The Cartesian Closed Bicategory of Thin Spans of Groupoids

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Abstract—Recently, there has been growing interest in bicategorical models of programming languages, which are “proof-relevant” in the sense that they keep distinct account of execution traces leading to the same observable outcomes, while assigning a formal meaning to reduction paths as isomorphisms.

In this paper we introduce a new model, a bicategory called thin spans of groupoids. Conceptually it is close to Fiore et al.’s generalized species of structures and to Melliès’ homotopy template games, but fundamentally differs as to how replication of resources and the resulting symmetries are treated. Where those models are saturated – the interpretation is inflated by the fact that semantic individuals may carry arbitrary symmetries – our model is thin, drawing inspiration from thin concurrent games: the interpretation of terms carries no symmetries, but semantic individuals satisfy a subtle invariant defined via biorthogonality, which guarantees their invariance under symmetry.

We first build the bicategory Thin of thin spans of groupoids. Its objects are certain groupoids with additional structure, its morphisms are spans composed via plain pullback with identities the identity spans, and its 2-cells are span morphisms making the induced triangles commute only up to natural isomorphism. We then equip Thin with a pseudocomonad !, and finally show that the Kleisli bicategory Thin is cartesian closed.

I. INTRODUCTION

The relational model [1] is one of the most basic and elementary denotational models for linear logic or the \( \lambda \)-calculus. At its heart, it is simply an interpretation of formulas / types as sets and proofs / programs as relations, i.e. in the category Rel. Despite its simplicity the relational model is ubiquitous: it is the basic substrate for the spectrum of so-called web-based models of linear logic, including coherence or finiteness spaces [2]. It faithfully predicts reduction time [3]. It supports quantitative extensions such as in probabilistic coherence spaces [4], the weighted relational model [5], and even up to quantum computation [6] – quantitative extensions which enjoy powerful full abstraction results [7], [8]. Presented syntactically, the relational model exactly corresponds to non-idempotent intersection types [9], a currently active research topic in its own right (see e.g. [10], [11]) which enables a syntactic methodology to addressing semantic questions.

Finally, it has a tight connection with game semantics [12], [13], of which it appears as a desequentialization (see e.g. [8], [14] – [16]). In short, it is at the crossroads of multiple topics, past and current, of the denotational semantics universe.

Another recent trend in denotational semantics is the adoption of bicategorical models [17] where the familiar categorical laws hold only up to certain 2-cells satisfying coherence conditions – in particular, Fiore and Saville have recently thoroughly explored cartesian closed bicategories [18]. In such models, the denotation is no longer an invariant of reduction: two convertible terms yield merely isomorphic objects, and reduction paths have a genuine interpretation as specific isomorphisms [19] – thus bringing reduction into the bicategorical model. There are still not many concrete bicategorical models, and we are aware of only three (families of) such models that can deal with non-linear computation, in chronological order: firstly, Fiore, Gambino, Hyland and Winskel’s cartesian closed bicategory of generalized species of structure [20]; secondly, Castellan, Clairambault and Winskel’s thin concurrent games [21] (as established by Paquet [22]); thirdly, Melliès’ homotopy template games [23]. Of these three, the first is by far the most studied with various works including generalizations and application to semantics [24], [25], links with intersection types and Taylor expansion [26], [27], or applications to the pure \( \lambda \)-calculus [28]. Beyond giving a non-degenerated interpretation to reduction paths, those concrete bicategorical models are “proof-relevant”, in the sense that they keep distinct semantic witnesses for the possibly multiple evaluation traces with the same observable behaviour and thus keep a clear, branching account of non-determinism.

These models have something else in common: in their construction, the main subtlety has to do with replication, i.e. the modality \(!\) of linear logic. In the relational model, \(!A\) is the set \( \mathcal{M}(A) \) of finite multisets of elements of \( A \), or alternatively, the free monoid \( A^* \) quotiented by permutations. In bicategorical models, this is replaced by a categorification of \( \mathcal{M}(A) \): a category (or groupoid) whose objects keep separate individual resource usages (e.g. \( A^* \)). Its morphisms are explicit permutations, often called symmetries in this paper. Individuals in the model must refer to specific resources (e.g. \( a_i \) in \( a_1 \ldots a_n \in A^* \)), but the categorical laws expected for models of programming languages requires that their behaviour should still be invariant under symmetry. In both generalized species of structure and template games, this is done by saturating the set of witnesses with respect to symmetries: intuitively, the behaviour of an individual cannot depend on the specific identity of resources, because those resources are seen through the “noise” of all possible symmetries – this shall be reviewed gently in Section II. This saturation complicates models and their construction, though for good reasons. But this contrasts with thin concurrent games, which...
handles symmetry with a mechanism inspired by Abramsky-Jagadeesan-Malacaria games [12] and Melliès’ orbital games [15]: strategies are not saturated, but their invariance under “Opponent’s symmetries” is ensured by a subtle bisimulation-like structure – we call this the thin approach.

We believe that the thin approach is helpful at least for applications to semantics: the absence of symmetries on witnesses allows a more concrete flavour which may help when ordering individuals allowing continuous reasoning\footnote{For instance, in [29], the generalization from finite to infinite computation is not simply by continuity as per usual in denotational semantics, because of the quotient involved in the management of saturation.} or simplify quantitative extensions such as [24]. But more fundamentally, there is a clear tension between these two worlds that deserves investigation. Are proof-relevant relational models inherently saturated? Is the thin approach only possible in games thanks to the presence of time and causality? These fundamental questions may be of interest beyond denotational semantics, as the handling of symmetry in such models is deeply connected to algebraic combinatorics [20] and homotopy theory [23].

a) Contributions: We introduce the bicategory Thin of thin spans of groupoids: its objects are certain groupoids with additional structure, its morphisms certain spans, and its 2-cells certain weak span morphisms, i.e. making the induced triangles commute up to chosen natural isos. Identities are identity spans, and composition of spans is by plain pullback.

Of course, plain pullbacks are too weak to support the horizontal composition of weak span morphisms. To address this, we first define uniform spans via a biorthogonality construction, ensuring that the composition pullbacks also satisfy the bipullback universal property. This allows us to compose 2-cells horizontally, but that horizontal composition is still not canonically defined and fails to give a bicategory.

For the next step, we import from thin concurrent games and from Melliès’ orbital games a decomposition of symmetries into positive symmetries (due to the program), and negative symmetries (due to the environment). We then define thin spans via a second biorthogonality construction, which ensures that the horizontal composition of weak span morphisms are canonically defined as long as we consider positive weak span morphisms, where the chosen iso only involves positive symmetries. We show this results in a bicategory Thin. Furthermore, we equip Thin with a pseudocomonad !, and show that the Kleisli bicategory Thin is cartesian closed.

b) Outline: In Section II, we start with a gentle introduction to the relational model and its proof-relevant extensions. In Section III we introduce the bicategory Thin, deploying first the uniform orthogonality and then the thin orthogonality. In Section IV we introduce the pseudocomonad !, and show that the Kleisli bicategory Thin is cartesian closed.

II. RELATIONAL MODELS, SPANS, SPECIES

A. The Relational Model

The relational model is one of the simplest denotational models of the $\lambda$-calculus, linear logic, or simple programming languages such as PCF. It consists in simply interpreting every type $A$ as a set $|A|$, and a program $\vdash M : A$ as a subset of $|A|$. This set $|A|$ is often called the web seeing that it is the first component of the so-called web-based models of linear logic such as coherence spaces and their extensions. One may think of elements of $|A|$ as completed executions (which is straightforward enough for ground types such as booleans or natural numbers but may be more complex for higher-order types), and of $\{M\} \subseteq |A|$ as simply the collection of all the completed executions that $M$ may achieve.

Example 1. The ground type for booleans is interpreted as $|\mathbb{B}| = \{\mathsf{tt}, \mathsf{ff}\}$, and the constant $\vdash \mathsf{tt} : \mathbb{B}$ as $|\mathsf{tt}| = \{\mathsf{tt}\}$.

The interpretation of a program $M$ is computed compositionally, following the methodology of denotational semantics, organized by the categorical structure of sets and relations.

1) Basic categorical structure: There is a category $\mathbb{Rel}$ with sets as objects, and as morphisms the relations from $A$ to $B$, i.e. subsets $R \subseteq A \times B$. The identity on $A$ is the diagonal relation $\{(a, a) \mid a \in A\} \subseteq A \times A$, and the composition of $R \subseteq A \times B$ and $S \subseteq B \times C$ consists in all pairs $(a, c) \in A \times B$ such that $(a, b) \in R$ and $(b, c) \in S$ for some $b \in B$.

Besides, $\mathbb{Rel}$ has a monoidal structure given by the cartesian product on objects, and for $R_1 \in \mathbb{Rel}(A_1, B_1)$, $R_1 \times R_2 \in \mathbb{Rel}(A_1 \times A_2, B_1 \times B_2)$ set as comprising all $((a_1, a_2), (b_1, b_2))$ when $(a_1, b_1) \in R_1$ – the unit $I$ is a fixed singleton set, say $\{\ast\}$. Additionally, $\mathbb{Rel}$ is compact closed: each set $A$ has a dual $A^*$ defined simply as $A$ itself, and there are relations $\eta_A \in \mathbb{Rel}(I, A \times A)$ and $\epsilon_A \in \mathbb{Rel}(A \times A, I)$, both diagonal relations, satisfying coherence conditions [30].

In particular, $\mathbb{Rel}$ is *-autonomous and as such a model of multiplicative linear logic, and the linear $\lambda$-calculus: the linear arrow type is interpreted as $|A \rightarrow B| = |A| \times |B|$. Finally, $\mathbb{Rel}$ has finite products, with the binary product of sets $A$ and $B$ given by the disjoint union $A + B = \{1\} \times A \uplus \{2\} \times B$.

2) The exponential modality: The exponential modality of $\mathbb{Rel}$ is based on finite multisets. If $A$ is a set, we write $\mathcal{M}(A)$ for the set of finite multisets on $A$. To denote specific multisets we use a list-like notation, as in e.g. $[0, 1, 1] \in \mathcal{M}(\mathbb{N})$ – we write $[\bullet] \in \mathcal{M}(A)$ for the empty multiset.

For $A$ a set, its bang !A is simply the set $\mathcal{M}(A)$. This extends to a comonad on $\mathbb{Rel}$, satisfying the required conditions to form a so-called Seely category – in particular, there is $\mathcal{M}(A + B) \cong \mathcal{M}(A) \times \mathcal{M}(B)$ a bijection providing the Seely isomorphism. Altogether, this makes $\mathbb{Rel}$ a model of intuitionistic linear logic; and this makes the Kleisli category $\mathbb{Rel}_!$ cartesian closed so that we may interpret (among others) the simply-typed $\lambda$-calculus.

Example 2. Considering the term $\vdash \lambda x : \mathbb{B} \rightarrow \mathbb{B}$ of PCF

\[
\lambda x : \mathbb{B}. \text{if } x \text{ then } x \text{ else if } x \text{ then } \mathsf{ff} \text{ else } \mathsf{tt} : \mathbb{B} \rightarrow \mathbb{B},
\]

we have $|\left[ M \right]| = \{(\mathsf{tt}, \mathsf{tt}), (\mathsf{tt}, \mathsf{ff}), (\mathsf{ff}, \mathsf{ff}), (\mathsf{ff}, \mathsf{tt})\}$. Here we can observe that the model is quantitative, in that it records how many resources each execution consumes: one
may observe output \( tt \) either with two evaluations of \( x \) to \( tt \), or with two evaluations of \( x \) to \( ff \). One may observe output \( ff \) with two evaluations of \( x \), one to \( tt \) and one to \( ff \). Recall that in \([tt, ff\rangle = [ff, tt]\), the order is irrelevant.

The relational model also supports the interpretation of non-determinism: if \( \vdash \text{choice} : B \) is a new primitive evaluating non-deterministically to \( tt \) or \( ff \), then we may simply set \([[\text{choice}]] = \{tt, ff\}\).

3) Extensions of the relational model: The relational model is extremely flexible, and can be extended in multiple different ways. In one direction one may add to the objects a coherence relation and restrict to compatible morphisms – we obtain in this way (multiset-based) coherence semantics.

Another extension is the weighted relational model \([5], [31]\) where a term \( \vdash M : A \) instead of denoting a subset of \([A] \) – i.e. a function \([M] : [A] \rightarrow \{0, 1\} \) – denotes a function

\[ [M] : [A] \rightarrow R \]

assigning to each point of the web \( a \in [A] \) a weight \( [M]_a \in R \). The weight may be used to record additional information about executions. One may record the number of distinct non-deterministic branches leading to a certain result: for instance, if \( R = \mathbb{N} \cup \{+\infty\} \), then \( [[\text{if choice then } tt \text{ else } tt]]_a = 2 \).

With \( R = \mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\} \), we may track the probability with which a certain result occurs, obtaining a model fully abstract for probabilistic PCF \([7]\). The paper \([5]\) contains other examples: resource consumption, must convergence, etc.

It is natural to go one step further and make the relational model “proof-relevant”. This means not merely recording a weight or counting non-deterministic branches, but keeping track of a set \([M]_a \in \text{Set} \) of witnesses of the execution of \( M \) to \( a \), for each \( \vdash M : A \) and \( a \in [A] \). There are well-documented ways to do that which we shall review later on, but for now let us attempt this naively.

B. The Bicategory of Spans

A first idea is simply replace relations with spans.

1) Spans: Recall that if \( C \) is a category with pullbacks, then we form \( \text{Span}(C) \) has having as objects those of \( C \), and as morphisms from \( A \) to \( B \) triples \((S, \partial^A_S, \partial^B_S)\) forming a diagram

\[
\begin{array}{ccc}
A & \overset{\partial^A_S}{\underset{\partial^B_S}{\rightrightarrows}} & S & \overset{\partial^B_S}{\underset{\partial^A_S}{\rightrightarrows}} & B,
\end{array}
\]

where intuitively \( S \) is a set of internal witnesses, projected to \( A \) and \( B \) via the maps \( \partial^A_S \) and \( \partial^B_S \). For \( C = \text{Set} \) one obtains a relation by collecting the pairs \((\partial^A_S(s), \partial^B_S(s))\) for \( s \in S \), but we have more: for each pair \((a, b) \in A \times B\) we have

\[ \text{wit}^S(a, b) = \{s \in S | \partial^A_S(s) = a \& \partial^B_S(s) = b\}, \]

a set of witnesses that \( a \) and \( b \) are related – hence this indeed provides a notion of a proof-relevant relational model.

Example 3. Writing \( B = \{tt, ff\} \) and \( 1 = \{*\} \), we may represent the program \( \vdash \text{if choice then } tt \text{ else } tt \) as

\[
1 \overset{\partial_1}{\underset{\partial_2}{\rightrightarrows}} \{a, b\} \overset{\partial_3}{\underset{\partial_4}{\rightrightarrows}} B
\]

a span, where \( \partial_1(a) = \partial_1(b) = \ast, \partial_2(a) = \partial_4(b) = \text{tt}. \)

Thus, the evaluation of the program \( \text{to } tt \) has two witnesses.

2) A bicategory: The exact identity of \( S \) does not matter – the same span above with \( S = \{a', b'\} \) should not be treated distinctly. A morphism between spans is \( f : S \rightarrow S' \) making

\[
\begin{array}{ccc}
A & \overset{\partial^A_S}{\underset{\partial^B_S}{\rightrightarrows}} & S & \overset{\partial^B_S}{\underset{\partial^A_S}{\rightrightarrows}} & B
\end{array}
\]

commute; an isomorphism of span is an invertible morphism.

The identity span on \( A \) is simply \( A \overset{1_A}{\rightrightarrows} A \) with two identity maps. The composition of \( A \overset{f}{\rightrightarrows} B \) and \( B \overset{g}{\rightrightarrows} C \) is obtained by first forming the pullback

\[
\begin{array}{ccc}
A & \overset{\partial^A_S}{\underset{\partial^B_S}{\rightrightarrows}} & S & \overset{\partial^B_S}{\underset{\partial^A_S}{\rightrightarrows}} & C
\end{array}
\]

and setting \( \partial'^{T \circ S}_S = \partial'^S \circ \iota \) and \( \partial'^C_{T \circ S} = \partial'^T \circ \rho \) – for \( \text{Span(Set)} \), this means that \( T \circ S \) has elements all pairs \((s, t)\) such that \( \partial'^S_S(s) = \partial'^T_T(t) \), projected to \( A \) and \( C \) via \( \partial'^{T \circ S}_{T}(s, t) = \partial'^S_S(s) \) and \( \partial'^{T \circ S}_{C}(s, t) = \partial'^T_T(t) \).

This composition need not be associative on the nose, but the universal property of pullbacks entails that it is associative up to canonical isomorphism – forming a bicategory:

**Theorem 1.** If \( C \) has pullbacks, then \( \text{Span}(C) \) defined with

objects: \( \text{objects of } C \),

morphisms: \( \text{spans } A \overset{f}{\rightrightarrows} B \),

2-cells: \( \text{morphisms of spans}, \)

forms a bicategory, denoted \( \text{Span}(C) \).

In fact, \( \text{Span}(C) \) is a compact closed bicategory \([32]\) and thus a model of the linear \( \lambda \)-calculus. In particular, \( \text{Span(Set)} \) shares much structure with \( \text{Rel} \): it has the same objects and the operation sending a span \( A \overset{f}{\rightrightarrows} B \) to the pairs \((\partial^A_S(s), \partial^B_S(s))\) for \( s \in S \) is a functor, establishing \( \text{Span(Set)} \) as a natural candidate for a proof-relevant relational model.

3) The exponential: However, the exponential of \( \text{Rel} \) does not directly transport to \( \text{Span} \). The operation \( \mathcal{M}(\_\_\_) \) does yield a functor on \( \text{Set} \) obtained by setting, for \( f : A \rightarrow B \),

\[
\mathcal{M}(f)((a_1, \ldots, a_n)) = [f(a_1), \ldots, f(a_n)],
\]

defining \( \mathcal{M}(f) : \mathcal{M}(A) \rightarrow \mathcal{M}(B) \). But \( \mathcal{M}(f) \) does not lift to \( \text{Span(Set)} \) as it does not preserve pullbacks. Indeed, the diagram obtained by image of the composition pullback

\[
\begin{array}{ccc}
\mathcal{M}(\iota) & \rightarrow & \mathcal{M}(T \circ S) & \rightarrow & \mathcal{M}(\rho) \\
\mathcal{M}(\iota) & \rightarrow & \mathcal{M}(S) & \rightarrow & \mathcal{M}(T) \\
\mathcal{M}(\partial^B_S) & \rightarrow & \mathcal{M}(B) & \rightarrow & \mathcal{M}(\partial'^B_C)
\end{array}
\]

is no pullback: this would need a bijection of \( \mathcal{M}(T \circ S) \) with \( \{(\mu, \nu) \in \mathcal{M}(S) \times \mathcal{M}(T) | \mathcal{M}(\partial^B_S)(\mu) = \mathcal{M}(\partial'^B_C)(\nu)\} \), which fails in general. If \( S = T = \mathbb{E} \) and \( B = 1 \), the pair of multisets \(([tt, ff], [tt, ff]) \) does not uniquely specify who is
synchronized with whom: it may correspond to both multisets \([\{t, t\}, \{f, f\}]\) and \([\{t, f\}, \{f, t\}]\) in \(\mathcal{M}(T \circ S)\).

This might be expected: a finite multiset only remembers the multiplicity of elements, but does not track distinct individual occurrences. This is in tension with the goal of a proof-relevant relational semantics, for which specific witnesses are naturally associated with individual resource occurrences.

4) Categorifying objects: If the exponential is to track individual resource occurrences, that means avoiding the quotient of finite multisets: an element of \(!A\) may for instance be a list, or a word \(a_1 \ldots a_n \in A^*\) of elements of \(A\). We must of course still account for reorderings, which turn \(A^*\) into a groupoid – in fact, it is an instance of the construction of the free symmetric monoidal category \(\text{Sym}(A)\) over a category \(A\): its objects are finite words \(a_1 \ldots a_n\) of objects of \(A\), and a morphism from \(a_1 \ldots a_n\) to \(a'_1 \ldots a'_n\) consists of a permutation \(\pi \in S_n\), and a family \((f_i \in A(a_i, a_{\pi(i)}))_{1 \leq i \leq n}\).

Thus, objects are not mere sets but categories, which means that we move from \(\text{Span}(\text{Set})\) to \(\text{Span}(\text{Cat})\). Indeed, \(\text{Cat}\) also has pullbacks, and so the exact same construction as above yields a bicategory \(\text{Sym}(\text{Cat})\) – except that now the functor \(\text{Sym} : \text{Cat} \rightarrow \text{Cat}\) preserves pullbacks and thus lifts to

\[
\text{Sym} : \text{Span}(\text{Cat}) \rightarrow \text{Span}(\text{Cat})
\]

However, in this categorification, the Seely isomorphism \(\mathcal{M}(A + B) \cong \mathcal{M}(A) \times \mathcal{M}(B)\) is lost. Instead, we only get

\[
\text{Sym}(A + B) \simeq \text{Sym}(A) \times \text{Sym}(B)
\]

an equivalence of categories. In order to lift it to spans, we observe that given a functor \(F : A \rightarrow B\) we get a span

\[
\tilde{F} = \begin{array}{ccc}
A & \xrightarrow{F} & B \\
\| \downarrow \| & \| \downarrow \| & \| \downarrow \|
\end{array} \in \text{Span}(\text{Cat})(A, B)
\]

so that lifting an equivalence \(F : A \simeq B : G\) to spans requires us to provide a family of 2-cells, i.e. for each category \(A:\)

\[
\begin{array}{ccc}
A & \xrightarrow{G} & A \\
\| \downarrow \| & \| \downarrow \| & \| \downarrow \|
\end{array}
\]

but this universal property is powerless to compose horizontally weak morphisms. We cannot have the cake and eat it too: if our method to compose spans ignores the 2-categorical nature of \(\text{Cat}\), then we cannot hope composition to preserve an equivalence between spans that relies on it, as required for a model of linear logic. So it seems that this road to a proof-relevant relational model is doomed – except that this is exactly what we shall do in this paper!

Before we delve into that, we review existing solutions.

C. Proof-Relevant Relational Models, and Other Related Work

As plain pullbacks are “too 1-dimensional”, it is natural to compose spans with a 2-dimensional version.

1) Bipullbacks: There are multiple variants for weakened versions of pullbacks in a 2-category. In this paper, a central notion will be that of a bipullback.\(^2\)

**Definition 2.** In a 2-category \(C\), a bipullback of the cospan \(S \xleftarrow{u} B \xrightarrow{v} T\) is a square commuting up to an invertible 2-cell as in\(\text{Fig. 1\)} such that for any square as in\(\text{Fig. 2\)}

(a) There is a morphism \(h : X \rightarrow P\) and 2-cells \(\alpha\) and \(\beta\) s.t.:

(b) \(h, \alpha, \beta\) are unique up to unique 2-cell – see\(\text{Appendix A\).}

The important observation is that this alternative universal property is sufficient to extend the definition of the horizontal composition in\(\text{2\)} to weak morphisms – with the proviso that this defines horizontal composition only up to iso; as (b) does not guarantee uniqueness of \(h\) on the nose.

\(^2\)According to the nlab, its proper name is a bi-iso-comma-object.
2) Hoffnung’s monoidal tricategory: Hoffnung [33] constructs a categorification of \( \text{Span} \text{(Cat)} \) following this idea. He exploits that \( \text{Cat} \) actually has pseudo-pullbacks\(^3\) which are a special case of the definition above where \( \alpha \) and \( \beta \) are required to be identities and \( h \) is unique on the nose – making horizontal composition of weak morphisms of spans a well-defined function once a choice of pseudo-pullbacks is fixed.

Concretely, a pseudo-pullback of a cospan \( S \rightarrow B \leftarrow T \) may be constructed as a category with objects triples \( (s, \theta, t) \) where \( \theta \in B(u(s), v(t)) \). So for instance, if \( S = T = \text{Sym}(\mathbb{B}) \) and \( B = \text{Sym}(1) \), the pseudo-pullback would have two objects synchronizing \( [tt, ff] \in S \) and \( [tt, ff] \in T \): \(([tt, ff], \text{id}, [tt, ff]) \) and \(([tt, ff], \text{swap}, [tt, ff]) \). The issue of Section II-B3 is avoided by adding new witnesses carrying all possible symmetries. This is a fundamental phenomenon in models of linear logic, which we refer to as saturation.

Because saturation inflates the number of witnesses at each composition, spans composed by pseudo-pullbacks no longer form a bicategory. In particular, the post-composition of a span composition, spans composed by pseudo-pullbacks no longer in models of linear logic, which we refer to as \((\text{Sym}, \text{Dist}, \text{op})\) and \((\text{tt}, \text{sw}) \). The issue of Section II-B3 is avoided by adding new witnesses carrying all possible symmetries. This is a fundamental phenomenon in models of linear logic, which we refer to as saturation.

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4) Generalized Species of Structures: Last but not least, the most well-studied proof-relevant extension of \( \text{Rel} \) is definitely Fiore, Gambino, Hyland and Winskel’s cartesian closed bicategory of generalized species of structure [20]. Relations from \( A \) to \( B \) are replaced with distributors or profunctors:

\[
F : A^{op} \times B \rightarrow \text{Set}
\]

for \( A \) and \( B \) categories. This forms a (compact closed) bicategory \( \text{Dist} \) of (small) categories, distributors and natural transformations between them. The free symmetric monoidal construction \( \text{Sym}(\_\_\_)\) yields a pseudocomonad on \( \text{Dist} \), whose Kleisli bicategory \( \text{Esp} \) is cartesian closed.

As for the span-based approaches above, the way in which \( \text{Dist} \) and \( \text{Esp} \) handle symmetries is saturated. This may first be seen in the identity distributor which is defined to be

\[
A[-,-] : A^{op} \times A \rightarrow \text{Set}
\]

the Yoneda embedding, which associates as witnesses over a pair \((a, a)\) the homset \( A[a,a]\), including all symmetries on \( a \). Composition of distributors is via the coend formula

\[
G \odot F = \int_{b \in B} F(-, b) \times G(b, -)
\]

which sets witnesses in \((G \odot F)(a, c)\) to be pairs \((s, t) \in F(a, b) \times G(b, c)\) quotiented by a relation identifying the action of a morphism in \( B \) on \( s \) or on \( t \).

Accordingly, when computing the interpretation of a program \( M : A \in \text{Esp} \), for every \( a \in \mathbb{A} \) we get \( \mathbb{M}(a) \) a set of witnesses carrying around explicit symmetries, quotiented by an equivalence relation letting symmetries flow around – this is described syntactically elegantly by Olimpieri [28]. The treatment of symmetry in \( \text{Esp} \) is, again, saturated.

5) Game semantics: To our knowledge, this saturation phenomenon in models of linear logic first appears in Baillot, Danos, Ehrhard and Regnier’s (BDER) version [35] of Abramsky-Jagadeesan-Malacaria (AJM) games [12].

In AJM games, the moves of a game \( A! \) are defined as pairs \((i, a)\) of \( i \in \mathbb{N} \) a copy index, and \( a \in A \) a move in \( A \) – a fundamental difficulty in setting up the games model, is that of uniformity: ensuring that the behaviour of strategies does not depend on the specific choice of copy indices (which is the game semantics analogue of composition preserving weak morphisms). In BDER, uniformity is guaranteed by requiring strategies to be saturated: they are morally wrapped by copycat processes exchanging non-deterministically all copy indices. This “noise” prevents strategies from seeing specific copy indices, let alone depending on them – this is analogous to the saturation phenomenon above.

But in AJM games there is another choice: in the original AJM setting [12], strategies carry a deterministic choice of copy indices. Instead of saturation, uniformity is guaranteed by requiring that strategies satisfy a bisimulation-like property, which ensures that whenever Opponent swaps their copy indices, Player can swap theirs accordingly, leaving the behaviour “up to copy indices” invariant. In contrast to the “saturated” approach to uniformity, we refer to this as the
“thin” approach. Similar ideas are at play in other game models based on copy indices: in Mellies’ orbital games [15], and more recently in thin concurrent games [21].

Thin concurrent games are a particularly striking related work, because just as Esp, they also form a cartesian closed bicategory as proved by Paquet [22], and also generalize the relational model [37]. In thin concurrent games, strategies are composed by pullback. But it is a theorem that this pullback is also a bipullback, which can be used to compose horizontally weak morphisms even though strategies are not saturated. But this bipullback property follows from a subtle interactive reindexing mechanism between strategies, relying crucially on the fact that we have access to time – it seems hard to replicate it purely statically as in a relational model.

III. THE BICATEGORY OF THIN SPANS

This long discussion lets us state the main question in this paper: can we construct a thin version of categorical spans?

A. Pullbacks and Bipullbacks in Groupoids

For simplicity, we focus on spans of groupoids rather than categories, which are sufficient for the interpretation of types – we write Gpd for the small 2-category of groupoids. So we aim to construct a bicategory whose objects are small groupoids, whose morphisms are spans $A \leftarrow S \rightarrow B$ with identity the identity span $A \leftarrow A \rightarrow B$, whose composition is plain pullback and yet, whose 2-cells are weak morphisms.

1) Notations and terminology: A span $A \leftarrow S \rightarrow B$ may be presented as a functor $S \rightarrow A \times B$, so it is convenient not to focus on spans, but on functors $S \rightarrow A$ over a groupoid $A$. We refer to those with terminology inspired from game semantics. A prestrategy on groupoid $A$ is a pair $(S, \partial^E)$ where $\partial^E : S \rightarrow A$ is called the display map. We often refer to the prestrategy only with $S$, and write PreStrat$(A)$ for the set of prestrategies on $A$. A prestrategy from $A$ to $B$ is a prestrategy on $A \times B$ – then, we write $\partial^E : S \rightarrow A$ and $\partial^F : S \rightarrow B$ for the two display maps. If $S$ is a prestrategy from $A$ to $B$ and $T$ a prestrategy from $B$ to $C$, we write $T \circ S$ for the prestrategy from $A$ to $C$ obtained as in Section II-B2.

We often refer to morphisms in groupoids as symmetries and write e.g. $\varphi : s \cong_S s'$ instead of $\varphi \in S(s, s')$.

We write 1 for the groupoid with one object $\bullet$, and only the identity morphism; and 0 for the groupoid with one object $\bullet$ and only the identity morphism. If $A, B$ are groupoids, then we use $A \leftarrow B$ and $A \rightarrow B$ as synonyms for $A \times B$, with objects respectively denoted by $a \leftarrow b$ and $a \rightarrow b$ – likewise, their morphisms have form $\theta_A \leftarrow \theta_B \in (A \rightarrow B)(a \rightarrow b, a' \rightarrow b')$ for $\theta_A \in A(a, a')$ and $\theta_B \in B(b, b')$ and likewise for $\theta_A \rightarrow \theta_B$. We find these purely notational distinctions useful to read, since they coincide with familiar type constructors.

2) Indexed families: As explained earlier, types of $\lambda$-calculi may be interpreted as groupoids – but in a linear language, these groupoids remain discrete: only the exponential introduces non-trivial morphisms. As those symmetries play a crucial role, we introduce early our version of the exponential construction. If $X$ is a set, then we write Fam$(X)$ the set of families indexed by finite sets of natural numbers, i.e. of $(x_i)_{i \in I}$ where $I \subseteq \mathbb{N}$ and for all $i \in I$, $x_i \in X$.

Definition 3. Consider $A$ a (small) groupoid. The (small) groupoid Fam$(A)$ has: objects, the set Fam$(A)$; morphisms from $(a_i)_{i \in I}$ to $(b_j)_{j \in J}$, pairs $(\pi, (f_i)_{i \in I})$ of a bijection $\pi : I \cong J$ and for each $i \in I$, $f_i \in A(a_i, b_{\pi(i)})$.

This yields a functor Fam : Gpd $\rightarrow$ Gpd in the obvious way. For $(A_i)_{i \in I} \in$ Fam$(A)$, we call elements of $I$ copy indices. A family $(a_i)_{i \in I} \in$ Fam$(A)$ is more “intensional” than $A^*$ (which is more intensional than $M(A)$): it gives a presentation of a multiset in $M(A)$ not only providing a sequence, but assigning to each element a distinct “address”.

Just as multisets are connected to non-idempotent intersection types, families are connected with Vial’s sequence types [38] – we thus often write families using Vial’s notation, e.g.

$$[2 \cdot a_2, 4 \cdot a_4, 12 \cdot a_{12}] \in \text{Fam}(A)$$

for $(a_i)_{i \in \{2, 4, 12\}}$ – in the particular case where $A = o$, we only write $[i_1, \ldots, i_n]$ for $[i_1 \cdot \bullet, \ldots, i_n \cdot \bullet]$.

For any groupoid $A$, Fam$(A)$ and Sym$(A)$ are equivalent. However, using Fam$(A)$ is crucial in our model construction: it allows the interpretation of programs to use copy indices as identifiers for resource accesses, that are independent of other concurrent resource accesses. We give a few examples:

Example 4. For a groupoid $A$, the dereliction span $\text{der}_A$ is

$$\text{Fam}(A) \xleftarrow{\text{der}_A} A \xrightarrow{id_A} A$$

where $\text{der}_A : A \rightarrow \text{Fam}(A)$ sends $a$ to $[0 \cdot a]$.

In models of linear logic, the role of dereliction is to extract a single instance of a replicable resource. In our model – as in AJM games [12] and thin concurrent games [21] – dereliction does so by picking a copy index (here 0), chosen in advance once and for all. The specific choice is irrelevant; in fact for any $n$ the span using $n$ instead of 0 will be turn out to be isomorphic to $\text{der}_A$. But, the span must comprise a choice.

Example 5. The interpretation of the term $M$ of Example 2 in thin spans shall have head groupoid that with four objects

$$[0 \cdot \text{tt}, 1 \cdot \text{tt}] \rightarrow \text{tt}, \quad [0 \cdot \text{ff}, 1 \cdot \text{ff}] \rightarrow \text{tt}, \quad [0 \cdot \text{tt}, 1 \cdot \text{ff}] \rightarrow \text{ff}, \quad [0 \cdot \text{ff}, 1 \cdot \text{tt}] \rightarrow \text{tt},$$

morphisms reduced to identities, and display map the identity.

The use of specific copy indices allows one to observe which occurrence of $x$ evaluates to $\text{tt}$ or $\text{ff}$, hence associating distinct points to the two evaluations leading to $\text{ff}$.

3) Bipullbacks of groupoids: If composition-by-pullback is to allow us to compose horizontally weak morphisms, we must ensure that every composition pullback is also a bipullback. It is useful to understand a bit better the shape of bipullbacks in Gpd. A first useful fact is that condition (b) of Definition 2 (uniqueness up to iso) automatically holds in the case of Gpd.

\[\text{The first version of concurrent games with symmetry was saturated [36].}\]
furthermore, we can characterise those pullbacks that are also bipullbacks (see [Appendix B]):

**Lemma 1.** A pullback square in $\text{Gpd}$, of the form

$$
\begin{array}{c}
P \\
S \\
\downarrow f \quad \downarrow \pi \\
B \\
\downarrow s \\
T
\end{array}
$$

is a bipullback if and only if it satisfies the following property: for all $s \in S$, $t \in T$ and $\theta \in B(f(s),gt)$, there is $\varphi \in S(s,s')$ and $\psi \in T(t',t)$ such that $f\varphi = gt\psi$ and $\theta = f\varphi \circ g\psi$.

Let us comment on this. We regard triples of the form

$$
s \in S, \quad \theta \in B(fs, gt), \quad t \in T
$$

as pairs of states $(s,t)$ that match up to symmetry – we call this a reindexing problem. The lemma above says that given a reindexing problem, we can always find $s'$ symmetric to $s$ and $t'$ symmetric to $t$ matching on the nose, in a way compatible with $\theta$ – called a solution to the reindexing problem. Thus, the lemma above may be reformulated to say that a pullback is a bipullback iff all its reindexing problems have a solution. We show a concrete example of this reindexing process:

**Example 6.** Take $B = \text{Fam}(o) \to \text{Fam}(o)$, with objects

$$
[i_1, \ldots, i_n] \to [j_1, \ldots, j_k].
$$

Take $S$ the sub-groupoid of $B$ with objects $[i_1, \ldots, i_n] \to [j_1, \ldots, j_n]$ and morphisms $\theta: [2] \to [2], [1] \to [0]$.

The pullback of $S \to B \leftarrow T$ is a bipullback. For instance,

$$
\theta \in B([2] \to [2], [1] \to [0])
$$

is a reindexing problem that may be solved by first applying

$$
\varphi \in S([2] \to [2], [0] \to [0])
$$

in $S$. We are reduced to finding morphisms in $S$ and $T$ w.r.t.

$$
\theta' \in B([0] \to [0], [1] \to [0])
$$

Now, applying $\psi \in T([0] \to [0], [1] \to [0])$ in $T$, we have

$$
\varphi \in S([2] \to [2], [0] \to [0]), \quad \psi \in T([0] \to [0], [1] \to [0])
$$

a solution to the reindexing problem, as in Lemma 1.

That the pullback of two prestrategies forms a bipullback is not a property of either: in this example neither strategy is a fibration as in \cite{ref23}, and solving the reindexing problem requires reindexing in both groupoids. So it is a property emerging from the harmonious interaction between two prestrategies. In an appropriate game semantics setting \cite{ref27}, one can prove that under reasonable assumptions, such interactive reindexing always succeeds. However, this is a gradual process progressing over time – which we do not have access to here.

**B. Orthogonality and Uniform Groupoids**

1) **Definition:** In the literature on models of linear logic, there is a technique for choreographing models where one only composes pairs of morphisms satisfying a given interactive property: biorthogonality. The first step is to specify the desired interactive property via an orthogonality relation:

**Definition 4.** Take $(S, \partial^S)$ and $(T, \partial^T)$ prestrategies on $B$. We say they are uniformly orthogonal, written $S \perp T$, iff the pullback of the cospan $S \to B \leftarrow T$ is also a bipullback.

If $S \subseteq \text{PreStrat}(B)$, then its uniform orthogonal is set to:

$$
S^\perp = \{ T \in \text{PreStrat}(B) \mid \forall S \in S, \quad S \perp T \}.
$$

As usual with orthogonality, this automatically entails a number of properties: for all $S \subseteq \text{PreStrat}(B)$, we have $S \subseteq S^{\perp \perp}$, and $S^{\perp \perp} = S^{\perp \perp \perp}$. We are particularly interested in sets of the form $S^{\perp}$, which are invariant under biorthogonal:

**Definition 5.** A uniform groupoid is a pair $(A, U_A)$ where $A$ is a groupoid and $U_A \subseteq \text{PreStrat}(A)$ is s.t. $U_A^{\perp \perp} = U_A$.

We often refer to a uniform groupoid $(A, U_A)$ just with $A$ when it is clear from the context that it is a uniform groupoid.

2) **Constructions:** The uniform groupoid 1 is the terminal groupoid equipped with $U_1 = \text{PreStrat}(1)$. If $A$ and $B$ are uniform groupoids, their tensor is the groupoid $A \times B$ equipped with the set $U_{A \otimes B} = (U_A \otimes U_B)^{\perp \perp}$, writing

$$
U_A \otimes U_B = \{(S \times T, \partial^S \times \partial^T) \mid S \in U_A, \quad T \in U_B \}
$$

with $\partial^S \times \partial^T : S \times T \to A \times B$. The dual $A^\perp$ of $A$ is $(A, U_A^{\perp \perp})$ with $U_A^{\perp \perp} = U_A$. The par of $A$ and $B$ has

$$
U_{A \otimes B} = (U_A^{\perp \perp} \otimes U_B^{\perp \perp})
$$

yielding the De Morgan duality $(A \otimes B)^\perp = A^\perp \otimes B^\perp$. From this we derive the linear arrow $A \to B = A^\perp \otimes B$.

A uniform prestrategy on uniform groupoid $A$ is simply any $S \subseteq U_A$. If $A$, $B$ are uniform groupoids, then a uniform prestrategy from $A$ to $B$ is a uniform prestrategy on $A \to B$.

3) **Uniform composition:** We claim that whenever composing $S \in U_{A \to B}$ with $T \in U_{B \to C}$, we have the orthogonality

$$
(S, \partial^S_B) \perp (T, \partial^T_C)
$$

so that the composition pullback is a bipullback.

If $S$ is a prestrategy on $A$ and $T$ is a prestrategy from $A$ to $B$, we write $T \circ S$ from the prestrategy on $B$ obtained by

$$
\begin{array}{c}
S \\
\downarrow \circ \\
T
\end{array}
$$

called the application of $T$ to $S$. This lets us state:

**Proposition 1.** Consider $(A, U_A)$ and $(B, U_B)$ uniform groupoids, and $T$ a prestrategy from $A$ to $B$; consider furthermore a class $S \subseteq U_A$ s.t. $(A, \{i_A\}) \in S$ and $U_A = S^{\perp \perp}$.

Then $T \in U_{A \to B}$ iff the following two conditions hold:

1) for all $S \in S$, $T \circ S \in U_B$. 


(2) \((T, \partial^T_B) \in U^+_A\).

**Proof.** Unfolding the definitions, one encounters a few diagram chasing lemmas on pullbacks that are also bipullbacks – themselves proved via [Lemma 1]. See [Appendix C].

The apparent asymmetry is intriguing: by definition \(A^+ \cong B + A^+ \cong B^+\), so that \(T \in U_{A \Rightarrow B}\) iff the span \(B \leftarrow T \rightarrow A\) denoted by \(T^*\) obtained by reversing the two legs, is in \(U_{B^+ \Rightarrow A^+}\). A similar phenomenon appears in the orthogonality used by Ehrhard for his extensional collapse [39].

Now, observe that \((A, id_A) \in U_A\) always – not the identity span, but the identity functor regarded as a prestrategy on \(A\). Indeed, if \(S \in U^+_A\), then the pullback of \(A \rightarrow A \leftarrow S\) is clearly a bipullback, so \((A, id_A) \in U^+_A = U_A\). But now this lets us instantiate [Proposition 1] with \(S = U_A\). Then given \(S \in U_{A \Rightarrow B}\), the application \(S \oplus (A, id_A)\) is (up to iso) the right leg \((S, \partial^S_B)\), which must by (1) be in \(U_B\). Likewise, if \(T \in U_{B \Rightarrow C}\), the left leg \((T, \partial^T_B)\) is in \(U^+_B\). Hence,

\[
(S, \partial^S_B) \perp (T, \partial^T_B)
\]

and thus the composition pullback of \(S\) and \(T\) is a bipullback. [Proposition 1] has more consequences, all obtained in the particular case where \(S = U_A\): we saw above that \((A, id_A) \in U_A\), but the same argument goes to show \((A, id_A) \in U^+_A\) as well – so the identity span satisfies condition (2). Since it also satisfies (1), we have \((A \rightarrow A) \in U_{A \Rightarrow A}\) as expected. Likewise, if \(A \leftarrow S \rightarrow B\) and \(B \leftarrow T \rightarrow C\) are uniform prestrategies, then it follows fairly easily that the composition \(A \leftarrow T \circ S \rightarrow C\) is indeed \(U_{A \Rightarrow C}\) (see [Appendix C]).

4) Horizontal composition of 2-cells: We have an identity uniform prestrategy in \(U_{A \Rightarrow A}\), and a well-defined composition of \(S \in U_{A \Rightarrow B}\) and \(T \in U_{B \Rightarrow C}\) such that the composition pullback is always a bipullback. So given weak morphisms

\[
A \xleftarrow{\partial^A_S} F^A S \xrightarrow{\partial^S_B} B \xleftarrow{\partial^B_S} S' \xrightarrow{\partial^B_B} C
\]

by the bipullback property of \(T' \circ S'\) there are a functor \(H\) and natural isos \(\alpha\) and \(\beta\) such that we have the equality

\[
\xymatrix{T \circ S \ar[r] & T' \circ S'}\]

altogether yielding a weak morphism as in the diagram:

\[
\xymatrix{A \ar[r]^{\partial^A_S} & F^A S \ar[r]^{\partial^S_B} & B \ar[r]^{\partial^B_S} & S' \ar[r]^{\partial^B_B} & C}
\]

However, \(H, \alpha, \beta\) are not unique: though [Lemma 1] guarantees the existence of solutions to all reindexing problems, those may not be unique. We only know by condition (b) of [Definition 2] that different choices of \(H, \alpha, \beta\) yield isomorphic weak morphisms of uniform prestrategies, by which we mean isomorphic morphisms of the 2-category \(\text{Unif}(A)\):

**Definition 6.** Consider \(A\) a uniform groupoid.

The 2-category \(\text{Unif}(A)\) has: objects \(U_A\), i.e. uniform prestrategies on \(A\); morphisms from \(S \Rightarrow T\) the weak morphisms, i.e. pairs \((F : S \Rightarrow T, \phi : \partial^S \Rightarrow \partial^T F)\); 2-cells from \((F, \phi)\) to \((G, \psi)\) the natural transformations \(\mu : F \Rightarrow G\) such that:

\[
\xymatrix{S \ar[r]^{G} & T}
\]

Thus, although bipullbacks guarantee the existence of a fitting weak morphism for horizontal composition, there is a priori no canonical choice. One could pick a choice of horizontal composition, but there is no reason why an arbitrary choice would satisfy the coherence conditions for a bicategory.

**C. Thin Spans of Groupoids**

In fact, if formulated in the adequate way, the reindexing problems that arise from the interpretation of programming languages do have a unique solution – as in [Example 6] But to prove that, we shall need to add more structure to uniform groupoids, starting with polarized sub-groupoids:

1) **Polarized sub-groupoids**: Consider the groupoid

\[\text{Fam}(o) \rightarrow \text{Fam}(o)\]

of [Example 6] interpreting the formula \(lo \rightarrow lo\) of intuitionistic linear logic. Here, the two occurrences of \(l\) are intuitively very different: on the left-hand side, as in [Example 4] the program performs the copying – in game semantics the copy index would be carried by a Player move. In contrast, for the right hand side exponential, the environment does the copying – in game semantics, the copy index would be carried by an Opponent move. This assigns a polarity to certain symmetries, very clear in game semantics: those reindexing copy indices only for exponentials in covariant position (resp. contravariant position) are negative (resp. positive). We enrich the groupoids interpreting types to keep track of these special symmetries:

**Definition 7.** A polarized groupoid is a groupoid \(A\) with two sub-groupoids \(A_-\) and \(A_+\), with the same objects as \(A\).

It would be natural to require additional conditions for this structure (in particular, see conditions (a) and (b) in [Lemma 3]). We omit them here, as they shall hold automatically once we introduce the more complete notion of a thin groupoid.

If \(\theta \in A_-(a_1, a_2)\), we write \(\theta : a_1 \equiv a_2\) and likewise for positive symmetries. Usual constructions on groupoids extend to polarized groupoids componentwise. The dual of \((A, A_-, A_+)\) is defined as \((A, A_+, A_-)\), exchanging the two sub-groupoids. Finally, we set \((!A)_- = \text{Fam}(A_-)\) and \((!A)_+ = \text{Fam}^{id}(A_+)\), which has morphisms from \((a_i)_{i \in I}\) to \((b_j)_{j \in J}\) those \((\pi_i, \theta_i)_{i \in I}\) such that \(I = J\) and \(\pi = \text{id}_I\) – thus we see indeed that Player cannot reindex copy indices from the outer ! in \(!A\), as it appears in covariant position.
2) **Thinness:** Solutions to reindexing problems may be computed interactively as in Example 6. Intuitively, the uniqueness of the solution relies on the fact that at each stage, there is a unique choice of reindexing. This is captured by the definition of thin below, imported from thin concurrent games:

**Definition 8.** Consider $A$ a polarized groupoid, and $S$ a prestrategy on $A$. We say that $S$ is *thin* iff for all $\phi : s \cong_S s'$, if $\partial^2 \phi$ is positive then $s = s'$ and $\phi = \text{id}_A$.

Intuitively, this captures that positive copy indices are selected deterministically from negative ones—so a non-trivial symmetry $\phi : s \cong_S s'$ cannot display to a purely positive symmetry on $A$. This is in contrast with the saturated case, where spans must be able to reach all positive symmetries.

We show how thinness addresses uniqueness for the resolution of reindexing problems. Call a solution to a reindexing problem $\varphi, \psi$ as in Lemma 1 positive if writing $\partial^S \varphi = \varphi_A \vdash \varphi_B$ and $\partial^T \psi = \psi_B \vdash \psi_C$, we have $\varphi_A \vdash \psi_C$ positive.

**Lemma 2.** Consider $A, B, C$ polarized uniform groupoids, $S \in \mathcal{U}_{A \rightarrow B}$ and $T \in \mathcal{U}_{B \rightarrow C}$ s.t. $T \circ S \in \mathcal{U}_{A \rightarrow C}$ is thin.

Then, any reindexing problem in the composition pullback of $S$ and $T$ has at most one positive solution.

**Proof.** Consider a reindexing problem $s \in S, t \in T, \theta : \partial^S t \cong_B \partial^T t$ with solutions $\varphi_1 : s \cong s'_1$ and $\psi_1 : t'_1 \cong T t$ with $\partial^S s'_1 = \partial^T t'_1$ and $\partial^T \varphi_1 \circ \partial^B \varphi_1 = \theta$, and $\varphi_2 : s \cong s'_2$ and $\psi_2 : t'_2 \cong T t$ with $\partial^S s'_2 = \partial^T t'_2$ and $\partial^B \psi_2 \circ \partial^T \varphi_1 = \theta$.

Then, $\partial^T (\varphi_2 \circ \varphi_1^{-1}, \psi_2 \circ \psi_1^{-1}) = (s'_1, t'_1) \cong_{T \circ S} (s'_2, t'_2)$, whose display to $A \vdash C$ is positive since $\varphi_1, \psi_1$ and $\varphi_2, \psi_2$ are positive solutions. Hence, by thin, $\Omega$ is an identity map which entails $\varphi_1 = \varphi_2$ and $\psi_1 = \psi_2$ as required.

Thus, thinness allows us to find canonical solutions to reindexing problems by insisting on finding positive solutions.

However, this relies on thinness not of $S$ and $T$, but of $T \circ S$. Again in thin concurrent games, this follows by induction on the causal structure. In the absence of a handle on causality, we must as for uniformity treat the fact that $T \circ S$ is thin as an interactive property, again handled by biorthogonality.

**D. Thin Spans**

1) **The thin orthogonality:** We observe that for $A$ a polarized groupoid, a prestrategy $S$ on $A$ is thin iff the pullback

$$S \perp T$$

is discrete, i.e. all the morphisms in $P$ are identities. We shall base our orthogonality on this observation, and set:

**Definition 9.** For $A$ a polarized uniform groupoid, $S \in \mathcal{U}_A$, and $T \in \mathcal{U}_A^-$, we say $S$ and $T$ are *thinly orthogonal*, written $S \perp T$.

iff the pullback $T \circ S$ is discrete.

Note that this is already assuming that $S$ and $T$ are uniformly orthogonal. If $S \subseteq \mathcal{U}_A$, then its thin orthogonal is

$$S^\perp = \{ T \in \mathcal{U}_A^\perp \mid \forall S \in S, S \perp T \},$$

and as before we have $S \subseteq S^\perp \perp$ (note that this typechecks only because $U^\perp = U\perp_A$) and $S = S^\perp \perp$ for all $S \subseteq \mathcal{U}_A$.

2) **Thin groupoids:** As before, we are interested in sets of uniform prestrategies closed under bi-thin-orthogonal:

**Definition 10.** A *thin groupoid* is a polarized uniform groupoid with a set $T_A \subseteq \mathcal{U}_A$ of strategies s.t. $T_A^\perp \perp = T_A$, and such that $(A_-, \text{id}_A) \in T_A$ and $(A_+, \text{id}_A) \in T_A^\perp$.

If $S \in T_A$ then $S$ is automatically thin in the sense of Definition 8, as $(A_+, \text{id}_A) \in T_A^\perp$ the pullback (3) is discrete.

This also entails properties of the polarized symmetries:

**Lemma 3.** Consider $A$ a thin groupoid. Then we have:

(a) if $\theta : a \cong a'$ and $\theta : a \cong a''$, then $a = a'$ and $\theta = \text{id}_A$.

(b) if $a \cong a'$, then there are unique $a''$ along with $\theta : a \cong a''$ such that $\theta = \theta \circ \theta$.

**Proof.** (a) follows from $A_+ \vdash A_-$, as then the pullback of the cospan $A_- \rightarrow A_+ \rightarrow A_-$ is discrete.

For (b), $A_- \in T_A \subseteq \mathcal{U}_A$ and $A_+ \in T_A^\perp \subseteq \mathcal{U}_A^\perp$, we also have $A_- \vdash A_+$. Hence, the pullback of the cospan $A_- \rightarrow A_+ \rightarrow A_-$ is a bipullback. But then any $\theta : a \cong a'$ forms a reindexing problem, whose solution is exactly the seeked reindexing. Uniqueness follows immediately from (a).

Thus, we get from the definition of thin groupoids some of the expected properties of the polarized sub-groupoids: if a symmetry is both positive and negative then it must be an identity, and any symmetry can be obtained by first “reindexing Opponent moves”, then “reindexing Player moves”.

3) **Further structure:** Constructions on uniform groupoids extend to thin groupoids in the expected way. The thin groupoid 1 has $T_1 = \text{PreStrat}(1)$. If $A$ and $B$ are thin groupoids, their tensor is the uniform groupoid $A \otimes B$ extended with $T_{A \otimes B} = (T_A \otimes T_B)^\perp \perp$. The dual of $A$ has $T_{A^*} = T_A^\perp$. The par of $A$ and $B$ has $T_{A \otimes B} = (T_A^\perp \otimes T_B)^\perp$, and the linear arrow is $A\rightarrow B = A^\perp \rightarrow B$.

To establish the compositional properties of strategies, we rely on the following analogue of Proposition 1.

**Proposition 2.** Consider $T \in \mathcal{U}_{A \rightarrow B}$ for $A, B$ thin groupoids, along with a class $S \subseteq T_A$ such that $S^\perp \perp = T_A$.

Then, $T \in T_{A \rightarrow B}$ iff $T \circ S \in T_B$ for all $S \in S$.

This follows from diagram chasing lemmas on situations where the pullbacks are discrete, see Appendix C. Interestingly, this is also equivalent to $T^* \circ S \in T_B^\perp \perp$ for all $S \in T_B^\perp$.

It is a direct consequence of Proposition 2 that the identity span on $A$ is on $T_{A \rightarrow A}$ for any thin groupoid $A$, and that if $S \in T_{A \rightarrow B}$ and $T \in T_{B \rightarrow C}$ then $T \circ S \in T_{A \rightarrow C}$. Together with Lemma 2 we have thus identified a compositional situation where the composition pullback of spans is a bipullback, and where all arising reindexing problems have a unique solution if one insists on this solution being positive.
4) Positive weak morphisms: Insisting on positive solutions amounts to relating strategies not via arbitrary weak morphisms, but with positive weak morphisms:

**Definition 11.** Consider a thin groupoid, $S, T \in T_A$, and $(F, \phi)$ a weak morphism from $S$ to $T$, i.e. $F : S \to T$ and $\phi : \partial S \to \partial T \circ F$. Then, $(F, \phi)$ is **positive** if $\phi$ is positive, that is, if $\forall s \in S$, $\phi_s : \partial s \Rightarrow_{\phi} \partial^T F(s)$ is a positive symmetry.

Intuitively, comparing strategies with positive morphisms amounts to relating them only via maps that do not reindex Opponent moves. This has the effect of making everything stricter, and cutting the higher dimension. More precisely:

**Proposition 3.** Let $A$ be a thin groupoid. Consider $\text{PreThin}(A)$ the sub-2-category of $\text{Unif}(A)$ with objects $T_A$, and $\text{Thin}(A)$ where additionally morphisms are positive.

Then, $\text{Thin}(A)$ is locally discrete, i.e. all 2-cells are identities. Moreover, $\text{PreThin}(A)$ and $\text{Thin}(A)$ are biequivalent.

**Proof.** The first is a direct consequence of thinness: if $\mu : (F, \phi) \Rightarrow (G, \psi) : S \to T$ for $\phi$ and $\psi$ positive, then by definition of 2-cells of $\text{Unif}(A)$, for all $s \in S$, $\mu_s \in T(Fs, Gs)$ is such that $\psi_s = \partial^T \mu_s \circ \phi_s$, i.e. $\partial^T \mu_s = \psi_s \circ \phi_s^{-1}$ positive. Thus, $\mu_s$ is an identity morphism by thinness.

For the biequivalence, the crux is that if $(F, \phi) : S \to T$ is a weak morphism, then there is a unique $(F_+, \phi_+) : S \to T$ positive isomorphic to $(F, \phi)$, and a unique 2-cell $\mu$ between them. Uniqueness follows from thinness. For existence, note that if $s \in S$ and $\theta : \partial S s \cong A a$, then there exist unique $\varphi : s \cong s'$ and $\theta' : \partial S s \cong A a$ such that $\theta = \theta' \circ \partial^T \varphi$ – this exploits thinness, and the reindexing problem from the fact that the pullback of the cospan $S \rightarrow A \leftarrow A +$ is a bipullback. We obtain $(F_+, \phi_+)$ by applying this lemma pointwise.

This proposition illustrates the situation well: thanks to the thin biorthogonality, the 2-category $\text{PreThin}(A)$ is represented up to biequivalence as a mere category $\text{Thin}(A)$. The higher dimensional structure simply vanishes.

5) The bicategory $\text{Thin}$: With this in place, we may finally define the components of our bicategory $\text{Thin}$. Its objects are thin groupoids. Its morphisms from $A$ to $B$ are strategies from $A$ to $B$, i.e. elements of $T_{A \to B}$ – recall that they are $(S, \partial S : S \to A \times B)$, in particular spans from $A$ to $B$

$$A \xleftarrow{\partial S_A} S \xrightarrow{\partial S_B} B.$$  

The identities are span identities, and composition is via the pullback $[1]$. If $S$ and $T$ are strategies from $A$ to $B$, the 2-cells from $S$ to $T$ are the positive morphisms $(F, \phi) : S \to T$. As $\phi : \partial S \Rightarrow \partial^T \circ F$ is a family of positive morphisms on $A \downarrow \downarrow B$ with underlying plain groupoid $A \times B$, it may be equivalently presented as pair of $F_A : \partial S \Rightarrow \partial^T \circ F$ and $F_B : \partial S_B \Rightarrow \partial^T B \circ F$, as in Definition 1. For horizontal composition of positive morphisms, we first proceed as in Section III-B[4] and obtain a connected groupoid of (non necessarily positive) horizontal compositions – which must all have the same image through the biequivalence of Proposition 3[4], providing our unique positive horizontal composition. Altogether, we have:

**Theorem 2.** Those components form $\text{Thin}$, a bicategory.

**Proof.** See details in Appendix D.

Next, we develop the further structure of $\text{Thin}$.

### IV. Cartesian Closed Structure

To construct a cartesian closed bicategory, we intend to follow [20]. We first turn the construction $\text{Fam}$ – thereafter denoted by $!$ – into a pseudomonad, and then equip the Kleisli bicategory $\text{Thin}_1$ with the cartesian closed structure.

**A. The Pseudomonad**

We first develop the action of $!$ on objects of $\text{Thin}$.  

1) The bang of thin groupoids: First, $!$ is defined on uniform groupoids via $U_A = (\langle U_A \rangle)^{1+}$, where we have

$$!U_A = \{ (S, [G]) \mid S \in U_A \}$$

using the functorial action $\partial S : S \to A$. For thin groupoids, the positive and negative symmetries of $!A$ were defined in Section III-C[1]. The thin structure is set as $T_{IA} = (\langle TA \rangle)^{1+} = !A$ is such that $!A \cong A$. This is a direct verification that this is a thin groupoid.

2) The bang of strategies: If $S \in T_{A \to B}$, we have $\partial S = (\partial S_A, \partial S_B)$ for $\partial S_A : S \to A$ and $\partial S_B : S \to B$ – its bang is

$$!A \cong A \xleftarrow{\partial S_B} !S \xrightarrow{\partial S_A} !B$$

packaged as $\langle !S, (\partial S_A, \partial S_B) \rangle$. That this is in $T_{A \to !B}$ relies on:

**Lemma 4.** Consider $A, B$ thin groupoids, and $T$ a prestrategy from $!A$ to $B$. Then, the following two properties hold:

1) $T \in U_{A \to !B} \iff (T, \partial S_A) \in U_A$ and for all $S \in U_A$, $T \circ S \in U_B$.

2) $T \in T_{A \to !B} \iff$ for all $S \in T_A$, $T \circ S \in T_B$.

This is an immediate application of Propositions 1[2] and 2. Since $U_{!A} = (\langle U_A \rangle)^{1+}$ and $T_{!A} = (\langle TA \rangle)^{1+}$. From this lemma, it is a rather direct verification that for any $S \in T_{A \to !B}$, we have $!S \in T_{A \to !B}$ as required.

3) A pseudofunctor: $!$ is a functor, it preserves the identity span on the nose. Since $!$ preserves pullbacks, for $S \in T_{A \to B}$ and $T \in T_{B \to C}$, the universal property gives us

$$m_{S,T} : !(T \circ S) \cong !(T \circ !S)$$

a strong invertible 2-cell in $\text{Thin}$. As expected, this 2-cell is natural in $S$ and $T$ (with respect to positive morphisms). Altogether, we obtain a pseudofunctor $!: \text{Thin} \to \text{See}$.

**Appendix E** for details.

4) A pseudomonad on groupoids: In fact we first turn $!$ into a pseudomonad on $\text{Gpd}$, from which its pseudocomonad structure on $\text{Thin}$ shall be derived. We noted earlier that we have a functor $\text{Fam} : \text{Gpd} \to \text{Gpd}$ – in fact, it is extended to a 2-endofunctor on the 2-category of small groupoids, noted

$$!: \text{Gpd} \to \text{Gpd},$$

defined on a 2-cell $\alpha : F \Rightarrow G : A \to B$ as the natural transformation $!\alpha : !F \Rightarrow !G$ with components all pairs

$$(!\alpha)_{(A_i)}_{i \in I} = (id_I, \langle \alpha_{A_i} \rangle)_{i \in I} \in !B((FA_i)_{i \in I}, (GA_i)_{i \in I}).$$


To turn this into a pseudomonad, we must adjoin a multiplication and a unit. The components of the unit are the functors

\[ \eta_A : A \to !A \]

with the obvious functorial action. The intuition is that the unit transports a single resource usage from \( A \) to \(!A\), arriving at a singleton family. In doing so, it must select a copy index. Any natural number will do – the rest of the paper does not depend on this choice – but for definiteness and compatibility with the traditional convention from AJM games, we pick 0.

For the multiplication \( \mu_A : !!A \to !A \), we must flatten a family of families into a family. For this purpose, we fix an injective function \((-,-) : \mathbb{N}^2 \to \mathbb{N}\) – again, the results of this paper do not depend on that choice. Given \( I \subseteq \mathbb{N} \) and a family \((J_i)_{i \in I}\) where \( J_i \subseteq \mathbb{N} \) for all \( i \in I \), let us write

\[ \Sigma_{i \in I} J_i = \{ (i,j) \mid i \in I, j \in J_i \}, \]

which is by definition still a finite subset of \( \mathbb{N} \). Then we set

\[ \mu_A : !!A \to !A \]

for any groupoid \( A \), along with the obvious functorial action.

Altogether this yields \( \eta : \text{id}_{\text{Gpd}} \Rightarrow ! \) and \( \mu : !! \Rightarrow ! \), two (strict 2-) natural transformations. The monad laws, if they were to hold on the nose, would mean that \( (0,i) = (i,0) = i \) and \( \langle (i,j), k \rangle = \langle i, (j,k) \rangle \) for all \( i,j,k \in \mathbb{N} \); and it is clear that no injection satisfying those laws exists. Nevertheless, for every groupoid \( A \) the coherence laws for a monad hold up to natural isomorphisms: we have \( \alpha_A, \beta_A, \gamma_A \) as indicated in Figures 3 and 4.

For instance, for any \((a_j)_{j \in J} \in !!A:\]

\[ (\alpha_A)(a_j)_{j \in J} : (a_j)_{j \in J} \cong_{\text{Gpd}} (a_j)_{(a_j) \in \Sigma_{i \in J} J} \]

reindexing along the bijection \( J \cong \Sigma_{i \in \{0\}} J \). The other components act similarly – note that they are all negative symmetries. The associated families \( (\alpha_A)_{A \in \text{Gpd}}, (\beta_A)_{A \in \text{Gpd}} \) and \( (\gamma_A)_{A \in \text{Gpd}} \) satisfy the conditions for modifications, and the additional coherence laws for a pseudomonad:

**Proposition 4.** The 2-functor \( ! : \text{Gpd} \to \text{Gpd} \) along with the components above yield a pseudomonad on \( \text{Gpd} \).

5) **Lifting functors to spans:** We shall turn \( ! \) into a pseudomonad on \( \text{Thin} \) by lifting the components above to spans. In general, if \( F : B \to A \) is a functor, then there is a span \( \bar{F} \) called the lifting of \( F \) – but we need sufficient conditions on \( F \) for this construction to yield morphisms in \( \text{Thin} \). For that purpose, if \( A \) and \( B \) are thin groupoids, we say that a functor \( F : A \to B \) is a renaming of the following conditions hold:

(1) for all \( \theta : a \cong A a' \), if \( \theta \) is positive then so is \( \theta' \),

(2) for all \( (T, \partial T) \in \text{U}^D_B, (T, F \circ \partial T) \in \text{U}^D_A \),

(3) for all \( (T, \partial T) \in \text{T}^D_B, (T, F \circ \partial T) \in \text{T}^D_A \).

Clearly, renamings compose – we consider the 2-category \( \text{Ren} \) whose objects are thin groupoids, whose morphisms are renamings, and whose 2-cells are natural transformations. As expected, lifting renamings yields thin spans (see Appendix E). Lifting can be extended to 2-cells: if \( \alpha : F \Rightarrow G : A \to B \) is a negative natural transformation, then \( \bar{\alpha} \) is the positive morphism described by the diagram:

Fig. 3. Unit natural isomorphisms

Fig. 4. Associativity natural isomorphism

\[ \begin{array}{ccc}
\text{A} & \xrightarrow{\mu_A} & \text{!A} \\
\downarrow{\eta_A} & & \downarrow{\mu_A} \\
\text{!!A} & \xrightarrow{\eta_A} & \text{!A}
\end{array} \]

noting that this is positive as negative \( \alpha \) is in contravariant position. Altogether, we get (see details in Appendix E).

**Proposition 5.** There is a pseudofunctor \( \bar{\cdot} : \text{Ren}^{op} \to \text{Thin} \).

Here, \( \text{Ren}^{op} \) is \( \text{Ren} \) with the morphisms reversed, but the 2-cells unchanged. It can be checked that for any thin groupoid \( A \), the functors \( \eta_A : A \to !A \) and \( \mu_A : !!A \to !A \) are renamings, in particular for every thin groupoid \( A \) we get

\[ \eta_A \in \text{Thin}(A, A), \quad \mu_A \in \text{Thin}(A, !!A) \]

the main components to turn \( ! \) into a pseudomonad. Unlike in \( \text{Gpd} \), the families \( \bar{\eta} \) and \( \bar{\mu} \) are not strict 2-natural transformations but only pseudonatural transformations, with 2-cells

\[ \begin{array}{ccc}
\eta_S & : & \bar{\eta}B \circ S \Rightarrow S \circ \bar{\eta}_A \\
\mu_S & : & \bar{\mu}B \circ S \Rightarrow !!S \circ \bar{\mu}_A
\end{array} \]

positive isomorphisms obtained for \( S \in \text{Thin}(A, B) \) from the universal property of pullbacks, via the observation that \( \eta : \text{id}_{\text{Gpd}} \Rightarrow ! \) and \( \mu : !! \Rightarrow ! \) are cartesian natural transformations. It may be checked that \( \eta_S \) and \( \mu_S \) are natural in \( S \) and satisfy the coherence conditions of pseudonatural transformations. Finally, the modifications \( \alpha, \beta, \gamma \) involved in the pseudomonad structure of \( ! \) on \( \text{Gpd} \) lift to the modifications required for the pseudomonad structure of \( ! \) on \( \text{Thin} \).

**Theorem 3.** We have a pseudomonad \( ! \) on \( \text{Thin} \).

**Proof.** See details in Appendix G.

5) We move on to studying the Kleisli bicategory \( \text{Thin} \); whose horizontal composition, denoted \( \circ \), is defined as expected.
B. Cartesian Closed Structure

1) Finite products: First, we show that \( \text{Thin}_1 \) has finite products, i.e. is a fp-bicategory in the sense of Fiore and Saville [13] – unlike them, we work with binary products.

a) Terminal object: Write \( \top \) for the empty groupoid, made a thin groupoid with \( U_\top = T_\top = \{ \text{id}_\top \} \). For any thin groupoid \( A \), \( \text{Thin}_1(A, \top) \) has exactly one element – the empty groupoid. Thus, \( \text{Thin}_1 \) has a (strict) terminal object.

b) Binary product: If \( A \) and \( B \) are thin groupoids, then the with \( A \& B \) has underlying groupoid \( A \& B \) the disjoint union, with \((A + B)_- = A_- + B_- \) and \((A + B)_+ = A_+ + B_+ \). We adjoin \( U_{A+B} = (U_A \cup U_B)_{\perp \perp} \) and \( T_{A+B} = (T_A + T_B)_\perp \perp \), where as usual, \( U_A + U_B \) comprises the set of all \((S + T, \partial^S + \partial^T)\) for \((S, \partial^S) \in U_A \) and \((T, \partial^T) \in U_B \), using the functorial action of \( + \) (and likewise for \( T_A + T_B \)).

c) Pairing and projections: The projections are simply set as \( L_i = \left( \eta_{A+B} \circ \bar{l} \right) \in \text{Thin}_1(A \& B, A) \) and \( R_i = \left( \eta_{A+B} \circ \bar{r} \right) \in \text{Thin}_1(A \& B, B) \) for \( i : \Gamma \to A + B \) and \( r : B \to A + B \) the obvious coprojections/renamings.

The pairing of \( S \in \text{Thin}_1(\Gamma, A) \) and \( T \in \text{Thin}_1(\Gamma, B) \) is
\[
(S + T, \partial^T : S + T \to \Gamma, \partial_{A+B} : S + T \to A + B)
\]
with \( \partial_T \) the co-pairing and \( \partial_{A+B} = \partial^S + \partial^T \). We have:

**Proposition 6.** For any thin groupoids \( \Gamma, A \) and \( B \), there is
\[
\text{Thin}_1(\Gamma, A \& B) \xrightarrow{\left( L_i \circ -,-, R_i \circ - \right)} \text{Thin}_1(\Gamma, A) \times \text{Thin}_1(\Gamma, B)
\]
an adjoint equivalence.

**Proof.** If \( S \in \text{Thin}_1(\Gamma, A) \) and \( T \in \text{Thin}_1(\Gamma, B) \) there are
\[
\omega^A_{S,T} : L_i \circ (S, T) \cong S \quad \omega^B_{S,T} : R_i \circ (S, T) \cong T
\]
positive isos, and for \( U \in \text{Thin}_1(\Gamma, A \& B) \) there is
\[
\bar{\omega}_U : U \cong \langle U_i \circ U, R_i \circ U \rangle
\]
a positive iso, defined in the obvious way. Those are all natural in \( S, T, U \), and satisfy the required triangle identities. \( \square \)

See Appendix I for more details. Altogether, this establishes that \( \text{Thin}_1 \) is a fp-bicategory in the sense of [18].

2) Cartesian closure: If \( A \) and \( B \) are thin groupoids, then we set \( A \Rightarrow B = !A \times B \). Before we describe the additional components, we must observe the Seely equivalence:
\[
!A \otimes !B \xleftarrow{s_{A,B}} !\langle A \& B \rangle
\]
where \( s_{A,B} \) sends \((a_i)_{i \in I}, (b_j)_{j \in J}\) to \((c_k)_{k \in I \times J}\), with \( I \times J = \cup (I \cup J) \) for some chosen bijection \( \varpi = (\varpi_I, \varpi_J) \) between \( I \) and \( J \), and where \( c_{\varpi_i(i)} = a_i \) and \( c_{\varpi_J(j)} = b_j \); and \( s_{A,B} \) sends \( (c_k)_{k \in K} \) to \((a_i)_{i \in I}, (b_j)_{j \in J}\) where \( I \subseteq K \) is the subset of those \( i \in I \) such that \( c_i = a_i \), and likewise for \( b_j \). Both functors are renamings, and the isomorphisms witnessing the equivalence are negative.

Via the Seely equivalence, we first define the evaluation as the span with basic groupoid !\( A \times B \), with left leg the functor !\( A \times B \to !\langle A \times B \rangle \times !A \times !B \times !\langle A \Rightarrow B \rangle \Rightarrow A \Rightarrow B \). This yields a thin span \( \text{ev}_{A,B} \in \text{Thin}_1(A \Rightarrow B, A, B) \). Now, we need
\[
\Lambda(-) : \text{Thin}_1(\Gamma \& A, B) \to \text{Thin}_1(\Gamma, A \Rightarrow B)
\]
the currying functor; given \( S \in \text{Thin}_1(\Gamma \& A, B) \), its currying \( \Lambda(S) \) is simply \( S \), with display map post-composed with
\[
(!\Gamma + A) \times B \cong (!\Gamma \times !A) \times B \cong !\Gamma \times (!A \times B).
\]

With this data in place, we may finally prove:

**Proposition 7.** For any groupoids \( \Gamma, A, B \), there is
\[
\text{Thin}_1(\Gamma, A \Rightarrow B) \xrightarrow{\text{ev}_{A,B} \otimes (- \& A)} \text{Thin}_1(\Gamma \& A, B)
\]
an adjoint equivalence.

**Proof.** One can first show the existence of adjoint equivalence between the currying operation \( \Lambda(-) \), and a symmetric uncurrying operation \( \Lambda(-) \). The unit and counit of this adjunction can be derived from the ones of the Seely (adjoint) equivalence. One can then prove that \( \Lambda(-) \) is in fact isomorphic to \( \text{ev}_{A,B} \otimes (- \& A) \) in order to get the wanted equivalence. \( \square \)

See Appendix I for details. Altogether, we have:

**Theorem 4.** We have \( \text{Thin}_1 \), a cartesian closed bicategory.

This entails that we can interpret types of the simply-typed \( \lambda \)-calculus as thin groupoids, morphisms as thin spans and rewrites between terms as certain positive isomorphisms [19].

V. Conclusion

This paper focuses on the construction of \( \text{Thin}_1 \), leaving for later its application to semantics of \( \lambda \)-calculi and programming languages. We believe this opens multiple perspectives for further research: firstly, we may explore the obtained interpretation of the \( \lambda \)-calculus, which syntactically should correspond to the sequence typing system of Vial [38] and to the non-uniform \( \lambda \)-calculus of Melliès [13]. We should explore links with other models of the literature, notably with the weighted relational model recasting ideas from [37], and with generalized species of structures and template games. Another related direction consists in accommodating another feature of template games, the mechanism to capture scheduling and synchronization [24], into thin spans.

In more semantic directions, we believe that with respect to generalized species of structures, the fact that operations on thin spans involve no quotient may be helpful in two ways: (1) individuals may be ordered concretely, and the model should support continuous reasoning allowing one to deal easily with infinite computation; and (2) adding “typed” weights coming from an SMCC as in [24] should be a lot simpler, since those weights no longer have to themselves be saturated.
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REFERENCES

APPENDIX

In the following, we will type spans $A \xrightarrow{u} S \xrightarrow{v} B$ between groupoids or thin groupoids $A$ and $B$ as $S: A \leftrightarrow B$.

A. Bipullbacks

In order to complete the equational definition of bipullbacks of Definition 2, it is useful to consider the intensional definition first: given a 2-category $C$ with invertible 2-cells (i.e., a $(2,1)$-category) and a cospan $S \xrightarrow{u} B \xleftarrow{v} T$ in $C$, a bipullback is a pseudocone $(P, l, r, \mu)$ as in Figure 1 such that, for every $X \in C$, the precomposition of the pseudocone $(P, l, r, \mu)$ by morphisms $X \to P$ induces an equivalence of categories between $C(X, P)$ and the category of pseudocones over the cospan $S \xrightarrow{u} B \xleftarrow{v} T$ and of vertex $X$ and pseudocone morphisms. The essential surjectiveness of this precomposition corresponds exactly to the condition (a) of Definition 2. Its full faithfulness can be expressed as the following condition: (b') given a 2-cell equality

![Diagram](https://via.placeholder.com/150)

for some $h, h': X \to P$ and 2-cells $\alpha: l \circ h \Rightarrow l \circ h'$ and $\beta: r \circ h' \Rightarrow r \circ h$, there is a unique $\theta: h \Rightarrow h'$ such that $\alpha = l\theta$ and $\beta = r\theta^{-1}$.

It is not too difficult to show that the latter property is equivalent to the one asserting that, given two decompositions of a pseudocone $\nu$

![Diagram](https://via.placeholder.com/150)

there exists a unique $\theta: h \Rightarrow h'$ such that $l\theta = \alpha' \circ \alpha^{-1}$ and $r\theta = \beta^{-1} \circ \beta'$, or equivalently $\alpha' = (l\theta) \circ \alpha$ and $\beta' = \beta \circ (r\theta^{-1})$, which is the complete form of the condition (b).

B. Pullbacks in Gpd

It happens that pullbacks in Gpd are well-behaved w.r.t 2-cells:

**Proposition 8.** A pullback of a cospan $S \xrightarrow{u} B \xleftarrow{v} T$ in Gpd is a strict 2-pullback, that is, also admits a universal factorization property w.r.t. morphisms of cones.

**Proof.** Let $I$ be the groupoid consisting of a walking isomorphism $u$ between two objects 0 and 1. Given two functors $F$ and $G$ between two groupoids $C$ and $D$, a 2-cell

$\alpha: F \Rightarrow G: C \to D \in \text{Gpd}$

is then exactly the data of a functor $H: I \times C \to D$ such that $H(0, -) = F$ and $H(1, -) = G$. Using this correspondence, the property that a pullback $(P, l, r)$ over the cospan is a 2-pullback easily reduces to the one that $(P, l, r)$ is a 1-pullback.

$\Box$

Note that a pullback is a cone, which is in particular a pseudocone with identity as inner 2-cell. One might then ask when a pullback is a bipullback, in which case we have the following characterization:

**Proposition 9.** Let $(P, l, r)$ be a pullback over a cospan $S \xrightarrow{u} B \xleftarrow{v} T$ in Gpd. Then the pseudocone induced by $(P, l, r)$ is a bipullback if and only if it satisfies the condition (a) of Definition 2.

**Proof.** By Proposition 8 $(P, l, r)$ is a 2-pullback, so that it satisfies a universal property w.r.t. cone morphisms. This condition is in fact exactly (b') which is equivalent to (b). Thus, only (a) is left to check for $(P, l, r)$ to be a bipullback.

$\Box$

We can then refine the previous proposition into a “pointwise” characterization in the form of already stated Lemma 1 for which we now provide a proof.

**Proof of Lemma 1.** The implication is immediate, since the data of an isomorphism $\vartheta: f(s) \to g(t)$ is equivalent to the one of a pseudocone on $S \xrightarrow{u} B \xleftarrow{v} T$ of vertex the terminal groupoid, whose factorization in the form of the condition of Proposition 9 is exactly the wanted property.

We now show the converse property. So let

![Diagram](https://via.placeholder.com/150)

be a pseudocone. By hypothesis, for every $z \in Z$, there exist $s_z \in S$, $\phi_z: h(z) \to s_z$, $t_z \in T$, $\psi_z: t_z \to k(z)$ such that $f(s_z) = g(t_z)$ and $\theta_z = g(\psi_z) \circ f(\phi_z)$. The collection of isomorphisms $(\phi_z)_{z \in Z}$ induces a functor $h'$ defined by $h'(z) = s_z$ for $z \in Z$, and $h'(w) = \phi_z \circ h(w) \circ \phi_z^{-1}$ for $w: z \to z' \in Z$. Similarly, we get a functor $k'$ defined by $k'(z) = t_z$ for $z \in Z$, and $k'(w) = \psi_z \circ k(w) \circ \psi_z^{-1}$ for $w: z \to z' \in Z$. Given $w: z \to z' \in Z$, we check that $f \circ h'(w)$ and $g \circ k'(w)$ are equal. Since there are only
isomorphisms involved, it is enough to check that the equality holds when in the context \( g(\psi) \circ (h \circ f(\phi)) \):

\[
g(\psi) \circ f(h'(w)) \circ f(\phi) = g(\psi) \circ f(h'(w) \circ f(\phi))
\]

\[
= g(\psi) \circ f(h(\phi) \circ f(\phi))
\]

\[
= g(\psi) \circ f(h(w))
\]

\[
= \theta \circ f(h(w))
\]

\[
= g(k(w)) \circ \theta_z
\]

\[
= g(k(w)) \circ g(\psi) \circ f(\phi)
\]

\[
= g(\psi) \circ g(k'(w)) \circ f(\phi).
\]

Thus, \((Z, h', k')\) is a cone on \( S \xrightarrow{i} B \xrightarrow{\partial_S} T\), so there exists \( m: Z \to P\) which factors \( h'\) and \( k'\) through \( i\) and \( r\). The collections \((\phi_z)_{z \in Z}\) and \((\psi_z)_{z \in Z}\) defines natural isomorphisms \( \phi: h \Rightarrow i \circ m\) and \( \psi: r \circ m \Rightarrow k\) which satisfy \( (g \psi) \circ (f \phi) = \theta\). Hence, the condition of Proposition 9 is satisfied and \((P, l, r)\) is a bipullback. \(\square\)

We also have the following criterion for rectangles of bipullbacks:

**Lemma 5.** Given a rectangle made of two squares which are pullbacks in \( \text{Gpd} \) as in

\[
\begin{array}{ccc}
L & \xrightarrow{\pi_L} & M \\
\downarrow & & \downarrow \pi_B \\
A & \xrightarrow{f} & B \\
\downarrow g & & \downarrow \pi_R \\
& & R
\end{array}
\]

the following hold:

(i) if the whole rectangle is a bipullback, then the left square is too;

(ii) if the left and right square are pullback, then the whole rectangle is a bipullback.

**Proof.** We first prove (i) For this purpose, we use Proposition 9. So let \( a \in A, m \in M \) and \( \theta: f(a) \to \pi_B(M)\). We have that \( g(\theta) \) is a morphism from \( g \circ f(a) \) to \( g \circ \pi_B(M) = h \circ \pi_B(M) \). Since the outer rectangle is assumed to be a bipullback, there exist \( a' \in A, r' \in R, u^A: a \to a' \in A, u^R: r' \to \pi_R(M) \) such that \( g(\theta) = h(u^R) \circ (g \circ f(u^A)) \). Thus, we have \( g(\theta) \circ (f(u^A))^{-1} = h(u^R) \). Since the right square is a pullback, there exists a unique \( u^M \) such that \( \pi_B(u^M) = \theta \circ (f(u^A))^{-1} \) and \( \pi_B(u^M) = u^M \). Moreover, by the right pullback again, the target of \( u^M \) is \( m \); its source is some \( m' \) such that \( \pi_B(m') = f(a') \). Then, we have that \( \theta = \pi_B(u^M) \circ f(u^A) \). We can conclude with Proposition 9 that the left pullback is a bipullback.

We now prove (ii) using Proposition 9 again. So let \( a \in A, r \in R \) and \( \theta: g \circ f(a) \to h(r) \). Since the right square is a bipullback, there exist \( b' \in B, r' \in R, u^B: f(a) \to b' \) and \( v^R: r' \to r \) such that \( \theta = h(v^R \circ g(u^B)) \). Since \( b' \) and \( r' \) have the same projection in \( C \) through \( g \) and \( h \) respectively, there exists \( m' \in M \) such that \( \pi_B(m') = b' \) and \( \pi_B(m') = r' \). Thus, we have \( u^B: f(a) \to \pi_B(m') \). Since the left square is a pullback, there exist \( a'' \in A, m'' \in M \), \( \tilde{u}^A: a \to a'' \), \( \tilde{v}^M: m'' \to m' \) such that \( u^B = \pi_B(\tilde{v}^M) \circ f(\tilde{u}^A) \). We thus have

\[
\theta = h(u^R) \circ g(u^B) = h(u^R) \circ g(\pi_B(\tilde{v}^M) \circ f(\tilde{u}^A))
\]

\[
= h(u^R) \circ g(\pi_B(\tilde{v}^M)) \circ f(\tilde{u}^A)
\]

\[
= h(u^R) \circ h(\pi_B(\tilde{v}^M)) \circ g(f(\tilde{u}^A))
\]

\[
= h(u^R) \circ \pi_B(\tilde{v}^M) \circ g(f(\tilde{u}^A))
\]

which is precisely the factorization required by Proposition 9 to conclude that the whole rectangle is a bipullback. \(\square\)

**C. Uniformity and thinness**

Several arguments concerning uniformity requires some sort of diagram chasing relative to bipullbacks. An important lemma for this is the following:

**Lemma 6.** Consider the diagram in \( \text{Gpd} \)

\[
\begin{array}{ccc}
P & \xrightarrow{i_P} & S \\
\downarrow & & \downarrow \pi_S \\
L & \xrightarrow{\pi_L} & M \\
\downarrow f_A & & \downarrow \pi_B \\
A & \xrightarrow{f_B} & B \\
\downarrow \pi_R & & \downarrow f_B \\
M & \xrightarrow{R} & Q
\end{array}
\]

where the square 1, 2 and 3 are pullbacks, and derive from it the following diagram using the product structure:

\[
\begin{array}{ccc}
S & \xrightarrow{r_S} & R \\
\downarrow r_M & & \downarrow r_R \\
M & \xrightarrow{\pi_M} & S \\
\downarrow (f_A \times f_B) & & \downarrow (f_A \times f_B) \\
A \times B & \xrightarrow{(f_A \times f_B)} & A \times B
\end{array}
\]

Then 4 is a pullback. Moreover, the following hold:

(i) if 1 and the rectangle made of 2 and 3 are bipullbacks, then 4 is a bipullback;

(ii) if 3 and the rectangle made of 1 and 2 are bipullbacks, then 4 is a bipullback;

(iii) if 4 is a bipullback, then the rectangle made of 1 and 2 (resp. 2 and 3) is a bipullback.

**Proof.** The fact that 4 is a pullback is an easy consequence of the fact that 1,2,3 are pullbacks.

One can then use Lemma 1 without too much trouble on the different bipullback hypotheses in order to deduce that the wanted pullbacks are bipullbacks. \(\square\)

With the above tool, we can now prove Proposition 1.

**Proof of Proposition 1** We first prove the first implication, and start by showing (1) So let \((S, \partial^U_S) \in S\). Given \((U, \partial^U_S) \in U_B\), we must show that \( T \circ S \perp U \). By hypothesis, we have that \( T \perp S \times U \), i.e., the pullback of \( \partial^U_S \times \partial^U_B \) is a bipullback. Thus, we conclude by Lemma 6(iii) that the
pullback of $\partial_T^{\top A}_S$ and $\partial_T^{U'}$ is a bipullback, i.e., $T \circ S \perp U$. Hence, $T \circ S \in U_B$.

We now show (2). Since $\text{id}_B$ is an isofibration, we have that the pullback of $\partial_T^{U'}$ and $\text{id}_B$ is a bipullback. Thus, given $U \in U_A$, by Lemma 6 we have that $\partial_T^U \perp \partial_T^{\top A}_S$ if and only if $\partial_{U \times B}^T \perp \partial_T^{U'} \times \text{id}_B$. But the latter holds, since $T \in U_{A \rightarrow B}$. Hence, $\partial_T^U \in U^A_A$.

We now show the converse implication. So assume that $T$ satisfies (1) and (2). First note that, since $(A, \text{id}_A) \in U_A$, we have that $\partial_T^U \in U_B$ by (1). Given $V \in U_B$, we must show that, for every $U \in U_A$, we have $T \perp U \times V$. Since we have $\partial_T^U \perp \partial_T^{\top A}_S$, by Lemma 6 this is equivalent to have $U \perp T^\top @V$ for every $U \in U_B$. By hypothesis, it is equivalent to only check the previous condition for $U \in S$. By Lemma 6 again, it is equivalent to check that $T \perp U$ for every $U \in S$. Since $U \perp \partial_T^U$, by Lemma 6 again, it is equivalent to check that $T \circ U \perp V$, but the latter holds by (1). Thus, $T \in U_{A \rightarrow B}$. □

Using Proposition 1, we can prove the compatibility of uniformity with composition:

**Proposition 10.** Given uniform groupoids $(A, U_A)$ and $(B, U_B)$ and prestrategies $S \in U_{A \rightarrow B}$ and $T \in U_{B \rightarrow C}$, we have $T \circ S \in U_{A \rightarrow C}$.

**Proof.** Recall that the composition of the two spans $S$ and $T$ is formed as in Equation (1). We show the uniformity of $T \circ S$ using Proposition 1 with $U_B$ taken as generating class of $U_A$. The fact that (1) is satisfied for $T \circ S$ is immediate from its validity for both $S$ and $T$. The fact that (2) holds, that is, that $\partial_T^{\top A}_S \circ l \in U^A_A$, is a consequence of the fact that $\partial_B^T \in U_B^B$ by (2) on $T$, and the dual of Proposition 1 for $S$, asserting in particular that $S^\top$ maps elements of $U_B^B$ to $U_A^A$. □

We handle thinness similarly, and start by proving Proposition 2.

**Proof of Proposition 2.** Assume that $T \in U_{A \rightarrow B}$ and let $S \in U_B$. By Proposition 1 we already have $T @ S \in U_{A \rightarrow B}$. Next, we use the fact that $T_B = T_{A \rightarrow B}^A$ to show that $T @ S \in U_B$. So let $U \in U_B$. By Lemma 6 we have $T @ S \perp U$ iff $T \perp U \times S \times U$. But the latter holds since $T \in U_{A \rightarrow B}$ and $S \in S \in U_A$. Thus, $T @ S \in U_B$.

Conversely, assume that $T @ S \in U_B$ for every $S \in S$. First observe that $T_{A \rightarrow B} = (T_A \times T_B)^A$. So we must show that, for every $U \in U_A$ and $U \in U_B$, $T \perp U \times U$. By Lemma 6 for a given $U$, the latter is equivalent to $S \perp T^\top @U$ for every $S \in U_A$, i.e., $T^\top @U \in U_A^A$. But since $T^\top = T^{\perp A}$, for a given $U$, it is equivalent to $S \perp T^\top @S$ for every $S \in U_B$, itself equivalent to $T \perp T^\top @S$ for every $S \in U_B$, and finally equivalent to $T @ S \perp U$ for every $S \in S$, which amounts to our initial assumption. □

Using Proposition 2, we can then prove the compatibility of thinness with composition:

**Proposition 11.** Given thin groupoids $A$ and $B$ and strategies $S \in U_{A \rightarrow B}$ and $T \in U_{B \rightarrow C}$, we have $T \circ S \in U_{A \rightarrow C}$.

**Proof.** The proof is similar to (in fact simpler than) the one of Proposition 10 and follows from the criterion given by Proposition 2. □

**D. Details about the bicategory Thin**

We have the following convenient characterization of 0-composition of 2-cells of Thin:

**Proposition 12 (Paved Characterization of Composition (PCC)).** Given thin groupoids $A$, $B$, $C$, strategies $R$, $R'$: $A \Rightarrow B$, $S$, $S'$: $B \Rightarrow C$ and weak morphisms $F$: $R \Rightarrow R'$ and $G$: $S \Rightarrow S'$ of Thin, if there exist a functor $H$: $S \circ R \Rightarrow S' \circ R'$ and two natural transformations $H^I$ and $H^r$ as in

$$
\begin{array}{c}
S \circ R \\
\downarrow H^I \Rightarrow \\
S' \circ R'
\end{array}
\quad \quad
\begin{array}{c}
S \circ R \\
\downarrow H^r \Rightarrow \\
S' \circ R'
\end{array}
$$

such that

$$
\begin{array}{c}
S \circ R \\
\downarrow H^I \Rightarrow \\
S' \circ R'
\end{array}
\quad \quad
\begin{array}{c}
S \circ R \\
\downarrow H^r \Rightarrow \\
S' \circ R'
\end{array}
$$

and such that the natural transformations

$$
\begin{array}{c}
S \circ R \\
\downarrow H^I \Rightarrow \\
S' \circ R'
\end{array}
\quad \quad
\begin{array}{c}
S \circ R \\
\downarrow H^r \Rightarrow \\
S' \circ R'
\end{array}
$$

are positive over $A^\perp$ and $C$ respectively, we have that $H \cong (H^I, H^C)$ is a positive morphism $S \circ R \Rightarrow S' \circ R'$ of Thin and that $H = G \circ F$.

**Proof.** The fact that $H$ is a 2-cell $S \circ R \Rightarrow S' \circ R'$ of Thin is immediate by the polarity assumption. The equality of 2-cells given by the hypothesis can be rewritten as

$$
\begin{array}{c}
S \circ R \\
\downarrow H^I \Rightarrow \\
S' \circ R'
\end{array}
\quad \quad
\begin{array}{c}
S \circ R \\
\downarrow H^r \Rightarrow \\
S' \circ R'
\end{array}
$$

so that $H^I$ and $H^r$ provide a factorization of the pseudocone on the right, and define an object of the groupoid of compositions mentioned in Section III-D5. The actual
horizontal composition in Thin is then obtained by applying the biequivalence of Proposition 3. But since \( \langle H^A, H^C \rangle \) is already a positive natural transformation on \( A \to C \), this biequivalence does nothing on this object and \( H = G \circ F \). □

**Lemma 7.** Given thin groupoids \( A, B, C \), we have a functor
\[
(\cdot \odot (\cdot)) : \text{Thin}(B, C) \times \text{Thin}(A, B) \to \text{Thin}(A, C).
\]

**Proof.** By the definition we took for the composition of the 2-cells of Thin, we already have that \((\cdot \odot (\cdot))\) respects the sources and targets of weak morphisms, so that we are left to verify functoriality.

Given \( R \in \text{Thin}(A, B) \) and \( S \in \text{Thin}(B, C) \), a solution in \( H, H^I \) and \( H^r \) for the equation
\[
\begin{align*}
S \odot R & \xrightarrow{H} S \odot R \\
R & \xrightarrow{\text{id}_R} R \\
\partial^S_B & = \partial^R_B \\
B & \xrightarrow{B} B
\end{align*}
\]
is given by \( H = \text{id}_{R \odot S} \), \( H^I = \text{id}_I \) and \( H^r = \text{id}_r \). Thus, since identities are member of \( A^- \) and \( C^+ \), the polarity condition of Proposition 12 is satisfied so that
\[
\text{id}_S \odot \text{id}_R = (\text{id}_{S \odot R}, \text{id}_{S \odot R^I}, \text{id}_{S \odot R^r}) = \text{id}_{S \odot R}.
\]

Now, given four positive morphisms organized as
\[
\begin{align*}
R & \xrightarrow{F} R' \xrightarrow{F'} R'' \in \text{Thin}(A, B), \\
S & \xrightarrow{G} S' \xrightarrow{G'} S'' \in \text{Thin}(B, C),
\end{align*}
\]
the procedure to compute \( F \circ G \) gives us \( H, H^I \) and \( H^r \) s.t.
\[
\begin{align*}
S \odot R & \xrightarrow{H} S' \odot R' \\
R & \xrightarrow{F} R' \\
\partial^S_B & = \partial^R_B \\
B & \xrightarrow{B} B
\end{align*}
\]
and \( \partial^F_B H^I \) and \( \partial^G_B H^r \) respectively negative on \( A \) and positive on \( C \); similarly, the procedure to compute \( F' \circ G' \) gives us \( H^I, H^r \) and \( H'' \) such that
\[
\begin{align*}
S' \odot R' & \xrightarrow{H'} S'' \odot R'' \\
R' & \xrightarrow{F'} R'' \\
\partial^{S'}_{B'} & = \partial^{R'}_{B'} \\
B & \xrightarrow{B} B
\end{align*}
\]
and with \( \partial^{F'}_{B'} H^I \) and \( \partial^{G'}_{C'} H^r \) respectively negative on \( A \) and positive on \( C \). On the one hand, we thus have that \((G' \odot F') \circ (G \odot F)\) is the span morphism \( K = (K, K^I, K^r) \) with \( K = H' H \) and
\[
\begin{align*}
\text{Thin}(A, B) & \xrightarrow{\text{id} \times \text{cc}^A} \text{Thin}(A, B) \times 1 \to \cdots \\
\text{id} & \xrightarrow{\text{cc}^A} \text{Thin}(A, B) \times \text{Thin}(A, A) \xrightarrow{(\cdot \odot (\cdot))} \text{Thin}(A, B)
\end{align*}
\]
On the other hand, we have
\[
\begin{align*}
S \odot R & \xrightarrow{H} S' \odot R' \xrightarrow{H'} S'' \odot R'' \\
R & \xrightarrow{F} R' \xrightarrow{F'} R'' \\
\partial_B^S & = \partial_B^R \\
B & \xrightarrow{B} B
\end{align*}
\]
Moreover, the two natural transformations obtained as the horizontal pasting of \( H^I \) and \( H^r \) along \( l \) and the horizontal pasting of \( H^I \) and \( H^r \) along \( r \) satisfy the polarity condition of the PCC.

Hence, by considering again the diagrammatic definition of \( K \), the PCC tells us that \( K \) is also \((G' \circ G) \circ (F' \circ F)\), which concludes functoriality. □

We now show the unitality of the horizontal composition. Given a thin groupoid \( A \), we write \( cc^A \) for the identity span \( A \xleftarrow{\text{id}_A} A \xrightarrow{\text{id}_A} \). We have

**Lemma 8.** Given a thin groupoid \( A \), we have \( cc^A \in T_{A \to A} \).

**Proof.** This is an easy consequence of Proposition 1 and Proposition 2. □

We also write \( cc^A \) for the corresponding functor \( 1 \to \text{Thin}(A, A) \). Given an additional thin groupoid \( B \), there is a transformation \( R \) between the functors
\[
\text{Thin}(A, B) \xrightarrow{\text{id}} \text{Thin}(A, B) \times 1 \to \cdots \\
\text{id} \xrightarrow{\text{cc}^A} \text{Thin}(A, B) \times \text{Thin}(A, A) \xrightarrow{(\cdot \odot (\cdot))} \text{Thin}(A, B)
\]
and
\[
\text{Thin}(A, B) \xrightarrow{\text{id}} \text{Thin}(A, B)
\]
whose component at \( S \in \text{Thin}(A, B) \) is defined as follows. Recall that the span \( S \odot \text{cc}^A \) is defined by the pullback

\[
\begin{array}{ccc}
A & \xleftarrow{\text{id}_A} & S \\
\downarrow & & \downarrow \\
B & \xleftarrow{\text{id}_B} & S \\
\end{array}
\]

Then, \( R_S \cong r \) is an isomorphism (as the pullback of an isomorphism) which moreover induces a strong isomorphism of strategies \( R_S: S \odot \text{cc}^A \Rightarrow S \in \text{Thin} \).

**Lemma 9.** \( R \cong (R_S)_{S \in \text{Thin}} \) is a natural isomorphism.

**Proof.** Let \( A, B, S, S': A \to B \) and \( F: S \to S' \) in \( \text{Thin}(A, B) \). We first picture the two compositions \( T \cong \text{cc}^A \odot S \) and \( T' \cong \text{cc}^A \odot S' \) on the diagram

\[
\begin{array}{ccc}
A & \xleftarrow{\text{id}_A} & S \\
\downarrow & & \downarrow \\
B & \xleftarrow{\text{id}_B} & S' \\
\end{array}
\]

We now compute the composition \( F \odot \text{id}_{cc}^A \). By the **PCC** it is the morphism \( F: S \odot \text{cc}^A \Rightarrow S' \odot \text{cc}^A \) defined by

\[ F = T \xrightarrow{r_T} S \xrightarrow{F} S' \xrightarrow{F(T')^{-1}} T' \]

and

\[ F_A = \begin{array}{ccc}
T & \xrightarrow{r_T} & S \\
\downarrow & = & \downarrow \\
A & \xrightarrow{F} & S' \\
\downarrow & = & \downarrow \\
A & \xrightarrow{\text{id}_A} & A \\
\end{array} \]

Moreover,

\[ \left( R_{S'} \circ (F \odot \text{id}_{cc}^A) \right)^A = T \xrightarrow{r_T} S \xrightarrow{F} S' \xrightarrow{(F(T')^{-1}} T' \]

and

\[ \left( R_{S'} \circ (F \odot \text{id}_{cc}^A) \right)^B = T \xrightarrow{r_T} S \xrightarrow{F} S' \xrightarrow{F(T')^{-1}} T' \]

Which concludes the proof that \( R \) defines a natural iso. \( \square \)

Similarly, there is a transformation \( L \) between

\[ \text{Thin}(A, B) \xrightarrow{\text{cc}^B \times \text{id}} \text{Thin}(B, B) \times \text{Thin}(A, B) \xrightarrow{(-) \odot (-)} \text{Thin}(A, B) \]

and

\[ \text{Thin}(A, B) \xrightarrow{\text{id}} \text{Thin}(A, B) \]

whose component at a strategy \( S \in \text{Thin}(A, B) \) is defined as follows. Recall that the span \( \text{cc}^B \odot S \) is defined by the pullback

\[
\begin{array}{ccc}
A & \xleftarrow{\text{id}_A} & S \\
\downarrow & & \downarrow \\
B & \xleftarrow{\text{id}_B} & S \\
\end{array}
\]

Then, \( L_S \cong l \) is an isomorphism (as the pullback of an isomorphism) which moreover induces a strong isomorphism of thin spans \( L_S: \text{cc}^B \odot S \Rightarrow S \in \text{Thin} \). As before, we have

**Lemma 10.** \( L \cong (L_S)_{S \in \text{Thin}} \) is a natural isomorphism.

Given thin groupoids \( A, B, C, D \), there is a transformation

\[ A: ((-) \odot (-)) \odot (-) \Rightarrow (-) \odot ((-) \odot (-)) \]

\[ \text{Thin}(C, D) \times \text{Thin}(B, C) \times \text{Thin}(A, B) \]

\[ \Rightarrow \text{Thin}(A, D) \]

whose component at \( S \in \text{Thin}(A, B) \), \( T \in \text{Thin}(B, C) \) and \( U \in \text{Thin}(C, D) \) is given by a strong morphism

\[ A_{S, T, U}: (U \odot T) \odot S \Rightarrow U \odot (T \odot S) \]
defined as expected between the two compositions using the different pullbacks involved, as in

\[ \begin{array}{c}
\xymatrix{
(U \circ T) \circ S 
\ar[rr]^{(U \circ T) \circ S} 
\ar[rr]_{i(U \circ T) \circ S} & & (U \circ T) \circ S
\ar[rr]_{i(U \circ T) \circ S} & & T \circ S 
\ar[rr]^{T \circ S} & & S 
\ar[rr]_{y(U \circ T) \circ S} & & T \circ S 
\ar[rr]_{T \circ S} & & T
} \\
\end{array} \]

An inverse for \( A_{S,T,U} \) is defined symmetrically, so that \( A \) is an isomorphic transformation.

**Lemma 11.** The transformation \( A \) is a natural isomorphism.

*Proof.* Let \( F: S \Rightarrow S' \): \( A \Rightarrow B \), \( G: T \Rightarrow T' \): \( B \Rightarrow C \) and \( H: U \Rightarrow U' \): \( C \Rightarrow D \) be weak morphisms in Thin. We compute \((U \circ T) \circ S\) as usual but moreover factor the projection \( l(U \circ T) \circ S\) canonically through \( T \circ S \) by a unique morphism \( l(U \circ T) \circ S \) so that we get a diagram

\[ \begin{array}{c}
\xymatrix{
T \circ S 
\ar[rr]^{U \circ T} & & S 
\ar[rr]^{T \circ S} & & B 
\ar[rr]^{U \circ T} & & A 
\ar[rr]^{T \circ S} & & C 
\ar[rr]^{U \circ T} & & D 
} \\
\end{array} \]

and we get a similar diagram for \((U \circ T) \circ S\). Symmetrically, the projection \( r(U \circ T) \circ S\) can be factored canonically through \( U \circ T \) by a morphism \( r(U \circ T) \circ S \), and the projection \( r(U' \circ T' \circ S') \) through \( U' \circ T' \) by a morphism \( r(U' \circ T' \circ S') \).

Note that \( l(U \circ T) \circ S \circ A_{S,T,U} = r(U \circ T) \circ S \) and other similar equalities hold. By computing \( K = G \circ F \), we get natural transformations \( K^A \) and \( K^C \) such that

\[ K^A = \begin{array}{c}
\begin{array}{c}
T \circ S 
\ar[rr]^{K} & & T' \circ S'
\ar[rr]^{i(T \circ S)} & & (T \circ S) 
\ar[rr]^{r(T \circ S)} & & T \circ S, 
\ar[rr]^{i(T \circ S)} & & T \circ S 
\ar[rr]^{r(T \circ S)} & & T
\end{array}
\end{array} \]

and

\[ K^C = \begin{array}{c}
\begin{array}{c}
T \circ S 
\ar[rr]^{K} & & T' \circ S'
\ar[rr]^{r(T \circ S)} & & (T \circ S) 
\ar[rr]^{i(T \circ S)} & & T \circ S, 
\ar[rr]^{r(T \circ S)} & & T \circ S 
\ar[rr]^{i(T \circ S)} & & T \circ S 
\ar[rr]^{r(T \circ S)} & & T
\end{array}
\end{array} \]

which satisfy moreover that

\[ \begin{array}{c}
\xymatrix{
T \circ S \ar[r]^{K} & T' \circ S' \\
S \ar[r]_{g} & S' 
\ar@{=}[r] & \\
A \ar[r]^{A} & A 
\end{array} \]

\[ \begin{array}{c}
\xymatrix{
T \circ S \ar[r]^{K} & T' \circ S' \\
T \ar[r]_{g} & T' 
\ar@{=}[r] & \\
C \ar[r]^{C} & C 
\end{array} \]

Similarly, by computing \( L = H \circ G \), we get natural transformations \( L^B \) and \( L^D \) such that

\[ \begin{array}{c}
\xymatrix{
U \circ T \ar[r]^{L} & U' \circ T' \\
T \ar[r]_{g} & T' 
\ar@{=}[r] & \\
B \ar[r]^{B} & B 
\end{array} \]

\[ \begin{array}{c}
\xymatrix{
U \circ T \ar[r]^{L} & U' \circ T' \\
U \ar[r]_{h} & U' 
\ar@{=}[r] & \\
C \ar[r]^{C} & C 
\end{array} \]

which satisfy moreover that

\[ \begin{array}{c}
\xymatrix{
U \circ T \ar[r]^{L} & U' \circ T' \\
T \ar[r]_{g} & T' 
\ar@{=}[r] & \\
C \ar[r]^{C} & C 
\end{array} \]

\[ \begin{array}{c}
\xymatrix{
U \circ T \ar[r]^{L} & U' \circ T' \\
U \ar[r]_{h} & U' 
\ar@{=}[r] & \\
C \ar[r]^{C} & C 
\end{array} \]

Since

\[ \begin{array}{c}
\xymatrix{
(U' \circ T') \circ S' \ar[r] & (U' \circ T') \circ S'
\ar@{=}[r] & \\
T' \circ S' \ar[r] & U' \circ T'
\ar@{=}[r] & \\
T' \ar[r] & U' 
\ar@{=}[r] & \\
C \ar[r]^{C} & C 
\end{array} \]

is a bipullback, by Lemma 3 and that the natural transformations \( K^C \) and \( L^B \) define a pseudocone of vertex \((U \circ T) \circ S\)
on the associated cospan, we get $M$, $M^A$ and $M^D$ such that
\[
(U \odot T) \circ S \xrightarrow{M} (U' \odot T') \circ S'
\]
which was the wanted naturality.

We can now prove \textbf{Theorem 2}

\textbf{Proof of Theorem 2} By Lemmas 10 to 11 the 0-composition is naturally left unital, right unital and associative. Moreover, the coherence conditions on the natural isomorphisms, required by the definition of bicategories, directly follow from their pullback definitions.

\textbf{E. Renamings}

\textbf{Proposition 13.} Given $F : A \to B \in \text{Ren}$, $\tilde{F} \in \text{Thin}$.

\textbf{Proof.} We first prove that $\tilde{F} \in U_{B \to A}$ and we use the dual version Proposition 1 for this purpose. We already have that id$_A \in U_A$ since it is an isomorphism. We are left to show that $\tilde{F} \ast S \in U_{B \to A}$ for every $S \in U_A$. Up to isomorphism of domain, $\tilde{F} \ast S$ is the composition $F \circ \partial^S$ and, by hypothesis on $F$, the latter is in $U_{B \to A}$. So $\tilde{F} \in U_{B \to A}$ by Proposition 1.

We are left to show that $\tilde{F} \in T_{B \to A}$. But it follows from Proposition 2 by the same arguments as for uniformity.

Given $F : A \to B$ and $G : B \to C$ in $\text{Ren}$, there is a strong morphism of $\text{Thin}$, defined by the universal property of the pullback as

\[
m_{F,G} : (\tilde{G} F) \Rightarrow \tilde{F} \circ \tilde{G}.
\]

\textbf{Lemma 12.} Let $A,B,C$ be thin groupoids, and $\phi : F \Rightarrow F' : A \to B$ and $\psi : G \Rightarrow G' : B \to C$ be two 2-cells of $\text{Ren}$. The composition $\phi \odot \psi : F \circ G \Rightarrow F' \circ G'$ is given by $((H, \chi^C, \chi^A))$ where $\chi^G$ and $\chi^A$ are respectively $\tilde{F} \circ \tilde{G} \xrightarrow{H} \tilde{F}' \circ \tilde{G}'$ and $\tilde{F} \circ \tilde{G} \xrightarrow{H} \tilde{F}' \circ \tilde{G}'$ with $H$ as on the top of $\phi$.

\textbf{Proof.} This is a consequence of the PCC.

\textbf{Proposition 14.} Given thin groupoids $A,B,C$, the 2-cells $m_{F,G}$ for $F : A \to B$ and $G : B \to C$ in $\text{Thin}$ define a natural iso $m$ of type

\[
((-)_{(2)} \odot (-)_{(1)}) \Rightarrow (-)_{(1)} \circ (-)_{(2)} : \text{Ren}(A,B) \times \text{Ren}(B,C) \to \text{Thin}(C,A).
\]

\textbf{Proof.} Let $\phi : F \Rightarrow F' : A \to B$ and $\psi : G \Rightarrow G' : B \to C$ be two 2-cells of $\text{Ren}$.

We must show that

\[
(GF) \xrightarrow{(\psi \circ)_{(2)}} (G' F')
\]

\[
m_{F,G} \downarrow \quad \tilde{F} \circ \tilde{G} \xrightarrow{\phi \circ \psi} \tilde{F}' \circ \tilde{G}'
\]
in Thin\( (C, A) \). But this equation can easily be deduced from Lemma 12 whose statement implies that
\[
\tilde{\phi} \circ \tilde{\psi} = m_{F', G'} \circ (\tilde{\psi} \tilde{\phi}) \circ m_{F, G}^{-1}
\]

We can now finish the proof of Proposition 5.

**Proof.** For every \( A \) and \( B \), \((\cdot, \cdot)\) can easily be seen to define a functor \((\cdot, \cdot)_{A, B} : \text{Ren}(A, B) \to \text{Thin}(B, A)\). We are just left to show that the usual coherence conditions for pseudofunctors are satisfied by \( m \). But the required coherence conditions follow directly from the universal property of the pullback. \( \square \)

**F. The \(!\) functor and its structure**

We now finish the definition of the pseudofunctor \(! : \ Thin \to \ Thin\). First, while we described weak morphisms between two spans \( S, S' : A \to B \) as triples \((F, F^+, F^-)\), often identifying \( F \) with the whole triple, we will in the following often refer to the first element of the triple \( F \) by \( E \) for disambiguation. We now start by proving the 2-cell naturality of the coherence \( m \).

**Lemma 13.** Let \( F = (E, F^+, F^-) : S \Rightarrow S' : A \to B \) and \( G = (G^+, G^-, G^C) : T \Rightarrow T' : B \to C \). Let \( \chi^S \) and \( \chi^T \) be two 2-cells given by the definition of horizontal composition so that \( G \circ F \) is given by the two 2-cells
\[
\begin{array}{ccc}
T \circ S & \xrightarrow{G \circ F} & T' \circ S' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\chi^S} & S'
\end{array}
\]
and
\[
\begin{array}{ccc}
T \circ S & \xrightarrow{G \circ F} & T' \circ S' \\
\downarrow & & \downarrow \\
C & \xrightarrow{\chi^C} & C
\end{array}
\]

The composition \( !G \circ !F \) is then given by the two 2-cells
\[
\begin{array}{ccc}
!T \circ !S & \xrightarrow{m_{S, T, U}^{-1}} & !((T' \circ S') \circ m_{S', T'}) \\
\downarrow & & \downarrow \\
!T' \circ !S' & \xrightarrow{\chi^{T'}} & !T' \circ !S'
\end{array}
\]
and
\[
\begin{array}{ccc}
!T \circ !S & \xrightarrow{m_{S, T, U}^{-1}} & !((T' \circ S') \circ m_{S', T'}) \\
\downarrow & & \downarrow \\
!T' \circ !S' & \xrightarrow{\chi^{T'}} & !T' \circ !S'
\end{array}
\]
Proof. We use the **PCC**. The respective positivity and negativity of the proposed 2-cells follow from the positivity of the vertical composition of \( \chi^S \) and \( F^A \) and the negativity of the vertical composition of \( \chi^T \) and \( G^C \). We are left to show the top row of the two proposed 2-cells satisfy the equality required by the **PCC** but it follows from that satisfied by \( \chi^S \) and \( \chi^T \) by the functoriality of \(!: \text{Gpd} \to \text{Gpd} \) on 2-cells. \( \square \)

We can now conclude naturality:

**Lemma 14.** The morphisms \( m_{A,B,C}^{S,T} \) define a natural iso

\[
m^{A,B,C}_{S,T} : !((\cdot) \circ (\cdot)) \Rightarrow !((\cdot) \circ !(\cdot))
\]

: \( \text{Thin}(A, B) \times \text{Thin}(B, C) \to \text{Thin}(A, C) \).

**Proof.** Let \( F : S \Rightarrow S' : A \to B \) and \( G : T \Rightarrow T' : B \to C \). We must show that

\[
m^{A,B,C}_{S,T} \circ !(G \circ F) = !(G \circ !F) \circ m^{A,B,C}_{S,T}.
\]

But it directly follows from **Lemma 13** whose conclusion states in particular that

\[
m^{A,B,C}_{S,T} \circ !(G \circ F) \circ (m^{A,B,C}_{S,T})^{-1} = !G \circ !F.
\]

We can thus conclude that \(! \) is a pseudofunctor:

**Proposition 15.** The functor \(!: \text{Gpd} \to \text{Gpd} \) induces a pseudofunctor \(!: \text{Thin} \to \text{Thin} \).

**Proof.** By **Lemma 14** we have an adequate natural isomorphism expressing the functoriality of \(! \) on \text{Thin}. The coherence laws for pseudofunctors can be directly verified by the universal properties of the pullbacks involved in the horizontal compositions appearing in these laws. \( \square \)

**G. The \(! \) pseudocomonad**

We are going to derive the \(! \) pseudocomonad on \text{Thin} from the \(! \) pseudomonad on \text{Gpd} through functoriality. Before using this functoriality argument, we need to describe what are the (higher) categories we are going to apply to. The domain (bi)category will be the one of endofunctors on \text{Gpd} with properties similar to the ones of \(!: \text{Gpd} \to \text{Gpd} \), while the codomain (bi)category will be the one of endopseudofunctors on \text{Thin}. We shall first discuss how to relate some functors on \text{Gpd} to pseudofunctors on \text{Thin}.

Given a functor \( H : \text{Gpd} \to \text{Gpd} \), there is a canonical uniform groupoid \( HA = (HA, U_{HA}) \) associated to any uniform groupoid \( A \), where \( U_{HA} = \{HS \mid S \in U_A\}^{\perp \perp} \).

**Proposition 16.** If \( H \) preserves pullbacks, and pullbacks which are bipullbacks, then given uniform groupoids \( A \) and \( B \), and \( S \in U_{A \rightarrow B} \), we have \( HS \in U_{HA \rightarrow HB} \).

**Proof.** This is a direct consequence of Proposition 1. \( \square \)

We call **bicartesian functors** the functors \( H \) which satisfy the hypothesis of the above property.

A \( \pm \)-functor is a tuple \( (H, H^+, \iota) \) with \( H, H^+ \) being functors \( \text{Gpd} \to \text{Gpd} \) where \( H \) and \( H^+ \) are bicartesian and preserve functors (between groupoids) that are bijective on objects (of the groupoids), and such that \( H^+ \) preserves discrete groupoids, and \( \iota : H^+ \Rightarrow H \) being a natural transformation which is pointwise (that is, such that each \( \iota_X \) is) monomorphic and surjective on objects (of the groupoids), satisfying moreover that it is **bicartesian**, meaning that its naturality squares are both pullbacks and bipullbacks. Intuitively, the definition of \( \pm \)-functor is an abstraction of the case of \(!: \text{Gpd} \to \text{Gpd} \), from which we derive a functor \( \text{Thin} \to \text{Thin} \). In the case of \(! \), given a thin groupoid \( A \), a positive sub-groupoid \( \{A\}_+ \) is defined from a construction which is not derivable from the definition of \(! \) and the data of \( A \) and \( A_+ \), so that we have to take it into account in our definition of \( \pm \)-functor, in the form of a functor \( H^+ \) and a natural transformation \( \iota : H^+ \Rightarrow H \). We should a **priori** also require similar data for the negative side, but it so happens that, in the case of \(! \), \( \{A\}_- = \{A\}_- \), so that it is in fact not necessary. In order to show that \(! \) induces a pseudocomonad on \text{Thin}, the \( \pm \)-functors we will consider will only be iterated compositions of \(! \).

Given a \( \pm \)-functor \( (H, H^+, \iota) \) and a thin groupoid \( A \), there is a canonical thin groupoid \( HA \) whose underlying uniform groupoid is defined as earlier, whose class of thin prestrategies is \( T_{HA} = \{HS \mid S \in T_A\}^{\perp \perp} \), and whose negative and positive sub-groupoids are \( \{HA\}_- = HA_- \) and \( \{HA\}_+ = H^+ A_+ \) with embeddings given by the compositions

\[
HA_- \xrightarrow{H(id_A)} HA \quad \text{and} \quad H^+ A_+ \xrightarrow{H^+(id_A)} H^+ A \xrightarrow{\iota} HA.
\]

By the conditions of \( \pm \)-functors, they are elements of \( T_{HA} \) and \( T_{HA}^{\perp \perp} \) as required (exercise to the reader).

**Proposition 17.** Given a \( \pm \)-functor \( (H, H^+, \iota) \) and thin groupoids \( A \) and \( B \), and \( S \in T_{A \rightarrow B} \), \( HS \in T_{HA \rightarrow HB} \).

**Proof.** This is an easy consequence of the hypotheses on a \( \pm \)-functor and **Propositions 1 and 2**. \( \square \)

**Proposition 18.** Given a \( \pm \)-functor \( (H, H^+, \iota) \) and a thin groupoid \( A \), \( H \) preserves negative (resp. positive) 2-cells.

**Proof.** Given a negative 2-cell \( \phi : F \Rightarrow F' : X \to A \), writing \( X_{(0)} \) for the discrete groupoid with the same object as \( X \), we have a commutative diagram

\[
\begin{array}{ccc}
X_{(0)} & \xrightarrow{e} & X \\
\downarrow F \circ \phi & & \downarrow F' \circ \phi \\
A_- & \xrightarrow{id_A} & A
\end{array}
\]

for some 2-cell \( \phi^- : F \Rightarrow F' \), where the top arrow \( e \) is the canonical embedding. By hypothesis on \( H \), the image of \( e \) by \( H \) is bijective on objects of \( X \). Moreover, by functoriality, we have \( H(id_A) \circ H(\phi^-) = H(\phi) \circ H(e) \). Thus, all the components of the natural transformation \( H(\phi) \) are in the image of \( H(id_A) \). Thus, it is negative.
Now, given a positive 2-cell $\phi: F \Rightarrow F': X \to A$, we have a similar commutative diagram

$$
\begin{array}{c}
X(0) \xrightarrow{\phi} X \\
\downarrow F \downarrow F' \\
A_+ \xrightarrow{id_A} A
\end{array}
\quad \begin{array}{c}
\phi \\
\begin{array}{c}
F' \\
F
\end{array}
\end{array}
$$

for some 2-cell $\phi': \bar{F} \Rightarrow F'$. We then have the commutative diagram

$$
\begin{array}{c}
H^+X(0) \xrightarrow{H^+(\phi)} H^+X \\
\downarrow H^+(F) \downarrow H^+(F') \\
H^+A_+ \xrightarrow{H^+(id_A)} H^+A
\end{array}
\quad \begin{array}{c}
\phi \\
\begin{array}{c}
F' \\
F
\end{array}
\end{array}
\xrightarrow{\phi'}
\begin{array}{c}
H+F \xrightarrow{H(F')} H(A') \\
\begin{array}{c}
H(A) \\
\phi \\
\phi'
\end{array}
\end{array}
$$

where the top arrow is bijective on objects by assumptions. Thus, all the components of $H(\phi)$ are in the image of $i_A \circ H^+(id_A)$, so that $H(\phi)$ is positive.

Given two $\pm$-functors $H = (H, H^+, \iota)$ and $K = (K, K^+, \kappa)$, a $\pm$-transformation is a pair $(\alpha, \alpha^+)$ of bicartesian natural transformations where $\alpha: H \Rightarrow K$ and $\alpha^+: H^+ \Rightarrow K^+$ are such that

$$
\kappa \circ \alpha^+ = \alpha \circ \iota.
$$

Now, a $\pm$-modification between two such $\pm$-transformations $(\alpha, \alpha^+)$ and $(\beta, \beta^+)$ is the data of a modification $m: \alpha \Rightarrow \beta$ in the 3-category of 2-categories.

**Proposition 19.** Given a thin modification $m: \alpha \Rightarrow \beta: H \Rightarrow K$ and a thin groupoid $A$, the 2-cell

$$
m_A: \alpha_A \Rightarrow \beta_A: HA \to KA
$$

is negative as a 2-cell on $KA$.

**Proof.** Since $m$ is a modification, we have that $m_A \circ H(id_A) = K(id_A) \circ m_A$. Since $H(id_A)$ is bijective on objects by hypotheses on $H$ and $id_A$, we have that all the components of $m_A$ are in the image of $K(id_A)$, so that they are negative.

$\pm$-functors, $\pm$-transformations and $\pm$-modifications can be equipped with the evident operations in order to form a strict 3-category $\pm$-$\text{Funct}$ with one object (which, morally, is $\text{Gpd}$, the domain and codomain of each $\pm$-functor). The important point to note here is that the data of $!$, $\eta$, $\mu$, $\alpha$, $\beta$ and $\gamma$ induces the expected way a pseudomonad in $\pm$-$\text{Funct}$.

We shall now describe an operation $(\cdot)$ relating $\pm$-$\text{Funct}$ and $\text{Thin}$. First, a $\pm$-functor $(H, H^+, \iota)$ induces an endofunctor $H: \text{Thin} \to \text{Thin}$ mapping a thin groupoid $A$ to the thin groupoid $HA$ defined earlier, and mapping spans and their weak morphisms to their images by $H$, which is well-defined by Propositions 17 and 18.

Now, given a $\pm$-transformation $(\alpha, \alpha^+): (H, H^+, \iota) \Rightarrow (K, K^+, \kappa)$, we define a pseudonatural transformation $\alpha_A: HA \to KA$ whose component at a thin groupoid $A$ is $(\alpha_A)$, that is, the image of $\alpha_A: HA \to KA$ by the pseudofunctor $(\cdot): \text{Ren}^{op} \to \text{Thin}$.

We can then extend the pseudofunctor $(\cdot): \text{Ren}^{op} \to \text{Thin}$ to some sort of 3-dimensional functor

$$(\cdot): \pm$-$\text{Funct}^{co} \to \text{Bicat}$$

sending the unique 0-cell to $\text{Thin}$ and the higher cells in the hom-bicategory $\text{Bicat}(\text{Thin}, \text{Thin})$ (here, $\pm$-$\text{Funct}^{co}$ denotes $\pm$-$\text{Funct}$ with 2-cells reversed).

While this is probably a trifunctor, it would be very tiresome to prove. Instead, we will only rely on the simpler proposition stating that

**Proposition 20.** Considering $\pm$-$\text{Funct}$ as a strict 2-category by forgetting the dimension 0, $(\cdot)$ induces a pseudofunctor

$$(\cdot): \pm$-$\text{Funct}^{co} \to \text{Bicat}(\text{Thin}, \text{Thin})$$

between bicategories.

**Proof.** By checking the axioms of pseudofunctors.

We can now briefly describe a proof of Theorem 3.

**Proof of Theorem 3** The most satisfying proof of this statement would rely on the fact that $(\cdot): \pm$-$\text{Funct}^{co} \to \text{Bicat}(\text{Thin}, \text{Thin})$ is a trifunctor and that a trifunctor sends any pseudocomonad to a pseudocomonad, but we do not know a proof for the latter fact (though it is probably true) and deem a full proof of the former tedious.

Instead, we can rely on the weaker Proposition 20 to prove the required coherences. Following \[40\], we are required to prove the equations of modifications of Figure 7 are verified. The idea is to relate each of these equations to the equations satisfied by the pseudomonad $!: \text{Gpd} \to \text{Gpd}$, and this is done through paving. For example, we use the following pavings for the two first modifications of the left hand-side of the first equation of Figure 7:

$$
\begin{array}{c}
!!\mu \circ (!!\mu \circ \mu) \\
\downarrow \\
(!!!\mu \circ (!!\mu \circ \mu)) = (!!!\mu \circ (!!\mu \circ \mu))
\end{array}
\quad \begin{array}{c}
(!!\mu \circ (!!\mu \circ \mu)) \\
\downarrow \\
(!!\mu \circ (!!\mu \circ \mu))
\end{array}
\quad \begin{array}{c}
(!!\mu \circ (!!\mu \circ \mu)) \\
\downarrow \\
(!!\mu \circ (!!\mu \circ \mu))
\end{array}
\quad \begin{array}{c}
(!!\mu \circ (!!\mu \circ \mu)) \\
\downarrow \\
(!!\mu \circ (!!\mu \circ \mu))
\end{array}
$$
The other elementary modifications of [Figure 7] are paved similarly, so that the first equation of [Figure 7] is reduced to the equation

\[(\mu \circ (\mu \circ (\mu))) \Rightarrow (\mu \circ (\mu \circ (\mu))) = (\mu \circ (\mu \circ (\mu))) \Rightarrow (\mu \circ (\mu \circ (\mu)))\]

which is the image by \((-\)) of an equation satisfied by the pseudomonad \((!\cdot \eta \cdot \mu)\) on \(\text{Gpd}\). and the second equation of [Figure 7] to the equation

\[(\mu \circ (\mu \circ (\mu))) = (\mu \circ (\mu \circ (\mu))) \Rightarrow (\mu \circ (\mu \circ (\mu))) = (\mu \circ (\mu \circ (\mu)))\]

also an image of an equation satisfied by the pseudomonad \((!\cdot \eta \cdot \mu)\) on \(\text{Gpd}\). So that \((!, \eta, \mu)\) is indeed a pseudomonad on \(\text{Thin}\). □

**H. The cartesian product**

Given thin groupoids \(A, B\), we write \(\tilde{r}^{A, B}\) and \(\bar{r}^{A, B}\), simply denoted \(l\) and \(\bar{r}\) as earlier when \(A, B\) can be deduced from the context, for the coprojections

\[\tilde{I}: A \twoheadrightarrow A + B \quad \text{and} \quad \tilde{r}: A \twoheadrightarrow A + B.\]

By applying the functor \((-\)) to the thin spans \(\tilde{I}: A \& B \twoheadrightarrow A\) and \(\bar{r}: A \& B \twoheadrightarrow B\). Given thin groupoids \(\Gamma, A, B\), we define a functor

\[(\sim, -\sim)\}_{\Gamma, A, B}: \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B) \rightarrow \text{Thin}(\Gamma, A \& B),\]

often abbreviated \((\sim, -\sim)\), as follows. Given \(S \in \text{Thin}(\Gamma, A)\) and \(T \in \text{Thin}(\Gamma, B)\), we define \(\langle S, T \rangle\) as the span

\[\langle S, T \rangle = \left[ \begin{array}{c} S + T \\ \Gamma \\ \downarrow \end{array} \right] \xrightarrow{\delta^A + \delta^B} A + B \]

**Proposition 21.** Given \(S \in \text{Thin}(\Gamma, A)\) and \(T \in \text{Thin}(\Gamma, B)\), we have \(\langle S, T \rangle \in \text{Thin}(\Gamma, A \& B)\).

**Proof.** By an adequate use of [Propositions 1] and [2] □

Given morphisms \(F: S \rightarrow S' \in \text{Thin}(\Gamma, A)\) and \(G: T \rightarrow T' \in \text{Thin}(\Gamma, B)\), \((F, G)\) is defined as the morphism \(H\) with \(H = F + G\) and

\[H = \left[ \begin{array}{c} S + T \\ \Gamma \\ \downarrow \end{array} \right] \xrightarrow{\delta^A + \delta^B} A + B \]

and

\[H_{A+B} = \left[ \begin{array}{c} S + T \\ \Gamma \\ \downarrow \end{array} \right] \xrightarrow{\delta^A + \delta^B} A + B \]
One immediately verifies that these two 2-cells have the adequate polarities, so that \((\langle F, G \rangle, \Gamma) \in \text{Thin}(\Gamma, A & B)\). Moreover, the functoriality of \((-,-)_{\Gamma, A, B}\) is immediately verified.

Given \(S \in \text{Thin}(\Gamma, A & B)\), we write \(S_A\) for the span

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\partial^\Gamma} & S_A \\
\downarrow & & \downarrow \partial^S_A \circ \delta^A_S \\
\Gamma & \xleftarrow{\partial^\Gamma} & A
\end{array}
\]

where \(S_A\) is the submonoid of \(S\) whose image by \(\partial^S_{A+B}\) is in \(A\), and where \(\partial^S_A\) is the induced map \(S_A \to A\) from \(\partial^S_{A+B}\).

We define a span \(S_B\) similarly.

**Proposition 22.** Given \(S \in \text{Thin}(\Gamma, A & B)\), we have \(S_A \in \text{Thin}(\Gamma, A)\) and \(S_B \in \text{Thin}(\Gamma, B)\).

*Proof.* As the result of the composition of two thin spans, we know that \(\bar{l} \circ S\) is in \(\text{Thin}(\Gamma, A)\). It is the span

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\partial^\Gamma} & S \xrightarrow{i} A \\
\downarrow & & \downarrow \partial^S_A \circ \delta^A_S \\
\Gamma & \xleftarrow{\partial^\Gamma} & A + B
\end{array}
\]

which, by an isomorphism of pullbacks, is isomorphic to

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\partial^\Gamma} & S \xrightarrow{i} A \\
\downarrow & & \downarrow \partial^S_A \circ \delta^A_S \\
\Gamma & \xleftarrow{\partial^\Gamma} & A + B
\end{array}
\]

which is exactly \(S_A\). Thus, the latter is in \(\text{Thin}(\Gamma, A)\). A similar argument holds for \(S_B\).

The mapping \(S \mapsto S_A\) extends to a functor

\((-)_A: \text{Thin}(\Gamma, A & B) \to \text{Thin}(\Gamma, A)\)

the expected way. Similarly, we obtain a functor \((-)_B: \text{Thin}(\Gamma, A & B) \to \text{Thin}(\Gamma, B)\).

**Proposition 23.** The functors \((-)_A\) and \((-)_B\) are isomorphic to the functors \(\bar{l} \circ (-)\) and \(\bar{r} \circ (-)\) respectively.

*Proof.* Using the [PCC](#) one can show the naturality of the family of isomorphisms of thin spans \(\bar{l} \circ S \cong S_A\) described in the proof of [Proposition 22](#) obtaining an isomorphism of functors. A similar argument holds for \((-)_B\) and \(\bar{l} \circ (-)\).

**Proposition 24.** Given thin groupoids \(\Gamma, A, B\), there is an adjoint equivalence

\[
\text{Thin}(\Gamma, A & B) \perp \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B).
\]

*Proof.* Given \(S \in \text{Thin}(\Gamma, A & B)\), write \(\iota^S_A: S_A \hookrightarrow S\) and \(\iota^S_B: S_B \hookrightarrow S\) for the canonical inclusions of \(\text{Gpd}\). There is a canonical \(\gamma_S: S \to \langle S_A, S_B \rangle\) defined as the inverse of

\[
S_A + S_B \xrightarrow{[\iota^S_A, \iota^S_B]} S
\]

and it induces a strong morphism of thin span \(\gamma_S: S \Rightarrow \langle S_A, S_B \rangle \in \text{Thin}(\Gamma, A & B)\). The naturality of \(\gamma\) can be shown using the [PCC](#) so that we obtain a natural isomorphism

\[
\gamma: \text{id}_{\text{Thin}(\Gamma, A & B)} \Rightarrow \langle (-)_A, (-)_B \rangle.
\]

Given \((R,T) \in \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B)\), we have a canonical isomorphism \(\delta^R_{A,T}: (R,T)_A \to R\) defined as the pullback isomorphism between

\[
\begin{array}{ccc}
(R,T)_A & \xleftarrow{\iota^R_{A,T}} & (R,T) \\
\downarrow & & \downarrow \iota^R_{A,A} & \circ & \iota^R_{B,B} \\
A & \xrightarrow{\partial^R} & A + B
\end{array}
\]

It extends to a morphism \(\delta^R_{R,T}: (R,T)_A \to R \in \text{Thin}(\Gamma, A)\). We define similarly a morphism \(\delta^B_{R,T}: (R,T)_B \to T \in \text{Thin}(\Gamma, B)\). Writing \(\delta_{R,T} = (\delta^A_{R,T}, \delta^B_{R,T})\), the naturality of \(\delta_{R,T}\) with respect to \(R\) and \(T\) can be checked using the [PCC](#) so that we obtain a natural isomorphism

\[
\delta: \langle((-)_A, (-)_B)_A, ((-)_A, (-)_B)_B \rangle \Rightarrow \text{id}_{\text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B)}.
\]

We thus have an equivalence, and we verify that it is adjoint. We check the first zigzag equation, namely

\[
(\delta((-)_A, (-)_B)) \circ ((-)_A, (-)_B)\gamma = \text{id}_{((-)_A, (-)_B)} \gamma.
\]

In order to verify the above equality, by symmetry, we just need to check its projection on \(\text{Thin}(\Gamma, A)\). So let \(S \in \text{Thin}(\Gamma, A & B)\). The component of the left-hand side of (4) at \(S\) is then

\[
S_A \xrightarrow{(\gamma_S)_A} \langle S_A, S_B \rangle_A \xrightarrow{\delta^R_{S_A, S_B}} S_A.
\]

By unfolding the definition of \(\gamma\) and \(\delta\), we compute that

\[
S_A \xrightarrow{(\delta^R_{S_A, S_B})^{-1}} (S_A + S_B)_A \xrightarrow{(\gamma_S)_A^{-1}} S_A \xrightarrow{\iota^S_A} S
\]

is precisely \(\iota^S_A\), which happens to be a monomorphism, so that \((\gamma_S)_A^{-1} \circ (\delta^A_{S_A, S_B})^{-1} = \text{id}_{S_A}\), which is, up to inverses, what we wanted to show. Thus, the first zigzag equation holds.
We now verify the second zigzag equation, namely

\[ \gamma(-,-) \circ (\gamma(-,-)) = \text{id}_{(-,-)}. \]  

(5)

So let \((R, T) \in \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B)\). The component of the left-hand side of (5) at \((R, T)\) is

\[ \langle R, T \rangle \gamma_{(R,T)} \mapsto \langle \langle R, T \rangle_A, \langle R, T \rangle_B \rangle \stackrel{\delta^A_{R,T} \delta^B_{R,T}}{\Rightarrow} \langle R, T \rangle \]

By unfolding the definition of \(\gamma\) and \(\delta\), we compute that

\[ R \mapsto \tilde{I} \mapsto \langle R, T \rangle \]

reduces to \(R \mapsto \langle R, T \rangle\) and similarly,

\[ T \mapsto \tilde{r} \mapsto \langle R, T \rangle \]

reduces to \(T \mapsto \langle R, T \rangle\) so that, since \(\tilde{I}\) and \(\tilde{r}\) are jointly surjective, \(\gamma_{(R,T)}^{-1} \circ (\delta^A_{R,T} \delta^B_{R,T})^{-1} = \text{id}_{(R,T)}\), which is, up to inverses, what we wanted. So the second zigzag holds.

Proposition 25. Thin groupoids \(\Gamma, A, B\) with an adjoint equivalence

\[ \tilde{I} \circ (-), \tilde{r} \circ (-) \]

\[ \text{Thin}(\Gamma, A \& B) \quad \text{ } \downarrow \quad \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B). \]

Proof. This is a consequence of Propositions 23 and 24.

Note that, given \((R, T) \in \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B)\), the component at \((R, T)\) of the counit associated to the adjoint equivalence of Proposition 25 is the composite

\[ (\tilde{I} \circ (-), \tilde{r} \circ (-)) \Rightarrow (\langle R, T \rangle_A, \langle R, T \rangle_B) \Rightarrow (R, T) \]

Now, we can conclude the proof of Proposition 6.

Proof of Proposition 6. We have the equalities \(\text{Thin}(\Gamma, A \& B) = \text{Thin}(\Gamma, A \& B)\) and \(\text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B) = \text{Thin}(\Gamma, A) \times \text{Thin}(\Gamma, B)\). Moreover, it is quite immediate that, up to these identifications, there is an isomorphism of functors \((L \circ \gamma, R \circ \gamma) \cong (\tilde{I} \circ (-), \tilde{r} \circ (-))\). Thus, the unit/counit pair of the adjoint equivalence of Proposition 25 can be adjusted to get a unit/counit pair witnessing that we have an adjoint equivalence as in the statement.

I. The evaluation adjunction

We give here some additional details for the proof of Proposition 7, stating the existence of an adjoint equivalence between the currying operation and the evaluation one. This adjoint equivalence will be derived from the Seely equivalence already introduced.

1) Properties of the Seely equivalence:

Proposition 26. The family of functors \(s_{A,B}\) for groupoids \(A, B\) form a 2-natural transformation

\[ s : ![(-) \times (-) \Rightarrow !((-) + (-)) : \text{Gpd} \times \text{Gpd} \Rightarrow \text{Gpd}. \]

Proof. The naturality w.r.t. 2-cells is checked by direct point-wise computation.

Proposition 27. The natural transformation \(s\) is bicartesian.

Proof. By a direct use of the point-wise characterization of pullbacks.

Similarly, we have the same kind of properties for \(\bar{s}\):

Proposition 28. The family of functors \(\bar{s}_{A,B}\) for groupoids \(A, B\) form a 2-natural transformation

\[ \bar{s} : ![((-) + (-)) \Rightarrow !((-) \times (-)) : \text{Gpd} \times \text{Gpd} \Rightarrow \text{Gpd}. \]

Proposition 29. The natural transformation \(\bar{s}\) is bicartesian.

2) The Seely coherence 2-cell: While the Seely isomorphisms of 1-categorical models of linear logic are required to satisfy an equality, in our 2-categorical setting we only have the following 2-cell

\[ !A \times !B \xrightarrow{s_{A,B}} !A \times !B \]

\[ !A + !B \xrightarrow{s_{A,B}} !A \times !B \]

\[ !A + B \xrightarrow{s_{A,B}} !A \times !B \]

We first compute the action of the two vertical morphisms on objects of \(!A \times !B\). So let \(a = \{(a_{i,j})_{i,j \in I,A} \in \text{!!} A\) and \(b = \{(b_{i,j})_{i,j \in I,B} \in \text{!!} B\). The mappings associated with the left vertical morphism are

\[ (a, b) \mapsto (\langle (a_{i,j})_{i,j \in I,A} \rangle)_{i \in \text{!!} A} \xrightarrow{\text{See}_{A,B}} (a_{i,j})_{i \in \text{!!} A, j \in I,A} \]

and the mappings associated with the right are

\[ (a, b) \mapsto (\langle (a_{i,j})_{i,j \in I,B} \rangle)_{i \in \text{!!} B} \xrightarrow{\text{See}_{A,B}} (a_{i,j})_{i \in \text{!!} B, j \in I,B} \]

where, given \(D \in \text{Gpd}\) and \((d_{i,j})_{i \in I, j \in J} \in \text{!!} D\) with \(I \cap J = \emptyset\), we write \((d'_{i,j})_{i \in I, j \in J} \in \text{!!} D\) for the evident family indexed by \(I \cup J\). We thus define \(\text{See}_{A,B} a_{i,j}\) as the map \((\pi, (\text{id})_{k \in K})\) where \(\pi\) is the bijection \(K \rightarrow K'\) with

\[ K = \sum_{i \in \pi(I,A)} J_i^A \cup \sum_{i \in \pi(I,B)} J_i^B \]
and
\[ K' = \varpi_l(\sum_{i \in I^A} J^A_i) \cup \varpi_r(\sum_{i \in I^B} J^B_i) \]
such that \( \pi \) maps \([\varpi_l(i), j] \in K\) to \([\varpi_l([i, j])] \in K'\), and \([\varpi_r(i), j] \in K\) to \([\varpi_r([i, j])] \in K'\). The naturality of \( \text{See}_{A,B} \) with respect to morphisms \((a, b) \to (a', b')\) of \(!A \times !B\) can be readily checked. So \( \text{See}_{A,B} \) is indeed a 2-cell of \( \text{Gpd} \).

We moreover verify that

**Proposition 30.** The family of 2-cells \( \text{See}_{A,B} \) for \( A, B \in \text{Gpd} \) is natural with respect to functors \( F: A \to A' \) and \( G: B \to B' \) of \( \text{Gpd} \). In other words, \( \text{See} = (\text{See}_{A,B})_{A,B \in \text{Gpd}} \) is a modification.

**Proof.** A direct point-wise computation of naturality.

3) Properties of the evaluation span: Given thin groupoids \( A, B \), recall that we introduced a span
\[ \text{ev}_{A,B}: (A 
Rightarrow B) \& A 
Rightarrow B \]
defined by
\[
eq (l,r) : (A \times B) = \begin{array}{c}
!A \times B \\
\eta_{A \times B} \times A \\
(\rho_{A \times B} \times A) \\
\delta^A_{(\Gamma \times A)} & !((A \times B) \& A) \end{array} \]

We have that

**Proposition 31.** We have \( \text{ev}_{A,B} \in \text{T}((A \Rightarrow B) \& A \Rightarrow B) \).

**Proof.** By an adequate use of [Propositions 1 and 2](#), using [Proposition 27](#) the bicartesianness of \( \eta \) and [Lemma 5](#) to get the required bipullbacks.

4) The currying operation: Recall that, given thin groupoids \( \Gamma, A, B \) and \( S \in \text{Thin}(\Gamma \& A, B) \), we defined \( \Lambda(S) \) as the span
\[
\Lambda(S) = \begin{array}{c}
S \circ \delta^A_{(\Gamma \times A)} \\
\Gamma \times !A \\
\delta^A_{\Gamma} & !A \times B \\
\end{array} \]

**Proposition 32.** Given thin groupoids \( \Gamma, A, B \) and \( S \in \text{Thin}(\Gamma \& A, B) \), we have \( \Lambda(S) \in \text{T}(\Gamma \Rightarrow (A \Rightarrow B)) \).

**Proof.** While more involved than the other instances, this still relies on [Propositions 1 and 2](#) using [Proposition 29](#) and [Lemma 5](#) to get the required intermediate bipullbacks.

The operation \( \Lambda(-) \) can be extended to weak morphisms the expected way, and it is compatible with the polarities, and we moreover have

**Proposition 33.** Given thin groupoids \( \Gamma, A, B \), the operation
\[ \Lambda(-): \text{Thin}(\Gamma \& A, B) \to \text{Thin}(\Gamma, A \Rightarrow B) \]
is functorial.

5) The uncurrying operation: We can define conversely an uncurrying operation. Given \( S \in \text{Thin}(\Gamma, A \Rightarrow B) \), we define \( \bar{\Lambda}(S) \) as the span
\[
\bar{\Lambda}(S) = \begin{array}{c}
S \circ \delta^A_{\Gamma} \\
\delta^A_{\Gamma} \\
\Gamma \times !A \\
!A \times B \\
\end{array} \]

As before, using similar methods, we can verify that

**Proposition 34.** Given thin groupoids \( \Gamma, A, B \) and \( \sigma \in \text{Thin}(\Gamma, A \Rightarrow B) \), we have \( \bar{\Lambda}(\sigma) \in \text{T}(\Gamma \& A \Rightarrow B) \).

Moreover, we can extend this uncurrying operation to the weak morphisms the expected way, and this operation is compatible with the polarities, and we moreover have

**Proposition 35.** Given thin groupoids \( \Gamma, A, B \), the operation
\[ \bar{\Lambda}(-): \text{Thin}(\Gamma, A \Rightarrow B) \to \text{Thin}(\Gamma \& A, B) \]
is functorial.

6) The adjoint equivalences: We have a first adjoint equivalence between the currying and uncurrying operations:

**Proposition 36.** There is an adjoint equivalence
\[
\begin{array}{c}
\text{Thin}(\Gamma, A \Rightarrow B) \quad \perp \quad \text{Thin}(\Gamma \& A, B) \end{array}
\]

**Proof.** The application of \( \bar{\Lambda}(-) \) and \( \Lambda(-) \) to spans essentially amounts to adequately postcompose the display maps of these spans by \( s_{\Gamma,A} \) and \( \bar{s}_{\Gamma,A} \). The unit/counit pair witnessing the adjoint equivalence are then easily derived from a unit/counit pair \((\Sigma_{\Gamma,A}, \bar{\Sigma}_{\Gamma,A})\) witnessing the adjoint equivalence \( s_{\Gamma,A} \Rightarrow \bar{s}_{\Gamma,A} \).

For example, given \( S \in \text{Thin}(\Gamma, A \Rightarrow B) \), the component of the unit of \( \Lambda(-) \Rightarrow \bar{\Lambda}(-) \) at \( S \) is the weak morphism \((\text{id}_S, \phi)\) with

\[
\begin{array}{c}
S = \begin{array}{c}
\phi^S \circ \delta^A_{\Gamma} & \phi^S \circ \delta^A_{\Gamma} \\
\Sigma_{\Gamma,A} & \bar{\Sigma}_{\Gamma,A} \\
\delta^A_{\Gamma} & \delta^A_{\Gamma} \\
\end{array} \]
\end{array}
\]

and

\[
\begin{array}{c}
S = \begin{array}{c}
\phi^{\Gamma \times A} \circ \delta^A_{\Gamma \times A} \\
\delta^A_{\Gamma \times A} \\
\delta^A_{\Gamma \times A} \\
\end{array} \]
\end{array}
\]
Let \( \bar{\Lambda} \) be a functor \( B \to S \). The functor evaluation operation: uncurrying functor is isomorphic to the uncurrying-through-evaluation operation:

\[
\phi^A = \partial^A_A = \begin{vmatrix}
S & S & S \\
(\partial^A_{\eta}, \partial^A_A) & = & (\partial^A_{\eta}, \partial^A_A) \\
!\Gamma \times !A & \downarrow \quad \downarrow \\
!\Gamma \times !A
\end{vmatrix}
\]

and

\[
\phi^B = \begin{vmatrix}
S & S \\
\partial^B_B & = \\
B & B
\end{vmatrix}
\]

where we write \( \partial^S_A \) and \( \partial^B_B \) for \( l \circ \partial^S_{A \times B} \) and \( r \circ \partial^S_{A \times B} \) respectively (and similarly for \( \partial^S_B \)). The counit is defined similarly. The fact that this unit/counit pair satisfies the zigzag equations is a consequence of the fact that \( (\Sigma \Gamma \cdot A \cdot \Sigma \Gamma \cdot A) \) satisfies the same equations, by vertical pasting of 2-cells. \( \square \)

In order to get another adjoint equivalence, we prove that the uncurrying functor is isomorphic to the uncurrying-through-evaluation operation:

**Proposition 37.** The functor \( \text{ev} \circ \gamma (\mathcal{C} & A) : \text{Thin}(\Gamma, A \Rightarrow B) \to \text{Thin}(\Gamma & A, B) \) is isomorphic to the uncurrying functor \( \bar{\Lambda}(\mathcal{C}) \).

**Proof.** Let \( S \in \text{Thin}(\Gamma, A \Rightarrow B) \). We compute \( \text{ev} \circ \gamma (S \& A) \).

It is the composition of the spans

\[
\begin{array}{c}
!{(S + A)} \\
\downarrow (\partial^A_{\eta}, \bar{\partial}^A_A + A) \\
!{(\Gamma + A)} \\
\downarrow (\bar{\partial}^A_A, \bar{\partial}^A_{A \times B} + A) \\
!{(\Gamma + A)} \\
\downarrow \mu^A + A \\
!{(\Gamma + A)} \\
\downarrow (r, \bar{r}) \\
\end{array}
\]

and

\[
\begin{array}{c}
!{S} \\
\downarrow \eta^A_{A \times B} \\
!(A \times B) \\
\downarrow s^A_{A \times B} \\
!{(A \times B)} \\
\downarrow s^A_{A \times B} \\
!{(!A \times B) + A} \\
\downarrow s^A_{A \times B} \\
!{(A + B)} + A \\
\end{array}
\]

We compute the inner pullback of this composition as the rectangle of pullbacks shown in Figure 8. Thus, up to a canonical isomorphism of pullbacks, \( \text{ev} \circ \gamma (S \& A) \) is \( S \in \text{Thin}(\Gamma & A, B) \) with \( S \) as the support of \( S \) and

\[
\phi^S = \begin{vmatrix}
S & S \\
\partial^S_B & = \\
A \times B & B
\end{vmatrix}
\]

The operation \( S \mapsto \bar{S} \) can be shown to extend naturally to weak morphisms, so that we obtain a functor \( (-) : \text{Thin}(\Gamma, A \Rightarrow B) \to \text{Thin}(\Gamma & A, B) \) which is naturally isomorphic to \( \text{ev} \circ \gamma (\mathcal{C} & A) \). So we are left to show that \( (-) \cong \bar{\Lambda}(\mathcal{C}) \). For this purpose, for \( S \in \text{Thin}(\Gamma, A \Rightarrow B) \), we define an isomorphism \( \theta_S = (\bar{\bar{S}}, \bar{\partial}^S_{(\Gamma + A)}, \bar{\partial}^S_B) : \bar{S} \to \bar{\Lambda}(S) \).

We take \( \bar{\theta}_S = \text{id}_S \) and \( \bar{\theta}_B = \text{id}_{\bar{\partial}^S_B} \), and define \( \bar{\partial}^S_{(\Gamma + A)} \) as the (vertically expressed) 2-cell of Figure 9. We directly observe that \( \bar{\theta}^S_{(\Gamma + A)} \) and \( \bar{\partial}^S_B \) have the adequate polarities, so that \( \theta_S \in \text{Thin}(\Gamma & A, B) \) and it is an isomorphism. The naturality of \( \theta_S \) w.r.t. \( S \) can be checked diagrammatically, by pasting. Thus, \( \theta \) defines an isomorphism \( (-) \cong \bar{\Lambda}(\mathcal{C}) \), so that we have

\[
\text{ev} \circ \gamma (\mathcal{C} & A) \cong (-) \Rightarrow \bar{\Lambda}(\mathcal{C}).
\]

We can now conclude the proof of Proposition 7.

**Proof of Proposition 7.** By Proposition 36, we have an adjoint equivalence between the currying and uncurrying operations. By Proposition 37, we can replace the uncurrying operation by \( \text{ev} \circ \gamma (\mathcal{C} & A) \); by adjusting the unit and counit the expected way, we keep the adjoint equivalence. \( \square \)
Fig. 8. The inner rectangle of the composition of the two spans

Fig. 9. The $\theta_d^{\Gamma + A}$ 2-cell
Recall the definition of $\alpha$ and $\beta$ from Figure 3.