An Analysis of Symmetry in Quantitative Semantics

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ABSTRACT

Drawing inspiration from linear logic, *quantitative semantics* aim at representing quantitative information about programs and their executions: they include the relational model and its numerous extensions, game semantics, and syntactic approaches such as non-idempotent intersection types and the Taylor expansion of λ-terms. The crucial feature of these models is that programs are interpreted as witnesses which consume "bags" of resources.

"Bags" are often taken to be finite multisets, *i.e.* quotiented structures. Another approach typically seen in categorifications of the relational model is to work with unquotiented structures (*e.g.* sequences) related with explicit morphisms referred to here as *symmetries*, which express the exchange of resources. Symmetries are obviously at the core of these categorified models, but we argue their interest reaches beyond those — notably, symmetry *leaks* in some non-categorified quantitative models (such as the weighted relational model, or Taylor expansion) under the form of numbers whose combinatorial interpretation is not always clear.

In this paper, we build on a recent bicategorical model called *thin spans of groupoids*, introduced by Clairambault and Forest. Notably, thin spans feature a decomposition of symmetry into two sub-groupoids of *polarized* — *positive* and *negative* — symmetries. We first construct a variation of the original exponential of thin spans, based on sequences rather than families. Then we give a syntactic characterisation of the interpretation of simply-typed λ-terms in thin spans, in terms of rigid intersection types and rigid resource terms. Finally, we formally relate thin spans with the weighted relational model and generalized species of structure. This allows us to show how some quantities in those models reflect polarized symmetries: in particular we show that the weighted relational model counts witnesses from generalized species of structure, *divided* by the cardinal of a group of positive symmetries.

CCS CONCEPTS

• **Theory of computation → Denotational semantics**.

KEYWORDS

Denotational semantics, quantitative semantics

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1 INTRODUCTION

Denotational semantics is an approach to the semantics of programming languages that consists in associating to every program a denotation in an adequate mathematical universe; crucially this is done compositionally, by induction on syntax. Most denotational models are *qualitative*: a term $\vdash M : A \rightarrow B$ is typically represented by a function from the denotation of A to the denotation of B, giving us the input/output behaviour of M, but omitting *quantitative* information, such as resources, time, probabilities…

Within denotational semantics, *quantitative semantics* is a family of models whose distinguishing feature is to record quantitative information — first and foremost, displaying *how many times* a function $\vdash M : A \rightarrow B$ must evaluate its argument in order to produce a given result. Originally prompted by Girard's linear logic [15], quantitative semantics has developed into a wide research topic with numerous models and approaches, including the relational model [15] and its weighted [8, 20, 21] or categorical [3, 12] extensions, resource terms and the Taylor expansion of λ-terms [10], non-idempotent intersection types [5, 14], game semantics [1, 17], and others. This is not merely a subjective methodological difference: quantitative models are well-suited to model quantitative features such as probabilistic [9] or quantum [27] primitives, reflecting quantitative property such as execution time [7], or the number of non-deterministic branches [20], and many others.

To keep track of quantitative information, quantitative models must represent all individual resource accesses, but this is trickier than it might seem. Linear logic decomposes the intuitionistic arrow $A \rightarrow B$ as $!A \multimap B$ where $\multimap$ is the *linear arrow* (for functions calling their argument exactly once), and $!$ is the *exponential modality*, allowing arbitrary duplications of resources. Typically, the difficulty in designing a quantitative model arises with handling the exponential: how to keep track of all individual resource accesses while ensuring the laws required for an exponential modality in models of linear logic?

Quotients. If resource accesses in $!A$ are ordered in a sequence

$$\langle \alpha_1, \ldots, \alpha_n \rangle ,$$

then this will generally fail the commutations laws for the exponential, which require a *commutative comonoid* [24][1]. So sequences are often quotiented out by commutativity, as in the relational model [15] (and in general the so-called web-based models of linear logic), where $!A = \mathcal{M}(A)$ the set of finite multisets. This quotient is also found in quantitative notions of program approximation: for instance, the *Taylor expansion* of λ-terms [10] approximates

[1]Though some games models, notably simple games with the Hyland exponential [18], get away with that exploiting that copy accesses are totally chronologically ordered.

λ-terms via the *resource calculus*, a strongly finitary calculus where an application $M\,N$ from the λ-calculus is approximated with

$$m\,[n_1, \ldots, n_k]$$

the application of a resource term m, approximating M, to a *finite multiset* of resource terms $n_1, \ldots, n_k$, all approximating N. This expresses one of the possible behaviours of $M\,N$, where M will call its argument *exactly k times*, each call associated to one of the n_i's.

This quotient, at the heart of quantitative semantics, is by no means innocent: in situations when quantitative semantics manipulate numerical coefficients, the underlying symmetries on multisets *leak*, yielding scalars which are not clearly related to the computational situation, but instead reflect some aspect of its underlying symmetries. For instance, the relational model weighted by (completed) natural numbers [20], which in this paper we refer to as $\mathbf{WRel}_!$, counts distinct execution branches for non-deterministic programs when applied at ground type. But at higher-order type it yields non-trivial coefficients, even for plain simply-typed λ-terms: what do these numbers mean? Are those numbers related to the coefficients appearing in the Taylor expansion of λ-terms?

Rigid structures. It is tempting to avoid these quotients: in the quantitative semantics literature, the corresponding structures are often called *rigid*. Developping rigid models is subtle; for instance naively replacing finite multisets with sequences in the resource calculus yields a non-confluent reduction [26]; while naive rigid non-idempotent intersection types fail subject reduction.

Proper treatments of rigid structures may be found in *categorifications* of the relational model, the prime example being the cartesian closed bicategory **Esp** of generalized species of structure [12]. There, types are interpreted as *categories* (or *groupoids*) and the exponential $!A$ is the *free strict symmetric monoidal category* $\mathbf{Sym}(A)$ on A, where objects are sequences $\langle a_1, \ldots, a_n\rangle$ of objects of A, and where a morphism from $\langle a_1, \ldots, a_n\rangle$ to $\langle a'_1, \ldots, a'_m\rangle$ is a bijection $\sigma : n \simeq m$ along with $f_i : a_i \to a'_{\sigma(i)}$ in A for all $1 \leq i \leq n$. A term $\Gamma \vdash M : A$ is interpreted as a *distributor* from $\mathbf{Sym}(\llbracket\Gamma\rrbracket)$ to $\llbracket A\rrbracket$, *i.e.*

$$\llbracket M\rrbracket_{\mathbf{Esp}} : \mathbf{Sym}(\llbracket\Gamma\rrbracket)^{\mathrm{op}} \times \llbracket A\rrbracket \to \mathbf{Set}\,,$$

a functor which to $\vec{\gamma} \in \mathbf{Ob}(\mathbf{Sym}(\llbracket\Gamma\rrbracket))$ and $a \in \mathbf{Ob}(\llbracket A\rrbracket)$ associates a set $\llbracket M\rrbracket_{\mathbf{Esp}}(\gamma, a)$ of *witnesses* — crucially, $\llbracket M\rrbracket_{\mathbf{Esp}}$ also has a functorial action, making the symmetries (morphisms) of $\mathbf{Sym}(\llbracket\Gamma\rrbracket)$ and $\llbracket A\rrbracket$ *act* on witnesses. Tsukada *et al.* [28] and Olimpieri [25] have studied the nature of these witnesses, showing that they can be regarded as terms of a rigid resource calculus. Their calculi are not the naive rigid resource calculus mentioned above: they refine it by letting resource terms carry *morphisms*/*symmetries* from the types — but the precise location of these symmetries in the term is irrelevant, and it must be forgotten by yet another quotient!

Nevertheless, as **Esp** is a generalization of **Rel** properly accounting for symmetries, it looks like a natural candidate to illuminate the scalars arising from the weighted relational model: we may expect

$$(\llbracket M\rrbracket_{\mathbf{WRel}_!})_{\gamma,a} \quad = \quad \#\,(\llbracket M\rrbracket_{\mathbf{Esp}})(\gamma, a) \qquad (1)$$

(conflating for now objects and symmetry classes). But this fails, and we shall see that the link between the two involves data that is missing from the theory of **Esp**: *polarized symmetries.*

Contributions. Recently, Clairambault and Forest have introduced a new bicategorical model **Thin**, called *thin spans of groupoids* [3], also a categorification of the relational model, inspired by concurrent game semantics [2] — our first contribution is to show that it supports an exponential based on sequences rather than families.

We then delve deeper into the interpretation of the simply-typed λ-calculus in the Kleisli bicategory $\mathbf{Thin}_!$. Just like for **Esp** [25, 28], we show that an intersection type system (and matching resource terms) is implicit in thin spans. Perhaps surprisingly, it turns out to be the naive rigid intersection type system discussed above, obtained by merely replacing finite multisets with sequences (or the similarly naive rigid resource calculus), not carrying any symmetries, and without any quotient. Though subject reduction fails on the nose, our results entail that it does hold in a relaxed sense, *up to symmetry*. Beyond just characterising the witnesses as in [25, 28], we go further and also give a syntactic description of *symmetries* between derivations, obtaining a syntactic description of the full groupoid obtained as the interpretation of a term.

A central feature of **Thin** is that objects are certain groupoids A admitting two sub-groupoids A_- and A_+, respectively of *negative* and *positive* symmetries. Those are reminiscent from ideas in game semantics: *negative* symmetries exchange resources controlled by the environment, while *positive* symmetries exchange resources controlled by the program. Not every symmetry is negative or positive, but every symmetry factors uniquely as a negative composed with a positive. Far from being a technicality of the model construction, we argue that these polarized sub-symmetries are *fundamental.* In particular, they are the key to illuminate some of the questions mentioned earlier: in this paper, we characterise the coefficients obtained by $\mathbf{WRel}_!$ as counting witnesses in $\mathbf{Thin}_!$ — *i.e.* rigid resource terms — *up to positive symmetry*, or symmetry classes of witnesses — *i.e.* standard resource terms — with a correcting coefficient involving *negative symmetries.* Drawing inspiration from recent work linking thin concurrent games with generalized species of structure [4], we also construct an interpretation-preserving pseudofunctor from $\mathbf{Thin}_!$ to **Esp**, allowing us overall to express the coefficients obtained through $\mathbf{WRel}_!$ directly in terms of **Esp**, correcting (1) — again, the correct equation involves polarized symmetries.

Related work. *Polarized symmetries* are central to the construction of thin spans of groupoids (and before that, thin concurrent games [2]), but they predate those models: to our knowledge, they first appear in Melliès' approach to uniformity by bi-invariance, in the setting of asynchronous games [23]. They also make an appearance in Tsukada *et al.*'s study of *weighted generalized species* [29], though they are not part of the general theory but computed *a posteriori* for groupoids arising from simple types.

This work is part of an ongoing effort from the community to refine our understanding of resources in quantitative models, replacing quotients with rigid structures related with explicit morphisms and explore the corresponding categorical structures. Aside from work on generalized species of structure, a work complementary to ours is Melliès' *homotopy template games* [22], also based on categorical spans, focusing on links with homotopy theory.

Outline. In Section 2 we recall the definition of **Thin** from [3], replacing their exponential with a new one based on **Sym**. In Section 3, we give our syntactic characterisation of the interpretation of the simply-typed λ-terms in $\mathbf{Thin}_!$. Finally, in Section 4 we explore the link between $\mathbf{Thin}_!$ and relational models: first the plain relational model **Rel**, then the weighted (by completed natural numbers) relational model **WRel**, and finally generalized species **Esp**.

2 THIN SPANS ON SEQUENCES

We start with a brief reminder on **Thin** [3], along with the definition of the new exponential based on sequences. In the following, we write **Gpd** for the 2-category of groupoids, functors between groupoids and natural transformations between such functors. We will also often call **symmetries** the morphisms of a groupoid.

2.1 Reminder on Thin Spans of Groupoids

A **span** from A to B in a category C is simply a diagram like

$$A \xleftarrow{\partial_l^S} S \xrightarrow{\partial_r^S} B$$

which in **Set** (or **Cat**, or **Gpd**) is regarded as a generalized relation: a pair (a, b) may be related via a number of distinct **witnesses**, *i.e.* elements $s \in S$ *s.t.* $\partial_l^S(s) = a$ and $\partial_r^S(s) = b$ — in this paper, we often write $\partial_l^S(s) = s_A$ and $\partial_r^S(s) = s_B$, keeping ∂_l^S and ∂_r^S implicit. Here we focus on spans over groupoids: those form a bicategory **Span** where objects are groupoids, and a morphism from A to B is a span $A \leftarrow S \rightarrow B$. The identity span Id_A is $A \leftarrow A \rightarrow A$ with two identity functors, and spans are composed by pullback.

In **Span**, the 2-cells from a span $A \leftarrow S \rightarrow B$ to $A \leftarrow T \rightarrow B$ are functors $S \rightarrow T$ making the two triangles commute, and their horizontal composition is given by the universal property of pullbacks. Unfortunately, these 2-cells are too strict for many purposes; in particular they are incompatible with the laws required for the exponential modality of linear logic. Alternative 2-cells relax the hypothesis that the two triangles commute, asking instead for

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[dd] \arrow[dl, phantom, "\Downarrow", shift right=6] \arrow[dr, phantom, "\Downarrow", shift left=6] & \\ A & & B \\ & T \arrow[ul] \arrow[ur] & \end{tikzcd}$$

two natural isomorphisms. This allows us to relate more spans and indeed supports the laws for the exponential modality. However, the universal property of pullbacks then fails to provide a definition of horizontal composition for those. This mismatch has different solutions, either replacing the pullbacks with adequate notions of homotopy pullbacks, or requiring additional fibrational conditions on spans — in almost all cases this concretely means importing the morphisms of groupoids inside witnesses, as in generalized species of structure or in template games [22].

In [3], an alternative idea was introduced. In **Span**, some pullbacks happen to behave well *w.r.t.* homotopy (they are *bipullbacks*, see below). The key observation is that as it turns out, the pullbacks arising from the denotational interpretation of programs actually *always are* bipullbacks! The bicategory **Thin** of *thin spans* captures this via a biorthogonality construction, morally cutting **Span** down and keeping only certain spans — those deemed "uniform" — ensuring that their composition pullbacks are always bipullbacks.

2.1.1 Uniformity. Given a groupoid A, a **prestrategy**[2] on A is a pair (S, ∂^S) of a groupoid S and a functor $\partial^S : S \rightarrow A$, the **display map**. We write $\mathsf{PreStrat}(A)$ for the class of prestrategies on A.

Given two prestrategies (S, ∂^S) and (T, ∂^T), we write $(S, \partial^S) \perp (T, \partial^T)$ (or, more simply, $S \perp T$), when the following pullback

$$\begin{tikzcd} & P \arrow[dl, dashed, "l"'] \arrow[dr, dashed, "r"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ S \arrow[dr, "\partial^S"'] & & T \arrow[dl, "\partial^T"] \\ & A & \end{tikzcd} \tag{2}$$

is a **bipullback**. In **Gpd**, this means that for every $s \in S, t \in T$ and $\theta : s_A \rightarrow t_A$, there is $u : s \rightarrow s' \in S$ and $v : t' \rightarrow t \in T$ such that $s'_A = t'_A$ and $\theta = v_A \circ u_A$ in A: when two states can synchronize up to symmetry, we can find symmetric states that can synchronize on the nose, coherently. Given a set, or even a class $\mathbf{S}$ of prestrategies on A, we write $\mathbf{S}^\perp$ for the class $\{T \in \mathsf{PreStrat}(A) \mid \forall S \in \mathbf{S},\ S \perp T\}$.

A **uniform groupoid** is a pair $A = (A, \mathbf{U}_A)$ where A is a groupoid and $\mathbf{U}_A \subseteq \mathsf{PreStrat}(A)$ is a class of prestrategies, called the **uniform prestrategies**, such that $\mathbf{S}^{\perp\perp} = \mathbf{S}$. One can define several constructions on uniform groupoids [3]. The **dual** $A^\perp$ of the uniform groupoid A has $(A, \mathbf{U}_A^\perp)$. Given another uniform groupoid $B = (B, \mathbf{U}_B)$, one can define binary constructions like the **tensor** $A \otimes B$ and its de Morgan dual the **par** $A ⅋ B$, both having underlying groupoid $A \times B$. From these two constructions, one then defines the **linear arrow** $A \multimap B$ as $A^\perp ⅋ B$. Finally, the **with** $A \,\&\, B$ has underlying groupoid $A + B$.

2.1.2 Spans. The underlying groupoid of $A \multimap B$ is $A \times B$ so that $S \in \mathbf{U}_{A \multimap B}$ is a prestrategy on $A \times B$, equivalently seen as a span

$$A \leftarrow S \rightarrow B$$

in **Gpd**. In the following, we call such an S a **uniform span** to emphasize that it is a prestrategy of $\mathbf{U}_{A \multimap B}$. Notably, the identity span on a uniform groupoid A, is uniform. Given uniform groupoids $A, B, C, S \in \mathbf{U}_{A \multimap B}$ and $T \in \mathbf{U}_{B \multimap C}$, the composition via the pullback

$$\begin{tikzcd} & & & T \odot S \arrow[dl, dashed, "l"'] \arrow[dr, dashed, "r"] & & & \\ & & S \arrow[dll, "\partial_l^S"'] \arrow[dr, "\partial_r^S"'] & & T \arrow[dl, "\partial_l^T"] \arrow[drr, "\partial_r^T"] & & \\ A & & & B & & & C. \end{tikzcd}$$

is uniform (*i.e.* in $U_{A \multimap C}$) by [3, Lem. 2] — and the composition pullback is a bipullback, as stated in our motivation for **Thin**.

2.1.3 Morphisms of spans. As introduced above, uniform spans must be related via adequate notions of morphisms between spans:

Definition 2.1 ([3, Def. 1]). A **weak morphism** from $A \leftarrow S \rightarrow B$ to $A \leftarrow S' \rightarrow B$ is (F, F^A, F^B), with F^A and F^B natural isos, and

$$\begin{tikzcd} & S \arrow[dl, "\partial_l^S"'] \arrow[dr, "\partial_r^S"] \arrow[dd, "F"] \arrow[dl, phantom, "F^A \Downarrow", shift right=8] \arrow[dr, phantom, "\Downarrow F^B", shift left=8] & \\ A & & B \\ & S' \arrow[ul, "\partial_l^{S'}"] \arrow[ur, "\partial_r^{S'}"'] & \end{tikzcd}$$

We call this a **strong morphism** if F^A and F^B are identities.

The bipullback property, for the composition pullback, ensures the existence of candidates for the horizontal composition of weak morphisms. However, it is not uniquely defined, and the bipullback property is insufficient to guarantee a canonical choice satisfying

[2] Some terminology in [3] is game-theoretic, reflecting the game semantics inspirations.

the laws of a bicategory (see [3, Par. III-B4]). We thus need additional structure in order to ensure the existence of a canonical choice.

2.1.4 Thinness. For this we must capture a more subtle property observed in the denotational interpretation of programs: non-trivial symmetries between states always originate from the environment — in a closed world interaction, no non-trivial symmetry is left. This is called *thinness*, and again is captured by orthogonality.

Given a uniform groupoid A, $S \in \mathbf{U}_A$ and $T \in \mathbf{U}_A^\perp$, we write $S \perp\!\!\!\perp T$ when the pullback vertex of (2) is a discrete groupoid. Given a class $\mathbf{S} \subseteq \mathbf{U}_A$, we write $\mathbf{S}^{\perp\!\!\!\perp}$ for the class $\{T \in \mathbf{U}_A^\perp \mid \forall S \in \mathbf{S},\ S \perp\!\!\!\perp T\}$.

Definition 2.2 ([3, Def. 10]). A **thin groupoid** is a tuple $A = (A, A_-, A_+, \mathbf{U}_A, \mathbf{T}_A)$ where $(A, \mathbf{U}_A)$ is a uniform groupoid, and

- A_- and A_+ are subgroupoids of A with the same objects as A, with embedding functors $\mathrm{id}_A^- \colon A_- \to A$ and $\mathrm{id}_A^+ \colon A_+ \to A$;
- $\mathbf{T}_A \subseteq \mathbf{U}_A$ is a class of prestrategies such that $\mathbf{T}_A^{\perp\!\!\!\perp\,\perp\!\!\!\perp} = \mathbf{T}_A$, satisfying that $(A_-, \mathrm{id}_A^-) \in \mathbf{T}_A$ and $(A_+, \mathrm{id}_A^+) \in \mathbf{T}_A^{\perp\!\!\!\perp}$.

As motivated earlier, the prestrategies in $\mathbf{U}_A$ are those that are well-behaved with respect to synchronization up to symmetry. In addition, for those prestrategies in $\mathbf{T}_A$ ($\mathbf{T}$ stands for "thin"), this synchronization enjoys a uniqueness property that is crucial in defining the horizontal composition of weak morphisms — see [3, Sec. III] for details.

In a groupoid G with $x, y \in G$, we often write $\theta : x \cong_G y$ to mean that $\theta \in G[x, y]$. For A a thin groupoid, $\theta : a \cong_A^+ a'$ indicates that $\theta \in A_+[a, a']$ — we say that θ is a **positive** symmetry — likewise, $\theta : a \cong_A^- a'$ indicates that $\theta \in A_-[a, a']$, and we say that θ is **negative**. Intuitively, this polarity tells us who, among the program or the environment, is responsible for a permutation. If it is a permutation among resources called upon by the environement (*e.g.*, coming from an occurrence of ! in covariant position), then the symmetry is *negative*. If it permutes resources controlled by the program (*e.g.* with a ! in contravariant position), then the symmetry is *positive*. In general a symmetry may mix the two and can be neither negative nor positive, but from Defininition 2.2 we get:

Lemma 2.3. *For any $\theta : a \cong_A a'$ in a thin groupoid A, there are unique $a'' \in A$ and $\theta^+ : a \cong_A^+ a''$, $\theta^- : a'' \cong_A^- a'$* s.t. $\theta = \theta^- \circ \theta^+$.

See [3, Lem. 3]. The constructions introduced before on uniform groupoids ($(-)^\perp, \otimes, ⅋, \&$) extend to thin groupoids [3].

2.1.5 Thin spans. Given thin groupoids A and B, a **thin span** is a prestrategy $S \in \mathbf{T}_{A \multimap B}$. As above the underlying groupoid of $A \multimap B$ is $A \times B$, so S can be seen as a span between A and B. Given a thin groupoid A, we have $\mathsf{Id}_A \in \mathbf{T}_{A \multimap A}$; and for thin spans $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, we have $T \odot S \in \mathbf{T}_{A \multimap C}$ (see [3, Prop. 2]).

Together, uniformity and thinness guarantee strong properties for the composition of thin spans. For thin spans $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, recall that (following the obvious pullback construction in **Gpd**) elements of $T \odot S$ are simply pairs (s, t) such that $s_B = t_B$. However, it is central in the construction of **Thin** (in particular for the horizontal composition of 2-cells that we shall not detail here) that thin spans may synchronize *up to symmetry*:

Lemma 2.4. *Consider $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$ thin spans, $s \in S, t \in T$, linked with a symmetry $\theta : s_B \cong_B t_B$.*

Then there are unique *$s' \in S, t' \in T$ and $\varphi : s \cong_S s'$, $\psi : t' \cong_T t$ such that φ_A negative, ψ_C positive, and $\theta = \psi_B \circ \varphi_B$.*

See [3, Lem. 2]. Another important consequence of the definition of thin spans is that symmetries *act* on thin spans:

Lemma 2.5. *Consider $A \leftarrow S \rightarrow B$ a thin span, $s \in S$, with $\theta_A : a \cong_A s_A$ and $\theta_B : s_B \cong_B b$. Then, there are unique $s' \in S$, $\varphi : s \cong_S s'$, ϑ_A^- and ϑ_B^+ such that the two triangles commute:*

$$
\begin{tikzcd}
 & s_A \arrow[dd, "\varphi_A"] \\
a \arrow[ur, "\theta_A"] \arrow[dr, "\vartheta_A^-"'] & \\
 & s'_A
\end{tikzcd}
\qquad
\begin{tikzcd}
s_B \arrow[dr, "\theta_B"] \arrow[dd, "\varphi_B"'] & \\
 & b \\
s'_B \arrow[ur, "\vartheta_B^+"'] &
\end{tikzcd}
$$

So $s \in S$ may be reindexed by symmetries θ_A and θ_B, though we will not exactly hit the targets a and b: only up to positive (or negative, depending on the variance) symmetry.

2.1.6 Positive weak morphisms. This additional structure may be leveraged to get the canonicity of horizontal composition of 2-cells — modulo a final fine-tuning of their definition:

Definition 2.6. Given two thin groupoids A and B, a weak morphism (F, F^A, F^B) between A and B as in Definition 2.1 is **positive** when, for every $s \in S$, $F_s^B \colon s_B \cong_B^+ F(s)_B$ and $F_s^A \colon s_A \cong_A^- F(s)_A$.

We call it *positive* since it is positive on $A \multimap B$. Positivity lets us use the uniqueness property of Lemma 2.4 to give a *unique* choice for horizontal composition of positive weak morphisms, and:

Theorem 2.7 ([3, Thm 2]). *There is a bicategory* **Thin** *of thin groupoids, thin spans, and positive weak morphisms. The identity on A is Id_A, and the composition of thin spans is given by plain pullbacks.*

2.2 The Sym Exponential on Thin

Thin was originally developed using the **Fam** functor as exponential, mapping a groupoid A to $\mathbf{Fam}(A)$ with objects families $(a_i)_{i \in I}$ indexed by finite sets of integers I. Instead, we consider here the **Sym** functor (used as an exponential modality on distributors to construct generalized species of structure), which extends to groupoids the list functor of **Set**. This seems a minor difference since **Fam** and **Sym** are equivalent as endofunctors of **Gpd**, but it is actually a non-trivial shift since thin spans do not respect the principle of equivalence, by relying on strict pullbacks in a 2-categorical setting[3].

2.2.1 The **Sym** *monad on* **Gpd**. We start by considering the functor

$$\mathbf{Sym} \colon \mathbf{Gpd} \to \mathbf{Gpd}$$

mapping A to the free strict symmetric monoidal groupoid $\mathbf{Sym}(A)$. Concretely, the objects of $\mathbf{Sym}(A)$ are sequences $\langle a_i \rangle_{i \in \{1,\dots,n\}} = \langle a_1, \dots, a_n \rangle$ of objects of A, and its morphisms from $\langle a_1, \dots, a_n \rangle$ to $\langle b_1, \dots, b_m \rangle$ are pairs $(\pi, \langle f_i \rangle_{i \in \{1,\dots,n\}})$ where π is a bijection between $\{1, \dots, n\}$ and $\{1, \dots, m\}$, and $\langle f_i \rangle_i$ is a sequence of morphisms $f_i \colon a_i \to b_{\pi(i)}$ for $i \in \{1, \dots, n\}$. **Sym** can be extended to a monad $(\mathbf{Sym}, \eta, \mu)$ on **Gpd**: on objects, the unit $\eta_A \colon A \to \mathbf{Sym}(A)$ maps $a \in A$ to $\langle a \rangle$, and $\mu_A \colon \mathbf{Sym}(\mathbf{Sym}(A)) \to \mathbf{Sym}(A)$ concatenates sequences — this extends to symmetries as expected.

[3] Indeed, strict pullbacks of equivalent categories need not be equivalent.

2.2.2 The pseudocomonad. The definition of a pseudocomonad ! for **Thin** based on **Sym** is done as in [3, Sec. IV-A], we recall the salient elements here. Given $A = (A, A_-, A_+, \mathbf{U}_A, \mathbf{T}_A)$, we set

$$!A = (\mathbf{Sym}(A), \mathbf{Sym}(A_-), \mathbf{Sym}^+(A_+), (\mathbf{Sym}\,\mathbf{U}_A)^{\perp\perp}, (\mathbf{Sym}\,\mathbf{T}_A)^{\perp\!\!\!\perp\perp\!\!\!\perp})$$

where $\mathbf{Sym}^+(A_+)$ is a subgroupoid of $\mathbf{Sym}(A_+)$ with the same objects but morphisms only the $(\mathrm{id}, \langle f_i\rangle_i)$; where $\mathbf{Sym}\,\mathbf{U}_A$ has all $(\mathbf{Sym}(S), \mathbf{Sym}(\partial^S))$ for all $(S, \partial^S) \in \mathbf{U}_A$, and likewise for $\mathbf{Sym}\,\mathbf{T}_A$.

Sym lifts to a pseudofunctor ! on **Thin** via the functorial action

$$!\left(\begin{tikzcd} & S \arrow[dl, "\partial_l^S"'] \arrow[dr, "\partial_r^S"] & \\ A & & B \end{tikzcd}\right) \quad = \quad \begin{tikzcd} & \mathbf{Sym}(S) \arrow[dl, "\mathrm{Sym}(\partial_l^S)"'] \arrow[dr, "\mathrm{Sym}(\partial_r^S)"] & \\ \mathbf{Sym}(A) & & \mathbf{Sym}(B) \end{tikzcd}$$

on thin spans, defining similarly the image of 2-cells as the image by **Sym** of their underlying components.

When instantiated on the underlying groupoid of a thin groupoid A, the natural transformations η_A and μ_A are not only functors, but **renamings** in the sense of [3]. Recall from there the pseudofunctor $\check{-} : \mathbf{Ren}^{\mathrm{op}} \to \mathbf{Thin}$ from the (dualized) 2-category of renamings to the bicategory of thin spans, mapping a renaming $f : A \to B$ to

$$B \quad \overset{f}{\longleftarrow} \quad A \quad \overset{\mathrm{id}_A}{\longrightarrow} \quad A$$

a thin span, yielding a counit $\check{\eta_A} \in \mathbf{Thin}[!A, A]$ and a comultiplication $\check{\mu_A} \in \mathbf{Thin}[!A, !!A]$ for !. We have:

Theorem 2.8. *We have a pseudocomonad* ! *on* **Thin** *based on* **Sym**.

2.2.3 The exponential. **Sym** enjoys a Seely equivalence in **Thin**, derived from an equivalence already existing in **Gpd**:

$$\mathbf{Sym}(A + B) \underset{\bar{s}_{A,B}}{\overset{s_{A,B}}{\rightleftarrows}} \mathbf{Sym}(A) \times \mathbf{Sym}(B) \qquad \in \mathbf{Gpd} \qquad (3)$$

for groupoids A, B, with $s_{A,B}$ mapping the sequence $\langle a_1, b_1, b_2, a_2\rangle$ to $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle)$, and with $\bar{s}_{A,B}$ mapping $(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle)$ to $\langle a_1, a_2, b_1, b_2\rangle$ for instance. When A and B are thin groupoids, $s_{A,B}$ and $\bar{s}_{A,B}$ are moreover renamings, so that we can take the image of the above equivalence by $\check{-}$ to obtain the Seely equivalence

$$!A \otimes !B \underset{\check{\bar{s}}_{A,B}}{\overset{\check{s}_{A,B}}{\rightleftarrows}} !(A \,\&\, B) \qquad \in \mathbf{Thin}.$$

2.2.4 The cartesian closed bicategory. Equipped with the pseudocomonad !, we derive a Kleisli bicategory $\mathbf{Thin}_!$, whose 1-morphisms are thus thin spans of the form $!A \leftarrow S \to B$, composed using the comonadic structure. By following the proofs in [3], which were mostly non-specific to the **Fam** pseudomonad used there, we get:

Theorem 2.9. $\mathbf{Thin}_!$ *is a cartesian closed bicategory.*

3 INTERPRETATION AND ITS SYNTACTIC CHARACTERISATION

By a straightforward generalisation of the categorical case, the bicategorical cartesian closure given by Theorem 2.9 provides a method to interpret simply-typed λ-terms into the bicategory $\mathbf{Thin}_!$. This interpretation, which we call the *Kleisli interpretation*, can be shown suitably equivalent to another one, the *direct interpretation*, which will be easier to characterise syntactically. Indeed, we will show how the latter interpretation can be syntactically characterised using a rigid variant of intersection type systems, at the levels of both objects and symmetries. Finally, we introduce a resource calculus as a more convenient way to represent the derivations in our intersection type system.

3.1 Interpreting programs as spans

Suppose fixed a countable set Var of **variables**.

The λ-terms are defined by the inductive grammar

$$M, N, \ldots \quad ::= \quad x \in \mathrm{Var} \quad | \quad M\,N \quad | \quad \lambda x.M,$$

and the **simple types** are $A, B, \ldots ::= o \mid A \to B$. A **context** is a sequence of bindings $x_1 : A_1, \ldots, x_n : A_n$ where the x_i are (distinct) elements of Var and the A_i are simple types. We write $x \in \Gamma$ when there is a binding $x : B$, for some B, appearing in the sequence of Γ. We consider the standard typing relation $\Gamma \vdash M : A$ for the simply-typed λ-calculus.

3.1.1 Kleisli interpretation. Given a simple type A we define inductively its **interpretation** $(\!|A|\!)$: we define $(\!|o|\!)$ to be the unique thin groupoid based on the terminal (singleton) groupoid 1 — which satisfies $\mathbf{U}_1 = \mathbf{U}_1^{\perp} = \mathrm{PreStrat}(1)$ and $\mathbf{T}_1 = \mathbf{T}_1^{\perp\!\!\!\perp}$ is the class of prestrategies on 1 with discrete domains — and we put $(\!|A \to B|\!) = !(\!|A|\!) \multimap (\!|B|\!)$. Given a context $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, we define its **Kleisli interpretation** $(\!|\Gamma|\!)$ as $(\!|A_1|\!) \,\&\, \cdots \,\&\, (\!|A_n|\!)$. The underlying groupoid of $!(\!|\Gamma|\!)$ has a monoid structure in the cartesian category **Gpd** giving resource management operations: the "multiplication" $\gamma \bullet \gamma'$ of γ and γ' in $!G$ is simply their concatenation as sequences; the neutral element of $!G$ is the empty sequence $\langle\rangle$.

A simply-typed λ-term $\Gamma \vdash M : A$ then admits an interpretation

$$(\!|M|\!) \qquad = \qquad !(\!|\Gamma|\!) \longleftarrow (\!|M|\!) \longrightarrow (\!|A|\!)$$

in $\mathbf{Thin}_!$ via the standard clauses of the interpretation of the simply-typed λ-calculus into a cartesian closed category — we call this the **Kleisli interpretation**. The soundness theorem of cartesian closed categories ensures that $\beta\eta$-equivalent terms map to positively isomorphic thin spans; the results of Fiore and Saville [13] even yield a coherent interpretation of reduction sequences as positive isos.

We now set to show that this interpretation is a rigid intersection type system in disguise; but this will be more visible after we cope with two aspects of the Kleisli interpretation: *(1)* elements of $!(\!|\Gamma|\!)$ are sequences *over the whole context*, interleaving accesses to all variables — whereas in intersection type systems it is more natural to have a distinct sequence for each variable; and *(2)* unfolding the categorical interpretation of λ-terms in a cartesian closed category itself constructed as a Kleisli category yields some heavy bureaucracy, involving compositions with many structural maps, blurring out the connection with syntax. To mitigate these, we first give a more syntax-directed characterisation of the interpretation.

3.1.2 Direct interpretation. We first change the interpretation of contexts: the interpretation of Γ as above is the thin groupoid $[\![\Gamma]\!] = ![\![A_1]\!] \otimes \cdots \otimes ![\![A_n]\!]$ — for A a type, we write $[\![A]\!]$ as a synonym for $(\!|A|\!)$. Note that $[\![\Gamma]\!]$ still has a monoid structure: the multiplication of $\gamma = (\alpha_1, \ldots, \alpha_n)$ and $\gamma' = (\alpha'_1, \ldots, \alpha'_n)$, two elements of $[\![\Gamma]\!]$, is

$$\gamma \bullet \gamma' \quad = \quad (\alpha_1 \bullet \alpha'_1, \ldots, \alpha_n \bullet \alpha'_n) \qquad \in [\![\Gamma]\!]$$

and the neutral element is the n-tuple of empty sequences.

Given a typed λ-term $\Gamma \vdash M : A$, we now describe its **direct interpretation** in $\mathbf{Thin}_!$ as a span $[\![\Gamma]\!] \leftarrow [\![M]\!] \to [\![A]\!]$ given by induction on the typing derivation. In the case of a variable x_i typed in a context $\Gamma = x_1 : A_1, \dots, x_n : A_n$, we define $\llbracket x_i \rrbracket$ as

$$!\llbracket A_1 \rrbracket \times \cdots \times !\llbracket A_n \rrbracket \xleftarrow{(\langle\rangle, \dots, \eta_{\llbracket A_i \rrbracket}, \dots, \langle\rangle)} \llbracket A_i \rrbracket \xrightarrow{\mathrm{id}_{\llbracket A_i \rrbracket}} \llbracket A_i \rrbracket.$$

For $\Gamma \vdash M\,N : B$ where $\Gamma \vdash M : A \to B$ and $\Gamma \vdash N : A$, we set:

$$\begin{tikzcd}
& & \llbracket M\,N \rrbracket \arrow[dl, dotted, "l'"'] \arrow[dr, dotted, "r'"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
& \llbracket M \rrbracket \times \llbracket N \rrbracket^! \arrow[dl, "(\bullet)\circ(\partial_l^{\llbracket M \rrbracket} \times \partial_l^{\llbracket N \rrbracket^!})"'] \arrow[dr, "\partial_r^{\llbracket M \rrbracket} \times \partial_r^{\llbracket N \rrbracket^!}"] & & !\llbracket A \rrbracket \times \llbracket B \rrbracket \arrow[dl, "((l,r),l)"'] \arrow[dr, "r"] & \\
\llbracket \Gamma \rrbracket & & (!\llbracket A \rrbracket \times \llbracket B \rrbracket) \times !\llbracket A \rrbracket & & \llbracket B \rrbracket
\end{tikzcd}$$

where we used $\llbracket M \rrbracket^!$, the **promotion** of $\llbracket M \rrbracket$, defined as the span

$$\llbracket \Gamma \rrbracket \xleftarrow{\tilde{\mu}_\Gamma} !\llbracket \Gamma \rrbracket \xleftarrow{!\partial_l^{\llbracket M \rrbracket}} !\llbracket M \rrbracket \xrightarrow{!\partial_r^{\llbracket M \rrbracket}} !\llbracket A \rrbracket$$

where $\tilde{\mu}_\Gamma : !\llbracket \Gamma \rrbracket \to \llbracket \Gamma \rrbracket$ is the obvious functor sending a sequence of tuples of sequences into the tuple of concatenated sequences.

Finally, for $\Gamma \vdash \lambda x.\, M : A \to B$, we set $\llbracket \lambda x.\, M \rrbracket$ to be the span

$$\llbracket \Gamma \rrbracket \xleftarrow{\partial_{ll}^{\llbracket M \rrbracket}} \llbracket M \rrbracket \xrightarrow{(\partial_{lr}^{\llbracket M \rrbracket}, \partial_r^{\llbracket M \rrbracket})} !\llbracket A \rrbracket \times \llbracket B \rrbracket$$

where $\partial_{ll}^{\llbracket M \rrbracket}$ and $\partial_{lr}^{\llbracket M \rrbracket}$ are obtained from $\partial_l^{\llbracket M \rrbracket}$ by adequately projecting from $\llbracket \Gamma, x : A \rrbracket \cong \llbracket \Gamma \rrbracket \times !\llbracket A \rrbracket$.

We relate the two interpretations: given a context Γ, we write

$$s_\Gamma : !(\llbracket A_1 \rrbracket + \cdots + \llbracket A_n \rrbracket) \to !\llbracket A_1 \rrbracket \times \cdots \times !\llbracket A_n \rrbracket$$

for the evident generalisation of the Seely functor from (3). Then:

Theorem 3.1. *Given a simply-typed term $\Gamma \vdash M : A$, the span*

$$!\llbracket A_1 \rrbracket \times \cdots \times !\llbracket A_n \rrbracket \xleftarrow{s_\Gamma \circ \partial_l^{(\!|M|\!)}} (\!|M|\!) \xrightarrow{\partial_r^{(\!|M|\!)}} \llbracket A \rrbracket$$

is thin and moreover strongly isomorphic to the span $\llbracket M \rrbracket$.

3.2 Intersection types for spans

As the direct interpretation is syntax-directed, it is fairly easy to represent it purely syntactically as an intersection type system.

3.2.1 Rigid intersection types. The **rigid intersection types** are:

$$\begin{array}{lcl} \alpha, \beta, \dots & ::= & \star \quad | \quad \vec{\alpha} \multimap \beta \\ \vec{\alpha}, \vec{\beta}, \dots & ::= & \langle \alpha_1, \dots, \alpha_n \rangle \quad (n \in \mathbb{N}). \end{array}$$

As we study the simply-typed λ-calculus, we shall not consider these intersection types as standalone objects but only as refinements of simple types — we now move to the refinement relation.

3.2.2 Refinement. The refinement relation is defined with

$$\frac{}{\star \triangleleft o} \qquad \frac{\vec{\alpha} \triangleleft A \qquad \beta \triangleleft B}{\vec{\alpha} \multimap \beta \triangleleft A \to B} \qquad \frac{\forall i \in \{1, \dots, n\} \qquad \alpha_i \triangleleft A}{\langle \alpha_1, \dots, \alpha_n \rangle \triangleleft A},$$

noting that both intersection and sequence types may refine simple types. This refinement judgement correctly captures the objects in the groupoid interpreting a type A, as expressed by the bijections of the following definition:

$$\frac{(\alpha \triangleleft A_i)}{\dots, x_i : \langle \alpha \rangle \triangleleft A_i, \dots \triangleleft x_1 : A_1, \dots, x_n : A_n \vdash x_i^\alpha \triangleleft x_i : \alpha \triangleleft A_i}$$

$$\frac{\Theta \triangleleft \Gamma \vdash m \triangleleft M : \vec{\alpha} \multimap \beta \triangleleft A \to B \qquad \Theta' \triangleleft \Gamma \vdash \vec{n} \triangleleft N : \vec{\alpha} \triangleleft A}{\Theta \bullet \Theta' \triangleleft \Gamma \vdash m\,\vec{n} \triangleleft M\,N : \beta \triangleleft B}$$

$$\frac{\forall i \in \{1, \dots, k\}, \quad \Theta_i \triangleleft \Gamma \vdash m_i \triangleleft M : \alpha_i \triangleleft A}{\Theta_1 \bullet \cdots \bullet \Theta_n \triangleleft \Gamma \vdash \langle m_1, \dots, m_k \rangle \triangleleft M : \langle \alpha_1, \dots, \alpha_k \rangle \triangleleft A}$$

$$\frac{(\Theta, x : \vec{\alpha} \triangleleft A) \triangleleft \Gamma, x : A \vdash m \triangleleft M : \beta \triangleleft B}{\Theta \triangleleft \Gamma \vdash \lambda x.\, m \triangleleft \lambda x.\, M : \vec{\alpha} \multimap \beta \triangleleft A \to B}$$

Figure 1: Intersection types and approximation

Definition 3.2. For every simple type A, there are bijections

$$K_A : \mathbf{Ob}(\llbracket A \rrbracket) \simeq \{\alpha \mid \alpha \triangleleft A\}, \quad K_A^! : \mathbf{Ob}(!\llbracket A \rrbracket) \simeq \{\vec{\alpha} \mid \vec{\alpha} \triangleleft A\}$$

straight-forwardly defined by induction on the type A:

- K_o sends the only object of $\llbracket o \rrbracket$ to $\star$;
- $K_{A \to B}$ sends the pair $(\langle a_1, \dots, a_n \rangle, b) \in \llbracket A \to B \rrbracket$ to the pair $(\langle K_A(a_1), \dots, K_A(a_n) \rangle, K_B(b))$.

3.2.3 Resource contexts. To extend this to contexts, it is convenient to introduce *resource contexts*. A **resource context** for $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is a sequence of bindings $\Theta = (x_1 : \vec{\alpha}_1 \triangleleft A_1, \dots, x_n : \vec{\alpha}_n \triangleleft A_n)$ — we then write $\Theta \triangleleft \Gamma$. Clearly, the bijections above extend to $K_\Gamma : \mathbf{Ob}(\llbracket \Gamma \rrbracket) \simeq \{\Theta \mid \Theta \triangleleft \Gamma\}$. Given resource contexts for Γ

$$\Sigma = (x_i : \vec{\alpha}_i \triangleleft A_i)_{1 \le i \le n} \qquad \text{and} \qquad \Theta = (x_i : \vec{\beta}_i \triangleleft A_i)_{1 \le i \le n},$$

their **concatenation** $\Sigma \bullet \Theta$ is the resource context $(x_i : (\vec{\alpha}_i \bullet \vec{\beta}_i) \triangleleft A_i)_i$, where $\vec{\alpha}_i \bullet \vec{\beta}_i$ is the **concatenation** of sequence types.

3.2.4 Intersection type judgements. We now introduce typing judgements for rigid intersection types. There are two kinds of judgements, respectively for single intersection types and for sequences:

$$\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A \quad \text{and} \quad \Theta \triangleleft \Gamma \vdash M : \vec{\alpha} \triangleleft A.$$

The rules appear in Figure 1 ignoring, for the moment, the $\cdots u/\vec{v} \triangleleft \cdots$ parts in the middle. In the variable rule, we only display variables with non-empty sequences. The rules may appear heavy due to the multiple components of jugdments as required for the simple type refinement. But ignoring simple type refinements, what remains is the standard ruleset for non-idempotent intersection types as appears *e.g.* in [6], just without commutativity.

Given a derivation $\Gamma \vdash M : A$, $\gamma \in \mathbf{Ob}(\llbracket \Gamma \rrbracket)$ and $a \in \mathbf{Ob}(\llbracket A \rrbracket)$, we write $\llbracket M \rrbracket_{\gamma, a}$ for the **witnesses** of γ, a, *i.e.* the objects of $\llbracket M \rrbracket$ that project on γ and a through $\partial_l^{\llbracket M \rrbracket}$ and $\partial_r^{\llbracket M \rrbracket}$. As the definition of $\llbracket M \rrbracket$ directly follows the syntax, it is relatively direct that:

Proposition 3.3. *Given a simply-typed $\Gamma \vdash M : A$, for every $\gamma \in \mathbf{Ob}(\llbracket \Gamma \rrbracket)$, for every $a \in \mathbf{Ob}(\llbracket A \rrbracket)$, we have a bijection*

$$\llbracket M \rrbracket_{\gamma, a} \simeq \{\pi \mid \pi \text{ is a derivation of } K_\Gamma(\gamma) \triangleleft \Gamma \vdash M : K_A(a) \triangleleft A\}.$$

Combined with Theorem 3.1, this shows that for any simply-typed λ-term $\Gamma \vdash M : A$, for any $\gamma \in \llbracket \Gamma \rrbracket$ and $a \in \llbracket A \rrbracket$, the set of $m \in \mathbf{Ob}(\llparenthesis M \rrparenthesis)$ mapping to γ, a may be regarded as the set of derivations of $K_\Gamma(\gamma) \triangleleft \Gamma \vdash M : K_A(a) \triangleleft A$ in our rigid intersection type system. This result is to be compared with existing works providing similar characterisations in generalized species of structure [25, 28], where the rigid intersection type systems considered are much more complex, in particular importing symmetries in derivations — and derivations must be quotiented by relations forgetting the exact position of symmetries in the derivations. In contrast, our derivations are the simple inductive structures they appear to be, no quotient is required to obtain our characterisation.

3.3 Extension to symmetries

Proposition 3.3 is analogous to earlier results of Tsukada *et al.* [28] and Olimpieri [25] set in generalized species of structures, but here we go further and characterise the full *groupoid* by also giving an inductive, syntax-directed presentation of the *symmetries*.

3.3.1 Intersection type morphisms. The linear, sequence and multilinear intersection type morphisms are defined by the grammar

$$\begin{array}{lcll} \phi, \psi, \dots & ::= & \mathrm{id}_\star \quad | \quad \widetilde{\phi} \multimap \psi & \\ \vec{\phi}, \vec{\psi}, \dots & ::= & \langle \phi_1, \dots, \phi_n \rangle & (n \in \mathbb{N}) \\ \widetilde{\phi}, \widetilde{\psi}, \dots & ::= & (\sigma, \vec{\phi}) & (\sigma \in \mathcal{S}_n, |\vec{\phi}| = n) \end{array}$$

where $\mathcal{S}_n$ is the symmetric group on n elements. Given two multilinear morphisms $\widetilde{\phi_1}$ and $\widetilde{\phi_2}$ where $\widetilde{\phi_i} = (\sigma_i, \langle \phi_{i,1}, \dots, \phi_{i,n_i} \rangle)$, we define their **concatenation** $\widetilde{\phi_1} \bullet \widetilde{\phi_2}$ as $(\sigma_1 \oplus \sigma_2, \langle \phi_{1,i} \rangle_i \bullet \langle \phi_{2,i'} \rangle_{i'})$.

3.3.2 Groupoids of refinements for types. We extend our refinement relations to morphisms and introduce the **linear and multilinear morphism refinement judgements**, of the form $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$ and $\widetilde{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A$. The former states that ϕ is a linear morphism from α to α' within refinements of simple type A, and likewise for the latter. Those are defined inductively through:

$$\frac{}{\mathrm{id}_\star :: \star \Rightarrow \star \triangleleft o} \qquad \frac{\widetilde{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A \qquad \psi :: \beta \Rightarrow \beta' \triangleleft B}{(\widetilde{\phi} \multimap \psi) :: (\vec{\alpha} \multimap \beta) \Rightarrow (\vec{\alpha}' \multimap \beta') \triangleleft A \to B}$$

$$\frac{n \in \mathbb{N} \qquad \sigma \in \mathcal{S}_n \qquad \forall i \in \{1, \dots, n\} \quad \phi_i :: \alpha_i \Rightarrow \alpha'_{\sigma(i)} \triangleleft A}{(\sigma, \langle \phi_1, \dots, \phi_n \rangle) :: \langle \alpha_1, \dots, \alpha_n \rangle \Rightarrow \langle \alpha'_1, \dots, \alpha'_n \rangle \triangleleft A}$$

It is immediate that if $\phi :: \alpha \Rightarrow \alpha' \triangleleft A$, then $\alpha \triangleleft A$ and $\alpha' \triangleleft A$, and that likewise, if $\widetilde{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A$, then $\vec{\alpha} \triangleleft A$ and $\vec{\alpha}' \triangleleft A$.

As suggested by the syntax, the linear (*resp.* multilinear) intersection types and the associated morphisms that refine a common simple type A organize into a groupoid $\mathbf{IT}(A)$ (*resp.* $\mathbf{IT}_!(A)$). The composition operation is defined by induction on derivations, with:

$$\begin{array}{rcl} \mathrm{id}_\star \circ \mathrm{id}_\star & = & \mathrm{id}_\star \\ (\widetilde{\phi}' \multimap \psi') \circ (\widetilde{\phi} \multimap \psi) & = & (\widetilde{\phi}' \circ \widetilde{\phi}) \multimap (\psi' \circ \psi) \\ (\sigma', \langle \phi'_i \rangle_{1 \le i \le n}) \circ (\sigma, \langle \phi_i \rangle_{1 \le i \le n}) & = & (\sigma' \circ \sigma, \langle \phi'_{\sigma(i)} \circ \phi_i \rangle_{1 \le i \le n}) \end{array}$$

The inverse of a morphism is defined by induction similarly. This allows us to extend the correspondence of Definition 3.2:

Proposition 3.4. *For A a simple type, there are groupoid isos:*

$$K_A : \llbracket A \rrbracket \cong \mathbf{IT}(A) \quad \textit{and} \quad K^!_A : !\llbracket A \rrbracket \cong \mathbf{IT}_!(A).$$

As $\llbracket A \rrbracket$ is a thin groupoid, it comes equipped with its two polarized sub-groupoids $\llbracket A \rrbracket_-$ and $\llbracket A \rrbracket_+$ — via the proposition above, they transport to two sub-groupoids $\mathbf{IT}_-(A)$ and $\mathbf{IT}_+(A)$ of $\mathbf{IT}(A)$.

3.3.3 Groupoids of refinements for contexts. Consider Γ a context and $\Theta, \Theta' \triangleleft \Gamma$. A **context morphism** from Θ to Θ' is a sequence

$$\Xi = (x_1 : \widetilde{\phi_1} :: \vec{\alpha}_1 \Rightarrow \vec{\alpha}'_1 \triangleleft A_1, \dots, x_n : \widetilde{\phi_n} :: \vec{\alpha}_n \Rightarrow \vec{\alpha}'_n \triangleleft A_n)$$

where $\Theta = (x_i : \vec{\alpha}_i \triangleleft A_i)_{1 \le i \le n}$ and $\Theta' = (x_i : \vec{\alpha}'_i \triangleleft A_i)_{1 \le i \le n}$ — we also write $\Xi :: \Theta \Rightarrow \Theta' \triangleleft \Gamma$ to mean that Ξ is a morphism of refinements of Γ from Θ to Θ'; in that case we write $\Theta = \mathrm{dom}(\Xi)$ and $\Theta' = \mathrm{cod}(\Xi)$. Given two such morphisms $\Xi_1 :: \Theta_1 \Rightarrow \Theta'_1 \triangleleft \Gamma$ and $\Xi_2 :: \Theta_2 \Rightarrow \Theta'_2 \triangleleft \Gamma$ for a common context Γ, their **concatenation**

$$\Xi_1 \bullet \Xi_2 :: \Theta_1 \bullet \Theta_2 \Rightarrow \Theta'_1 \bullet \Theta'_2 \triangleleft \Gamma$$

is defined by componentwise concatenation. The resource contexts and resource context morphisms form a groupoid $\mathbf{IT}(\Gamma)$ which can be seen as the product of the $\mathbf{IT}_!(A_i)$, so we have a groupoid iso

$$K_\Gamma : \llbracket \Gamma \rrbracket \cong \mathbf{IT}(\Gamma) .$$

3.3.4 Morphisms between derivations. We finally set to construct a *groupoid* of derivations in our rigid intersection type system. The morphisms will be given by two kinds of judgements, of the form

$$\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A \quad \text{and} \quad \Xi \triangleleft \Gamma \vdash M : \widetilde{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A$$

read as stating that ϕ is a morphism from $\mathrm{dom}(\Xi) \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ to $\mathrm{cod}(\Xi) \triangleleft \Gamma \vdash M : \alpha' \triangleleft A$, and likewise for multilinear refinements.

The rules appear in Figure 2, where we omit the $\cdots \triangleleft \Gamma$ part for conciseness. The most subtle case is the last, corresponding to *promotion* and introducing new symmetries following an arbitrary permutation σ. In particular, swapping derivations for M by σ requires swapping accordingly the resource accesses in the context. This uses an operation that to a family $(\widetilde{\phi_i} :: \vec{\alpha}_i \Rightarrow \vec{\alpha}'_i \triangleleft A)_{1 \le i \le n}$ of morphisms of refinements of A associates a single morphism

$$\sigma \circledast (\widetilde{\phi_i})_{1 \le i \le n} :: \vec{\alpha}_1 \bullet \dots \bullet \vec{\alpha}_n \Rightarrow \vec{\alpha}'_{\sigma^{-1}(1)} \bullet \dots \bullet \vec{\alpha}'_{\sigma^{-1}(n)} \triangleleft A$$

defined in the obvious way. This generalizes to context refinement morphisms transparently, variable by variable.

Now, given a derivation $\Gamma \vdash M : A$, its associated intersection type derivations $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ and intersection type morphism derivations $\Xi \triangleleft \Gamma \vdash M : \phi :: \alpha \Rightarrow \alpha' \triangleleft A$ organize into a groupoid $\mathbf{IT}(M)$, whose composition is directly derived from those for refinement types and resource contexts. By considering the two projection functors defined in the obvious way, we get a span

$$\mathbf{IT}(\Gamma) \xleftarrow{\partial^M_l} \mathbf{IT}(M) \xrightarrow{\partial^M_r} \mathbf{IT}(A)$$

which can be seen as a syntactic description of $\llbracket M \rrbracket$ by the result:

Theorem 3.5. *For any simply-typed λ-term $\Gamma \vdash M : A$, there is an iso of groupoids $K_M : \llbracket M \rrbracket \to \mathbf{IT}(M)$ making the diagram commute:*

$$\begin{tikzcd} \llbracket \Gamma \rrbracket \arrow[d, "K_\Gamma"'] & \llbracket M \rrbracket \arrow[l, "\partial^{\llbracket M \rrbracket}_l"'] \arrow[r, "\partial^{\llbracket M \rrbracket}_r"] \arrow[d, "K_M"] & \llbracket A \rrbracket \arrow[d, "K_A"] \\ \mathbf{IT}(\Gamma) & \mathbf{IT}(M) \arrow[l, "\partial^M_l"] \arrow[r, "\partial^M_r"'] & \mathbf{IT}(A) \end{tikzcd}$$

By Theorem 3.1, this also applies to the Kleisli interpretation. From this connection to the interpretation in the cartesian closed bicategory $\mathbf{Thin}_!$, we immediately get the following corollary:

Corollary 3.6. *Consider $\Gamma \vdash M, M' : A$ simply-typed λ-terms, s.t. $M \to_\beta M'$. Then, there is a weak iso of spans $\mathbf{IT}(M) \cong \mathbf{IT}(M')$.*

$$\frac{(\phi :: \alpha \Rightarrow \alpha' \triangleleft A_i)}{\ldots, x_i : (\mathrm{id}_{\{1\}}, \langle\phi\rangle) :: \langle\alpha\rangle \Rightarrow \langle\alpha'\rangle \triangleleft A_i, \ldots \vdash x_i : \phi :: \alpha \Rightarrow \alpha' \triangleleft A_i} \qquad \frac{\Xi \vdash M : (\widetilde{\phi} \multimap \psi) :: (\vec{\alpha} \multimap \beta) \Rightarrow (\vec{\alpha}' \multimap \beta') \triangleleft A \to B \quad \Xi' \vdash N : \widetilde{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A}{\Xi \bullet \Xi' \vdash M\,N : \psi :: \beta \Rightarrow \beta' \triangleleft B}$$

$$\frac{\Xi, x : \widetilde{\phi} :: \vec{\alpha} \Rightarrow \vec{\alpha}' \triangleleft A \vdash M : \psi :: \beta \Rightarrow \beta' \triangleleft B}{\Xi \vdash \lambda x.\, M : (\widetilde{\phi} \multimap \psi) :: (\vec{\alpha} \multimap \beta) \Rightarrow (\vec{\alpha}' \multimap \beta') \triangleleft A \to B} \qquad \frac{n \in \mathbb{N} \quad \sigma \in \mathcal{S}_n \quad \forall i \in \{1, \ldots, n\},\ \Xi_i \vdash M : \phi_i :: \alpha_i \Rightarrow \alpha'_i \triangleleft A}{\sigma \circledcirc (\Xi_i)_{1 \le i \le n} \vdash M : (\sigma, \langle\phi_1, \ldots, \phi_n\rangle) :: \langle\alpha_1, \ldots, \alpha_n\rangle \Rightarrow \langle\alpha'_{\sigma^{-1}(1)}, \ldots, \alpha'_{\sigma^{-1}(n)}\rangle \triangleleft A}$$

Figure 2: The rules for rigid intersection type morphisms

This shows that although rigid intersection types do not enjoy subject reduction as observed in the introduction, the interpretation in $\mathbf{Thin}_!$ associates to every β-reduction $M \to_\beta M'$ a bijective *transport* between derivations of M and M' "correcting" the error, up to some residual symmetries in the groupoids for Γ and A.

3.4 Rigid Resource Calculus

As derivations are somewhat heavy, it seems helpful to remark that they can be equivalently presented as certain *rigid resource terms.*

3.4.1 Resource terms. The grammar for **rigid resource terms** is:

$$\begin{array}{lcl} m, n, \ldots & ::= & x^\alpha \mid \lambda x.m \mid m\ \vec{n} \\ \vec{m}, \vec{n} \ldots & ::= & \langle m_1, \ldots, m_k\rangle\,, \end{array}$$

where x^α is the data of a variable $x \in \mathrm{Var}$ and of a **labelling** intersection type α. Our resource terms depart from standard resource terms [11] in two significant ways. Firstly, as in [26] our calculus is *rigid*: argument subterms are sequences rather than finite multisets. Secondly, we label variable occurrences with intersection types, so as to guarantee the correspondence with derivations.

3.4.2 Approximation relations. Those resource terms are already implicitly present in our derivations. To formalize that, we introduce the **linear and multilinear approximation judgements**

$$\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A \quad \text{and} \quad \Theta \triangleleft \Gamma \vdash \vec{m} \triangleleft M : \vec{\alpha} \triangleleft A$$

which are defined by the (full) rules of Figure 1. We have a canonical forgetful function U mapping a derivation π of $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$ to the corresponding derivation $U(\pi)$ of $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ and similarly for multilinear judgements. We easily check that:

Proposition 3.7. *The following two properties hold:*

(a) Given a term $\Gamma \vdash M : A$ and resource term m, there is at most one (Θ, α, π) with π a derivation of $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$,

(b) For a derivation π of $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$, there is a unique $(u, \tilde{\pi})$ s.t. $\tilde{\pi}$ is a derivation of $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$ and $U(\tilde{\pi}) = \pi$.

For a term $\Gamma \vdash M : A$, we write $\mathbf{Res}(M)$ for the set of resource terms m such that $\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A$ is derivable, for some rigid intersection types / contexts a, Θ. The proposition above gives

$$\mathbf{Res}(M) \simeq \mathbf{Ob}(\mathbf{IT}(M))$$

a bijection showing that up to isomorphism, $\mathbf{Thin}_!$ interprets a simply-typed λ-term as a set of rigid resource terms.

3.4.3 Resource terms and reduction. This representation lets us examine the action of the interpretation of reduction steps given by Corollary 3.6. Consider a β-redex $\vdash (\lambda x.\, M)\, N$. There is an iso

$$(\!|(\lambda x.\, M)\, N \to_\beta M[N/x]|\!) : (\!|(\lambda x.\, M)\, N|\!) \cong (\!|M[N/x]|\!)$$

obtained via the cartesian closed bicategorical structure of $\mathbf{Thin}_!$ [13], and through our results it yields a bijection $\Omega : \mathbf{Res}((\lambda x.\, M)\, N) \simeq \mathbf{Res}(M[N/x])$ which we can compute. Considering a resource term $u = (\lambda x.\, m)\, \langle n_1, \ldots, n_k\rangle \in \mathbf{Res}((\lambda x.\, M)\, N)$ for $m \triangleleft M$, $\vec{n} \triangleleft N$, we get

$$\Omega((\lambda x.\, m)\, \langle n_1, \ldots, n_k\rangle) \quad = \quad m[n_1/x_1, \ldots, n_k/x_k] \tag{4}$$

where $x_1, \ldots, x_k$ are the occurrences of x in m, *in order from left to right* — there must indeed be k occurrences with the right intersection types, because u matches an intersection type derivation.

But this apparent simplicity for toplevel β-reductions is misleading: $\mathbf{Thin}_!$ interprets reduction as *weak* span isos. If we have

$$\Theta \triangleleft \Gamma \vdash m \triangleleft M : \alpha \triangleleft A\,,$$

for $\Gamma \vdash M : A$ with $M \to_\beta M'$, then we do not have $\Theta \triangleleft \Gamma \vdash M' : \alpha \triangleleft A$ but only $\Theta' \triangleleft \Gamma \vdash M' : \alpha' \triangleleft A$ for $\Theta' \cong^-_\Gamma \Theta$ and $\alpha' \cong^+_A \alpha$; so we cannot directly perform (4) deep within m as the resulting resource term would fail to typecheck in our rigid intersection type system.

$\mathbf{Thin}_!$ *does* provide some $m' = (\!|M \to_\beta M'|\!)(m)$, obtained through an interactive reindexing of all components of m, correcting the typing mismatches. But its construction fully exploits the bicategorical structure of $\mathbf{Thin}_!$ and in particular the horizontal composition of 2-cells (via the uniqueness property of Lemma 2.4), and it does not seem to have a simple syntactic presentation.

3.4.4 Link with multiset resource terms. To conclude this section, we show how our rigid resource terms do not have a self-contained rewriting theory; however we show here how they can be used as representatives for more standard (multiset-based) resource terms.

We consider **multiset resource terms** generated by the grammar:

$$\begin{array}{lcl} \mathbf{u}, \mathbf{v}, \ldots & ::= & x^{\boldsymbol{\alpha}} \mid \lambda x.\mathbf{u} \mid \mathbf{u}\ \mathbf{v}^* \\ \mathbf{u}^*, \mathbf{v}^* \ldots & ::= & [\mathbf{u}_1, \ldots, \mathbf{u}_n] \end{array}$$

using the (multiset) non-idempotent intersection types defined by

$$\begin{array}{lcl} \boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots & ::= & \star \quad \mid \quad \boldsymbol{\alpha}^* \multimap \boldsymbol{\beta} \\ \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \ldots & ::= & [\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n] \quad (n \in \mathbb{N}) \end{array}$$

where, as expected, we use multisets $[\cdots]$ instead of sequences $\langle\cdots\rangle$. Given a rigid intersection type α, one can obtain a multiset intersection type $\overline{\alpha}$ by replacing inductively the sequences $\langle\cdots\rangle$ with multisets $[\cdots]$. Similarly, given a rigid resource term m, one obtains a multiset resource term $\overline{m}$ with the same operation. Then:

Proposition 3.8. *Take β-normal $\Gamma \vdash M : A$, and $m, n \in \mathbf{Res}(M)$. Then, $m \cong n$ if and only if $\overline{m} = \overline{n}$.*

This is direct by induction — here $m \cong n$ is defined via the correspondence with derivations. This shows that standard resource terms fit in the theory of thin spans of groupoids as symmetry classes in the interpretation of terms, albeit for β-normal terms. For non-normal terms this correspondence fails: we have

$$(\lambda y.\, x\, y\, y)\,\langle z, w\rangle \quad \not\cong \quad (\lambda y.\, x\, y\, y)\,\langle w, z\rangle$$

though they both map to $(\lambda y.\, x\, y\, y)\,[w, z]$ — in rigid resource terms, β-redexes explicitly match variable occurrences and resources in the argument sequence, while usual resource terms do not.

4 THIN SPANS AND RELATIONAL MODELS

Now, we relate thin spans and other extensions of the relational model. This shall let us re-interpret what these compute in terms of rigid resource terms and symmetries of rigid intersection types.

4.1 The Relational Model

First of all, we start by describing the relationship between thin spans of groupoids and the relational model [16]. It is fairly straightforward, but is hopefully helpful for the generalizations to come.

4.1.1 Introducing the relational model. The relational model builds on the category **Rel** of *sets* and *relations*. **Rel** has a symmetric monoidal structure, obtained by defining the tensor $A \otimes B = A \times B$ as the cartesian product of sets — the unit is any singleton set. **Rel** is actually compact closed: the *dual* A^* of a set A is itself, and there are a unit $I \to A \otimes A^*$ and co-unit $A^* \otimes A \to I$ given by the obvious diagonal relations. This turns **Rel** into a symmetric monoidal closed category, and as such a model of the linear λ-calculus — in particular, it supports a linear arrow defined as $A \multimap B = A \times B$.

But **Rel** also has an *exponential modality*, given by $!A = \mathcal{M}(A)$ the set of finite multisets of elements of A. This extends to a comonad $!$ on **Rel** and for each A, B there is an isomorphism $!(A\&B) \cong\, !A\otimes !B$, the *Seely isomorphism.* Together with additional coherence conditions [24], this makes **Rel** a *Seely category*, a model of intuitionistic linear logic, and the Kleisli category $\mathbf{Rel}_!$ is cartesian closed.

4.1.2 From **Thin** *to* **Rel**. It seems clear how to relate **Thin** and **Rel**: on objects, simply send a thin groupoid A to $|A| = A/\cong$ its **symmetry classes** (or connected components) — clearly, $|\mathbf{Sym}(A)| = \mathcal{M}(|A|)$. Likewise, given a thin span $A \leftarrow S \to B$, we can obtain

$$|S| = \{(\overline{s_A}, \overline{s_B}) \mid s \in S\} \in \mathbf{Rel}[|A|, |B|]$$

called its **relational collapse**, for $\overline{(-)}$ the equivalence class. Then:

Proposition 4.1. *This yields a functor* $|-| : \mathbf{Thin} \to \mathbf{Rel}$.

Proof. This requires us to compose witnesses *up to symmetry*, which we do thanks to Lemma 2.4. □

4.1.3 Preservation of further structure. From the definition, it is straightforward that we have bijection yielding isos in **Rel**:

$$\begin{array}{cccc} t^{\otimes}_{A,B} & : & |A| \otimes |B| & \cong & |A \otimes B| \\ t^{\&}_{A,B} & : & |A| \,\&\, |B| & \cong & |A \,\&\, B| \\ t^{!}_{A} & : & !|A| & \cong & |!A| \end{array}$$

for A and B thin groupoids; in particular the third amounts to $|!A| \simeq \mathcal{M}(|A|)$ for A any thin groupoid. It is a routine verification that these components satisfy the coherence conditions required to make $|-| : \mathbf{Thin} \to \mathbf{Rel}$ a Seely functor, so that:

Theorem 4.2. *Setting, for any* $!A \leftarrow S \to B$ *in* $\mathbf{Thin}_![A, B]$,

$$|S|_! = |S| \circ t^!_A \in \mathbf{Rel}_![|A|, |B|]\,,$$

yields $|-|_! : \mathbf{Thin}_! \to \mathbf{Rel}_!$ *a cartesian closed functor.*

It follows that this preserves the interpretation of the simply-typed λ-calculus: for every simple type A there is a bijection $t_A : [\![A]\!]_{\mathbf{Rel}_!} \simeq |(\!|A|\!)|$ — and likewise for contexts — so that if $\Gamma \vdash M : A$, $\gamma \in [\![\Gamma]\!]_{\mathbf{Rel}_!}$, $a \in [\![A]\!]_{\mathbf{Rel}_!}$, $(\gamma, a) \in [\![M]\!]_{\mathbf{Rel}_!}$ iff $(t_\Gamma\, \gamma, t_A\, a) \in |(\!|M|\!)|_!$.

4.2 Weighted Relations

The *weighted relational model* is due to Lamarche [21], though its application to semantics was fleshed out by Laird *et al.* [19, 20]. In full generality, its construction is parametrized by a complete semiring; but for the purposes of this paper we will only work with the semiring $\mathbb{N}_\infty = \mathbb{N} \cup \{+\infty\}$ of completed natural numbers.

4.2.1 The weighted relational model. Rather than merely collecting the completed executions, the weighted relational assigns a *weight* — here, an element of $\mathbb{N}_\infty$ — to any execution. In other words, a weighted relation from set A to set B is a function $A \times B \to \mathbb{N}_\infty$.

This lets us *count* properties of execution: for instance, it is shown in [20] how the relational model weighted by $\mathbb{N}_\infty$ counts how many distinct executions may lead to a given result at ground type, for a non-deterministic extension of PCF. But even for purely deterministic programs (in fact, simply-typed λ-terms), the weighted relational model computes non-trivial coefficients.

Example 4.3. Considering the simply-typed λ-term

$$f : o \to o \to o, x : o, y : o \vdash f\,(f\,y\,x)\,(f\,x\,y) : o\,,$$

then the point of the web written in intersection type notation as

$$f : [[\star] \multimap [] \multimap \star, [] \multimap [\star] \multimap \star], x : [\star], y : [] \vdash \star$$

has a weight of 2 in the weighted relational model — this reflects the fact that this point can be realized in two distinct ways, depending on which occurrence of f calls which argument; seemingly corresponding to two distinct normal resource terms:

$$f\,[f\,[]\,[x^\star]]\,[] \qquad f\,[]\,[f\,[x^\star]\,[]]\,,$$

or (via Section 3.4.4) to two symmetry classes of rigid terms.

This suggests that, maybe, the weighted relational model counts the number of resource terms inhabiting a certain intersection type. But that is not actually the case, as illustrated by this next example.

Example 4.4. Considering now the simply-typed λ-term

$$f : o \to o, g : o \to o, y : o \vdash f\,(g\,y) : o\,,$$

then the point of the web written in intersection type notation as

$$f : [[\star, \star] \multimap \star], g : [[] \multimap \star, [\star] \multimap \star], y : [\star] \vdash \star$$

is *also* assigned a weight of 2 by the weighted relational model, even though the reader can check that there is only one resource term inhabiting that type. Clearly here we are somehow accounting for the *symmetries* of this resource term — but which symmetries?

4.2.2 Categorical structure. The weighted relational model is structured around the category **WRel**: its objects are sets, and a morphism from A to B is $\alpha \in \mathbb{N}_\infty^{A\times B}$ — for $a \in A$ and $b \in B$, we write $\alpha_{a,b} \in \mathbb{N}_\infty$ for $\alpha(a,b)$. Identity is $(\mathrm{id}_A)_{a,a'} = \delta_{a,a'}$. Composition is

$$(\beta \circ \alpha)_{a,c} = \sum_{b\in B} \alpha_{a,b} \cdot \beta_{b,c}$$

for $\alpha \in \mathbf{WRel}[A,B]$, $\beta \in \mathbf{WRel}[B,C]$, $a \in A$ and $c \in C$. This potentially infinite sum always "converges" because our set of weights $\mathbb{N}_\infty$ includes the infinity. Just like **Rel**, **WRel** is a compact closed category with biproducts, see [20] for details.

Finally, there is an exponential modality $!A = \mathcal{M}(A)$ on sets. On morphisms, the critical definition is that of *functorial promotion*:

$$(!\alpha)_{\mu,[b_1,\ldots,b_n]} = \sum_{\substack{(a_1,\ldots,a_n)\\ s.t.\ \mu=[a_1,\ldots,a_n]}} \prod_{i=1}^{n} \alpha_{a_i,b_i}\,.$$

Altogether, just like **Rel**, **WRel** is a Seely category, and thus the associated Kleisli category $\mathbf{WRel}_!$ is cartesian closed.

4.2.3 Positive witnesses. We must make the functor of Section 4.1.2 quantitative — from a thin span $A \leftarrow S \rightarrow B$ and symmetry classes $\mathbf{a} \in |A|$, $\mathbf{b} \in |B|$, we must assign a number $|S|_{\mathbf{a},\mathbf{b}} \in \mathbb{N}_\infty$. We naturally expect this number to be the cardinal of a set of *witnesses*

$$|S|_{\mathbf{a},\mathbf{b}} = \#\,\mathsf{wit}_S(\mathbf{a},\mathbf{b})\,,$$

thus our question boils down to the following: what is the adequate notion of witnesses, in a thin span, for symmetry classes $\mathbf{a}, \mathbf{b}$? It is tempting to count symmetry classes in S, however we have seen in Section 3.4.4 that (for normal terms) those correspond to resource terms, and Example 4.4 shows that it is not what the weighted relational model counts; in fact we shall see it accounts for

$$f\,\langle \lambda x.\, g\,\langle y\rangle, \lambda x\, g\,\langle\rangle\rangle\,, \qquad f\,\langle \lambda x.\, g\,\langle\rangle, \lambda x.\, g\,\langle y\rangle\rangle\,, \tag{5}$$

the *two* rigid resource terms that intuitively inhabit the intersection type of Example 4.4 — even though the two are symmetric. But it is not the case that we are simply counting rigid resource terms! If we were to replace y with x in Example 4.4, then the weight given by **WRel** becomes one and thus the two rigid resource terms displayed in (5) with x instead of y should suddenly just account for one...

Thin will help sort this out. Assume that all groupoids interpreting types come equipped with a function $\underline{(-)}$ associating to each symmetry class $\mathbf{a} \in |A|$ a representative $\underline{\mathbf{a}} \in \mathbf{a}$. Then we set

$$\mathsf{wit}^+_S(\mathbf{a},\mathbf{b}) \quad = \quad \{s \in S \mid \underline{\mathbf{a}} \cong^-_A s_A \,\&\, s_B \cong^+_B \underline{\mathbf{b}}\} \tag{6}$$

where $a \cong^+_A a'$ means there is $\theta^+ \in A_+[a,a']$ and likewise for $\cong^-_A$; we call those the **positive witnesses** of $\mathbf{a}$ and $\mathbf{b}$ in S. This depends on a choice of representatives for symmetry classes — our development will apply for thin groupoids equipped with representatives:

Definition 4.5. A **representation** for a thin groupoid A is a function $\underline{(-)} : (\mathbf{a} \in |A|) \to \mathbf{a}$ such that for all $\mathbf{a} \in \overline{A}$, $\underline{\mathbf{a}}$ is **canonical**, in the sense that for all $\theta \in A[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$, the unique factorization $\theta = \theta^- \circ \theta^+$ given by Lemma 2.3 satisfies $\theta^- \in A_-[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$ and $\theta^+ \in A_+[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$.

If A is a thin groupoid with a representation and $\mathbf{a} \in A$, we write $\mathsf{m}(\mathbf{a}) = \#A[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$ the **symmetry degree** of $\mathbf{a}$. Likewise, we write $\mathsf{m}_+(\mathbf{a}) = \#A_+[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$ (resp. $\mathsf{m}_-(\mathbf{a}) = \#A_-[\underline{\mathbf{a}}, \underline{\mathbf{a}}]$) the **positive symmetry degree** (resp. negative) of $\mathbf{a}$. From Definition 4.5, we then have

$$\mathsf{m}(\mathbf{a}) \quad = \quad \mathsf{m}_+(\mathbf{a}) \cdot \mathsf{m}_-(\mathbf{a}) \tag{7}$$

reflecting quantitatively the factorization of Lemma 2.3.

One can build a representation for all constructions on thin groupoids so far. The non-trivial case is the exponential, which is handled by the following proposition:

Proposition 4.6. *Assume A is a thin groupoid and $a_1, \ldots, a_n$ are canonical objects of A such that $a_i \cong a_j$ implies $a_i = a_j$ for every $1 \le i, j \le n$.*

Then, $\langle a_1, ..., a_n\rangle$ is a canonical object of $!A$.

Assuming A is equipped with a total ordering on its objects, this allows us to build a representation for $!A$ as follows: given $\mathbf{a} = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in |!A|$, we consider $\langle \underline{\mathbf{a}_1}, \ldots, \underline{\mathbf{a}_n}\rangle$, which we present in a sequential ordering, following the total order on objects of A. It is canonical thanks to the above proposition. Until the end of this section, we consider all thin groupoids equipped with a canonical representation and a total ordering of their objects.

Summing up, to any thin span $A \leftarrow S \rightarrow B$ we associate $|S|_{\mathbf{a},\mathbf{b}} = \#\mathsf{wit}^+_S(\mathbf{a},\mathbf{b})$, and we now aim to prove that this extends to a functor.

4.2.4 Functoriality. Preservation of the identity is obvious by the factorization property of Lemma 2.3. Composition is more subtle. Naturally, for $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$ we expect a bijection

$$\mathsf{wit}^+_{T\odot S}(\mathbf{a},\mathbf{c}) \quad \simeq \quad \sum_{\mathbf{b}\in|B|} \mathsf{wit}^+_S(\mathbf{a},\mathbf{b}) \times \mathsf{wit}^+_T(\mathbf{b},\mathbf{c})\,, \tag{8}$$

and while our results imply that such a bijection exists for cardinality reasons, it is not actually what we shall build directly. In fact, there appears to be no natural function from the right-hand side to the left-hand side. We must assemble $s \in \mathsf{wit}^+_S(\mathbf{a},\mathbf{b})$ and $t \in \mathsf{wit}^+_T(\mathbf{b},\mathbf{c})$ into an element of $\mathsf{wit}^+_{T\odot S}(\mathbf{a},\mathbf{c})$ but we cannot do that directly, as we only have $s_B \cong_B t_B$ and not $s_B = t_B$. We can, as in the proof of Proposition 4.15, compose s and t via any symmetry $\theta_B : s_B \cong_B t_B$ to obtain an element of $\mathsf{wit}^+_{T\odot S}(\mathbf{a},\mathbf{c})$; but this does not yield a function as the result depends on the choice of θ_B.

To address this dependency in the undetermined mediating symmetry, we consider instead the composition of witnesses carrying explicit symmetries: the **$\sim$-witnesses** from $\mathbf{a}$ to $\mathbf{b}$ are triples

$$\sim\text{-}\mathsf{wit}^+_S(\mathbf{a},\mathbf{b}) = \{(\theta^-_A, s, \theta^+_B) \mid \theta^-_A : \underline{\mathbf{a}} \cong^-_A s_A \,\&\, \theta^+_B : s_B \cong^+_B \underline{\mathbf{b}}\}\,;$$

so $(\theta^-_A, s, \theta^+_B) \in \sim\text{-}\mathsf{wit}^+_S(\mathbf{a},\mathbf{b})$ and $(\vartheta^-_B, t, \vartheta^+_C) \in \sim\text{-}\mathsf{wit}^+_T(\mathbf{b},\mathbf{c})$ providing $\vartheta^-_B \circ \theta^+_B$ used to compose s and t via Lemma 2.4.

While in a thin span $A \leftarrow S \rightarrow B$ the display $S \rightarrow A \times B$ is not a fibration, $\sim$-witnesses do enjoy a fibration-like property:

Proposition 4.7. *Consider $A \leftarrow S \rightarrow B$ a thin span, $s \in S$, and*

$$\theta^-_A : a \cong^-_A s_A \qquad \theta^+_B : s_B \cong^+_B b\,.$$

For $\Omega_A : a' \cong_A a$ and $\Omega_B : b \cong_B b'$, there are unique $\varphi^S : s \cong_S s'$ and $\vartheta^-_A : a' \cong^-_A s'_A$, $\vartheta^+_B : s'_B \cong^+_B b'$ s.t. the diagrams commute:

$$\begin{tikzcd} a \arrow[r, "\theta^-_A"] & s_A \arrow[d, "\varphi^S_A"] \\ a' \arrow[u, "\Omega_A"] \arrow[r, "\vartheta^-_A"'] & s'_A \end{tikzcd} \qquad \begin{tikzcd} s_B \arrow[r, "\theta^+_B"] \arrow[d, "\varphi^S_B"'] & b \arrow[d, "\Omega_B"] \\ s'_B \arrow[r, "\vartheta^+_B"'] & b' \end{tikzcd}$$

This follows from Lemma 2.5. We can now establish the bijection patching (8). Consider $A \leftarrow S \rightarrow B$ and $B \leftarrow T \rightarrow C$, $\mathbf{a} \in |A|, \mathbf{b} \in |B|$ and $\mathbf{c} \in |C|$, we write $\sim\text{-wit}^+_{S,T}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ for the **$\sim$-interaction witnesses**, *i.e.* tuples $(\theta^-_A, s, \Theta, t, \theta^+_C)$ where $\theta^-_A : \underline{\mathbf{a}} \cong^-_A s_A, s_B = t_B = b$ and $\theta^+_C : t_C \cong^+_C \underline{\mathbf{c}}$ so that $(s, t) \in T \odot S$; and $\Theta : \underline{\mathbf{b}} \cong_B b$.

Proposition 4.8. *For $S, T, \mathbf{a}, \mathbf{b}, \mathbf{c}$ as above, there is a bijection*

$$\Upsilon \ : \ \sim\text{-wit}^+_S(\mathbf{a}, \mathbf{b}) \times \sim\text{-wit}^+_T(\mathbf{b}, \mathbf{c}) \ \simeq \ \sim\text{-wit}^+_{S,T}(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

s.t. *for any $\Upsilon((\theta^-_A, s, \theta^+_B), (\Omega^-_B, t, \Omega^+_B)) = (\psi^-_A, s', \Theta, t', \psi^+_C)$, there are unique $\omega^S : s \cong_S s'$ and $\nu^T : t \cong_T t'$ making the diagrams commute:*

$$\begin{tikzcd}
 & s_A \arrow[d, "\omega^S_A"] \\
\underline{\mathbf{a}} \arrow[ur, "\theta^-_A"] \arrow[dr, "\psi^-_A"'] & \\
 & s'_A
\end{tikzcd}
\quad
\begin{tikzcd}
s_B \arrow[r, "\theta^+_B"] \arrow[d, "\omega^S_B"'] & \underline{\mathbf{b}} \arrow[r, "\Omega^-_B"] \arrow[d, "\Theta"] & t_B \arrow[d, "\nu^T_B"] \\
s'_B \arrow[r, equal] & b \arrow[r, equal] & t'_B
\end{tikzcd}
\quad
\begin{tikzcd}
t_C \arrow[d, "\nu^T_C"'] \arrow[dr, "\Omega^+_C"] & \\
 & \underline{\mathbf{c}} \\
t'_C \arrow[ur, "\psi^+_C"'] &
\end{tikzcd}$$

This is direct from Lemma 2.4 and Proposition 4.7.

We now have a bijection that somewhat looks like (8), but we must sum over all symmetry classes in B and check that the cardinality of added symmetries cancels out. Indeed, it is easy that

$$\# \sim\text{-wit}^+_S(\mathbf{a}, \mathbf{b}) \quad = \quad \mathsf{m}_-(\mathbf{a}) \cdot \# \,\text{wit}^+_S(\mathbf{a}, \mathbf{b}) \cdot \mathsf{m}_+(\mathbf{b})\,;$$

from the definition, and since $\sim$-interaction witnesses carry a symmetry class in B and an endo-symmetry, it is also direct that

$$\# \sim\text{-wit}^+_{T \odot S}(\mathbf{a}, \mathbf{c}) = \sum_{\mathbf{b} \in |B|} \frac{1}{\mathsf{m}(\mathbf{b})} \cdot \# \sim\text{-wit}^+_{S,T}(\mathbf{a}, \mathbf{b}, \mathbf{c})\,.$$

From there and (7), (8) follows from a simple computation. So:

Corollary 4.9. *This yields a functor $| - | : \mathbf{Thin} \rightarrow \mathbf{WRel}$.*

4.2.5 Exponential. The crucial point remaining is that the functorial action of ! is preserved. For this section, we adopt notations inlining the bijections of Section 4.1.3: in particular, we write elements of $|!A|$ as finite multisets of elements of $|A|$. We must give

$$\text{wit}^+_{!S}(\boldsymbol{\mu}, [\mathbf{b}_1, \dots, \mathbf{b}_n]) \quad \simeq \quad \sum_{\substack{\langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \\ s.t. [\mathbf{a}_1, \dots, \mathbf{a}_n] = \boldsymbol{\mu}}} \prod_{i=1}^{n} \text{wit}^+_S(\mathbf{a}_i, \mathbf{b}_i) \qquad (9)$$

a bijection, for any thin span $A \leftarrow S \rightarrow B$.

From left to right, recall that writing $\boldsymbol{\nu} = [\mathbf{b}_1, \dots, \mathbf{b}_n]$, $\text{wit}^+_{!S}(\boldsymbol{\mu}, \boldsymbol{\nu})$ comprises those $\vec{s}$ such that $\underline{\boldsymbol{\mu}} \cong^-_A \vec{s}_{!A}$ and $\vec{s}_{!B} \cong^+_B \underline{\boldsymbol{\nu}}$. Let us write $\underline{\boldsymbol{\nu}} = \langle b_1, \dots, b_n \rangle$. On the right-hand side, as positive symmetries cannot exchange elements of a sequence, we have $\vec{s} = \langle s^1, \dots, s^n \rangle$ where $s^i_B \cong^+_B b_i$. However on the left-hand side symmetries *can* exchange elements, so that there must exist an (unspecified) permutation $\sigma \in \varsigma(n)$ such that $\underline{\mathbf{a}}_{\sigma(i)} \cong^-_A s^i_A$, informing $\langle \mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)} \rangle$ satisfying $[\mathbf{a}_{\sigma(1)}, \dots, \mathbf{a}_{\sigma(n)}] = \boldsymbol{\mu}$ as needed. Reciprocally, it is clear that data on the right-hand side can be assembled into an element of $\text{wit}^+_{!S}(\boldsymbol{\mu}, \boldsymbol{\nu})$ and that those operations are inverse of one another.

This shows that modulo the bijection $t^!_A$ of Section 4.1.3, the functorial action of ! is preserved. The other bijections of Section 4.1.3 still yield isomorphisms in $\mathbf{WRel}$ — for which, by a slight abuse, we keep the same notation. All necessary coherence conditions are satisfied, so that this operation lifts to the Kleisli (bi)categories.

Theorem 4.10. *We have $| - |_! : \mathbf{Thin}_! \rightarrow \mathbf{WRel}_!$ cartesian closed.*

4.2.6 Consequences. Since a cartesian closed functor preserves the interpretation of the simply-typed λ-calculus, this gives us a combinatorial description of the coefficients computed by $\mathbf{WRel}_!$:

Corollary 4.11. *Consider $\Gamma \vdash M : A$ a simply-typed λ-term. For every $\boldsymbol{\gamma} \in [\![\Gamma]\!]_{\mathbf{WRel}_!}$ and $\mathbf{a} \in [\![A]\!]_{\mathbf{WRel}_!}$, we have*

$$([\![M]\!]_{\mathbf{WRel}_!})_{\boldsymbol{\gamma}, \mathbf{a}} = \#\text{wit}^+_{(\!|M|\!)}(t_\Gamma\, \boldsymbol{\gamma}, t_A\, \mathbf{a})\,.$$

By the results in Section 3.2, this is also the number of derivations $\Theta \triangleleft \Gamma \vdash M : \alpha \triangleleft A$ (or their representations as rigid resource terms) where Θ is negatively symmetric (*resp.* α is positively symmetric) to the intersection type matching a chosen canonical rigid representative for $\boldsymbol{\gamma}$ (*resp.* for $\mathbf{a}$). Note that we can also derive:

Proposition 4.12. *Consider $\Gamma \vdash M : A$ a simply-typed λ-term. For every $\boldsymbol{\gamma} \in [\![\Gamma]\!]_{\mathbf{WRel}_!}$ and $\mathbf{a} \in [\![A]\!]_{\mathbf{WRel}_!}$, we have*

$$([\![M]\!]_{\mathbf{WRel}_!})_{\boldsymbol{\gamma}, \mathbf{a}} = \sum_{\mathbf{w} \in W} \frac{\mathsf{m}_+(t_\Gamma\, \boldsymbol{\gamma}) \cdot \mathsf{m}_-(t_A\, \mathbf{a})}{\mathsf{m}(\mathbf{w})}$$

where W is the set of symmetry classes in $(\!|M|\!)$ mapping to $(t_\Gamma\, \boldsymbol{\gamma}, t_A\, \mathbf{a})$, and $\mathsf{m}(\mathbf{w})$ is the size of the group of symmetries on $\mathbf{w}$.

This is because to each symmetry class $\mathbf{w}$ correspond a number of positive witnesses equal to the negative symmetries of the matching rigid intersection type, divided by the symmetries of $\mathbf{w}$. Thus, one can obtain the right coefficient from symmetry classes (and therefore for normal standard resource terms following Section 3.4.4), but the weight of each symmetry class must be corrected, suitably accounting for symmetries.

4.3 Distributors and Generalized Species

We now establish a link between thin spans and the bicategory of distributors. We keep this section succinct; to a large extent, it is a simplification of the construction in [4].

4.3.1 The bicategory of groupoids and distributors. A **distributor** from groupoid A to B is a functor $\alpha : A^{\text{op}} \times B \rightarrow \mathbf{Set}$ giving, for all $a \in A, b \in B$, a set $\alpha(a, b)$ of *witnesses*, along with an action of symmetries: if $x \in \alpha(a, b)$ and $\theta \in B(b, b')$, we write $\theta \cdot x$ for the functorial action $\alpha(\text{id}, \theta)(x) \in \alpha(a, b')$. Similarly, if $\vartheta \in A(a', a)$, we write $x \cdot \vartheta \in \alpha(a', b)$ for $\alpha(\vartheta, \text{id})$.

The bicategory $\mathbf{Dist}$ has groupoids as objects, distributors as morphisms, and natural transformations as 2-cells. The **identity distributor** on A is the hom-set functor $\text{id}_A = A[-,-] : A^{\text{op}} \times A \rightarrow \mathbf{Set}$. The **composition** of two distributors $\alpha : A^{\text{op}} \times B \rightarrow \mathbf{Set}$ and $\beta : B^{\text{op}} \times C \rightarrow \mathbf{Set}$ is defined in terms of the coend formula:

$$(\beta \bullet \alpha)(a, c) = \int^{b \in B} \alpha(a, b) \times \beta(b, c)\,,$$

meaning that concretely, $(\beta \bullet \alpha)(a, c)$ consists in pairs (x, y), where $x \in \alpha(a, b)$ and $y \in \beta(b, c)$ for some $b \in B$, quotiented by $(g \cdot x, y) \sim (x, y \cdot g)$ for $x \in \alpha(a, b), g \in B(b, b')$ and $y \in \beta(b', c)$. The bicategory $\mathbf{Dist}$ has cartesian products given by the disjoint union $A + B$.

4.3.2 Extracting distributors from thin spans. On objects, we send a thin groupoid $(A, A_-, A_+, \mathbf{U}_A, \mathbf{T}_A)$ to its underlying groupoid A.

On morphisms, given a thin span $A \leftarrow S \rightarrow B$, for all $a \in A$ and $b \in B$ we must specify a set $\|S\|(a, b)$. It is tempting to set simply the pre-image $(\partial^S)^{-1}(a, b)$, but there is no functorial action

$$\|S\|(\theta_A, \theta_B) : \|S\|(a, b) \rightarrow \|S\|(a', b')$$

for $\theta_A \in A(a', a)$ and $\theta_B \in B(b, b')$ as ∂^S is not a fibration. We need a finer symmetry lifting property of thin spans — and we have one, seen in Proposition 4.7. Thus, we set instead $\|S\|(a, b)$ as the set $\sim\text{-wit}^+_S(a, b)$ of $\sim$-**witnesses** of (a, b) in S, *i.e.* triples $(\theta^-_A, s, \theta^+_B)$ *s.t.* $s \in S$, $\theta^-_A \in A_-(a, s_A)$ and $\theta^+_B \in B_+(s_B, b)$. Though we keep the same terminology and notation as in Section 4.2.4, those are $\sim$-witnesses of *specific* objects of the groupoids A and B, not symmetry classes.

We get a functorial action by setting $\|S\|(\Omega_A, \Omega_B)(\theta^-_A, s, \theta^+_B)$ as the positive witness $(\vartheta^-_A, s', \vartheta^+_B)$ as in the statement of Proposition 4.7, yielding a distributor for every thin span $A \leftarrow S \to B$:

Proposition 4.13. *We have a distributor* $\|S\| : A^{\mathrm{op}} \times B \to \mathbf{Set}$.

4.3.3 Constructing natural transformations. Consider thin spans S, T from A to B, and $(F, F^A, F^B) : S \to T$ a positive morphism; consisting for each $s \in S$ of two symmetries $F^A_s \in A_-(s_A, (Ft)_A)$ and $F^B_s \in B_+(s_B, (Fs)_B)$.

To each $\mathrm{w} = (\theta^-_A, s, \theta^+_B) \in \|S\|(a, b)$, we set $\|S\|(F, F^A, F^B)(\mathrm{w})$ to

$$(a \overset{\theta^-_A}{\to} s_A \overset{F^A_s}{\to} (Ft)_A, \qquad Ft, \qquad (Ft)_B \overset{F^B_s}{\to} s_B \overset{\theta^+_B}{\to} b)$$

which by the uniqueness property of Proposition 4.7 can be easily verified to give a natural transformation from $\|S\|$ to $\|T\|$.

4.3.4 Further components. To complete the pseudofunctor, we need two natural isomorphisms, the *unitor* and the *compositor*.

Proposition 4.14. *Given a thin span A, there is a natural iso*

$$\mathrm{pid}^A : \|\mathrm{Id}_A\| \overset{\cong}{\Rightarrow} A[-,-] : A^{\mathrm{op}} \times A \to \mathbf{Set}\,.$$

This is straightforward from the factorization result of Lemma 2.3. Now, we focus on the preservation of composition. For two thin spans $A \leftarrow S \to B$ and $B \leftarrow T \to C$, we have the **compositor**:

Proposition 4.15. *There is a natural isomorphism:*

$$\mathrm{pcomp}^{S,T} : \|T \odot S\| \Rightarrow \|T\| \bullet \|S\| : A^{\mathrm{op}} \times B \to \mathbf{Set}\,.$$

Proof. The map $\mathrm{pcomp}^{S,T}_{a,c}$ sends $(\theta^-_A, (s, t), \theta^+_C) \in \|T \odot S\|(a, c)$ (with $s_B = t_B = b$) to (the equivalence class of) the pair

$$((\theta^-_A, s, \mathrm{id}_b), (\mathrm{id}_b, t, \theta^+_C)) \in (\|T\| \bullet \|S\|)(a, c)\,.$$

For each $a \in A$ and $c \in C$, this forms a bijection. Consider indeed

$$\mathrm{w}^S = (\theta^-_A, s, \theta^+_B) \in \|S\|(a, b) \qquad \mathrm{w}^T = (\theta^-_B, t, \theta^+_C) \in \|T\|(b, c)$$

composable witnesses. By Lemma 2.4 we compose s and t through $\theta^-_B \circ \theta^+_B$, yielding unique $\varphi^S \in S[s, s'], \varphi^T \in T[t, t'], \vartheta^-_A, \vartheta^+_C$ s.t.:

$$\begin{tikzcd}
 & s_A \arrow[dd, "\varphi^S_A"] & s_B \arrow[r, "\theta^+_B"] \arrow[dd, "\varphi^S_B"] & b \arrow[r, "\theta^-_B"] & t_B \arrow[dd, "\varphi^T_B"] & t_C \arrow[dd, "\varphi^T_C"'] \arrow[dr, "\theta^+_C"] & \\
a \arrow[ur, "\theta^-_A"] \arrow[dr, "\vartheta^-_A"'] & & & & & & c \\
 & s'_A & s'_B \arrow[r, equal] & b' \arrow[r, equal] & t'_B & t'_C \arrow[ur, "\vartheta^+_C"'] &
\end{tikzcd}$$

which, writing $\Theta_B = \varphi^S_B \circ {\theta^+_B}^{-1} = \varphi^T_B \circ \theta^-_B$, entails

$$\begin{array}{rcccl} \mathrm{v}^S & = & (\vartheta^-_A, s', \mathrm{id}_{y_B}) & = & \Theta_B \cdot (\theta^-_A, s, \theta^+_B) \\ \mathrm{v}^T & = & (\mathrm{id}_{y_B}, t', \vartheta^+_C) & = & (\theta^-_B, t, \theta^+_C) \cdot \Theta_B \end{array}$$

so $(\mathrm{v}^S, \mathrm{v}^T) = (\Theta_B \cdot \mathrm{w}^S, \mathrm{v}^T) \sim (\mathrm{w}^S, \mathrm{v}^T \cdot \Theta_B) = (\mathrm{w}^S, \mathrm{w}^T)$. Now $(\mathrm{v}^S, \mathrm{v}^T) = \mathrm{pcomp}^{S,T}(\vartheta^-_A, t' \odot s', \vartheta^+_C)$, showing surjectivity — injectivity also follows from the uniqueness clause in Lemma 2.4. □

The naturality and coherence requirements hold, and altogether:

Theorem 4.16. *This yields a peudofunctor* $\|-\| : \mathbf{Thin} \to \mathbf{Dist}$.

4.3.5 Lifting to Kleisli bicategories. Recall that $\mathbf{Esp}$ is the Kleisli bicategory $\mathbf{Dist}_{\mathbf{Sym}}$. Composition of $F : \mathbf{Sym}(A)^{\mathrm{op}} \times B \to \mathbf{Set}$ and $G : \mathbf{Sym}(B)^{\mathrm{op}} \times C \to \mathbf{Set}$ is $G \bullet F^{\mathbf{Sym}}$, where the **promotion** is

$$F^{\mathbf{Sym}}(\vec{a}, \langle b_1, \ldots, b_n\rangle) = \int^{\vec{a}'_1, \ldots, \vec{a}'_n} A[\vec{a}, \vec{a}'_1 \ldots \vec{a}'_n] \times \Pi^n_{i=1} F(\vec{a}'_i, b_i)$$

comprising a morphism in $A[\vec{a}, \vec{a}'_1, \ldots, \vec{a}'_n]$ along with a family in $\Pi^n_{i=1} F(\vec{a}'_i, b_i)$, quotiented by an equivalence relation.

Likewise, the promotion $S^{\mathbf{Sym}}$ of a thin span, constructed as

$$\mathbf{Sym}(A) \leftarrow \mathbf{Sym}(\mathbf{Sym}(A)) \leftarrow \mathbf{Sym}(S) \to \mathbf{Sym}(B)\,,$$

yields by $\|-\|$ the distributor associating to $(\vec{a}, \langle b_1, \ldots, b_n\rangle)$, triples

$$(\theta^-_{\mathbf{Sym}(A)}, \langle s_1, \ldots, s_n\rangle, \theta^+_{\mathbf{Sym}(B)}) \in \|S^{\mathbf{Sym}}\|(\vec{a}, \vec{b})\,, \tag{10}$$

but $\theta^+_{\mathbf{Sym}(B)}$ is positive, so cannot reindex the b_is and must be $(\mathrm{id}_{1\ldots n}, (\theta^+_i)_{1 \le i \le n})$ for θ^+_i is positive in B. So we map (10) to

$$(\theta^-_{\mathbf{Sym}(A)}, \langle(\mathrm{id}, s_i, \theta^+_i) \mid 1 \le i \le n\rangle) \in \|S\|^{\mathbf{Sym}}(\vec{a}, \vec{b})$$

inducing a natural bijection $\|S^{\mathbf{Sym}}\|_{\vec{a},\vec{b}} \simeq \|S\|^{\mathbf{Sym}}_{\vec{a},\vec{b}}$.

Combined with $\mathrm{pcomp}^{S,T}$ this provides a natural iso for preservation of Kleisli composition. Together with a straightforward natural isomorphism for Kleisli identity laws and lengthy verifications for coherence, we obtain a pseudofunctor $\|-\| : \mathbf{Thin}_! \to \mathbf{Esp}$.

4.3.6 A cartesian closed pseudofunctor. We check that this extends to a *cc-pseudofunctor* [13]. First, $\|-\|$ preserves constructions on objects strictly. The notion of a *fp-pseudofunctor* [13] requires that for each $(A_i)_{1 \le i \le n}$, $\langle\|\pi_1\|, \ldots, \|\pi_n\|\rangle$ is part of an adjoint equivalence

$$\begin{tikzcd}
\prod^n_{i=1} A_i \arrow[rr, bend left, "\langle\|\pi_1\|, \ldots, \|\pi_n\|\rangle"] \arrow[rr, phantom, "\perp"] & & \prod^n_{i=1} A_i \arrow[ll, bend left, "\mathrm{q}^\times_{\vec{A}}"]
\end{tikzcd}$$

in $\mathbf{Esp}$: here $\mathrm{q}^\times$ can be taken to be the identity in $\mathbf{Esp}$, completed to an adjoint equivalence in the obvious way. On top of that, the definition of a *cc-pseudofunctor* [13] then additionally requires that $\mathrm{e}_{A,B} = \Lambda(\|\mathbf{ev}_{A,B}\| \bullet_{\mathbf{Sym}} \mathrm{q}^\times) : A \Rightarrow B \to A \Rightarrow B$ is also part of

$$\begin{tikzcd}
A \Rightarrow B \arrow[rr, bend left, "\mathrm{e}_{A,B}"] \arrow[rr, phantom, "\perp"] & & A \Rightarrow B \arrow[ll, bend left, "\mathrm{q}^\Rightarrow_{A,B}"]
\end{tikzcd}$$

an adjoint equivalence. But $\mathrm{e}_{A,B}$ can be computed to be naturally isomorphic to the identity on $A \Rightarrow B$ in $\mathbf{Esp}$; constructing the adjoint equivalence is then straightforward. Altogether:

Theorem 4.17. $\|-\| : \mathbf{Thin}_! \to \mathbf{Esp}$ *is a cc-pseudofunctor.*

4.3.7 Consequences. Fix a simply-typed λ-term $\Gamma \vdash M : A$.

By Theorem 4.17, we have a natural isomorphism $I : [\![M]\!]_{\mathbf{Esp}} \cong \|(\!|M|\!)\|$ showing that up to iso, generalized species of structure compute positive witnesses in the sense of thin spans of groupoids.

By the results of Section 3, this can be reformulated as:

Corollary 4.18. *For* $\gamma \in (\!|\Gamma|\!)$ *and* $a \in (\!|A|\!)$*, we have a bijection*

$$[\![M]\!]_{\mathbf{Esp}}(\gamma, a) \cong \left\{ (\theta^-_\Gamma, \mathbf{w}, \theta^+_A) \;\middle|\; \begin{array}{l} \theta^-_\Gamma \in \mathbf{IT}_-(\Gamma)[K_\Gamma(s_\Gamma\,\gamma), \Theta], \\ \mathbf{w} \in \mathbf{IT}(M)_{\Theta,\alpha}, \\ \theta^+_A \in \mathbf{IT}_+(A)[\alpha, K_A\,a] \end{array} \right\}.$$

This captures the interpretation of simply-typed λ-terms in **Esp** syntactically. This is analogous to results by Tsukada *et al.* [28] and Olimpieri [25], except our derivations are simpler, without quotient.

Finally, altogether, the isomorphism I and Corollary 4.11 entail:

Corollary 4.19. *For any $\boldsymbol{\gamma} \in [\![\Gamma]\!]_{\mathbf{WRel}_!}$ and $\mathbf{a} \in [\![A]\!]_{\mathbf{WRel}_!}$,*

$$([\![M]\!]_{\mathbf{WRel}_!})_{\boldsymbol{\gamma},\mathbf{a}} = \frac{\#[\![M]\!]_{\mathbf{Esp}}(t_\Gamma\, \boldsymbol{\gamma}, t_A\, \mathbf{a})}{\mathsf{m}_-(t_\Gamma\, \boldsymbol{\gamma}) \cdot \mathsf{m}_+(t_A\, \mathbf{a})}$$

where $\#[\![M]\!]_{\mathbf{Esp}}(t_\Gamma\, \boldsymbol{\gamma}, t_A\, \mathbf{a})$ is defined for any representative.

This is independent of $\mathbf{Thin}_!$, though it does require the positive and negative symmetries — this shows that these are fundamental in quantitative semantics, independently of their role in **Thin**.

5 CONCLUSION

We have illustrated our results on the simply-typed λ-calculus for the economy of presentation and since it already features the phenomena of interest, but **Thin** readily supports non-determinism and can be easily extended with quantitative (probabilistic and quantum) primitives, for which we expect our results still hold.

Our results show that the interpretation of the simply-typed λ-calculus in **Thin** can be regarded as a rigid Taylor expansion. Section 3.4.4 then suggests a link with the standard Taylor expansion of λ-terms which may illuminate the coefficients appearing there; however we could not find an exposition of the simply-typed Taylor expansion in the literature, so we had to omit this by lack of space. Detailing that, and the untyped case, is left for future work.

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