps. If S is a prestrategy from A to B and T a prestrategy from B to C, we write $T \odot S$ for the prestrategy from A to C obtained as in Section II-B2. We often refer to morphisms in groupoids as **symmetries** and write *e.g.* $\varphi : s \cong_S s'$ instead of $\varphi \in S(s, s')$.

We write 1 for the groupoid with one object $*$, and only the identity morphism; and o for the groupoid with one object $\bullet$ and only the identity morphism. If A, B are groupoids, then we use $A \vdash B$ and $A \multimap B$ as synonyms for $A \times B$, with objects respectively denoted by $a \vdash b$ and $a \multimap b$ – likewise, their morphisms have form $\theta_A \vdash \theta_B \in (A \multimap B)(a \multimap b, a' \multimap b')$ for $\theta_A \in A(a, a')$ and $\theta_B \in B(b, b')$ and likewise for $\theta_A \multimap \theta_B$. We find these purely notational distinctions useful to read examples, since they coincide with familiar type constructors.

2) Indexed families: As explained earlier, types of λ-calculi may be interpreted as groupoids – but in a linear language, these groupoids remain *discrete*: only the exponential introduces non-trivial morphisms. As those *symmetries* play a crucial role, we introduce early our version of the exponential construction. If X is a set, then we write $\mathrm{Fam}(X)$ the set of families indexed by *finite sets of natural numbers*, *i.e.* of $(x_i)_{i \in I}$ where $I \subseteq_f \mathbb{N}$ and for all $i \in I$, $x_i \in X$.

Definition 3. *Consider A a (small) groupoid. The (small) groupoid $\mathbf{Fam}(A)$ has:* objects*, the set $\mathrm{Fam}(A)$;* morphisms *from $(a_i)_{i \in I}$ to $(b_j)_{j \in J}$, pairs $(\pi, (f_i)_{i \in I})$ of a bijection $\pi : I \simeq J$ and for each $i \in I$, $f_i \in A(a_i, b_{\pi(i)})$.*

This yields a functor $\mathbf{Fam} : \mathbf{Gpd} \rightarrow \mathbf{Gpd}$ in the obvious way. For $(A_i)_{i \in I} \in \mathrm{Fam}(A)$, we call elements of I **copy indices**. A family $(a_i)_{i \in I} \in \mathrm{Fam}(A)$ is more "intensional" than A^* (which is more intensional than $\mathcal{M}(A)$): it gives a presentation of a multiset in $\mathcal{M}(A)$ not only providing a sequence, but assigning to each element a distinct "address".

Just as multisets are connected to non-idempotent intersection types, families are connected with Vial's *sequence types* [38] – thus we often write families using Vial's notation, *e.g.*

$$[2 \cdot a_2, 4 \cdot a_4, 12 \cdot a_{12}] \in \mathrm{Fam}(A)$$

for $(a_i)_{i \in \{2,4,12\}}$ – in the particular case where $A = o$, we only write $[i_1, \dots, i_n]$ for $[i_1 \cdot \bullet, \dots, i_n \cdot \bullet]$.

For any groupoid A, $\mathbf{Fam}(A)$ and $\mathbf{Sym}(A)$ are equivalent. However, using $\mathbf{Fam}(A)$ is crucial in our model construction: it allows the interpretation of programs to use copy indices as *identifiers* for resource accesses, that are independent of other concurrent resource accesses. We give a few examples:

Example 4. *For a groupoid A, the **dereliction** span $\mathbf{der}_A$ is*

$$\mathbf{Fam}(A) \xleftarrow{\mathrm{der}_A} A \xrightarrow{\mathrm{id}_A} A$$

where $\mathrm{der}_A : A \rightarrow \mathbf{Fam}(A)$ sends a to $[0 \cdot a]$.

In models of linear logic, the role of dereliction is to extract a *single* instance of a replicable resource. In our model – as in AJM games [12] and thin concurrent games [21] – dereliction does so by picking a copy index (here 0), chosen in advance once and for all. The specific choice is irrelevant; in fact for any n the span using n instead of 0 will turn out to be isomorphic to $\mathbf{der}_A$. But, the span must comprise a choice.

Example 5. *The thin span interpreting the term M of Example 2 shall have a "head" groupoid with four objects*

$$[0 \cdot \mathbf{tt}, 1 \cdot \mathbf{tt}] \multimap \mathbf{tt}\,, \qquad [0 \cdot \mathbf{ff}, 1 \cdot \mathbf{ff}] \multimap \mathbf{tt}\,,$$
$$[0 \cdot \mathbf{tt}, 1 \cdot \mathbf{ff}] \multimap \mathbf{ff}\,, \qquad [0 \cdot \mathbf{ff}, 1 \cdot \mathbf{tt}] \multimap \mathbf{tt}\,,$$

morphisms reduced to identities, and display map the identity.

The use of specific copy indices allows one to observe which occurrence of x evaluates to $\mathbf{tt}$ or $\mathbf{ff}$, hence associating distinct points to the two evaluations leading to $\mathbf{ff}$.

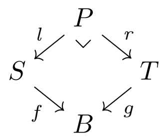
3) Bipullbacks of groupoids: If composition-by-pullback is to allow us to compose horizontally weak morphisms, we must ensure that every composition pullback is *also* a bipullback.

It is useful to understand a bit better the shape of bipullbacks in $\mathbf{Gpd}$. A first useful fact is that condition (b) of Definition 2 (uniqueness up to iso) automatically holds in the case of $\mathbf{Gpd}$;

[4] The first version of concurrent games with symmetry was saturated [36].

furthermore, we can characterise those pullbacks that are also bipullbacks (see **??**):

Lemma 1. *A pullback square in* $\mathbf{Gpd}$*, of the form*



is a bipullback if and only if it satisfies the following property: for all $s \in S$*,* $t \in T$ *and* $\theta \in B(fs, gt)$*, there is* $\varphi \in S(s, s')$ *and* $\psi \in T(t', t)$ *such that* $fs' = gt'$ *and* $\theta = f\psi \circ g\varphi$*.*

Let us comment on this. We regard triples of the form

$$s \in S\,, \qquad \theta \in B(fs, gt)\,, \qquad t \in T$$

as pairs of states (s, t) that match *up to symmetry* – we call this a **reindexing problem**. The lemma above says that given a reindexing problem, we can always find s' symmetric to s and t' symmetric to t matching *on the nose*, in a way compatible with θ – called a **solution** to the reindexing problem. Thus, the lemma above may be reformulated to say that a pullback is a bipullback iff all its reindexing problems have a solution.

We show a concrete example of this reindexing process:

Example 6. *Take* $B = \mathbf{Fam}(o) \multimap \mathbf{Fam}(o)$*, with objects*

$$[i_1, \ldots, i_n] \multimap [j_1, \ldots, j_k]\,.$$

Take S *the sub-groupoid of* B *with objects* $[i_1, \ldots, i_n] \multimap [i_1, \ldots, i_n]$ *and morphisms all* $\theta \multimap \theta$ *for* θ *in* $\mathbf{Fam}(o)$*; and* T *the full sub-groupoid of* B *with objects* $[j_1, \ldots, j_n] \multimap [0]$*.*

The pullback of $S \to B \leftarrow T$ *is a bipullback. For instance,*

$$\theta \in B([2] \multimap [2], [1] \multimap [0])$$

is a reindexing problem that may be solved by first applying

$$\varphi \in S([2] \multimap [2], [0] \multimap [0])$$

in S*. We are reduced to finding morphisms in* S *and* T w.r.t.

$$\theta' \in B([0] \multimap [0], [1] \multimap [0])$$

Now, applying $\psi \in T([0] \multimap [0], [1] \multimap [0])$ *in* T*, we have*

$$\varphi \in S([2] \multimap [2], [0] \multimap [0])\,, \quad \psi \in T([0] \multimap [0], [1] \multimap [0])$$

a solution to the reindexing problem, as in Lemma 1.

That the pullback of two prestrategies forms a bipullback is not a property of either: in this example neither strategy is a fibration as in [23], and solving the reindexing problem requires reindexing in *both* groupoids. So it is a property emerging from the harmonious interaction between two prestrategies. In an appropriate game semantics setting [21], one can *prove* that under reasonable assumptions, such interactive reindexing always succeeds. However, this is a gradual process progressing over time – which we do not have access to here.

B. Orthogonality and Uniform Groupoids

1) Definition: In the literature on models of linear logic, there is a technique for choreographing models where one only composes pairs of morphisms satisfying a given interactive property: *biorthogonality*. The first step is to specify the desired interactive property via an orthogonality relation:

Definition 4. *Take prestrategies* (S, ∂^S) *and* (T, ∂^T) *on* B*.*

We say they are ***uniformly orthogonal****, written* $S \perp T$*, iff the pullback of the cospan* $S \to B \leftarrow T$ *is also a bipullback.*

If $\mathbf{S} \subseteq \mathsf{PreStrat}(B)$, then its **uniform orthogonal** is set to:

$$\mathbf{S}^\perp = \{T \in \mathsf{PreStrat}(B) \mid \forall S \in \mathbf{S},\ S \perp T\}.$$

As usual with orthogonality, this automatically entails a number of properties: for all $\mathbf{S} \subseteq \mathsf{PreStrat}(B)$, we have $\mathbf{S} \subseteq \mathbf{S}^{\perp\perp}$, and $\mathbf{S}^\perp = \mathbf{S}^{\perp\perp\perp}$. We are particularly interested in sets of the form $\mathbf{S}^\perp$, which are *invariant under biorthogonal*:

Definition 5. *A* ***uniform groupoid*** *is a pair* $(A, \mathbf{U}_A)$ *where* A *is a groupoid and* $\mathbf{U}_A \subseteq \mathsf{PreStrat}(A)$ *is s.t.* $\mathbf{U}_A^{\perp\perp} = \mathbf{U}_A$*.*

We often refer to a uniform groupoid $(A, \mathbf{U}_A)$ just with A when it is clear from the context that it is a uniform groupoid.

2) Constructions: The uniform groupoid 1 is the terminal groupoid equipped with $\mathbf{U}_1 = \mathsf{PreStrat}(1)$. If A and B are uniform groupoids, their **tensor** $A \otimes B$ is the groupoid $A \times B$ equipped with the set $\mathbf{U}_{A\otimes B} = (\mathbf{U}_A \otimes \mathbf{U}_B)^{\perp\perp}$, writing

$$\mathbf{U}_A \otimes \mathbf{U}_B = \{(S \times T, \partial^S \times \partial^T) \mid S \in \mathbf{U}_A,\ T \in \mathbf{U}_B\}$$

with $\partial^S \times \partial^T : S \times T \to A \times B$. The **dual** $A^\perp$ of A is $(A, \mathbf{U}_{A^\perp})$ with $\mathbf{U}_{A^\perp} = \mathbf{U}_A^\perp$. The **par** of A and B has

$$\mathbf{U}_{A \parr B} = (\mathbf{U}_A^\perp \otimes \mathbf{U}_B^\perp)^\perp$$

yielding the De Morgan duality $(A \otimes B)^\perp = A^\perp \parr B^\perp$. From this we derive the **linear arrow** $A \multimap B = A^\perp \parr B$.

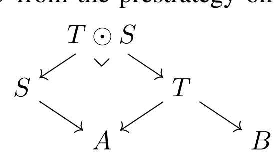
A **uniform prestrategy** on uniform groupoid A is simply any $S \in \mathbf{U}_A$. If A, B are uniform groupoids, then a **uniform prestrategy from** A **to** B is a uniform prestrategy on $A \multimap B$.

3) Uniform composition: We claim that whenever composing $S \in \mathbf{U}_{A\multimap B}$ with $T \in \mathbf{U}_{B\multimap C}$, we have the orthogonality

$$(S, \partial^S_B) \perp (T, \partial^T_B)$$

so that the composition pullback is a bipullback.

If S is a prestrategy on A and T is a prestrategy from A to B, we write $T@S$ from the prestrategy on B obtained by



called the **application** of T to S. This lets us state:

Proposition 1. *Consider* $(A, \mathbf{U}_A)$ *and* $(B, \mathbf{U}_B)$ *uniform groupoids, and* T *a prestrategy from* A *to* B*; consider furthermore a class* $\mathbf{S} \subseteq \mathbf{U}_A$ *s.t.* $(A, \mathrm{id}_A) \in \mathbf{S}$ *and* $\mathbf{U}_A = \mathbf{S}^{\perp\perp}$*.*

Then $T \in \mathbf{U}_{A\multimap B}$ *iff the following two conditions hold:*

(1) for all $S \in \mathbf{S}$*,* $T@S \in \mathbf{U}_B$*,*

(2) $(T, \partial^T_A) \in \mathbf{U}_A^\perp$.

Proof. Unfolding the definitions, one encounters a few diagram chasing lemmas on pullbacks that are also bipullbacks – themselves proved via Lemma 1. See **??**. ☐

The apparent asymmetry is intriguing: by definition $A^\perp \parr B = A^\perp \parr B^{\perp\perp}$, so that $T \in \mathbf{U}_{A \multimap B}$ iff the span $B \leftarrow T \rightarrow A$ denoted by $T^\star$ obtained by reversing the two legs, is in $\mathbf{U}_{B^\perp \multimap A^\perp}$. A similar phenomenon appears in the orthogonality used by Ehrhard for his extensional collapse [39].

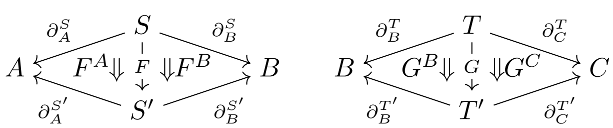
Now, observe that $(A, \mathrm{id}_A) \in \mathbf{U}_A$ always – not the identity span, but the identity functor regarded as a prestrategy on A. Indeed, if $S \in \mathbf{U}_A^\perp$, then the pullback of $A \rightarrow A \leftarrow S$ is clearly a bipullback, so $(A, \mathrm{id}_A) \in \mathbf{U}_A^{\perp\perp} = \mathbf{U}_A$. But now this lets us instantiate Proposition 1 with $\mathbf{S} = \mathbf{U}_A$. Then given $S \in \mathbf{U}_{A \multimap B}$, the application $S@(A, \mathrm{id}_A)$ is (up to iso) the right leg (S, ∂^S_B), which must by *(1)* be in $\mathbf{U}_B$. Likewise, if $T \in \mathbf{U}_{B \multimap C}$, the left leg (T, ∂^T_B) is in $\mathbf{U}_B^\perp$. Hence,

$$(S, \partial^S_B) \quad \perp \quad (T, \partial^T_B)$$

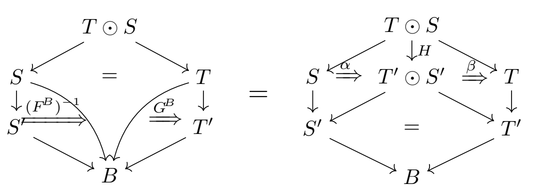
and thus the composition pullback of S and T is a bipullback.

Proposition 1 has more consequences, all obtained in the particular case where $\mathbf{S} = \mathbf{U}_A$: we saw above that $(A, \mathrm{id}_A) \in \mathbf{U}_A$, but the same argument goes to show $(A, \mathrm{id}_A) \in \mathbf{U}_A^\perp$ as well – so the identity span satisfies condition *(2)*. Since it also satisfies *(1)*, we have $(A \leftarrow A \rightarrow A) \in \mathbf{U}_{A \multimap A}$ as expected. Likewise, if $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$ are uniform prestrategies, then it follows fairly easily that the composition $A \leftarrow T \odot S \rightarrow C$ is indeed in $\mathbf{U}_{A \multimap C}$ (see **??**).

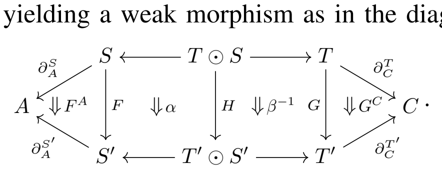
4) Horizontal composition of 2-cells: We have an identity uniform prestrategy in $\mathbf{U}_{A \multimap A}$, and a well-defined composition of $S \in \mathbf{U}_{A \multimap B}$ and $T \in \mathbf{U}_{B \multimap C}$ such that the composition pullback is always a bipullback. So given weak morphisms



by the bipullback property of $T' \odot S'$ there are a functor H and natural isos α and β such that we have the equality



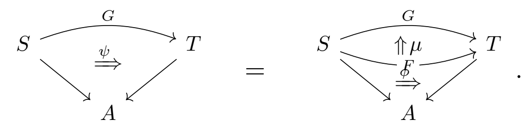
altogether yielding a weak morphism as in the diagram:



However, H, α, β are not unique: though Lemma 1 guarantees the existence of solutions to all reindexing problems, those may not be unique. We only know by condition *(b)* of Definition 2 that different choices of H, α, β yield *isomorphic* weak morphisms of uniform prestrategies, by which we mean isomorphic morphisms of the 2-category $\mathbf{Unif}(A)$:

Definition 6. *Consider A a uniform groupoid.*

The 2-category $\mathbf{Unif}(A)$ has: objects $\mathbf{U}_A$, i.e. *uniform prestrategies on A;* morphisms *from S to T the weak morphisms,* i.e. *pairs $(F : S \rightarrow T, \phi : \partial^S \Rightarrow \partial^T F)$; 2-cells from (F, ϕ) to (G, ψ) the natural transformations $\mu : F \Rightarrow G$ such that:*



Thus, although bipullbacks guarantee the existence of a fitting weak morphism for horizontal composition, there is a priori no canonical choice. One could pick a choice of horizontal composition, but there is no reason why an arbitrary choice would satisfy the coherence conditions for a bicategory.

C. Thin Spans of Groupoids

In fact, if formulated in the adequate way, the reindexing problems that arise from the interpretation of programming languages *do have* a unique solution – as in Example 6. But to prove that, we shall need to add more structure to uniform groupoids, starting with *polarized sub-groupoids*:

1) Polarized sub-groupoids: Consider the groupoid

$$\mathbf{Fam}(o) \multimap \mathbf{Fam}(o)$$

of Example 6, interpreting the formula $!o \multimap !o$ of intuitionistic linear logic. Here, the two occurrences of ! are intuitively very different: on the left-hand side, as in Example 4 the *program* performs the copying – in game semantics the copy index would be carried by a Player move. In contrast, for the right hand side exponential, the *environment* does the copying – in game semantics, the copy index would be carried by an Opponent move. This assigns a polarity to certain symmetries, very clear in game semantics: those reindexing copy indices only for exponentials in covariant position (resp. contravariant position) are *negative* (resp. *positive*). We enrich the groupoids interpreting types to keep track of these special symmetries:

Definition 7. *A **polarized groupoid** is a groupoid A with two sub-groupoids A_- and A_+, with the same objects as A.*

It would be natural to require additional conditions for this structure (in particular, see conditions *(a)* and *(b)* in Lemma 3). We omit them here, as they shall hold automatically once we introduce the more complete notion of a *thin groupoid*.

If $\theta \in A_-(a_1, a_2)$, we write $\theta : a_1 \cong_A^- a_2$ and likewise for positive symmetries. Usual constructions on groupoids extend to polarized groupoids componentwise. The **dual** of (A, A_-, A_+) is defined as (A, A_+, A_-), exchanging the two sub-groupoids. Finally, we set $(!A)_- = \mathbf{Fam}(A_-)$ and $(!A)_+ = \mathbf{Fam}^{\mathrm{id}}(A_+)$, which has morphisms from $(a_i)_{i \in I}$ to $(b_j)_{j \in J}$ those $(\pi, (\theta_i)_{i \in I})$ such that $I = J$ and $\pi = \mathrm{id}_I$ – thus we see indeed that Player cannot reindex copy indices from the outer ! in $!A$, as it appears in covariant position.

2) Thinness: Solutions to reindexing problems may be computed interactively as in Example 6. Intuitively, the uniqueness of the solution relies on the fact that at each stage, there is a unique choice of reindexing. This is captured by the definition of *thin* below, imported from thin concurrent games:

Definition 8. *Consider A a polarized groupoid, and S a prestrategy on A. We say that S is **thin** iff for all $\varphi : s \cong_S s'$, if $\partial^S \varphi$ is positive then $s = s'$ and $\varphi = \mathrm{id}_s$.*

Intuitively, this captures that positive copy indices are selected deterministically from negative ones – so a non-trivial symmetry $\varphi : s \cong_S s'$ cannot display to a purely positive symmetry on A. This is in contrast with the saturated case, where spans must be able to reach *all* positive symmetries.

We show how thinness addresses uniqueness for the resolution of reindexing problems. Call a solution to a reindexing problem φ, ψ as in Lemma 1 **positive** if writing $\partial^S \varphi = \varphi_A \vdash \varphi_B$ and $\partial^T \psi = \psi_B \vdash \psi_C$, we have $\varphi_A \vdash \psi_C$ positive.

Lemma 2. *Consider A, B, C polarized uniform groupoids, $S \in \mathbf{U}_{A \multimap B}$ and $T \in \mathbf{U}_{B \multimap C}$ s.t. $T \odot S \in \mathbf{U}_{A \multimap C}$ is thin.*

Then, any reindexing problem in the composition pullback of S and T has at most one positive *solution.*

Proof. Consider a reindexing problem $s \in S, t \in T, \theta : \partial^S_B s \cong_B \partial^T_B t$ with solutions $\varphi_1 : s \cong_S s'_1$ and $\psi_1 : t'_1 \cong_T t$ with $\partial^S_B s'_1 = \partial^T_B t'_1$ and $\partial^T_B \psi_1 \circ \partial^S_B \varphi_1 = \theta$, and $\varphi_2 : s \cong_S s'_2$ and $\psi_2 : t'_2 \cong_T t$ with $\partial^S_B s'_2 = \partial^T_B t'_2$ and $\partial^T_B \psi_2 \circ \partial^S_B \varphi_1 = \theta$.

Then, $\partial^S(\varphi_2 \circ \varphi_1^{-1}) = \partial^T(\psi_2 \circ \psi_1^{-1})$, so that we have

$$\Omega = (\varphi_2 \circ \varphi_1^{-1}, \psi_2 \circ \psi_1^{-1}) : (s'_1, t'_1) \cong_{T \odot S} (s'_2, t'_2)\,,$$

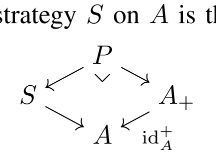
whose display to $A \vdash C$ is positive since φ_1, ψ_1 and φ_2, ψ_2 are positive solutions. Hence, by *thin*, Ω is an identity map which entails $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$ as required. $\square$

Thus, thinness allows us to find canonical solutions to reindexing problems by insisting on finding *positive* solutions.

However, this relies on thinness not of S and T, but of $T \odot S$. Again in thin concurrent games, this follows by induction on the causal structure. In the absence of a handle on causality, we must as for uniformity treat the fact that $T \odot S$ is thin as an interactive property, again handled by biorthogonality.

D. Thin Spans

1) The thin orthogonality: We observe that for A a polarized groupoid, a prestrategy S on A is thin iff the pullback



(3)

is *discrete*, *i.e.* all the morphisms in P are identities. We shall base our orthogonality on this observation, and set:

Definition 9. *For A a polarized uniform groupoid, $S \in \mathbf{U}_A$, and $T \in \mathbf{U}_A^\perp$, we say S and T are **thinly orthogonal**, written*

$$S \perp\!\!\!\perp T$$

iff the pullback $T \odot S$ is discrete.

Note that this is already assuming that S and T are uniformly orthogonal. If $\mathbf{S} \subseteq \mathbf{U}_A$, then its **thin orthogonal** is

$$\mathbf{S}^{\perp\!\!\!\perp} = \{T \in \mathbf{U}_A^\perp \mid \forall S \in \mathbf{S},\ S \perp\!\!\!\perp T\}\,,$$

and as before we have $\mathbf{S} \subseteq \mathbf{S}^{\perp\!\!\!\perp\perp\!\!\!\perp}$ (note that this typechecks only because $\mathbf{U}_A^{\perp\perp} = \mathbf{U}_A$) and $\mathbf{S}^{\perp\!\!\!\perp} = \mathbf{S}^{\perp\!\!\!\perp\perp\!\!\!\perp\perp\!\!\!\perp}$ for all $\mathbf{S} \subseteq \mathbf{U}_A$.

2) Thin groupoids: As before, we are interested in sets of uniform prestrategies *closed under bi-thin-orthogonal*:

Definition 10. *A **thin groupoid** is a polarized uniform groupoid with a set $\mathbf{T}_A \subseteq \mathbf{U}_A$ of **strategies** s.t. $\mathbf{T}_A^{\perp\!\!\!\perp\perp\!\!\!\perp} = \mathbf{T}_A$, and such that $(A_-, \mathrm{id}_A) \in \mathbf{T}_A$ and $(A_+, \mathrm{id}_A) \in \mathbf{T}_A^{\perp\!\!\!\perp}$.*

If $S \in \mathbf{T}_A$ then S is automatically thin in the sense of Definition 8: as $(A_+, \mathrm{id}_A) \in \mathbf{T}_A^{\perp\!\!\!\perp}$ the pullback (3) is discrete.

This also entails properties of the polarized symmetries:

Lemma 3. *Consider A a thin groupoid. Then we have:*
(a) if $\theta : a \cong_A^- a'$ and $\theta : a \cong_A^+ a'$, then $a = a'$ and $\theta = \mathrm{id}_a$.
(b) if $\theta : a \cong_A a'$, then there are unique a'' along with $\theta_- : a \cong_A^- a''$ and $\theta_+ : a'' \cong_A^+ a'$ such that $\theta = \theta_+ \circ \theta_-$.

Proof. For *(a)*, this follows from $A_- \perp\!\!\!\perp A_+$, as then the pullback of the cospan $A_- \hookrightarrow A \hookleftarrow A_+$ is discrete.

For *(b)*, $A_- \in \mathbf{T}_A \subseteq \mathbf{U}_A$ and $A_+ \in \mathbf{T}_A^{\perp\!\!\!\perp} \subseteq \mathbf{U}_A^\perp$, we also have $A_- \perp A_+$. Hence, the pullback of the cospan $A_- \hookrightarrow A \hookleftarrow A_+$ is a bipullback. But then any $\theta : a \cong_A a'$ forms a reindexing problem, whose solution is exactly the seeked reindexing. Uniqueness follows immediately from *(a)*. $\square$

Thus, we get from the definition of thin groupoids some of the expected properties of the polarized sub-groupoids: if a symmetry is both positive and negative then it must be an identity, and any symmetry can be obtained by first "reindexing Opponent moves", then "reindexing Player moves".

3) Further structure: Constructions on uniform groupoids extend to thin groupoids in the expected way. The thin groupoid 1 has $\mathbf{T}_1 = \mathsf{PreStrat}(1)$. If A and B are thin groupoids, their **tensor** is the uniform groupoid $A \otimes B$ extended with $\mathbf{T}_{A \otimes B} = (\mathbf{T}_A \otimes \mathbf{T}_B)^{\perp\!\!\!\perp\perp\!\!\!\perp}$. The **dual** of A has $\mathbf{T}_{A^\perp} = \mathbf{T}_A^{\perp\!\!\!\perp}$. The **par** of A and B has $\mathbf{T}_{A \parr B} = (\mathbf{T}_A^{\perp\!\!\!\perp} \otimes \mathbf{T}_B^{\perp\!\!\!\perp})^{\perp\!\!\!\perp}$, and the **linear arrow** is $A \multimap B = A^\perp \parr B$.

To establish the compositional properties of strategies, we rely on the following analogue of Proposition 1:

Proposition 2. *Consider $T \in \mathbf{U}_{A \multimap B}$ for A, B thin groupoids, along with a class $\mathbf{S} \subseteq \mathbf{T}_A$ such that $\mathbf{S}^{\perp\!\!\!\perp\perp\!\!\!\perp} = \mathbf{T}_A$.*
Then, $T \in \mathbf{T}_{A \multimap B}$ iff $T @ S \in \mathbf{T}_B$ for all $S \in \mathbf{S}$.

This follows from diagram chasing lemmas on situations where the pullbacks are discrete, see **??**. Interestingly, this is also equivalent to $T^\star @ S \in \mathbf{T}_A^{\perp\!\!\!\perp}$ for all $S \in \mathbf{T}_B^{\perp\!\!\!\perp}$.

It is a direct consequence of Proposition 2 that the identity span on A is in $\mathbf{T}_{A \multimap A}$ for any thin groupoid A, and that if $S \in \mathbf{T}_{A \multimap B}$ and $T \in \mathbf{T}_{B \multimap C}$ then $T \odot S \in \mathbf{T}_{A \multimap C}$. Together with Lemma 2, we have thus identified a compositional situation where the composition pullback of spans is a bipullback, and where all arising reindexing problems have a *unique* solution if one insists on this solution being *positive*.

4) Positive weak morphisms: Insisting on *positive* solutions amounts to relating strategies not via arbitrary weak morphisms, but with *positive* weak morphisms:

Definition 11. *Consider A a thin groupoid, $S, T \in \mathbf{T}_A$, and (F, ϕ) a weak morphism from S to T,* i.e. $F : S \to T$ *and* $\phi : \partial^S \Rightarrow \partial^T \circ F$. *Then, (F, ϕ) is* ***positive*** *if ϕ is positive, that is, if $\forall s \in S$, $\phi_s : \partial^S s \cong_A^+ \partial^T F(s)$ is a positive symmetry.*

Intuitively, comparing strategies with positive morphisms amounts to relating them only via maps that do not reindex Opponent moves. This has the effect of making everything stricter, and cutting the higher dimension. More precisely:

Proposition 3. *Let A be a thin groupoid. Consider $\mathbf{PreThin}(A)$ the sub-2-category of $\mathbf{Unif}(A)$ with objects $\mathbf{T}_A$, and $\mathbf{Thin}(A)$ where additionally morphisms are positive.*

Then, $\mathbf{Thin}(A)$ is locally discrete, i.e. *all 2-cells are identities. Moreover, $\mathbf{PreThin}(A)$ and $\mathbf{Thin}(A)$ are biequivalent.*

Proof. The first is a direct consequence of *thinness*: if $\mu : (F, \phi) \Rightarrow (G, \psi) : S \to T$ for ϕ and ψ positive, then by definition of 2-cells of $\mathbf{Unif}(A)$, for all $s \in S$, $\mu_s \in T(Fs, Gs)$ is such that $\psi_s = \partial^T \mu_s \circ \phi_s$, *i.e.* $\partial^T \mu_s = \psi_s \circ \phi_s^{-1}$ positive. Thus, μ_s is an identity morphism by thinness.

For the biequivalence, the crux is that if $(F, \phi) : S \to T$ is a weak morphism, then there is a unique $(F_+, \phi_+) : S \to T$ positive isomorphic to (F, ϕ), and a unique 2-cell μ between them. Uniqueness follows from thinness. For existence, note that if $s \in S$ and $\theta : \partial^S s \cong_A a$, then there exist unique $\varphi : s \cong_S s'$ and $\theta_+ : \partial^S s \cong_A^+ a$ such that $\theta = \theta_+ \circ \partial^S \varphi$ – this exploits thinness, and the reindexing problem from the fact that the pullback of the cospan $S \hookrightarrow A \hookleftarrow A_+$ is a bipullback. We obtain (F_+, ϕ_+) by applying this lemma pointwise. □

This proposition illustrates the situation well: thanks to the thin biorthogonality, the 2-category $\mathbf{PreThin}(A)$ is represented up to biequivalence as a mere category $\mathbf{Thin}(A)$. The higher dimensional structure simply vanishes.

5) The bicategory $\mathbf{Thin}$*:* With this in place, we may finally define the components of our bicategory $\mathbf{Thin}$. Its *objects* are thin groupoids. Its *morphisms* from A to B are strategies from A to B, *i.e.* elements of $\mathbf{T}_{A \multimap B}$ – recall that they are $(S, \partial^S : S \to A \times B)$, in particular spans from A to B

$$A \xleftarrow{\partial_A^S} S \xrightarrow{\partial_B^S} B\,.$$

The *identities* are identity spans, and *composition* is via the pullback (1). If S and T are strategies from A to B, the 2-*cells* from S to T are the positive morphisms $(F, \phi) : S \to T$. As $\phi : \partial^S \Rightarrow \partial^T \circ F$ is a family of positive morphisms on $A^\perp \parr B$ with underlying plain groupoid $A \times B$, it may be equivalently presented as pair of $F^A : \partial_A^S \Rightarrow \partial_A^T \circ F$ and $F^B : \partial_B^S \Rightarrow \partial_B^T \circ F$, as in Definition 1. For horizontal composition of positive morphisms, we first proceed as in Section III-B4 and obtain a connected groupoid of (non necessarily positive) horizontal compositions – which must all have the same image through the biequivalence of Proposition 3, providing our unique positive horizontal composition. Altogether, we have:

Theorem 2. *Those components form $\mathbf{Thin}$, a bicategory.*

Proof. See details in **??**. □

Next, we develop the further structure of $\mathbf{Thin}$.

IV. Cartesian Closed Structure

To construct a cartesian closed bicategory, we intend to follow [20]. We first turn the construction $\mathbf{Fam}$ – thereafter denoted by ! – into a pseudocomonad, and then equip the Kleisli bicategory $\mathbf{Thin}_!$ with the cartesian closed structure.

A. The Pseudocomonad

We first develop the action of ! on objects of $\mathbf{Thin}$.

1) The bang of thin groupoids: First, ! is defined on uniform groupoids via $\mathbf{U}_{!A} = (!\mathbf{U}_A)^{\perp\perp}$, where we have used

$$!\mathbf{U}_A = \{(!S, !\partial^S) \mid S \in \mathbf{U}_A\}$$

using the functorial action $!\partial^S : !S \to !A$. For thin groupoids, the positive and negative symmetries of $!A$ were defined in Section III-C1. The thin structure is set as $\mathbf{T}_{!A} = (!\mathbf{T}_A)^{\perp\perp}$ – it is a direct verification that this is a thin groupoid.

2) The bang of strategies: If $S \in \mathbf{T}_{A \multimap B}$, we have $\partial^S = \langle \partial_A^S, \partial_B^S \rangle$ for $\partial_A^S : S \to A$ and $\partial_B^S : S \to B$ – its bang is

$$!A \xleftarrow{!\partial_A^S} !S \xrightarrow{!\partial_B^S} !B$$

packaged as $(!S, \langle !\partial_A^S, !\partial_B^S \rangle)$. That this is in $\mathbf{T}_{!A \multimap !B}$ relies on:

Lemma 4. *Consider A, B thin groupoids, and T a prestrategy from $!A$ to B. Then, the following two properties hold:*

(1) *$T \in \mathbf{U}_{!A \multimap B}$ iff $(T, \partial_{!A}^T) \in \mathbf{U}_{!A}^\perp$ and for all $S \in \mathbf{U}_A$, $T @ !S \in \mathbf{U}_B$,*
(2) *$T \in \mathbf{T}_{!A \multimap B}$ iff for all $S \in \mathbf{T}_A$, $T @ !S \in \mathbf{T}_B$.*

This is an immediate application of Propositions 1 and 2. Since $\mathbf{U}_{!A} = (!\mathbf{U}_A)^{\perp\perp}$ and $\mathbf{T}_{!A} = (!\mathbf{T}_A)^{\perp\perp}$. From this lemma, it is a rather direct verification that for any $S \in \mathbf{T}_{A \multimap B}$, we have $!S \in \mathbf{T}_{!A \multimap !B}$ as required.

3) A pseudofunctor: Since ! is a functor, it preserves the identity span on the nose. Since ! preserves pullbacks, for $S \in \mathbf{T}_{A \multimap B}$ and $T \in \mathbf{T}_{B \multimap C}$, the universal property gives us

$$m_{S,T} : !(T \odot S) \cong !T \odot !S$$

a strong invertible 2-cell in $\mathbf{Thin}$. As expected, this 2-cell is natural in S and T (with respect to positive morphisms). Altogether, we obtain a pseudofunctor $! : \mathbf{Thin} \to \mathbf{Thin}$. See **??** for details.

4) A pseudomonad on groupoids: In fact we first turn ! into a pseudomonad on $\mathbf{Gpd}$, from which its pseudocomonad structure on $\mathbf{Thin}$ shall be derived. We noted earlier that we have a functor $\mathbf{Fam} : \mathbf{Gpd} \to \mathbf{Gpd}$ – in fact, it is extended to a 2-endofunctor on the 2-category of small groupoids, noted

$$! : \mathbf{Gpd} \to \mathbf{Gpd}\,,$$

defined on a 2-cell $\alpha : F \Rightarrow G : A \to B$ as the natural transformation $!\alpha : !F \Rightarrow !G$ with components all pairs

$$(!\alpha)_{(A_i)_{i \in I}} = (\mathrm{id}_I, (\alpha_{A_i})_{i \in I}) \in !B((FA_i)_{i \in I}, (GA_i)_{i \in I})\,.$$

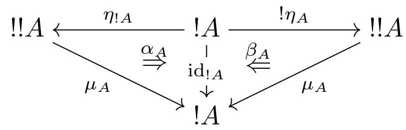


Fig. 3. Unit natural isomorphisms

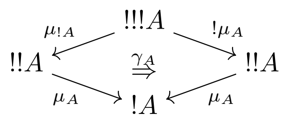


Fig. 4. Associativity natural isomorphism

To turn this into a pseudomonad, we must adjoin a multiplication and a unit. The components of the unit are the functors

$$\begin{array}{rcrcl} \eta_A & : & A & \to & !A \\ & & a & \mapsto & (a)_{\{0\}} = [0 \cdot a] \end{array}$$

with the obvious functorial action. The intuition is that the unit transports a single resource usage from A to $!A$, arriving at a singleton family. In doing so, it must select a copy index. Any natural number will do – the rest of the paper does not depend on this choice – but for definiteness and compatibility with the traditional convention from AJM games, we pick 0.

For the multiplication $\mu_A : !!A \to !A$, we must flatten a family of families into a family. For this purpose, we fix an injective function $\langle -, - \rangle : \mathbb{N}^2 \to \mathbb{N}$ – again, the results of this paper do not depend on that choice. Given $I \subseteq_f \mathbb{N}$ and a family $(J_i)_{i \in I}$ where $J_i \subseteq_f \mathbb{N}$ for all $i \in I$, let us write

$$\Sigma_{i \in I} J_i = \{\langle i, j \rangle \mid i \in I,\ j \in J_i\}\,,$$

which is by definition still a finite subset of $\mathbb{N}$. Then we set

$$\begin{array}{rcrcl} \mu_A & : & !!A & \to & !A \\ & & ((a_{i,j})_{j \in J_i})_{i \in I} & \mapsto & (a_{i,j})_{\langle i,j \rangle \in \Sigma_{i \in I} J_i} \end{array}$$

for any groupoid A, along with the obvious functorial action.

Altogether this yields $\eta : \mathrm{id}_{\mathbf{Gpd}} \Rightarrow\, !$ and $\mu : !! \Rightarrow\, !$, two (strict 2-) natural transformations. The monad laws, if they were to hold on the nose, would mean that $\langle 0, i \rangle = \langle i, 0 \rangle = i$ and $\langle \langle i, j \rangle, k \rangle = \langle i, \langle j, k \rangle \rangle$ for all $i, j, k \in \mathbb{N}$; and it is clear that no injection satisfying those laws exists. Nevertheless, for every groupoid A the coherence laws for a monad hold up to *natural isomorphisms*: we have α_A, β_A and γ_A as indicated in Figures 3 and 4. For instance, for any $(a_j)_{j \in J} \in !A$:

$$(\alpha_A)_{(a_j)_{j \in J}} : (a_j)_{j \in J} \cong_{!A} (a_j)_{\langle 0, j \rangle \in \Sigma_{i \in \{0\}} J}$$

reindexing along the bijection $J \simeq \Sigma_{i \in \{0\}} J$. The other components act similarly – note that they are all *negative* symmetries. The associated families $(\alpha_A)_{A \in \mathbf{Gpd}}$, $(\beta_A)_{A \in \mathbf{Gpd}}$ and $(\gamma_A)_{A \in \mathbf{Gpd}}$ satisfy the conditions for *modifications*, and the additional coherence laws for a pseudomonad:

Proposition 4. *The* 2*-functor* $! : \mathbf{Gpd} \to \mathbf{Gpd}$ *along with the components above yield a pseudomonad on* $\mathbf{Gpd}$.

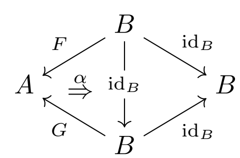
5) Lifting functors to spans: We shall turn ! into a pseudocomonad on $\mathbf{Thin}$ by lifting the components above to spans. In general, if $F : B \to A$ is a functor, then there is a span $\check{F}$

$$A \xleftarrow{F} B \xrightarrow{\mathrm{id}_B} B\,,$$

called the **lifting** of F – but we need sufficient conditions on F for this construction to yield morphisms in $\mathbf{Thin}$. For that purpose, if A and B are thin groupoids, we say that a functor $F : A \to B$ is a **renaming** iff the following conditions hold:

(1) for all $\theta : a \cong_A a'$, if θ is positive then so is $F\theta$,
(2) for all $(T, \partial^T) \in \mathbf{U}_B^{\perp}$, $(T, F \circ \partial^T) \in \mathbf{U}_A^{\perp}$,
(3) for all $(T, \partial^T) \in \mathbf{T}_B^{\perp\!\!\perp}$, $(T, F \circ \partial^T) \in \mathbf{T}_A^{\perp\!\!\perp}$.

Clearly, renamings compose – we consider the 2-category $\mathbf{Ren}$ whose objects are thin groupoids, whose morphisms are renamings, and whose 2-cells are negative natural transformations. As expected, lifting renamings yields thin spans (see **??**). Lifting can be extended to 2-cells: if $\alpha : F \Rightarrow G : A \to B$ is a negative natural transformation, then $\check{\alpha}$ is the positive morphism described by the diagram:



noting that this is positive as negative α is in contravariant position. Altogether, we get (see details in **??**):

Proposition 5. *There is a pseudofunctor* $\check{-} : \mathbf{Ren}^{\mathrm{op}} \to \mathbf{Thin}$.

Here, $\mathbf{Ren}^{\mathrm{op}}$ is $\mathbf{Ren}$ with the morphisms reversed, but the 2-cells unchanged. It can be checked that for any thin groupoid A, the functors $\eta_A : A \to !A$ and $\mu_A : !!A \to !A$ are renamings, in particular for every thin groupoid A we get

$$\check{\eta_A} \in \mathbf{Thin}(!A, A) \qquad \check{\mu_A} \in \mathbf{Thin}(!A, !!A)$$

the main components to turn ! into a pseudocomonad. Unlike in $\mathbf{Gpd}$, the families $\check{\eta}$ and $\check{\mu}$ are not strict 2-natural transformations but only pseudonatural transformations, with 2-cells

$$\begin{array}{rcrcl} \eta_S & : & \check{\eta}_B \odot !S & \Rightarrow & S \odot \check{\eta}_A \\ \mu_S & : & \check{\mu}_B \odot !S & \Rightarrow & !!S \odot \check{\mu}_A\,, \end{array}$$

positive isomorphisms obtained for $S \in \mathbf{Thin}(A, B)$ from the universal property of pullbacks, via the observation that $\eta : \mathrm{id}_{\mathbf{Gpd}} \Rightarrow\, !$ and $\mu : !! \Rightarrow\, !$ are *cartesian* natural transormations. It may be checked that η_S and μ_S are natural in S and satisfy the coherence conditions of pseudonatural transformations. Finally, the modifications α, β, γ involved in the pseudomonad structure of ! on $\mathbf{Gpd}$ lift to the modifications required for the pseudocomonad structure of ! on $\mathbf{Thin}$.

Theorem 3. *We have a pseudocomonad* ! *on* $\mathbf{Thin}$.

Proof. See details in **??**. □

We move on to studying the Kleisli bicategory $\mathbf{Thin}_!$ whose horizontal composition, denoted $\odot_!$, is defined as expected.

B. Cartesian Closed Structure

1) Finite products: First, we show that $\mathbf{Thin}_!$ has finite products, *i.e.* is a *fp-bicategory* in the sense of Fiore and Saville [18] – unlike them, we work with binary products.

a) Terminal object: Write $\top$ for the empty groupoid, made a thin groupoid with $\mathbf{U}_\top = \mathbf{T}_\top = \{\mathrm{id}_\emptyset\}$. For any thin groupoid A, $\mathbf{Thin}_!(A, \top)$ has exactly one element – the empty groupoid. Thus, $\mathbf{Thin}_!$ has a (strict) terminal object.

b) Binary product: If A and B are thin groupoids, then the **with** $A \,\&\, B$ has underlying groupoid $A + B$ the disjoint union, with $(A+B)_- = A_- + B_-$ and $(A+B)_+ = A_+ + B_+$. We adjoin $\mathbf{U}_{A\&B} = (\mathbf{U}_A + \mathbf{U}_B)^{\perp\perp}$ and $\mathbf{T}_{A\&B} = (\mathbf{T}_A + \mathbf{T}_B)^{\perp\perp}$, where as usual, $\mathbf{U}_A + \mathbf{U}_B$ comprises the set of all $(S+T, \partial^S + \partial^T)$ for $(S, \partial^S) \in \mathbf{U}_A$ and $(T, \partial^T) \in \mathbf{U}_B$, using the functorial action of $+$ (and likewise for $\mathbf{T}_A + \mathbf{T}_B$).

c) Pairing and projections: The **projections** are simply set as $L_! = (\eta_{A+B} \circ \bar{l}) \in \mathbf{Thin}_!(A \,\&\, B, A)$ and $R_! = (\eta_{A+B} \circ \bar{r}) \in \mathbf{Thin}_!(A \,\&\, B, B)$ for $\bar{l}\colon A \to A + B$ and $\bar{r}\colon B \to A + B$ the obvious coprojections/renamings. The **pairing** of $S \in \mathbf{Thin}_!(\Gamma, A)$ and $T \in \mathbf{Thin}_!(\Gamma, B)$ is

$$(S + T, \partial_{!\Gamma} : S + T \to {!\Gamma}, \partial_{A\&B} : S + T \to A + B)$$

with $\partial_{!\Gamma}$ the co-pairing and $\partial_{A\&B} = \partial^S_A + \partial^T_B$. We have:

Proposition 6. *For any thin groupoids* Γ, A *and* B*, there is*

$$\mathbf{Thin}_!(\Gamma, A \,\&\, B) \;\underset{\langle -,- \rangle}{\overset{(L_! \odot_! -,\, R_! \odot_! -)}{\rightleftarrows}}\; \mathbf{Thin}_!(\Gamma, A) \times \mathbf{Thin}_!(\Gamma, B) \qquad (\perp)$$

an adjoint equivalence.

Proof. If $S \in \mathbf{Thin}_!(\Gamma, A)$ and $T \in \mathbf{Thin}_!(\Gamma, B)$ there are

$$\omega^A_{S,T} : L_! \odot_! \langle S, T\rangle \cong S \qquad \omega^B_{S,T} : R_! \odot_! \langle S, T\rangle \cong T$$

positive isos, and for $U \in \mathbf{Thin}_!(\Gamma, A \,\&\, B)$ there is

$$\bar{\omega}_U : U \cong \langle L_! \odot_! U, R_! \odot_! T\rangle$$

a positive iso, defined in the obvious way. Those are all natural in S, T, U, and satisfy the required triangle identities. ☐

See **??** for more details. Altogether, this establishes that $\mathbf{Thin}_!$ is a *fp-bicategory* in the sense of [18].

2) Cartesian closure: If A and B are thin groupoids, then we set $A \Rightarrow B = {!A} \parr B$. Before we describe the additional components, we must observe the Seely *equivalence*:

$$!A \otimes {!B} \;\underset{\bar{s}_{A,B}}{\overset{s_{A,B}}{\rightleftarrows}}\; !(A \,\&\, B)$$

where $s_{A,B}$ sends $(a_i)_{i\in I}, (b_j)_{j\in J}$ to $(c_k)_{k\in I\bowtie J}$, with $I \bowtie J = \varpi(I \sqcup J)$ for some chosen bijection $\varpi = [\varpi_l, \varpi_r]$ between $\mathbb{N} \sqcup \mathbb{N}$ and $\mathbb{N}$, and where $c_{\varpi_l(i)} = a_i$ and $c_{\varpi_r(j)} = b_j$; and $\bar{s}_{A,B}$ sends $(c_k)_{k\in K}$ to $(a_i)_{i\in I}, (b_j)_{j\in J}$ where $I \subseteq K$ is the subset of those $i \in K$ such that $c_i = a_i \in A$, and likewise for b_j. Both functors are renamings, and the isomorphisms witnessing the equivalence are negative.

Via the Seely equivalence, we first define the **evaluation** as the span with basic groupoid $!A \times B$, with left leg the functor

$$!A \times B \to (!A \times B) \times {!A} \to {!(!A \times B)} \times {!A} \xrightarrow{s_{A,B}} {!((A \Rightarrow B) \,\&\, A)}$$

and right leg the projection $!A \times B \to B$. This yields a thin span $\mathbf{ev}_{A,B} \in \mathbf{Thin}_!((A \Rightarrow B) \,\&\, A, B)$. Now, we need

$$\Lambda(-) : \mathbf{Thin}_!(\Gamma \,\&\, A, B) \to \mathbf{Thin}_!(\Gamma, A \Rightarrow B)$$

the **currying** functor: given $S \in \mathbf{Thin}_!(\Gamma \& A, B)$, its currying $\Lambda(S)$ is simply S, with display map post-composed with

$$!(\Gamma + A) \times B \overset{\bar{s}_{\Gamma,A}}{\simeq} (!\Gamma \times {!A}) \times B \cong {!\Gamma} \times (!A \times B)\,.$$

With this data in place, we may finally prove:

Proposition 7. *For any groupoids* Γ, A, B*, there is*

$$\mathbf{Thin}_!(\Gamma, A \Rightarrow B) \;\underset{\Lambda(-)}{\overset{\mathbf{ev}_{A,B} \odot_! (- \& A)}{\rightleftarrows}}\; \mathbf{Thin}_!(\Gamma \,\&\, A, B) \qquad (\perp)$$

an adjoint equivalence.

Proof. One can first show the existence of adjoint equivalence between the currying operation $\Lambda(-)$, and a symmetric uncurrying operation $\bar{\Lambda}(-)$. The unit and counit of this adjunction can be derived from the ones of the Seely (adjoint) equivalence. One can then prove that $\bar{\Lambda}(-)$ is in fact isomorphic to $\mathbf{ev}_{A,B} \odot_! (- \,\&\, A)$ in order to get the wanted equivalence. ☐

See **??** for details. Altogether, we have:

Theorem 4. $\mathbf{Thin}_!$ *is a cartesian closed bicategory.*

This entails that we can interpret types of the simply-typed λ-calculus as thin groupoids, morphisms as thin spans and rewrites between terms as certain positive isomorphisms [19].

V. Conclusion

This paper focuses on the construction of $\mathbf{Thin}_!$, leaving for later its application to semantics of λ-calculi and programming languages. We believe this opens multiple perspectives for further research: firstly, we may explore the obtained interpretation of the λ-calculus, which syntactically should correspond to the sequence typing system of Vial [38] and to the non-uniform λ-calculus of Melliès [15]. We should explore links with other models of the literature, notably with the weighted relational model recasting ideas from [37], and with generalized species of structures and template games. Another related direction consists in accommodating another feature of template games, the mechanism to capture scheduling and synchronization [34], into thin spans.

In more semantic directions, we believe that with respect to generalized species of structures, the fact that operations on thin spans involve no quotient may be helpful in two ways: *(1)* individuals may be ordered concretely, and the model should support continuous reasoning allowing one to deal easily with infinite computation; and *(2)* adding "typed" weights coming from an SMCC as in [24] should be a lot simpler, since those weigths no longer have to themselves be saturated.

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