The Biequivalence of Locally Cartesian Closed Categories and Martin-Löf Type Theories

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Abstract. Seely's paper *Locally cartesian closed categories and type theory* contains a well-known result in categorical type theory: that the category of locally cartesian closed categories is equivalent to the category of Martin-Löf type theories with Π, Σ, and extensional identity types. However, Seely's proof relies on the problematic assumption that substitution in types can be interpreted by pullbacks. Here we prove a corrected version of Seely's theorem: that the Bénabou-Hofmann interpretation of Martin-Löf type theory in locally cartesian closed categories yields a biequivalence of 2-categories. To facilitate the technical development we employ categories with families as a substitute for syntactic Martin-Löf type theories. As a second result we prove that if we remove Π-types the resulting categories with families are biequivalent to left exact categories.

1 Introduction

It is "well-known" that locally cartesian closed categories (lcccs) are equivalent to Martin-Löf's intuitionistic type theory [8,9]. But how well-known is it really? Seely's original proof [11] contains a flaw, and the papers by Curien [3] and Hofmann [5] who address this flaw only show that Martin-Löf type theory can be interpreted in locally cartesian closed categories, but not that this interpretation is an equivalence of categories provided the type theory has Π, Σ, and extensional identity types. Here we complete the work and fully rectify Seely's result except that we do not prove an equivalence of categories but a *biequivalence* of 2-categories. In fact, a significant part of the endeavour has been to find an appropriate formulation of the result, and in particular to find a suitable notion analogous to Seely's "interpretation of Martin-Löf theories".

Categories with families and democracy. Seely turns a given Martin-Löf theory into a category where the objects are *closed* types and the morphisms from type A to type B are functions of type $A \to B$. Such categories are the objects of Seely's "category of Martin-Löf theories".

Instead of syntactic Martin-Löf theories we shall employ *categories with families (cwfs)* [4]. A cwf is a pair $(\mathbb{C}, T)$ where $\mathbb{C}$ is the category of contexts and explicit substitutions, and $T : \mathbb{C}^{op} \to \mathbf{Fam}$ is a functor where the $T(\Gamma)$ is the family of terms indexed by types in context Γ and $T(\gamma)$ performs the substitution of γ in types and terms. Cwf is an appropriate substitute for syntax for dependent types: its definition unfolds to a variable-free calculus of explicit substitutions [4], which is like Martin-Löf's [10,12] except that variables are encoded by projections. One advantage of this approach compared to Seely's is that we get a natural definition of morphism of cwfs, which preserves the structure of cwfs up to isomorphism. In contrast Seely's notion of "interpretation of Martin-Löf theories" is defined indirectly via the interpretation of a Martin-Löf theory in an lccc and basically amounts to a functor preserving structure between the corresponding lcccs, rather than directly as something which preserves all the "structure" of Martin-Löf theories.

To prove our biequivalences we must require that our cwfs are *democratic*. This means that each context is represented by a type. Our results require us to build local cartesian closed structure in the category of contexts. To this end we use available constructions on types and terms, and by democracy such constructions can be moved back and forth to the category of contexts. Since Seely works with closed types only he has no need for democracy.

The coherence problem. Seely interprets type substitution in Martin-Löf theories as pullbacks in lcccs. However, this is problematic, since type substitution is already defined by induction on the structure of types, and thus fixed by the interpretation of the other constructs of type theory. It is not clear that the pullbacks can be chosen to coincide with this interpretation.

In the paper *Substitution up to isomorphism* [3] Curien described the fundamental nature of this coherence problem. He sets out to

> ... to solve a difficulty arising from a mismatch between syntax and semantics: in locally cartesian closed categories, substitution is modelled by pullbacks (more generally pseudo-functors), that is, only up to isomorphism, unless split fibrational hypotheses are imposed. ... but not all semantics do satisfy them, and in particular not the general description of the interpretation in an arbitrary locally cartesian closed category. In the general case, we have to show that the isomorphisms between types arising from substitution are *coherent* in a sense familiar to category theorists.

To solve the problem Curien introduced a calculus with explicit substitutions for Martin-Löf type theory, with special terms witnessing applications of the type equality rule. In this calculus type equality can be interpreted as isomorphism in lcccs. The remaining coherence problem is to show that Curien's calculus is equivalent to the usual formulation of Martin-Löf type theory, and Curien proves this result by cut-elimination.

Somewhat later, Hofmann [5] gave an alternative solution based on a construction of Bénabou [1] which was used to prove that each fibration is equivalent to a split fibration. Hofmann showed how to construct a model of Martin-Löf type theory (category with attributes in the sense of Cartmell [2]) with Π-types,

Σ-types, and (extensional) identity types from a locally cartesian closed category. Categories with attributes (cwas) provide a split notion of model, hence the relevance of Bénabou's construction.

However, Seely wanted to prove an equivalence of categories. Hofmann conjectures [5]:

> We have now constructed a cwa over $\mathcal{C}$ which can be shown to be equivalent to $\mathcal{C}$ in some suitable 2-categorical sense.

Here we spell out and prove this result, and thus fully rectify Seely's theorem. It should be apparent from what follows that this is not a trivial exercise.

While carrying out the proof we noticed that if we remove Π-types the resulting 2-category of cwfs is biequivalent to left exact (or finitely complete) categories. We present this result in parallel with the main result.

Plan of the paper. An equivalence of categories consists of a pair of functors which are inverses up to natural isomorphism. Biequivalence is the appropriate notion of equivalence for bicategories [7]. Instead of functors we have *pseudofunctors* which only preserve identity and composition up to isomorphism. Instead of natural isomorphisms we have *pseudonatural transformations* which are inverses up to *invertible modification.*

A 2-category is a strict bicategory, and the remainder of the paper consists of constructing two biequivalences of 2-categories. In Section 2 we introduce cwfs and show how to turn a cwf into an indexed category. In Section 3 we define the 2-categories $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$ of democratic cwfs which support extensional identity types and Σ-types and $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$ which also support Π-types. Here we also define the notions of pseudo cwf-morphism and pseudo cwf-transformation. In Section 4 we define the 2-categories $\mathbf{FL}$ of left exact categories and $\mathbf{LCC}$ of locally cartesian closed categories. We show that there are forgetful 2-functors $U : \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma} \to \mathbf{FL}$ and $U : \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi} \to \mathbf{LCC}$. In section 5 we construct the pseudofunctors $H : \mathbf{FL} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$ and $H : \mathbf{LCC} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$ based on the Bénabou-Hofmann construction. In section 6 we prove that H and U give rise to the biequivalences of $\mathbf{FL}$ and $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$ and of $\mathbf{LCC}$ and $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$.

2 Categories with Families

2.1 Definition

Definition 1. *Let* **Fam** *be the category of families of sets defined as follows. An object is a family* $(B(x))_{x\in A}$*, and a morphism with source* $(B(x))_{x\in A}$ *and target* $(B'(x))_{x\in A'}$ *is a pair consisting of a function* $f : A \to A'$ *and a family of functions* $g(x) : B(x) \to B'(f(x))$ *indexed by* $x \in A$*.*

Definition 2. *A* ***category with families (cwf)*** *consists of the following data:*

- *A base category* $\mathbb{C}$*. Its objects are called* contexts *and its morphisms are called* substitutions*. The identity map is denoted by* $id : \Gamma \to \Gamma$ *and the* *composition of maps* $\gamma : \Delta \to \Gamma$ *and* $\delta : \Xi \to \Delta : \Xi \to \Gamma$ *is denoted by* $\gamma \circ \delta$ *or more briefly by* $\gamma\delta : \Xi \to \Gamma$.

– *A functor* $T : \mathbb{C}^{op} \to \mathbf{Fam}$. *We write* $T(\Gamma) = (\Gamma \vdash A)_{A \in \mathrm{Type}(\Gamma)}$, *where* Γ *is an object of* $\mathbb{C}$, *and call it the family of* terms *indexed by* types A *in context* Γ. *Moreover, if* γ *is a morphism in* $\mathbb{C}$, *we write* $A[\gamma]$ *for* type substitution, *the application of the first component of* $T(\gamma)$ *to a type* A, *and* $a[\gamma]$ *for* term substitution, *the application of the second component on a term* a.
– *A* terminal object $[]$ *of* $\mathbb{C}$ *called* the empty context *with terminal map* $\langle\rangle : \Delta \to []$ *called* the empty substitution.
– *A* context comprehension *which to an object* Γ *in* $\mathbb{C}$ *and a type* $A \in \mathrm{Type}(\Gamma)$ *associates an object* $\Gamma \cdot A$ *of* $\mathbb{C}$, *a morphism* $\mathrm{p}_A : \Gamma \cdot A \to \Gamma$ *of* $\mathbb{C}$ *and a term* $\mathrm{q} \in \Gamma \cdot A \vdash A[\mathrm{p}]$ *such the following universal property holds: for each object* Δ *in* $\mathbb{C}$, *morphism* $\gamma : \Delta \to \Gamma$, *and term* $a \in \Delta \vdash A[\gamma]$, *there is a unique morphism* $\theta = \langle \gamma, a \rangle : \Delta \to \Gamma \cdot A$, *such that* $\mathrm{p}_A \circ \theta = \gamma$ *and* $\mathrm{q}[\theta] = a$.

The definition of cwf can be presented as a system of axioms and inference rules for a variable-free generalized algebraic formulation of the most basic rules of dependent type theory [4]. The correspondence with standard syntax is discussed by Hofmann [6]. We shall now define what it means that a cwf supports extra structure corresponding to the rules for the various type formers of Martin-Löf type theory.

Definition 3. *A cwf* supports (extensional) identity types *iff the following conditions hold:*

Form. *If* $A \in \mathrm{Type}(\Gamma)$ *and* $a, a' : \Gamma \vdash A$, *there is* $\mathrm{I}_A(a, a')$;
Intro. *If* $a : \Gamma \vdash A$, *there is* $\mathrm{r}_{A,a} : \Gamma \vdash \mathrm{I}_A(a, a)$;
Elim. *If* $c : \Gamma \vdash \mathrm{I}_A(a, a')$ *then* $a = a'$ *and* $c = \mathrm{r}_{A,a}$.

Moreover, we have stability under substitution: if $\delta : \Delta \to \Gamma$ *then*

$$\mathrm{I}_A(a, a')[\delta] = \mathrm{I}_{A[\delta]}(a[\delta], a'[\delta])$$
$$\mathrm{r}_{A,a}[\delta] = \mathrm{r}_{A[\delta],a[\delta]}$$

Definition 4. *A cwf* supports Σ-types *iff the following conditions hold:*

Form. *If* $A \in \mathrm{Type}(\Gamma)$ *and* $B \in \mathrm{Type}(\Gamma \cdot A)$, *there is* $\Sigma(A, B) \in \mathrm{Type}(\Gamma)$,
Intro. *If* $a : \Gamma \vdash A$ *and* $b : \Gamma \vdash B[\langle \mathrm{id}, a \rangle]$, *there is* $\mathrm{pair}(a, b) : \Gamma \vdash \Sigma(A, B)$,
Elim. *If* $a : \Gamma \vdash \Sigma(A, B)$, *there are* $\pi_1(a) : \Gamma \vdash A$ *and* $\pi_2(a) : \Gamma \vdash B[\langle \mathrm{id}, \pi_1(a) \rangle]$ *such that*[1]*:*

$$\pi_1(\mathrm{pair}(a, b)) = a$$
$$\pi_2(\mathrm{pair}(a, b)) = b$$
$$\mathrm{pair}(\pi_1(c), \pi_2(c)) = c$$

[1] Note that in a cwf which supports extensional identity types and Σ-types surjective pairing follows from the other conditions [9].

Moreover, we have stability under substitution:

$$
\begin{aligned}
\Sigma(A,B)[\delta] &= \Sigma(A[\delta], B[\langle \delta \circ \mathrm{p}, \mathrm{q}\rangle]) \\
\mathrm{pair}(a,b)[\delta] &= \mathrm{pair}(a[\delta], b[\delta]) \\
\pi_1(c)[\delta] &= \pi_1(c[\delta]) \\
\pi_2(c)[\delta] &= \pi_2(c[\delta])
\end{aligned}
$$

Definition 5. *A cwf* supports Π-types *iff the following conditions hold:*

Form. *If* $A \in \mathrm{Type}(\Gamma)$ *and* $B \in \mathrm{Type}(\Gamma \cdot A)$*, there is* $\Pi(A,B) \in \mathrm{Type}(\Gamma)$*.*
Intro. *If* $b : \Gamma \cdot A \vdash B$*, there is* $\lambda(b) : \Gamma \vdash \Pi(A,B)$*.*
Elim. *If* $c : \Gamma \vdash \Pi(A,B)$ *and* $a : \Gamma \vdash A$ *then there is a term* $\mathrm{ap}(c,a) : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$ *such that*

$$
\begin{aligned}
\mathrm{ap}(\lambda(b), a) &= b[\langle \mathrm{id}, a\rangle] : \Gamma \vdash B[\langle \mathrm{id}, a\rangle] \\
c &= \lambda(\mathrm{ap}(c[\mathrm{p}], \mathrm{q})) : \Gamma \vdash \Pi(A,B)
\end{aligned}
$$

Moreover, we have stability under substitution:

$$
\begin{aligned}
\Pi(A,B)[\gamma] &= \Pi(A[\gamma], B[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\
\lambda(b)[\gamma] &= \lambda(b[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\
\mathrm{ap}(c,a)[\gamma] &= \mathrm{ap}(c[\gamma], a[\gamma])
\end{aligned}
$$

Definition 6. *A cwf* $(\mathbb{C}, T)$ *is* democratic *iff for each object* Γ *of* $\mathbb{C}$ *there is* $\overline{\Gamma} \in \mathrm{Type}([])$ *and an isomorphism* $\Gamma \cong_{\gamma_\Gamma} [] \cdot \overline{\Gamma}$*. Each substitution* $\delta : \Delta \to \Gamma$ *can then be represented by a term* $\overline{\delta} = q[\gamma_\Gamma \delta \gamma_\Delta^{-1}] : [] \cdot \overline{\Delta} \vdash \overline{\Gamma}[p]$*.*

Democracy is needed to prove our results, but does not correspond to a construction of Martin-Löf type theory.

2.2 The Indexed Category of Types in Context

We shall now define the indexed category associated with a cwf. This will play a crucial role and in particular introduce the notion of *isomorphism* of types.

Proposition 7 (The Context-Indexed Category of Types). *If* $(\mathbb{C}, T)$ *is a cwf, then we can define a functor* $\boldsymbol{T} : \mathbb{C}^{op} \to \mathbf{Cat}$ *as follows:*

- *The objects of* $\boldsymbol{T}(\Gamma)$ *are types in* $\mathrm{Type}(\Gamma)$*. If* $A, B \in \mathrm{Type}(\Gamma)$*, then a morphism in* $\boldsymbol{T}(\Gamma)(A,B)$ *is a morphism* $\delta : \Gamma \cdot A \to \Gamma \cdot B$ *in* $\mathbb{C}$ *such that* $\mathrm{p}\delta = \mathrm{p}$*.*
- *If* $\gamma \in \mathbb{C}(\Delta, \Gamma)$*, then* $\boldsymbol{T}(\gamma) : \mathrm{Type}(\Gamma) \to \mathrm{Type}(\Delta)$ *maps an object* $A \in \mathrm{Type}(\Gamma)$ *to* $A[\gamma]$ *and a morphism* $\delta : \Gamma \cdot A \to \Gamma \cdot B$ *to* $\langle \mathrm{p}, \mathrm{q}[\delta \langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]\rangle : \Delta \cdot A[\gamma] \to \Delta \cdot B[\gamma]$*.*

We write $A \cong_\theta B$ if $\theta : A \to B$ is an isomorphism in $\boldsymbol{T}(\Gamma)$. If $a \in \Gamma \vdash A$, we write $\{\theta\}(a) = q[\theta\langle id, a\rangle] : \Gamma \vdash B$ for the *coercion* of a to type B and $a =_\theta b$ if $a = \{\theta\}(b)$. Moreover, we get an alternative formulation of democracy.

Proposition 8. $(\mathbb{C}, T)$ *is democratic iff the functor from* $\boldsymbol{T}([])$ *to* $\mathbb{C}$*, which maps a closed type* A *to the context* $[] \cdot A$*, is an equivalence of categories.*

Seely's category **ML** of Martin-Löf theories [11] is essentially the category of categories $\boldsymbol{T}([])$ of closed types.

Fibres, slices and lcccs. Seely's interpretation of type theory in lcccs relies on the idea that a type $A \in \mathrm{Type}(\Gamma)$ can be interpreted as its *display map*, that is, a morphism with codomain Γ. For instance, the type $\mathtt{list}(n)$ of lists of length $n : \mathtt{nat}$ would be mapped to the function $l : \mathtt{list} \to \mathtt{nat}$ which to each list associates its length. Hence, types and terms over context Γ are interpreted in the *slice category* $\mathbb{C}/\Gamma$, since terms are interpreted as global sections. Syntactic types are connected with types-as-display-maps by the following result, an analogue of which was one of the cornerstones of Seely's paper.

Proposition 9. *If* $(\mathbb{C}, T)$ *is democratic and supports extensional identity and* Σ*-types, then* $\boldsymbol{T}(\Gamma)$ *and* $\mathbb{C}/\Gamma$ *are equivalent categories for all* Γ.

Proof. To each object (type) A in $\boldsymbol{T}(\Gamma)$ we associate the object p_A in $\mathbb{C}/\Gamma$. A morphism from A to B in $\boldsymbol{T}(\Gamma)$ is by definition a morphism from p_A to p_B in $\mathbb{C}/\Gamma$.

Conversely, to each object (morphism) $\delta : \Delta \to \Gamma$ of $\mathbb{C}/\Gamma$ we associate a type in $\mathrm{Type}(\Gamma)$. This is the inverse image $x : \Gamma \vdash \mathrm{Inv}(\delta)(x)$ which is defined type-theoretically by

$$\mathrm{Inv}(\delta)(x) = \Sigma y : \overline{\Delta}.\mathrm{I}_{\overline{\Gamma}}(\overline{x}, \overline{\delta}(y))$$

written in ordinary notation. In cwf combinator notation it becomes

$$\mathrm{Inv}(\delta) = \Sigma(\overline{\Delta}[\langle\rangle], \mathrm{I}_{\overline{\Gamma}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}], \overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \in \mathrm{Type}(\Gamma)$$

These associations yield an equivalence of categories since $\mathrm{p}_{\mathrm{Inv}(\delta)}$ and δ are isomorphic in $\mathbb{C}/\Gamma$.

It is easy to see that $\boldsymbol{T}(\Gamma)$ has binary products if the cwf supports Σ-types and exponentials if it supports Π-types. Simply define $A \times B = \Sigma(A, B[\mathrm{p}])$ and $B^A = \Pi(A, B[\mathrm{p}])$. Hence by Proposition 9 it follows that $\mathbb{C}/\Gamma$ has products and $\mathbb{C}$ has finite limits in any democratic cwf which supports extensional identity types and Σ-types. If it supports Π-types too, then $\mathbb{C}/\Gamma$ is cartesian closed and $\mathbb{C}$ is locally cartesian closed.

3 The 2-Category of Categories with Families

3.1 Pseudo Cwf-Morphisms

A notion of *strict cwf-morphism* between cwfs $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ was defined by Dybjer [4]. It is a pair (F, σ), where $F : \mathbb{C} \to \mathbb{C}'$ is a functor and $\sigma : T \xrightarrow{\bullet} T'F$ is a natural transformation of family-valued functors, such that terminal objects and context comprehension are preserved on the nose. Here we need a weak version where the terminal object, context comprehension, and substitution of types and terms of a cwf are only preserved up to isomorphism. The pseudo-natural transformations needed to prove our biequivalences will be families of cwf-morphisms which do not preserve cwf-structure on the nose.

The definition of pseudo cwf-morphism will be analogous to that of *strict* cwf-morphism, but cwf-structure will only be preserved up to coherent isomorphism.

Definition 10. *A **pseudo cwf-morphism** from* $(\mathbb{C}, T)$ *to* $(\mathbb{C}', T')$ *is a pair* (F, σ) *where:*

- $F : \mathbb{C} \to \mathbb{C}'$ *is a functor,*
- *For each context* Γ *in* $\mathbb{C}$*,* σ_Γ *is a* **Fam***-morphism from* $T\Gamma$ *to* $T'F\Gamma$*. We will write* $\sigma_\Gamma(A) : \mathrm{Type}'(F\Gamma)$ *for the type component and* $\sigma_\Gamma^A(a) : F\Gamma \vdash \sigma_\Gamma(A)$ *for the term component of this morphism.*

The following preservation properties must be satisfied:

- *Substitution is preserved: For each context* $\delta : \Delta \to \Gamma$ *in* $\mathbb{C}$ *and* $A \in \mathrm{Type}(\Gamma)$*, there is an isomorphism of types* $\theta_{A,\delta} : \sigma_\Gamma(A)[F\delta] \to \sigma_\Delta(A[\delta])$ *such that substitution on terms is also preserved, that is,* $\sigma_\Delta^{A[\gamma]}(a[\gamma]) =_{\theta_{A,\gamma}} \sigma_\Gamma^A(a)[F\gamma]$.
- *The terminal object is preserved:* $F[]$ *is terminal.*
- *Context comprehension is preserved:* $F(\Gamma \cdot A)$ *with the projections* $F(\mathrm{p}_A)$ *and* $\{\theta^{-1}_{A,\mathrm{p}}\}(\sigma_{\Gamma\cdot A}^{A[p]}(\mathrm{q}_A))$ *is a context comprehension of* $F\Gamma$ *and* $\sigma_\Gamma(A)$*. Note that the universal property on context comprehensions provides an unique isomorphism* $\rho_{\Gamma,A} : F(\Gamma\cdot A) \to F\Gamma\cdot\sigma_\Gamma(A)$ *which preserves projections.*

These data must satisfy naturality and coherence laws which amount to the fact that if we extend σ_Γ *to a functor* $\boldsymbol{\sigma}_\Gamma : \boldsymbol{T}(\Gamma) \to \boldsymbol{T'}F(\Gamma)$*, then* $\boldsymbol{\sigma}$ *is a pseudo natural transformation from* $\boldsymbol{T}$ *to* $\boldsymbol{T'}F$*. This functor is defined by* $\boldsymbol{\sigma}_\Gamma(A) = \sigma_\Gamma(A)$ *on an object* A *and* $\boldsymbol{\sigma}_\Gamma(f) = \rho_{\Gamma,B}F(f)\rho^{-1}_{\Gamma,A}$ *on a morphism* $f : A \to B$*.*

A consequence of this definition is that all cwf structure is preserved.

Proposition 11. *Let* (F, σ) *be a pseudo cwf-morphism from* $(\mathbb{C}, T)$ *to* $(\mathbb{C}', T')$*.*

(1) Then substitution extension is preserved: for all $\delta : \Delta \to \Gamma$ *in* $\mathbb{C}$ *and* $a : \Delta \vdash A[\delta]$*, we have* $F(\langle\delta, a\rangle) = \rho^{-1}_{\Gamma,A}\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma_\Delta^{A[\delta]}(a))\rangle$.
(2) Redundancy terms/sections: for all $a \in \Gamma \vdash A$*,* $\sigma_\Gamma^A(a) = \mathrm{q}[\rho_{\Gamma,A}F(\langle \mathrm{id}, a\rangle)]$.

If $(F, \sigma) : (\mathbb{C}_0, T_0) \to (\mathbb{C}_1, T_1)$ and $(G, \tau) : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ are two weak cwf-morphisms, we define their composition $(G, \tau)(F, \sigma)$ as $(GF, \tau\sigma)$ where:

$$(\tau\sigma)_\Gamma(A) = \tau_{F\Gamma}(\sigma_\Gamma(A))$$
$$(\tau\sigma)_\Gamma^A(a) = \tau_{F\Gamma}^{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a))$$

The families θ^{GF} and ρ^{GF} are obtained from θ^F, θ^G and ρ^F and ρ^G in the obvious way. The fact that these data satisfy the necessary coherence and naturality conditions basically amounts to the stability of pseudonatural transformation under composition. There is of course an identity pseudo cwf-morphisms whose components are all identities, which is obviously neutral for composition. So, there is a category of cwfs and pseudo cwf-morphisms.

Since the isomorphism $(\Gamma\cdot A)\cdot B \cong \Gamma\cdot\Sigma(A, B)$ holds in an arbitrary cwf which supports Σ-types, it follows that pseudo cwf-morphisms automatically preserve Σ-types, since they preserve context comprehension. However, if cwfs support other structure, we need to define what it means that cwf-morphisms preserve this extra structure up to isomorphism.

Definition 12. *Let (F, σ) be a pseudo cwf-morphism between cwfs $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ which support identity types, Π-types, and democracy, respectively.*

- (F, σ) preserves identity types *provided* $\sigma_\Gamma(\mathrm{I}_A(a, a')) \cong \mathrm{I}_{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a), \sigma_\Gamma^A(a))$*;*
- (F, σ) preserves Π-types *provided* $\sigma_\Gamma(\Pi(A, B)) \cong \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])$*;*
- (F, σ) preserves democracy *provided* $\sigma_{[]}(\overline{\Gamma}) \cong_{d_\Gamma} \overline{F\Gamma}[\langle\rangle]$*, and the following diagram commutes:*

$$\begin{tikzcd} F\Gamma \arrow[rr, "F\gamma_\Gamma"] \arrow[d, "\gamma_{F\gamma}"'] & & F([]\cdot\overline{\Gamma}) \arrow[d, "\rho_{[],\overline{\Gamma}}"] \\ {[]\cdot\overline{F\Gamma}} & F[]\cdot\overline{F\Gamma}[\langle\rangle] \arrow[l, "\langle\langle\rangle,\mathrm{q}\rangle"'] & F[]\cdot\sigma_{[]}(\overline{\Gamma}) \arrow[l, "d_\Gamma"'] \end{tikzcd}$$

These preservation properties are all stable under composition and thus yield several different 2-categories of structure-preserving pseudo cwf-morphisms.

3.2 Pseudo Cwf-Transformations

Definition 13 (Pseudo cwf-transformation). *Let (F, σ) and (G, τ) be two cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$. A* pseudo cwf-transformation *from (F, σ) to (G, τ) is a pair (ϕ, ψ) where $\phi : F \xrightarrow{\bullet} G$ is a natural transformation, and for each Γ in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$, a morphism $\psi_{\Gamma,A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ in $\boldsymbol{T'}(F\Gamma)$, natural in A and such that the following diagram commutes:*

$$\begin{tikzcd} \sigma_\Gamma(A)[F\delta] \arrow[r, "\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A})"] \arrow[d, "\theta_{A,\delta}"] & \tau_\Gamma(A)[\phi_\Gamma F(\delta)] \arrow[d, "\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})"] \\ \sigma_\Delta(A[\delta]) \arrow[r, "\psi_{\Delta,A[\delta]}"'] & \tau_\Delta(A[\delta])[\phi_\Delta] \end{tikzcd}$$

where θ and θ' are the isomorphisms witnessing preservation of substitution in types in the definition of pseudo cwf-morphism.

Pseudo cwf-transformations can be composed both vertically (denoted by $(\phi', \psi')(\phi, \psi)$) and horizontally (denoted by $(\phi', \psi') \star (\phi, \psi)$), and these compositions are associative and satisfy the interchange law. Note that just as coherence and naturality laws for pseudo cwf-morphisms ensure that they give rise to pseudonatural transformations (hence morphisms of indexed categories) $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$, this definition exactly amounts to the fact that pseudo cwf-transformations between (F, σ) and (F, τ) correspond to modifications from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$.

3.3 2-Categories of Cwfs with Extra Structure

Definition 14. *Let $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ be the 2-category of small democratic categories with families which support extensional identity types and Σ-types. The 1-cells are cwf-morphisms preserving democracy and extensional identity types (and Σ-types automatically) and the 2-cells are pseudo cwf-transformations.*

Moreover, let $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ be the sub-2-category of $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ where also Π-types are supported and preserved.

4 Forgetting Types and Terms

Definition 15. *Let* **FL** *be the 2-category of small categories with finite limits (left exact categories). The* 1*-cells are functors preserving finite limits (up to isomorphism) and the* 2*-cells are natural transformations.*

Let **LCC** *be the 2-category of small locally cartesian closed categories. The* 1*-cells are functors preserving local cartesian closed structure (up to isomorphism), and the* 2*-cells are natural transformations.*

FL is a sub(2-)category of the 2-category of categories: we do not provide a choice of finite limits. Similarly, **LCC** is a sub(2-)category of **FL**. The first component of our biequivalences will be *forgetful* 2-functors.

Proposition 16. *The forgetful* 2*-functors*

$$U : \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma} \to \mathbf{FL}$$
$$U : \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma\Pi} \to \mathbf{LCC}$$

where the effect on 0-, 1-, and 2-cells are as follows:

$$U(\mathbb{C}, T) = \mathbb{C}$$
$$U(F, \sigma) = F$$
$$U(\phi, \psi) = \phi$$

are well-defined.

Proof. By definition U is a 2-functor from **CwF** to **Cat**, it remains to prove that it sends a cwf in $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma}$ to **FL** and a cwf in $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I}_{\mathrm{ext}}\Sigma\Pi}$ to **LCC**, along with the corresponding properties for 1-cells and 2-cells.

For 0-cells we already proved as corollaries of Proposition 9 that if $(\mathbb{C}, T)$ supports Σ-types, identity types and democracy, then $\mathbb{C}$ has finite limits; and if $(\mathbb{C}, T)$ also supports Π-types, then $\mathbb{C}$ is lccc.

For 1-cells we need to prove that if (F, σ) preserves identity types and democracy, then F preserves finite limits; and if (F, σ) also preserves Π-types then F preserves local exponentiation. Since finite limits and local exponentiation in $\mathbb{C}$ and $\mathbb{C}'$ can be defined by the inverse image construction, these two statements boil down to the fact that if (F, σ) preserves identity types and democracy then inverse images are preserved. Indeed we have an isomorphism $F(\Gamma \cdot \mathrm{Inv}(\delta)) \cong F\Gamma \cdot \mathrm{Inv}(F\delta)$, as can be proved by long but mostly direct calculations involving all components and coherence laws of pseudo cwf-morphisms.

There is nothing to prove for 2-cells.

5 Rebuilding Types and Terms

Now, we turn to the reverse construction. We use adapt the Bénabou-Hofmann construction to build a cwf from any finitely complete category, then generalize this operation to functors and natural transformations, and show that this gives rise to a pseudofunctor.

Proposition 17. *There are pseudofunctors*

$$H : \mathbf{FL} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$$
$$H : \mathbf{LCC} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$$

defined by

$$H\mathbb{C} = (\mathbb{C}, T_{\mathbb{C}})$$
$$HF = (F, \sigma_F)$$
$$H\phi = (\phi, \psi_\phi)$$

on 0-cells, 1-cells, and 2-cells, respectively, and where $T_{\mathbb{C}}, \sigma_F$, *and* ψ_ϕ *are defined in the following three subsections.*

Proof. The remainder of this Section contains the proof. We will in turn show the action on 0-cells, 1-cells, 2-cells, and then prove pseudo functoriality of H.

5.1 Action on 0-Cells

As explained before, it is usual (going back to Cartmell [2]) to represent a type-in-context $A \in \mathrm{Type}(\Gamma)$ in a category as a *display map* [13], that is, as an object p_A in $\mathbb{C}/\Gamma$. A term $\Gamma \vdash A$ is then represented as a section of the display map for A, that is, a morphism a such that $\mathrm{p}_A \circ a = \mathrm{id}_\Gamma$. Substitution in types is then represented by pullback. This is essentially the technique used by Seely for interpreting Martin-Löf type theory in lcccs. However, as we already mentioned, it leads to a coherence problem.

To solve this problem Hofmann [5] used a construction due to Bénabou [1], which from any fibration builds an equivalent *split* fibration. Hofmann used it to build a category with attributes (cwa) [2] from a locally cartesian closed category. He then showed that this cwa supports Π, Σ, and extensional identity types. This technique essentially amounts to associating to a type A, not only a display map, but a whole family of display maps, one for each substitution instance $A[\delta]$. In other words, we choose a pullback square for every possible substitution and this choice is split, hence solving the coherence problem. As we shall explain below this family takes the form of a functor, and we refer to it as a *functorial family.*

Here we reformulate Hofmann's construction using cwfs. See Dybjer [4] for the correspondence between cwfs and cwas.

Lemma 18. *Let* $\mathbb{C}$ *be a category with terminal object. Then we can build a democratic cwf* $(\mathbb{C}, T_{\mathbb{C}})$ *which supports* Σ*-types. If* $\mathbb{C}$ *has finite limits, then* $(\mathbb{C}, T_{\mathbb{C}})$ *also supports extensional identity types. If* $\mathbb{C}$ *is locally cartesian closed, then* $(\mathbb{C}, T_{\mathbb{C}})$ *also supports* Π*-types.*

Proof. We only show the definition of types and terms in $T_{\mathbb{C}}(\Gamma)$. This construction is essentially the same as Hofmann's [5].

A *type* in $\mathrm{Type}_{\mathbb{C}}(\Gamma)$ is a *functorial family*, that is, a functor $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ such that $\mathrm{cod} \circ \overrightarrow{A} = \mathrm{dom}$ and if $\begin{tikzcd}[column sep=small] \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, then $\overrightarrow{A}(\alpha)$ is a pullback square:

$$\begin{tikzcd} {} \arrow[r, "\overrightarrow{A}(\delta,\alpha)"] \arrow[d, "\overrightarrow{A}(\delta\alpha)"'] & {} \arrow[d, "\overrightarrow{A}(\delta)"] \\ \Omega \arrow[r, "\alpha"'] & \Delta \end{tikzcd}$$

Following Hofmann, we denote the upper arrow of the square by $\overrightarrow{A}(\delta, \alpha)$.

A *term* $a : \Gamma \vdash \overrightarrow{A}$ is a section of $\overrightarrow{A}(\mathrm{id}_\Gamma)$, that is, a morphism $a : \Gamma \to \Gamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\Gamma)a = \mathrm{id}_\Gamma$, where we have defined context extension by $\Gamma \cdot \overrightarrow{A} = \mathrm{dom}(\overrightarrow{A}(\mathrm{id}_\Gamma))$. Interpreting types as functorial families makes it easy to define substitution in types. Substitution in terms is obtained by exploiting the universal property of pullback squares, yielding a functor $T_{\mathbb{C}} : \mathbb{C}^{op} \to \mathbf{Fam}$.

Note that $(\mathbb{C}, T_{\mathbb{C}})$ is a *democratic* cwf since to any context Γ we can associate a functorial family $\widehat{\langle\rangle} : \mathbb{C}/[] \to C^{\to}$, where $\langle\rangle : \Gamma \to []$ is the terminal projection. The isomorphism $\gamma_\Gamma : \Gamma \to [] \cdot \widehat{\langle\rangle}$ is just id_Γ.

5.2 Action on 1-Cells

Suppose that $\mathbb{C}$ and $\mathbb{C}'$ have finite limits and that $F : \mathbb{C} \to \mathbb{C}'$ preserves them. As described in the previous section, $\mathbb{C}$ and $\mathbb{C}'$ give rise to cwfs $(\mathbb{C}, T_{\mathbb{C}})$ and $(\mathbb{C}', T_{\mathbb{C}'})$. In order to extend F to a pseudo cwf-morphism, we need to define, for each object Γ in $\mathbb{C}$, a $\mathbf{Fam}$-morphism $(\sigma_F)_\Gamma : T_{\mathbb{C}}(\Gamma) \to T_{\mathbb{C}'}F(\Gamma)$. Unfortunately, unless F is full, it does not seem possible to embed faithfully a functorial family $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ into a functorial family over $F\Gamma$ in $\mathbb{C}'$. However, there is such an embedding for display maps (just apply F) from which we will freely regenerate a functorial family from the obtained display map.

The "hat" construction. As remarked by Hofmann, any morphism $f : \Delta \to \Gamma$ in a category $\mathbb{C}$ with a (not necessarily split) choice of finite limits generates a functorial family $\widehat{f} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$. If $\delta : \Delta \to \Gamma$ then $\widehat{f}(\delta) = \delta^*(f)$, where $\delta^*(f)$ is obtained by taking the pullback of f along δ (δ^* is known as the *pullback functor*):

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "\delta^*(f)"'] & {} \arrow[d, "f"] \\ \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

Note that we can always choose pullbacks such that $\widehat{f}(\mathrm{id}_\Gamma) = \mathrm{id}_\Gamma^*(f) = f$. If $\begin{tikzcd}[column sep=small] \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, we define $\widehat{f}(\alpha)$ as the left square in the following diagram:

$$\begin{tikzcd}
{} \arrow[r, "\widehat{f}(\delta,\alpha)"] \arrow[d, "\widehat{f}(\delta\alpha)"'] & {} \arrow[r] \arrow[d, "\widehat{f}(\delta)"] & {} \arrow[d, "f"] \\
\Delta' \arrow[r, "\alpha"'] & \Delta \arrow[r, "\delta"'] & \Gamma
\end{tikzcd}$$

This is a pullback, since both the outer square and the right square are pullbacks.

Translation of types. The hat construction can be used to extend F to types:

$$\sigma_F(\overrightarrow{A}) = \widehat{F(\overrightarrow{A}(\mathrm{id}))}$$

Note that $F(\Gamma \cdot \overrightarrow{A}) = F(\mathrm{dom}(\overrightarrow{A}(\mathrm{id}))) = \mathrm{dom}(F(\overrightarrow{A}(\mathrm{id}))) = \mathrm{dom}(\sigma_\Gamma(\overrightarrow{A})(\mathrm{id})) = F\Gamma{\cdot}\sigma_\Gamma(\overrightarrow{A})$, so context comprehension is preserved on the nose. However, substitution on types is *not* preserved on the nose. Hence we have to define a coherent family of isomorphisms $\theta_{\overrightarrow{A},\delta}$.

Completion of cwf-morphisms. Fortunately, whenever F preserves finite limits there is a canonical way to generate all the remaining data.

Lemma 19 (Generation of isomorphisms). *Let $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ be two cwfs, $F : \mathbb{C} \to \mathbb{C}'$ a functor preserving finite limits, $\sigma_\Gamma : \mathrm{Type}(\Gamma) \to \mathrm{Type}'(F\Gamma)$ a family of functions, and $\rho_{\Gamma,A} : F(\Gamma{\cdot}A) \to F\Gamma{\cdot}\sigma_\Gamma(A)$ a family of isomorphisms such that $\mathrm{p}\rho_{\Gamma,A} = F\mathrm{p}$. Then there exists an unique choice of functions σ_Γ^A on terms and of isomorphisms $\theta_{A,\delta}$ such that (F, σ) is a pseudo cwf-morphism.*

Proof. By item *(2)* of Proposition 11, the unique way to extend σ to terms is to set $\sigma_\Gamma^A(a) = \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]$. To generate θ, we use the two squares below:

$$\begin{tikzcd}
F\Delta{\cdot}\sigma_\Gamma(A)[F\delta] \arrow[r, "\langle (F\delta)\mathrm{p}\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"'] & F\Gamma{\cdot}\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
F\Delta{\cdot}\sigma_\Delta(A[\delta]) \arrow[r, "\rho_{\Gamma,A} F(\langle \delta\mathrm{p}, \mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}"] \arrow[d, "\mathrm{p}"'] & F\Gamma{\cdot}\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}$$

The first square is a substitution pullback. The second is a pullback because F preserves finite limits and $\rho_{\Gamma,A}$ and $\rho_{\Delta,A[\delta]}$ are isomorphisms. The isomorphism $\theta_{A,\delta}$ is defined as the unique mediating morphism from the first to the second. It follows from the universal property of pullbacks that the family θ satisfies the necessary naturality and coherence conditions. There is no other choice for $\theta_{A,\delta}$, because if (F, σ) is a pseudo cwf-morphism with families of isomorphisms θ and ρ, then $\rho_{\Gamma,A} F(\langle \delta\mathrm{p}, \mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} = \langle (F\delta)\mathrm{p}, \mathrm{q}\rangle$. Hence if F preserves finite limits, $\theta_{A,\delta}$ must coincide with the mediating morphism.

Preservation of additional structure. As a pseudo cwf-morphism, (F, σ_F) automatically preserves Σ-types. Since the democratic structure of $(\mathbb{C}, T_{\mathbb{C}})$ and $(\mathbb{C}', T_{\mathbb{C}'})$ is trivial it is clear that it is preserved by (F, σ_F). To prove that it also preserves type constructors, we use the following proposition.

Proposition 20. *Let (F, σ) be a pseudo cwf-morphism between $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ supporting Σ-types and democracy. Then:*

- *If $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ both support identity types, then (F, σ) preserves identity types provided F preserves finite limits.*
- *If $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ both support Π-types, then (F, σ) preserves Π-types provided F preserves local exponentiation.*

Proof. For the first part it remains to prove that if F preserves finite limits, then (F, σ) preserves identity types. Since $a, a' \in \Gamma \vdash A$, $\mathrm{p}_{\mathrm{I}_A(a,a')} : \Gamma \cdot \mathrm{I}_A(a, a') \to \Gamma$ is an equalizer of $\langle \mathrm{id}, a \rangle$ and $\langle \mathrm{id}, a' \rangle$ and F preserves equalizers, it follows that $F(\mathrm{p}_{\mathrm{I}_A(a,a')})$ is an equalizer of $\langle \mathrm{id}, \sigma^A_\Gamma(a) \rangle$ and $\langle \mathrm{id}, \sigma^A_\Gamma(a') \rangle$, and by uniqueness of equalizers it is isomorphic to $\mathrm{I}_{\sigma_\Gamma(A)}(\sigma^A_\Gamma(a), \sigma^A_\Gamma(a'))$.

The proof of preservation of Π-types exploits in a similar way the uniqueness (up to iso) of "Π-objects" of $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$.

5.3 Action on 2-Cells

Similarly to the case of 1-cells, under some conditions a natural transformation $\phi : F \xrightarrow{\bullet} G$ where (F, σ) and (G, τ) are pseudo cwf-morphisms can be completed to a pseudo cwf-transformation (ϕ, ψ_ϕ), as stated below.

Lemma 21 (Completion of pseudo cwf-transformations). *Suppose (F, σ) and (G, τ) are pseudo cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T)$ such that F and G preserve finite limits and $\phi : F \xrightarrow{\bullet} G$ is a natural transformation, then there exists a family of morphisms $(\psi_\phi)_{\Gamma, A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ such that (ϕ, ψ_ϕ) is a pseudo cwf-transformation from (F, σ) to (G, τ).*

Proof. We set $\psi_{\Gamma, A} = \langle \mathrm{p}, \mathrm{q}[\rho'_{\Gamma, A} \phi_{\Gamma \cdot A} \rho^{-1}_{\Gamma, A}] \rangle : F\Gamma \cdot \sigma_\Gamma A \to F\Gamma \cdot \tau_\Gamma(A)[\phi_\Gamma]$. To check the coherence law, we apply the universal property of a well-chosen pullback square (exploiting the fact that G preserves finite limits).

This completion operation on 2-cells commutes with units and both notions of composition, as will be crucial to prove pseudofunctoriality of H:

Lemma 22. *If $\phi : F \xrightarrow{\bullet} G$ and $\phi' : G \xrightarrow{\bullet} H$, then*

$$
\begin{aligned}
(\phi', \psi_{\phi'})(\phi, \psi_\phi) &= (\phi'\phi, \psi_{\phi'\phi}) \\
(\phi, \psi_\phi) \star 1 &= (\phi \star 1, \psi_{\phi \star 1}) \\
1 \star (\phi, \psi_\phi) &= (1 \star \phi, \psi_{1 \star \phi}) \\
(\phi', \psi_{\phi'}) \star (\phi, \psi_\phi) &= (\phi' \star \phi, \psi_{\phi' \star \phi})
\end{aligned}
$$

whenever these expressions typecheck.

Proof. Direct calculations.

5.4 Pseudofunctoriality of H

Note that H is *not* a functor, because for any $F : \mathbb{C} \to \mathbb{D}$ with finite limits and functorial family $\overrightarrow{A}$ over Γ (in $\mathbb{C}$), $\sigma_\Gamma(\overrightarrow{A})$ forgets all information on $\overrightarrow{A}$ except its display map $\overrightarrow{A}(\mathrm{id})$, and later extends $F(\overrightarrow{A}(\mathrm{id}))$ to an independent functorial family. However if $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{E}$ preserve finite limits, the two pseudo cwf-morphisms $(G, \sigma^G) \circ (F, \sigma^F) = (GF, \sigma^G \sigma^F)$ and (GF, σ^{GF}) are related by the pseudo cwf-transformation $(1_{GF}, \psi_{1_{GF}})$, which is obviously an isomorphism. The coherence laws only involve vertical and horizontal compositions of units and pseudo cwf-transformations obtained by completion, hence they are easy consequences of Lemma 22.

6 The Biequivalences

Theorem 23. *We have the following biequivalences of* 2*-categories.*

$$\mathbf{FL} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}} \qquad\qquad \mathbf{LCC} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$$

Proof. Since $UH = \mathrm{Id}$ (the identity 2-functor) it suffices to construct pseudonatural transformations of pseudofunctors:

$$\mathrm{Id} \underset{\epsilon}{\overset{\eta}{\rightleftarrows}} HU$$

which are inverse up to invertible modifications. Since $HU(\mathbb{C}, T) = (\mathbb{C}, T^{\mathbb{C}})$, these pseudonatural transformations are families of equivalences of cwfs:

$$(\mathbb{C}, T) \underset{\epsilon_{(\mathbb{C},T)}}{\overset{\eta_{(\mathbb{C},T)}}{\rightleftarrows}} (\mathbb{C}, T^{\mathbb{C}})$$

which satisfy the required conditions for pseudonatural transformations.

Construction of $\eta_{(\mathbb{C},T)}$. Using Lemma 19, we just need to define a base functor, which will be $\mathrm{Id}_{\mathbb{C}}$, and a family σ^η_Γ which translates types (in the sense of T) to functorial families. This is easy, since types in the cwf $(\mathbb{C}, T)$ come equipped with a chosen behaviour under substitution. Given $A \in \mathrm{Type}(\Gamma)$, we define:

$$\sigma^\eta_\Gamma(A)(\delta) = \mathrm{p}_{A[\delta]}$$
$$\sigma^\eta_\Gamma(A)(\delta, \gamma) = \langle \gamma\mathrm{p}, \mathrm{q} \rangle$$

For each pseudo cwf-morphism (F, σ), the pseudonaturality square relates two pseudo cwf-morphisms whose base functor is F. Hence, the necessary invertible pseudo cwf-transformation is obtained using Lemma 21 from the identity natural transformation on F. The coherence conditions are straightforward consequences of Lemma 22.

Construction of $\epsilon_{(\mathbb{C},T)}$. As for η, the base functor for $\epsilon_{(\mathbb{C},T)}$ is $Id_{\mathbb{C}}$. Using Lemma 19 again we need, for each context Γ, a function $\sigma^{\epsilon}_{\Gamma}$ which given a functorial family $\overrightarrow{A}$ over Γ will build a syntactic type $\sigma^{\epsilon}_{\Gamma}(\overrightarrow{A}) \in \mathrm{Type}(\Gamma)$. In other terms, we need to find a syntactic representative of an arbitrary display map, that is, an arbitrary morphism in $\mathbb{C}$. We use the inverse image:

$$\sigma^{\epsilon}_{\Gamma}(\overrightarrow{A}) = \mathrm{Inv}(\overrightarrow{A}(\mathrm{id})) \in \mathrm{Type}(\Gamma)$$

The family ϵ is pseudonatural for the same reason as η above.

Invertible modifications. For each cwf $(\mathbb{C}, T)$, we need to define invertible pseudo cwf-transformations $m_{(\mathbb{C},T)} : (\epsilon\eta)_{(\mathbb{C},T)} \to id_{(\mathbb{C},T)}$ and $m'_{(\mathbb{C},T)} : (\eta\epsilon)_{(\mathbb{C},T)} \to id_{(\mathbb{C},T)}$. As pseudo cwf-transformations between pseudo cwf-morphisms with the same base functor, their first component will be the identity natural transformation, and the second will be generated by Lemma 21. The coherence law for modifications is a consequence of Lemma 22.

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A Proofs of Section 2

Lemma 24 (Composition of coercions). *If $(\mathbb{C}, T)$ is a cwf, Γ a context in $\mathbb{C}$ and $\theta_1 : A \to B$ and $\theta_2 : B \to C$ are isomorphisms in $\boldsymbol{T}(\Gamma)$, then for all $a : \Gamma \vdash A$,*

$$\{\theta_2\}(\{\theta_1\}(a)) = \{\theta_2\theta_1\}(a)$$

Proof. Direct calculation, using the definition of coercions and manipulation of cwf combinators.

$$\begin{aligned}
\{\theta_2\}(\{\theta_1\}(a)) &= \mathrm{q}[\theta_2\langle \mathrm{id}, \mathrm{q}[\theta_1\langle \mathrm{id}, a\rangle]\rangle] \\
&= \mathrm{q}[\theta_2\langle \mathrm{p}\theta_1\langle \mathrm{id}, a\rangle, \mathrm{q}[\theta_2\langle \mathrm{id}, a\rangle]\rangle] \\
&= \mathrm{q}[\theta_2\langle \mathrm{p}, \mathrm{q}\rangle\theta_1\langle \mathrm{id}, a\rangle] \\
&= \mathrm{q}[\theta_2\theta_1\langle \mathrm{id}, a\rangle] \\
&= \{\theta_2\theta_1\}(a)
\end{aligned}$$

Lemma 25. *Let $(\mathbb{C}, T)$ be a democratic cwf with Σ-types and identity types, then for each $\delta : \Delta \to \Gamma$ there is an isomorphism α_δ in $\mathbb{C}/\Gamma$:*

$$\begin{tikzcd}
\Gamma \cdot \mathrm{Inv}(\delta) \arrow[rr, bend left, "\alpha_\delta"] \arrow[dr, "\mathrm{p}"'] & & \Delta \arrow[ll, bend left, "\alpha_\delta^{-1}"] \arrow[dl, "\delta"] \\
& \Gamma &
\end{tikzcd}$$

Proof. Recall that $\mathrm{Inv}(\delta) = \Sigma(\overline{\Delta}[\langle\rangle], \mathrm{I}_{\overline{\Gamma}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}], \overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle])) \in \mathrm{Type}(\Gamma)$. We define:

$$\begin{aligned}
\alpha_\delta &= \gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle \\
\alpha_\delta^{-1} &= \langle \delta, \mathrm{pair}(\mathrm{q}[\gamma_\Delta], \mathrm{r}_{\overline{\Gamma}[\langle\rangle]})\rangle
\end{aligned}$$

A straightforward calculation proves that this typechecks, and that $\alpha_\delta\alpha_\delta^{-1} = \mathrm{id}_\Delta$. For the other equality, we have $\alpha_\delta^{-1}\alpha_\delta = \langle \delta\gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle, \mathrm{pair}(\pi_1(\mathrm{q}), \mathrm{r}_{\overline{\Gamma}[\langle\rangle]})\rangle$. But by property of extensional identity types $\mathrm{q}[\gamma_\Gamma \mathrm{p}]$ and $\overline{\delta}[\langle\langle\rangle, \pi_1(\mathrm{q})\rangle]$ are equal terms in context $\Gamma \cdot \mathrm{Inv}(\delta)$, so $\gamma_\Gamma^{-1}\langle\langle\rangle, \mathrm{q}[\gamma_\Gamma \mathrm{p}]\rangle = \mathrm{p}$ and $\gamma_\Gamma^{-1}\langle\langle\rangle, \overline{\delta}[\langle\langle\rangle, \pi_1(\mathrm{q})\rangle]\rangle = \delta\gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle$ are equal substitutions. Likewise, $\mathrm{r}_{\overline{\Gamma}[\langle\rangle]} = \pi_2(\mathrm{q})$ by uniqueness of identity proofs, therefore $\alpha_\delta^{-1}\alpha_\delta = \mathrm{id}_{\Gamma \cdot \mathrm{Inv}(\delta)}$.

Proposition 26. *Let $(\mathbb{C}, T)$ be a democratic cwf with Σ-types and identity types, then for all context Γ the categories $\boldsymbol{T}(\Gamma)$ and $\mathbb{C}/\Gamma$ are equivalent.*

Proof. For each context Γ, the functor $F_\Gamma : \mathbf{T}(\Gamma) \to \mathbb{C}/\Gamma$ defined by $F_\Gamma(A) = \mathrm{p}_A$ on objects and $F_\Gamma(f) = f$ on morphisms in clearly full and faithful, but it is also essentially surjective. Indeed, for any object $\delta : \Delta \to \Gamma$ in $\mathbb{C}/\Gamma$, we have by Lemma 25 an isomorphic $F_\Gamma(\mathrm{Inv}(\delta)) = \mathrm{p}_{\mathrm{Inv}(\delta)}$, hence F_Γ is an equivalence of categories.

B Proofs of Section 3

B.1 Properties of pseudo cwf-morphisms

Let us first mention that in the definition of a pseudo cwf-morphism (F, σ) from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$, the fact that $\boldsymbol{\sigma}$ is a pseudonatural transformation from $\boldsymbol{T}$ to $\boldsymbol{T'}$ amounts to the satisfaction of the following coherence and naturality laws.

- *Identity.* For all $A \in \mathrm{Type}(\Gamma)$, we have $\theta_{A,\mathrm{id}} = \mathrm{id}_{F\Gamma\sigma_\Gamma(A)}$,
- *Coherence.* For all $\delta : \Xi \to \Delta$ and $\gamma : \Delta \to \Gamma$, the following diagram commutes.

$$\begin{tikzcd}
F\Xi\cdot\sigma_\Gamma(A)[F(\gamma\delta)] \arrow[rr, "\theta_{A,\gamma\delta}"] \arrow[dr, "\boldsymbol{T'}(F\delta)(\theta_{A,\gamma})"'] & & F\Xi\cdot\sigma_\Xi(A[\gamma\delta]) \\
& F\Xi\cdot\sigma_\Delta(A[\gamma])[F(\delta)] \arrow[ur, "\theta_{A[\gamma],\delta}"'] &
\end{tikzcd}$$

- *Naturality.* For all $\delta : \Delta \to \Gamma$ in $\mathbb{C}$, $A, B \in \mathrm{Type}(\Gamma)$ and $f : A \to B$ in $\boldsymbol{T}(\Gamma)$, the following diagram commutes in $\boldsymbol{T'}(F\Delta)$.

$$\begin{tikzcd}
\sigma_\Gamma(A)[F\delta] \arrow[r, "\theta_{A,\delta}"] \arrow[d, "\boldsymbol{T'}(F\delta)(\boldsymbol{\sigma}_\Gamma(f))"] & \sigma_\Delta(A[\delta]) \arrow[d, "\boldsymbol{\sigma}_\Delta(\boldsymbol{T}(\delta)(f))"] \\
\sigma_\Gamma(B)[F\delta] \arrow[r, "\theta_{B,\delta}"] & \sigma_\Delta(B[\delta])
\end{tikzcd}$$

This can be checked by simply unfolding the definition of a pseudonatural transformation.

Proposition 27. *Any pseudo cwf-morphism (F, σ) from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ preserves substitution extension, in the following sense: For all $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and $a : \Delta \vdash A[\delta]$, we have*

$$F(\langle\delta, a\rangle) = \rho^{-1}_{\Gamma,A}\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle$$

Proof. First note that for each context Γ in $\mathbb{C}$ and type $A \in \mathrm{Type}(\Gamma)$, the isomorphism $\rho_{\Gamma,A}$ is defined as the unique morphism preserving projections between the two context comprehensions of $F\Gamma$ and $\sigma_\Gamma A$, in other terms $\rho_{\Gamma,A} = \langle F(\mathrm{p}_A), \{\theta^{-1}_{A,\mathrm{p}}\}(\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q}_A))\rangle$, which implies that the projections are preserved in the following sense.

$$F(\mathrm{p}_A) = \mathrm{p}_{\sigma_\Gamma A}\rho_{\Gamma,A}$$
$$\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q}_A) = \{\theta_{A,\mathrm{p}}\}(\mathrm{q}_{\sigma_\Gamma A}[\rho_{\Gamma,A}])$$

We now use it to prove the announced property. Clearly, the required equality boils down to the following two equations.

$$\begin{aligned} \mathrm{p}\rho_{\Gamma,A} F(\langle \delta, a\rangle) &= F\delta \\ \mathrm{q}[\rho_{\Gamma,A} F(\langle \delta, a\rangle)] &= \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a)) \end{aligned}$$

The proof of the first equality is completely straightforward:

$$\begin{aligned} \mathrm{p}\rho_{\Gamma,A} F(\langle \delta, a\rangle) &= F(\mathrm{p}) F(\langle \delta, a\rangle) \\ &= F\delta \end{aligned}$$

However, the proof of the second is far more subtle and many aspects of pseudo cwf-morphisms and cwf combinators:

$$\begin{aligned} \mathrm{q}[\rho_{\Gamma,A} F(\langle \delta, a\rangle)] &=_1 \{\theta^{-1}_{A,\mathrm{p}}\}(\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q}))[F(\langle \delta, a\rangle)] \\ &=_2 \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle \mathrm{id}, \sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q})\rangle][F(\langle \delta, a\rangle)] \\ &= \ \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle \delta, a\rangle), \sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q})[F(\langle \delta, a\rangle)]\rangle] \\ &=_3 \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle \delta, a\rangle), \{\theta^{-1}_{A[\mathrm{p}],\langle \delta, a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(\mathrm{q}[\langle \delta, a\rangle]))\rangle] \\ &= \ \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle \delta, a\rangle), \{\theta^{-1}_{A[\mathrm{p}],\langle \delta, a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\ &= \ \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle \delta, a\rangle)\mathrm{p}, \mathrm{q}\rangle]\rangle\langle \mathrm{id}, \{\theta^{-1}_{A[\mathrm{p}],\langle \delta, a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\ &=_4 \mathrm{q}[\boldsymbol{T'}(F(\langle \delta, a\rangle))(\theta^{-1}_{A,\mathrm{p}})\langle \mathrm{id}, \{\theta^{-1}_{A[\mathrm{p}],\langle \delta, a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\ &=_2 \{\boldsymbol{T'}(F(\langle \delta, a\rangle))(\theta^{-1}_{A,\mathrm{p}})\}(\{\theta^{-1}_{A[\mathrm{p}],\langle \delta, a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))) \\ &=_5 \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a)) \end{aligned}$$

Equality (1) is by preservation of q, equalities (2) by definition of coercions, equality (3) by preservation of substitution on terms, equality (4) by definition of $\boldsymbol{T'}$, equality (5) by the coherence requirement on θ and Lemma 24. All the other steps are by simple manipulations on cwf combinators.

Lemma 28. *If (F, σ) is a pseudo cwf-morphism from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$, then its action on terms and sections is redundant: for all $a \in \Gamma \vdash A$,*

$$\sigma^{A}_{\Gamma}(a) = \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]$$

Proof. This is a direct consequence of preservation of substitution extension, as follows:

$$F(\langle \mathrm{id}, a\rangle) = \rho^{-1}_{\Gamma,A}\langle \mathrm{id}, \{\theta^{-1}_{A,\mathrm{id}}\}\sigma^{A}_{\Gamma}(a)\rangle$$

but $\theta_{A,\mathrm{id}} = \mathrm{id}$ by coherence of θ, hence the result is proved.

Lemma 29. *If (F, σ) is a pseudo cwf-morphism from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ and $\theta : A \to B$ is a morphism in $\boldsymbol{T}(\Gamma)$, then the coercion $\{\theta\}$ commutes with σ in the following way, for each $a \in \Gamma \vdash A$:*

$$\sigma_\Gamma^B(\{\theta\}(a)) = \{\boldsymbol{\sigma}_\Gamma(\theta)\}(\sigma_\Gamma^A(a))$$

Proof. Direct calculation.

$$\begin{aligned}
\sigma_\Gamma^B(\{\theta\}(a)) &=_1 \mathrm{q}[\rho_{\Gamma,B} F(\langle \mathrm{id}, \{\theta\}(a)\rangle)] \\
&=_2 \mathrm{q}[\rho_{\Gamma,B} F(\langle \mathrm{id}, \mathrm{q}[\theta\langle \mathrm{id}, a\rangle]\rangle)] \\
&=_3 \mathrm{q}[\rho_{\Gamma,B} F(\theta\langle \mathrm{id}, a\rangle)] \\
&=_4 \mathrm{q}[\boldsymbol{\sigma}_\Gamma(\theta)\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)] \\
&=_2 \{\boldsymbol{\sigma}_\Gamma(\theta)\}(\mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]) \\
&=_1 \{\boldsymbol{\sigma}_\Gamma(\theta)\}(\sigma_\Gamma^A(a))
\end{aligned}$$

Where (1) is by Lemma 28, (2) by definition of coercions, (3) by basic manipulation of cwf combinators and (4) by definition of $\boldsymbol{\sigma}$.

B.2 Composition of pseudo cwf-morphisms

Proposition 30. *Pseudo cwf-morphisms are stable under composition.*

Proof. Let us first give a bit more details about how pseudo cwf-morphisms are composed. If $(F, \sigma) : (\mathbb{C}_0, T_0) \to (\mathbb{C}_1, T_1)$ and $(G, \tau) : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ are two pseudo cwf-morphisms, we define their composition $(G, \tau)(F, \sigma)$ as $(GF, \tau\sigma)$ where:

$$\begin{aligned}
(\tau\sigma)_\Gamma(A) &= \tau_{F\Gamma}(\sigma_\Gamma(A)) \\
(\tau\sigma)_\Gamma^A(a) &= \tau_{F\Gamma}^{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a))
\end{aligned}$$

If the other components of (F, σ) are denoted by θ^F, ρ^F and those of (G, τ) by θ^G, ρ^G, we define:

$$\theta_{A,\delta} = \boldsymbol{\tau}_{F\Delta}(\theta_{A,\delta}^F)\theta_{\sigma_\Gamma(A),F\delta}^G$$

All the components of $(G, \tau)(F, \sigma)$ are now defined, but we still have a number of conditions to prove.

– *Preservation of substitution on terms.* Direct calculation, if $a : \Gamma \vdash A$ and $\delta : \Delta \to \Gamma$ in $\mathbb{C}_0$.

$$\begin{aligned}
(\tau\sigma)^{A[\delta]}_{\Delta}(a[\delta]) &=_1 \tau^{\sigma_\Delta(A[\delta])}_{F\Delta}(\sigma^{A[\delta]}_{\Delta}(a[\delta])) \\
&=_2 \tau^{\sigma_\Delta(A[\delta])}_{F\Delta}(\{\theta^F_{A,\delta}\}(\sigma^A_\Gamma(a)[F\delta])) \\
&=_3 \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\langle \mathrm{id}, \{\theta^F_{A,\delta}\}(\sigma^A_\Gamma(a)[F\delta])\rangle)] \\
&=_4 \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\langle \mathrm{id}, \mathrm{q}[\theta^F_{A,\delta}\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle]\rangle) \\
&=_5 \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\theta^F_{A,\delta}\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)] \\
&=_6 \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)] \\
&=_7 \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\langle \mathrm{id}, \mathrm{q}[\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)]\rangle] \\
&=_4 \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\mathrm{q}[\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma^A_\Gamma(a)[F\delta]\rangle)]) \\
&=_3 \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\tau^{\sigma_\Gamma(A)[F\delta]}_{F\Delta}(\sigma^A_\Gamma(a)[F\delta])) \\
&=_2 \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\{\theta^G_{\sigma_\Gamma(A),F\delta}\}(\tau^{\sigma_\Gamma(A)}_{F\Gamma}(\sigma^A_\Gamma(a))[GF\delta])) \\
&=_8 \{\theta_{A,\delta}\}(\tau^{\sigma_\Gamma(A)}_{F\Gamma}(\sigma^A_\Gamma(a))[GF\delta]) \\
&=_1 \{\theta_{A,\delta}\}((\tau\sigma)^A_\Gamma(a)[GF\delta])
\end{aligned}$$

Equalities annotated by (1) come from the definition of $\tau\sigma$, (2) is preservation of substitution for σ or τ, (3) is Lemma 28, (4) is by definition of coercions, (5) uses $\mathrm{p}\theta^F_{A,\delta} = \mathrm{p}$ and basic manipulations with cwf combinators, (6) is by definition of $\boldsymbol{\tau}$, (7) uses preservation of p by (G, τ) and basic manipulations with cwf combinators, and (8) is by definition of θ.

– *Preservation of the terminal object.* Trivial from the preservation of the terminal object by F and G.

– *Preservation of context comprehension.* Using preservation of context comprehension from (F, σ) and (G, τ) we define:

$$GF(\Gamma\cdot A) \xrightarrow{G(\rho^F_{\Gamma,A})} G(F\Gamma\cdot\sigma_\Gamma A) \xrightarrow{\rho^G_{F\Gamma,\sigma_\Gamma A}} GF\Gamma\cdot(\tau\sigma)_\Gamma(A)$$

As a composition of isomorphisms it is an isomorphism so $GF(\Gamma\cdot A)$ is also a context comprehension of $GF\Gamma$ and $(\tau\sigma)_\Gamma(A)$. We must still check that the corresponding projections are those required by the definition. It is obvious for the first projection:

$$\begin{aligned}
\mathrm{p}\rho^G_{F\Gamma,\sigma_\Gamma A}G(\rho^F_{\Gamma,A}) &= G(\mathrm{p})G(\rho^F_{\Gamma,A}) \\
&= GF\mathrm{p}
\end{aligned}$$

But more intricate for the second.

$$
\begin{aligned}
\mathrm{q}[\rho^G_{F\Gamma,\sigma_\Gamma A} G(\rho^F_{\Gamma,A})] &=_1 \{(\theta^G_{\sigma_\Gamma A,\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[\mathrm{p}]}_{F\Gamma\sigma_\Gamma A}(\mathrm{q}))[G(\rho^F_{\Gamma,A})] \\
&=_2 \{(\theta^G_{\sigma_\Gamma A,\mathrm{p}})^{-1}\}(\{(\theta^G_{\sigma_\Gamma(A)[\mathrm{p}],\rho^F_{\Gamma,A}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma\cdot A)}(\mathrm{q}[\rho^F_{\Gamma,A}]))) \\
&=_3 \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma\cdot A)}(\mathrm{q}[\rho^F_{\Gamma,A}])) \\
&=_4 \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma\cdot A)}(\{(\theta^F_{A,\mathrm{p}})^{-1}\}(\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q})))) \\
&=_5 \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\{\boldsymbol{\tau}_{F(\Gamma\cdot A)}((\theta^F_{A,\mathrm{p}})^{-1})\}(\tau^{\sigma_{\Gamma\cdot A}(A[\mathrm{p}])}_{F(\Gamma\cdot A)}(\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q})))) \\
&=_6 \{\theta^{-1}_{A,[p]}\}((\tau\sigma)^{A[\mathrm{p}]}_{\Gamma\cdot A}(q))
\end{aligned}
$$

Where (1) is preservation of the second projection by ρ^G, (2) is preservation of substitution on terms, (3) is coherence for θ^G, (4) is preservation of the second projection by ρ^F, (5) is Lemma 29 and (6) is by definition of θ and $\tau\sigma$.

Finally note that, as can be checked by unfolding the definitions, we have for all context Γ in $\mathbb{C}$

$$(\boldsymbol{\tau\sigma})_\Gamma = \boldsymbol{\tau}_{F\Gamma} \circ \boldsymbol{\sigma}_\Gamma$$

Hence the necessary coherence and naturality conditions amounts to the stability of pseudonatural transformations under composition.

Lemma 31. *Any pseudo cwf-morphism (F, σ) from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ where both cwfs support Σ-types automatically preserves them, in the sense that*

$$\sigma_\Gamma(\Sigma(A, B)) \cong \Sigma(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])$$

Proof. We exploit the fact that in any cwf $(\mathbb{C}, T)$ with Σ-types we have $\Gamma\cdot\Sigma(A, B) \cong \Gamma\cdot A\cdot B$ (obvious). Therefore:

$$
\begin{aligned}
F\Gamma\cdot\sigma_\Gamma(\Sigma(A, B)) &\cong F(\Gamma\cdot\Sigma(A, B)) \\
&\cong F(\Gamma\cdot A\cdot B) \\
&\cong F(\Gamma\cdot A)\cdot\sigma_{\Gamma\cdot A}(B) \\
&\cong F\Gamma\cdot\sigma_\Gamma(A)\cdot\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}] \\
&\cong F\Gamma\cdot\Sigma(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])
\end{aligned}
$$

It is easy to see that the resulting morphism is a type isomorphism.

Lemma 32. *Preservation of democracy, identity types and Π-types are all stable under composition.*

Proof. This is trivial for identity types and Π-types (just apply the hypothesis on both pseudo cwf-morphism). For democracy, we must check that the isomorphism $d^{GF}_{\Gamma} : GF[]\cdot(\tau\sigma)_{[]}(\overline{\Gamma}) \to GF[]\cdot\overline{GF\Gamma}[\langle\rangle]$ defined by:

$$\begin{aligned}
d^{GF}_{\Gamma} = \; GF[]\cdot\tau_{F[]}(\sigma_{[]}(\overline{\Gamma})) &\xrightarrow{(\rho^{G}_{F[],\sigma_{[]}\overline{\Gamma}})^{-1}} F(F[]\cdot\sigma_{[]}\overline{\Gamma}) \xrightarrow{G(d^{F}_{\Gamma})} G(F[]\cdot\overline{F\Gamma}[\langle\rangle]) \\
&\xrightarrow{G(\langle\langle\rangle,\mathrm{q}\rangle)} G([]\cdot\overline{F\Gamma}) \xrightarrow{\rho^{G}_{[],\overline{F\Gamma}}} G[]\cdot\tau_{[]}\overline{F\Gamma} \\
&\xrightarrow{d^{G}_{F\Gamma}} G[]\cdot\overline{GF\Gamma}[\langle\rangle]
\end{aligned}$$

The coherence law can then be checked by a simple diagram chasing.

B.3 Properties and composition of pseudo cwf-transformations

Let us just remark that if (F,σ) and (G,τ) are pseudo cwf-morphisms from $(\mathbb{C},T)$ to $(\mathbb{D},T')$, pairs (ϕ,m) where $\phi : F \overset{\bullet}{\to} G$ is a natural transformation and $m : (\boldsymbol{T'} \star \phi) \circ \boldsymbol{\sigma} \overset{\bullet\bullet}{\to} \boldsymbol{\tau}$ is a modification (where $\boldsymbol{T'} \star \phi$ denotes the *vertical composition* of the natural transformations $\mathrm{id}_{\boldsymbol{T'}}$ and ϕ) exactly correspond to pseudo cwf-transformations from (F,σ) to (G,τ) (as can be checked by unfolding the definition of a modification).

It is folklore that there is a 2-category **Ind** of indexed categories over arbitrary bases, which objects are pairs $(\mathbb{C},\boldsymbol{T})$ (where $\mathbb{C}$ is a category and $\boldsymbol{T} : \mathbb{C}^{op} \to \mathbf{Cat}$ is a pseudofunctor), 1-cells are pairs $(F,\boldsymbol{\sigma}) : (\mathbb{C},\boldsymbol{T}) \to (\mathbb{D},\boldsymbol{T'})$ (where $F : \mathbb{C} \to \mathbb{D}$ is a functor and $\boldsymbol{\sigma} : \boldsymbol{T} \to \boldsymbol{T'}F$ is a pseudonatural transformation) and 2-cells are pairs $(\phi,m) : (F,\boldsymbol{\sigma}) \to (G,\boldsymbol{\tau}) : (\mathbb{C},\boldsymbol{T}) \to (\mathbb{D},\boldsymbol{T'})$ (where $\phi : F \overset{\bullet}{\to} G$ is a natural transformation and $m : (\boldsymbol{T'} \star \phi) \circ \boldsymbol{\sigma} \overset{\bullet\bullet}{\to} \boldsymbol{\tau}$ is a modification).

Here we rely on **Ind** to define vertical and horizontal composition of pseudo cwf-transformations, so we get various 2-categories of cwfs supporting structure, structure-preserving pseudo cwf-morphisms and pseudo cwf-transformations between them, which can all be seen as sub-2-categories of **Ind**. In particular, we will be interested in the 2-category $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ of cwfs supporting democracy, Σ-types and identity types, pseudo cwf-morphisms preserving democracy and identity types and pseudo cwf-transformations. We will also be interested in the 2-category $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ where Π-types are additionally supported and preserved.

C Proofs of Section 4

For this section, the main thing to prove is that the inverse image is preserved (up to isomorphism) by structure-preserving pseudo cwf-morphisms: This relies mostly on long and intricate calculations involving all the components of pseudo cwf-morphisms. Let us first prove a few preliminary lemmas.

Lemma 33 (Propagation of isomorphisms). *Isomorphisms propagate through types in several different ways. Suppose that you have $A, A' \in Type(\Gamma)$, $B, B' \in Type(\Gamma\cdot A)$, then*

(1) If $B \cong B'$, then $\Sigma(A,B) \cong \Sigma(A,B')$

(2) If $A \cong_\theta A'$, then $\Sigma(A, B) \cong \Sigma(A', B[\theta^{-1}])$
(3) If $A \cong_\theta A'$ and $a, a' \in \Gamma \vdash A$, then $\mathrm{I}_A(a, a') \cong \mathrm{I}_{A'}(\{\theta\}(a), \{\theta\}(a'))$

Proof. (1) is obvious, since $\Gamma \cdot \Sigma(A, B)$ is isomorphic to $\Gamma \cdot A \cdot B$. For (2), we give the following two isomorphisms:

$$\langle p, \mathrm{pair}(\mathrm{q}[\theta\langle \mathrm{p}, \pi_1(\mathrm{q})\rangle], \pi_2(\mathrm{q}))\rangle : \Gamma \cdot \Sigma(A, B) \to \Gamma \cdot \Sigma(A', B[\theta^{-1}])$$
$$\langle p, \mathrm{pair}(\mathrm{q}[\theta^{-1}\langle \mathrm{p}, \pi_1(\mathrm{q})\rangle], \pi_2(\mathrm{q}))\rangle : \Gamma \cdot \Sigma(A', B[\theta^{-1}]) \to \Gamma \cdot \Sigma(A, B)$$

A simple calculation shows that they typecheck and that they are inverse of one another. It is obvious that they are isomorphisms of types. (3) is also obvious since by extensionality, $\langle \mathrm{p}, \mathrm{r}\rangle$ typechecks in both directions and is its own inverse.

Lemma 34 (Preservation of inverse image). *Let $(\mathbb{C}, T)$ and $(\mathbb{D}, T')$ be cwfs supporting democracy, Σ-types and identity types and let (F, σ) be a pseudo cwf-morphism preserving them. Moreover, suppose that $\delta : \Delta \to \Gamma$ is a morphism in $\mathbb{C}$, then there is an isomorphism in $\mathbb{D}$:*

$$F(\Gamma \cdot \mathrm{Inv}(\delta)) \cong F\Gamma \cdot \mathrm{Inv}(F\delta)$$

Proof. Exploiting Lemma 33 and preservation of substitution on types and terms, a careful (but rather straightforward) calculation allows to derive the following type isomorphism:

$$\sigma_\Gamma(\mathrm{Inv}(\delta)) \cong \Sigma(\overline{F\Delta}[\langle\rangle], \mathrm{I}_{\overline{F\Gamma}[\langle\rangle]}(C(\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle])), C(\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}]))))$$

where $C(-)$ is an invertible context given by:

$$C(M) = \{\boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma)\theta^{-1}_{\overline{\Gamma}, \langle\rangle}\}(M)[\rho_{\Gamma, \overline{\Delta}[\langle\rangle]}\theta_{\overline{\Delta}, \langle\rangle}\boldsymbol{T'}(\iota\langle\rangle)(d^{-1}_{\Delta})]$$

Here, ι denotes the inverse of the terminal morphism $\langle\rangle : F[] \to []$ whose existence is asserted by the definition of a pseudo cwf-morphism. Hence, the goal of the remaining part of this proof will be to show the following equalities:

$$\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) = C^{-1}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \quad (1)$$
$$\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}]) = C^{-1}(\mathrm{q}[\gamma_{F\Gamma} \mathrm{p}]) \quad (2)$$

Let us focus on (1). Using preservation of substitution on terms, coherence of θ and the basic computation laws in cwfs, we derive:

$$\begin{aligned}
\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) &= \sigma^{\overline{\Gamma}[\mathrm{p}][\langle\langle\rangle, \mathrm{q}\rangle]}_{\Gamma \cdot \overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \\
&= \{\theta_{\overline{\Gamma}[\mathrm{p}], \langle\langle\rangle, \mathrm{q}\rangle}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[] \cdot \overline{\Delta}}(\overline{\delta})[F(\langle\langle\rangle, \mathrm{q}\rangle)]) \\
&= \{\theta_{\overline{\Gamma}, \langle\rangle}\}(\{\boldsymbol{T'}(\langle\langle\rangle, \mathrm{q}\rangle)(\theta^{-1}_{\overline{\Gamma}, \mathrm{p}})\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[] \cdot \overline{\Delta}}(\overline{\delta})[F(\langle\langle\rangle, \mathrm{q}\rangle)])) \\
&= \{\theta_{\overline{\Gamma}, \langle\rangle}\}(\{\theta^{-1}_{\overline{\Gamma}, \mathrm{p}}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[] \cdot \overline{\Delta}}(\overline{\delta}))[F(\langle\langle\rangle, \mathrm{q}\rangle)])
\end{aligned}$$

Let us now focus on $\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})$, to see how terms created from substitution using democracy are transformed by the action of the cwf-morphism. Here, we are only going to use the coherence of θ, preservation of q and the basic computation laws in cwfs.

$$\begin{aligned}
\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta}) &= \sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\mathrm{q}[\gamma_\Gamma\delta\gamma_\Delta^{-1}]) \\
&= \sigma^{\overline{\Gamma}[\mathrm{p}][\gamma_\Gamma\delta\gamma_\Delta^{-1}]}_{[]\cdot\overline{\Delta}}(\mathrm{q}[\gamma_\Gamma\delta\gamma_\Delta^{-1}]) \\
&= \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma_\Delta^{-1}}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Gamma}}(\mathrm{q})[F(\gamma_\Gamma\delta\gamma_\Delta^{-1})]) \\
&= \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma_\Delta^{-1}}\}(\{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}])[F(\gamma_\Gamma\delta\gamma_\Delta^{-1})]) \\
&= \{\theta_{\overline{\Gamma},\mathrm{p}}\}(\{\boldsymbol{T'}(F(\gamma_\Gamma\delta\gamma_\Delta^{-1}))(\theta^{-1}_{\overline{\Gamma},\mathrm{p}})\}(\{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}])[F(\gamma_\Gamma\delta\gamma_\Delta^{-1})])) \\
&= \{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma_\Delta^{-1})])
\end{aligned}$$

Using preservation of democracy and the terminal object, we can now conclude:

$$\begin{aligned}
\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle,\mathrm{q}\rangle]) &= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma_\Delta^{-1}\langle\langle\rangle,\mathrm{q}\rangle)]) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d_r^{-1}\langle\iota\mathrm{p},\mathrm{q}\rangle\gamma_{F\Gamma}F\delta\gamma_{F\Delta}^{-1}\langle\langle\rangle,\mathrm{q}\rangle\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta)\theta^{-1}_{\overline{\Delta},\langle\rangle}\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma^{-1})\}(\overline{F\delta}[\langle\langle\rangle,\mathrm{q}\rangle])[\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta)\theta^{-1}_{\overline{\Delta},\langle\rangle}\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}] \\
&= C^{-1}(\overline{F\delta}[\langle\langle\rangle,\mathrm{q}\rangle])
\end{aligned}$$

We get the required expression. The case of Equation (2) is similar but less intricate, so we skip the details.

Lemma 35 (Propagation of isomorphisms under Π). *Suppose that we have $A, A' \in Type(\Gamma)$ and $B, B' \in Type(\Gamma\cdot A)$, then*

1. *If $B \cong_\theta B'$, then $\Pi(A,B) \cong \Pi(A,B')$*
2. *If $A \cong_\theta A'$, then $\Pi(A,B) \cong \Pi(A',B[\theta^{-1}])$*

Proof. (1). Using θ and the combinators of Π-types, it is straightforward to build an morphism:

$$\langle \mathrm{p}, \lambda(\{\boldsymbol{T}(\langle \mathrm{pp},\mathrm{q}\rangle)(\theta)\}(app(\mathrm{q}[\mathrm{p}],\mathrm{q})))\rangle : \Gamma\cdot\Pi(A,B) \to \Gamma\cdot\Pi(A,B')$$

Its inverse is the corresponding expression with θ^{-1} in place of θ. (2). Likewise, the following expression provides the required isomorphism:

$$\lambda(app(\mathrm{q}[\mathrm{pp}], \{\boldsymbol{T}(\mathrm{pp})(\theta^{-1})\}(\mathrm{q}))) : \Gamma\cdot\Pi(A,B) \to \Gamma\cdot\Pi(A',B[\theta^{-1}])$$

Proposition 36. *Let (F,σ) be a pseudo cwf-morphism between $(\mathbb{C},T)$ and $(\mathbb{C}',T')$ supporting Σ-types and democracy. Then:*

– *If* $(\mathbb{C}, T)$ *and* $(\mathbb{C}', T')$ *both support identity types and* (F, σ) *preserves them, then* F *preserves finite limits.*
– *If* $(\mathbb{C}, T)$ *and* $(\mathbb{C}', T')$ *both support* Π*-types and* (F, σ) *preserves them, then* F *preserves local exponentiation.*

Proof. Since finite limits and local exponentiation can be defined using σ-types, Π-types and the inverse image type, their preservation by F directly boils down to the Lemmas 33, 34 and 35.

D Proofs of Section 5

D.1 Action on 0-cells

This section is the exact analogue for cwfs of Hofmann's work [5] with cwas. For the sake of self-completeness we will give the full details of the construction. However, we will skip some of the proofs whenever they are not significantly different from the case of cwas, for which we refer to [5].

Base cwf structure. We will start by proving that given any category $\mathbb{C}$ with a terminal object, we can equip $\mathbb{C}$ of a cwf structure. This means that we have to define a functor $T_{\mathbb{C}} : \mathbb{C} \to \mathbf{Fam}$ (in other terms types, terms, and substitution on both of them), and a context comprehension operation.

Types. A *type* over Γ is a *functorial family*, *i.e.* a functor $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ such that:

(i) $cod \circ \overrightarrow{A} = dom$

(ii) If $\begin{tikzcd} \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, $\overrightarrow{A}(\alpha)$ is a pullback square, with the naming convention below:

$$\begin{tikzcd} dom(\overrightarrow{A}(\delta \circ \alpha)) \arrow[r, "\overrightarrow{A}(\delta,\alpha)"] \arrow[d, "\overrightarrow{A}(\delta\alpha)"'] & dom(\overrightarrow{A}(\delta)) \arrow[d, "\overrightarrow{A}(\delta)"] \\ \Omega \arrow[r, "\alpha"'] & \Delta \end{tikzcd}$$

Let $Type(\Gamma)$ denote the set of functorial families over Γ.

Remark. The functoriality of $\overrightarrow{A}$ means that the assignment of $\overrightarrow{A}(\delta, \alpha)$ satisfies the following equations:

– $\overrightarrow{A}(\delta, \mathrm{id}_{\Delta}) = \mathrm{id}_{dom(\overrightarrow{A}(\delta))}$
– $\overrightarrow{A}(\delta, \alpha\beta) = \overrightarrow{A}(\delta, \alpha)\overrightarrow{A}(\delta\alpha, \beta)$

Terms. Let $\varGamma \in \mathbb{C}$, and $\overrightarrow{A} \in \mathit{Type}(\varGamma)$. We define $\varGamma \cdot \overrightarrow{A} = \mathit{dom}(\overrightarrow{A}(\mathrm{id}_\varGamma))$. (This will later give us context comprehension.) Then, a term $a : \varGamma \vdash \overrightarrow{A}$ is a morphism $a : \varGamma \to \varGamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\varGamma)a = \mathrm{id}_\varGamma$.

Substitution in types. Let $\gamma : \varDelta \to \varGamma$ in $\mathbb{C}$ and $\overrightarrow{A} \in \mathit{Type}(\varGamma)$. We define $\overrightarrow{A}[\gamma] \in \mathit{Type}(\varDelta)$ as follows.

$$\begin{aligned}\overrightarrow{A}[\gamma](\delta) &= \overrightarrow{A}(\gamma\delta)\\ \overrightarrow{A}[\gamma](\delta, \alpha) &= \overrightarrow{A}(\gamma\delta, \alpha)\end{aligned}$$

where $\delta : \varOmega \to \varDelta$ and $\alpha : \varXi \to \varOmega$. We check that $\overrightarrow{A}[\gamma]$ satisfies the two conditions for types:

(i) $\mathit{cod} \circ \overrightarrow{A}[\gamma] = \mathit{dom}$.

$$\begin{aligned}\mathit{cod} \circ \overrightarrow{A}[\gamma](\delta) &= \mathit{cod} \circ \overrightarrow{A}(\gamma\delta)\\ &= \mathit{dom}(\gamma\delta)\\ &= \mathit{dom}(\delta)\end{aligned}$$

(ii) The image of the morphism
$$\begin{tikzcd} \varXi \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \varOmega \arrow[dl, "\delta"] \\ & \varDelta & \end{tikzcd}$$
in $\mathbb{C}/\varDelta$ is

$$\overrightarrow{A}[\gamma](\delta) \quad = \quad \begin{tikzcd} \mathit{dom}(\overrightarrow{A}(\gamma\delta\alpha)) \arrow[r, "\overrightarrow{A}(\gamma\delta{,}\alpha)"] \arrow[d, "\overrightarrow{A}(\gamma\delta\alpha)"'] & \mathit{dom}(\overrightarrow{A}(\gamma\delta)) \arrow[d, "\overrightarrow{A}(\gamma\delta)"] \\ \varXi \arrow[r, "\alpha"'] & \varOmega \end{tikzcd}$$

This is a pullback square by property (ii) of $\overrightarrow{A}$.

Substitution in terms. Let $\delta : \varDelta \to \varGamma$, and $a : \varGamma \vdash \overrightarrow{A}$, *i.e.* $a : \varGamma \to \varGamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\varGamma) \circ a = \mathrm{id}_\varGamma$. Then $a[\delta]$ is defined as the unique mediating arrow in the following diagram:

$$\begin{tikzcd} \varDelta \arrow[dr, dotted, "a[\delta]"] \arrow[drr, bend left, "a\circ\delta"] \arrow[ddr, bend right, "\mathrm{id}_\varDelta"'] & & \\ & \varDelta \circ \overrightarrow{A}[\delta] \arrow[r, "\overrightarrow{A}(\mathrm{id}_\varGamma{,}\delta)"] \arrow[d, "\overrightarrow{A}[\delta](\mathrm{id}_\varDelta)"] & \varGamma \cdot \overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\varGamma)"] \\ & \varDelta \arrow[r, "\delta"'] & \varGamma \end{tikzcd}$$

It is a term of type $\overrightarrow{A}[\delta]$ by commutativity of the left triangle.

Functoriality. Since substitution in types is defined by composition, the cwf-laws for it follow immediately. It is also immediate that $a[\mathrm{id}_\Gamma] = a$ since the defining pullback must be the identity in $\mathbb{C}^{\rightarrow}$. It remains to show that if we have $\delta_1 : \Delta' \to \Delta$ and $\delta_2 : \Delta \to \Gamma$, then $a[\delta_2 \circ \delta_1] = a[\delta_2][\delta_1]$. Consider the following diagram for pullback composition:

$$\begin{tikzcd}
\Delta' \arrow[dr, "{a[\delta_2][\delta_1]}"] \arrow[drrr, bend left, "a\circ\delta_2\circ\delta_1"] \arrow[ddr, bend right, "\mathrm{id}_{\Delta'}"'] & & & \\
& \Delta'\cdot\overrightarrow{A}[\delta_2\circ\delta_1] \arrow[r, "{\overrightarrow{A}[\delta_2](\mathrm{id}_\Delta,\delta_1)}"] \arrow[d, "{\overrightarrow{A}[\delta_2\circ\delta_1](id_{\Delta'})}"] & \Delta\cdot\overrightarrow{A}[\delta_2] \arrow[r, "{\overrightarrow{A}(\mathrm{id}_\Gamma,\delta_2)}"] \arrow[d, "{\overrightarrow{A}[\delta_2](\mathrm{id}_\Delta)}"] & \Gamma\cdot\overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\
& \Delta' \arrow[r, "\delta_1"'] & \Delta \arrow[r, "\delta_2"'] & \Gamma
\end{tikzcd}$$

The external pullback square is obtained by substitution of $\delta_2 \circ \delta_1$. By definition of substitution in terms, $a[\delta_2 \circ \delta_1]$ is the unique mediating arrow. But the external square is equal (not only up to isomorphism) to the composition of the smaller squares, because of the functoriality conditions for $\overrightarrow{A}$, more precisely the fact that $\overrightarrow{A}(s, \alpha \circ \beta) = \overrightarrow{A}(s, \alpha) \circ \overrightarrow{A}(s \circ \alpha, \beta)$. This implies that $a[\delta_2][\delta_1]$ also makes the two triangles commute. Hence $a[\delta_2][\delta_1] = a[\delta_2 \circ \delta_2]$ by uniqueness of the mediating arrow.

Putting all this together, we now have built a functor $T_{\mathbb{C}} : \mathbb{C}^{op} \to \mathbf{Fam}$. We still have to define context comprehension.

Context comprehension. Let $\Gamma \in \mathbb{C}$, and $\overrightarrow{A} \in Type(\Gamma)$. As mentioned above, we define:

$$\Gamma\cdot\overrightarrow{A} = dom(\overrightarrow{A}(\mathrm{id}_\Gamma))$$

The first projection is $\mathrm{p}_{\overrightarrow{A}} = \overrightarrow{A}(\mathrm{id}_\Gamma) : \Gamma\cdot\overrightarrow{A} \to \Gamma$. The second projection $\mathrm{q}_{\overrightarrow{A}}$ is defined as the unique mediating arrow of the following pullback diagram:

$$\begin{tikzcd}
\Gamma\cdot\overrightarrow{A} \arrow[dr, dotted, "\mathrm{q}_{\overrightarrow{A}}"] \arrow[drr, bend left, "\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}"] \arrow[ddr, bend right, "\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}"'] & & \\
& \Gamma\cdot\overrightarrow{A}\cdot\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}] \arrow[r, "{\overrightarrow{A}(\mathrm{id}_\Gamma,\mathrm{p}_{\overrightarrow{A}})}"] \arrow[d, "{\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}})}"] & \Gamma\cdot\overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\
& \Gamma\cdot\overrightarrow{A} \arrow[r, "\mathrm{p}_{\overrightarrow{A}}"'] & \Gamma
\end{tikzcd}$$

Suppose now we have $\delta : \Delta \to \Gamma$ and $a : \Delta \vdash \overrightarrow{A}[\delta]$. By definition of terms we have in fact $a : \Delta \to \Delta\cdot\overrightarrow{A}[\delta]$. We define:

$$\langle \delta, a\rangle = \overrightarrow{A}(id_\Gamma, \delta) \circ a : \Delta \to \Gamma\cdot\overrightarrow{A}$$

We must prove that these definitions satisfy the cwf-laws for context comprehension.

$$\begin{aligned}
\mathrm{p}_{\overrightarrow{A}} \circ \langle \delta, a\rangle &= \mathrm{p}_{\overrightarrow{A}} \circ \overrightarrow{A}(\mathrm{id}_\Gamma, \delta) \circ a && \text{def.of } \langle \delta, a\rangle \\
&= \overrightarrow{A}(\mathrm{id}_\Gamma) \circ \overrightarrow{A}(\mathrm{id}_\Gamma, \delta) \circ a && \text{def. of } \mathrm{p}_{\overrightarrow{A}} \\
&= \delta \circ \overrightarrow{A}[\delta](\mathrm{id}_\Delta) \circ a && \text{comm. of pullback square} \\
&= \delta && a \text{ is a term}
\end{aligned}$$

Proving the equation $\mathrm{q}_{\overrightarrow{A}}[\langle \delta, a\rangle] = a$ is a bit more involved. Let us first prove the following lemma, stating (intuitively) that the action of $\mathrm{q}_{\overrightarrow{A}}$ is to duplicate the last element of the context.

Lemma 37. *Let $\overrightarrow{A} \in Type(\Gamma)$, $\delta : \Delta \to \Gamma$, and $a : \Delta \vdash \overrightarrow{A}[\delta]$. Then $\mathrm{q}_{\overrightarrow{A}} \circ \langle \delta, a\rangle = \langle\langle \delta, a\rangle, a\rangle$.*

Proof. Consider the following pullback diagram:

$$\begin{tikzcd}
\Delta \arrow[dr, bend left=10, "\mathrm{q}_{\overrightarrow{A}}\circ\langle\delta,a\rangle"] \arrow[dr, bend right=10, "\langle\langle\delta,a\rangle,a\rangle"'] \arrow[drr, bend left, "\langle\delta,a\rangle"] \arrow[ddr, bend right, "\langle\delta,a\rangle"'] \arrow[drr, phantom, "(1)"] & & \\
(2) & \Gamma\cdot\overrightarrow{A}\cdot\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}] \arrow[r, "\overrightarrow{A}(\mathrm{id}_\Gamma,\mathrm{p}_{\overrightarrow{A}})"] \arrow[d, "\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}})"] & \Gamma\cdot\overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\
& \Gamma\cdot\overrightarrow{A} \arrow[r, "\mathrm{p}_{\overrightarrow{A}}"'] & \Gamma
\end{tikzcd}$$

It is clear that $\mathrm{q}_{\overrightarrow{A}}\circ\langle \delta, a\rangle$ makes (1) and (2) commute, by definition of $\mathrm{q}_{\overrightarrow{A}}$. It is also easy to see that $\langle\langle \delta, a\rangle, a\rangle$ makes (2) commute, because $\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}) = \mathrm{p}_{\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}]}$ and by the property of the first projection. We prove now that $\langle\langle \delta, a\rangle, a\rangle$ also makes (1) commute:

$$\begin{aligned}
\overrightarrow{A}(\mathrm{id}_\Gamma, \mathrm{p}_{\overrightarrow{A}}) \circ \langle\langle \delta, a\rangle, a\rangle &= \overrightarrow{A}(\mathrm{id}_\Gamma, \mathrm{p}_{\overrightarrow{A}}) \circ \overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](id_{\Gamma\cdot\overrightarrow{A}}, \langle \delta, a\rangle) \circ a && \text{def. of } \langle\langle \delta, a\rangle, a\rangle \\
&= \overrightarrow{A}(\mathrm{id}_\Gamma, \mathrm{p}_{\overrightarrow{A}}) \circ \overrightarrow{A}(\mathrm{p}_{\overrightarrow{A}}, \langle \delta, a\rangle) \circ a && \text{def. of } \overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}] \\
&= \overrightarrow{A}(\mathrm{id}_\Gamma, \mathrm{p}_{\overrightarrow{A}} \circ \langle \delta, a\rangle) \circ a && \text{funct. or } \overrightarrow{A}(s, \delta) \\
&= \overrightarrow{A}(\mathrm{id}_\Gamma, \delta) \circ a && \text{property of } \mathrm{p}_{\overrightarrow{A}} \\
&= \langle \delta, a\rangle && \text{def. of } \langle \delta, a\rangle
\end{aligned}$$

This concludes the proof.

From this lemma we deduce that $\mathrm{q}_{\overrightarrow{A}}[\langle\delta, a\rangle] = a$ in the following way. Consider the pullback diagram:

$$\begin{tikzcd}
\Delta \arrow[dr, bend left, "a"] \arrow[dr, bend right, "\mathrm{q}_{\overrightarrow{A}}[\langle\delta,a\rangle]"'] \arrow[drr, bend left, "\mathrm{q}_{\overrightarrow{A}} \circ \langle\delta,a\rangle"] \arrow[ddr, bend right, "\mathrm{id}_\Delta"'] & & \\
& \Delta \cdot \overrightarrow{A}[\delta] \arrow[r, "\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}, \langle\delta,a\rangle)"] \arrow[d, "\mathrm{p}_{\overrightarrow{A}[\delta]}"] & \Gamma \cdot \overrightarrow{A} \cdot \overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}] \arrow[d, "\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}})"] \\
& \Delta \arrow[r, "\langle\delta,a\rangle"'] & \Gamma \cdot \overrightarrow{A}
\end{tikzcd}$$

This diagram is an instance of the pullback used for the definition of substitution in terms. Hence, both triangles commute for $\mathrm{q}_{\overrightarrow{A}}[\langle\delta, a\rangle]$. The left triangle commutes for a since a is term of type $\overrightarrow{A}[\delta]$. The right triangle commutes because of the lemma above, since by definition $\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}, \langle\delta, a\rangle) = \langle\langle\delta, a\rangle, a\rangle$. Thus, by uniqueness of the mediating arrow $\mathrm{q}_{\overrightarrow{A}}[\langle\delta, a\rangle] = a$. This concludes the cwf construction, hence the proof of the following proposition.

Proposition 38. *Let $\mathbb{C}$ be a category with terminal object, then we can extend $\mathbb{C}$ to a cwf $(\mathbb{C}, T_{\mathbb{C}})$.*

Democracy. The cwf $(\mathbb{C}, T_{\mathbb{C}})$ is democratic: the idea is that each context Γ is represented by any functorial family having its terminal projection $\langle\rangle : \Gamma \to I$ as display map. We can easily build such a functorial family by $\overline{\Gamma} = \widehat{\langle\rangle} \in \mathit{Type}(I)$. We have then $I \cdot \overline{\Gamma} = \mathit{dom}(\widehat{\langle\rangle}(\mathrm{id})) = \Gamma$, thus the isomorphism between them is trivial.

Proposition 39. *If $\mathbb{C}$ is a category with a terminal object, then the cwf $(\mathbb{C}, T_{\mathbb{C}})$ is democratic.*

***Σ*-types.** For the sake of completeness we recall the definitions, but we refer the reader to [5] for some of the proofs, in particular when the distinction between cwas and cwfs does not change anything.

Formation. Let $A \in \mathit{Type}(\Gamma)$ and $B \in \mathit{Type}(\Gamma \cdot A)$. At each $s : \Delta \to \Gamma$, the image of $\Sigma(A, B)$ is given by the composition of the images of A and B. More formally, we define:

$$\Sigma(A, B)(s) = A(s) \circ B(A(\mathrm{id}, s))$$
$$\Sigma(A, B)(s, \alpha) = B(A(\mathrm{id}, s), A(s, \alpha))$$

The construction of the corresponding pullback square can be illustrated by the following diagram. Intuitively, the chosen pullbacks for $\Sigma(A, B)$ are directly obtained by composition the chosen pullbacks for A and for B.

$$\begin{tikzcd}
\cdot \arrow[r, "B(A(\mathrm{id},s),A(s,\alpha))"] \arrow[d] & \cdot \arrow[r, "B(\mathrm{id},A(\mathrm{id},s))"] \arrow[d, "B(A(\mathrm{id},s))"] & \Gamma\cdot A\cdot B \arrow[d, "B(\mathrm{id})"] \\
\cdot \arrow[r, "A(s,\alpha)"] \arrow[d, "A(s\alpha)"'] & \cdot \arrow[r, "A(\mathrm{id},s)"] \arrow[d, "A(s)"] & \Gamma\cdot A \arrow[d, "A(\mathrm{id})"] \\
\cdot \arrow[r, "\alpha"'] & B \arrow[r, "s"'] & \Gamma
\end{tikzcd}$$

It is easy to check that this defines a functor $\Sigma(A, B) : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ and that the necessary equations are satisfied so that we get a type $\Sigma(A, B) \in Type(\Gamma)$.

Introduction. If $a : \Gamma \vdash A$ and $b : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$, then $a : \Gamma \to \Gamma\cdot A$ is a section of $A(id_\Gamma)$ and $b : \Gamma \to \Gamma\cdot B[\langle \mathrm{id}, a\rangle]$ is a section of $B[\langle \mathrm{id}, a\rangle](id_\Gamma) = B(a)$ as illustrated by following diagram:

$$\begin{tikzcd}
\Gamma\cdot B[\langle \mathrm{id}, a\rangle] \arrow[r, "B(\mathrm{id},a)"] \arrow[d, "B(a)"] & \Gamma\cdot A\cdot B \arrow[d, "B(\mathrm{id})"] \\
\Gamma \arrow[r, "a"'] \arrow[u, bend left, "b"] & \Gamma\cdot A
\end{tikzcd}$$

We define $pair(a, b) = B(\mathrm{id}, a) \circ b$. It follows that $pair(a, b)$ is a section of $\Sigma(A, B)(\mathrm{id}_\Gamma) = \mathrm{p}_A \circ \mathrm{p}_B$.

Elimination. Let $c : \Gamma \vdash \Sigma(A, B)$. Thus c is a section of $\mathrm{p}_A \circ \mathrm{p}_B : \Gamma\cdot A\cdot B \to \Gamma$. We define the first projection $\pi_1(c) = \mathrm{p}_B \circ c$ which is clearly a section of p_A. The second projection $\pi_2(c)$ is given by the universal property of the following pullback:

$$\begin{tikzcd}
\Gamma \arrow[dr, dotted, "\pi_2(c)"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "\mathrm{id}_\Gamma"'] & & \\
& \Gamma\cdot B[\langle \mathrm{id}, a\rangle] \arrow[r, "B(\mathrm{id},a)"] \arrow[d, "B(a)"] & \Gamma\cdot A\cdot B \arrow[d, "B(\mathrm{id})"] \\
& \Gamma \arrow[r, "a"'] & \Gamma\cdot A
\end{tikzcd}$$

It is immediate from the diagram that it is a section of $\mathrm{p}_{B[\langle \mathrm{id},a\rangle]}$.

Equations. The equality rules for Σ-types are proved just as in [5].

This concludes the proof of the following proposition.

Proposition 40. *If $\mathbb{C}$ is a category with a terminal object, then $(\mathbb{C}, T)$ supports Σ-types.*

Extensional identity types. To improve readability, we will now sometimes omit the subscripts of the projections, when they can be recovered from the context. To build identity types, we require that the base category has finite limits.

Formation rule. Let $\Gamma \in \mathbb{C}$, $A \in Type(\Gamma)$, and $a, a' : \Gamma \vdash A$. If $s : \Delta \to \Gamma$, we define $\mathrm{I}_A(a, a')(s)$ as the equalizer of $a[s]$ and $a'[s]$ (seen as morphisms $\Delta \to \Delta \cdot A[s]$). If

$$\begin{tikzcd} \Delta' \arrow[rr, "\delta"] \arrow[dr, "s'"'] & & \Delta \arrow[dl, "s"] \\ & \Gamma & \end{tikzcd}$$

is a morphism in $\mathbb{C}/\Gamma$, we define $\mathrm{I}_A(a, a')(\delta)$ as the upper square in the following diagram:

$$\begin{tikzcd} dom(\mathrm{I}_A(a,a')(s\delta)) \arrow[r, dotted, "\gamma"] \arrow[d, "\mathrm{I}_A(a,a')(s\delta)"] & dom(\mathrm{I}_A(a,a')(s)) \arrow[d, "\mathrm{I}_A(a,a')(s)"] \\ \Delta' \arrow[r, "\delta"] \arrow[d, shift right, "a[s\delta]"'] \arrow[d, shift left, "a'[s\delta]"] & \Delta \arrow[d, shift right, "a[s]"'] \arrow[d, shift left, "a'[s]"] \\ \Delta' \cdot A[s\delta] \arrow[r, "\langle \delta \mathrm{p}, \mathrm{q} \rangle"'] & \Delta \cdot A[s] \end{tikzcd}$$

where γ is yet to be defined. For this purpose, and to prove that the obtained square is a pullback, we need the following:

Lemma 41. *In the diagram above, if $f : \mathrm{dom}(f) \to \Delta'$, then f equalizes $a[s\delta]$ and $a'[s\delta]$ iff δf equalizes $a[s]$ and $a'[s]$.*

Proof. First note that by construction of this cwf, we have the surprising equality $a = \langle \mathrm{id}_\Gamma, a \rangle$ for any term $a : \Gamma \vdash A$. Indeed, $\langle \mathrm{id}_\Gamma, a \rangle = A(\mathrm{id}_\Gamma, \mathrm{id}_\Gamma) \circ a = a$. Thus, we have that

$$\begin{aligned} \langle \delta \mathrm{p}, \mathrm{q} \rangle \circ a[s\delta] &= \langle \delta \mathrm{p}, \mathrm{q} \rangle \circ \langle \mathrm{id}, a[s\delta] \rangle \\ &= \langle \delta, a[s\delta] \rangle \\ &= \langle \mathrm{id}_\Delta, a[s] \rangle \circ \delta \\ &= a[s] \circ \delta \end{aligned}$$

For the same reason, we have $\langle \delta \mathrm{p}, \mathrm{q}\rangle \circ a'[s\delta] = a'[s] \circ \delta$. Suppose now that f equalizes $a[s\delta]$ and $a'[s\delta]$. Then:

$$\begin{aligned} a[s] \circ \delta \circ f &= \langle \delta \mathrm{p}, \mathrm{q}\rangle \circ a[s\delta] \circ f \\ &= \langle \delta \mathrm{p}, \mathrm{q}\rangle \circ a'[s\delta] \circ f \\ &= a'[s] \circ \delta \circ f \end{aligned}$$

Thus as claimed, δf equalizes $a[s]$ and $a'[s]$. The same equational reasoning gives the converse implication.

We use this lemma as follows. We know that $\mathrm{I}_A(a, a')(s\delta)$ equalizes $a[s\delta]$ and $a'[s\delta]$, thus $\delta \circ \mathrm{I}_A(a, a')(s\delta)$ equalizes $a[s]$ and $a'[s]$. Thus by the equalizer property, $\delta \circ \mathrm{I}_A(a, a')(s\delta)$ factors in a unique way through $\mathrm{I}_A(a, a')(s)$, and we define γ to be the unique morphism. Since we already know that the square commutes, it only remains to prove that it is a pullback square.

Let $h_1 : X \to \Delta'$ and $h_2 : X \to \mathrm{dom}(\mathrm{I}_A(a, a')(s))$ be two morphisms which make the outer square commute. Necessarily, $\mathrm{I}_A(a, a')(s) \circ h_2$ equalizes $a[s]$ and $a'[s]$. Since the outer square commutes, δh_1 equalizes them as well. By Lemma 41, h_1 equalizes $a[s\delta]$ and $a'[s\delta]$. Thus it factors uniquely through $\mathrm{I}_A(a, a')(s\delta)$. Let h be the mediating arrow. It makes the left triangle commute by the factorisation property and the right triangle commute because γh defines another unique factorisation of δh_1 through $\mathrm{I}_A(a, a')(s)$.

We must check that this construction is functorial. Both conditions (for id_s and $\delta_1 \circ \delta_2$) follow immediately by uniqueness of the factorisation through the equalizer. Thus we have shown that $\mathrm{I}_A(a, a') \in \mathit{Type}(\Gamma)$.

Reflexivity. For each $a \in \Gamma \vdash A$, we define the term $\mathrm{r}_{A,a} : \Gamma \vdash \mathrm{I}_A(a, a)$ as follows:

$$\begin{tikzcd} \Gamma \arrow[dr, dotted, "\mathrm{r}_{A,a}"] \arrow[ddr, bend right, "\mathrm{id}_\Gamma"'] & \\ & \mathit{dom}(\mathrm{I}_A(a, a)(\mathrm{id}_\Gamma)) \arrow[d] \\ & \Gamma \arrow[d, shift right, bend right, "a"'] \arrow[d, shift left, bend left, "a"] \\ & \Gamma \cdot A \end{tikzcd}$$

Stability under substitution. First we prove

$$\mathrm{I}_A(a, a')[\delta] = \mathrm{I}_{A[\delta]}(a[\delta], a'[\delta])$$

It suffices to note that for any s, the arrows $\mathrm{I}_A(a, a')[\delta](s)$ and $\mathrm{I}_{A[\delta]}(a[\delta], a'[\delta])(s)$ both equalize $a[s\delta]$ and $a'[s\delta]$. The image of arrows is determined uniquely by the factorisation under this equalizer, hence must also be unchanged.

Then we prove

$$\mathrm{r}_{A,a}[\delta] = \mathrm{r}_{A[\delta],a[\delta]}$$

This is because $\mathrm{r}_{A,a}[\delta]$ is a correct factorisation of id_Δ through $\mathrm{I}_A(a,a')[\delta] = \mathrm{I}_{A[\delta]}(a[\delta],a'[\delta])$ and $\mathrm{r}_{A[\delta],a[\delta]}$ is defined as the unique such factorisation.

Extensionality. Here, the judgement $\Gamma \vdash a = a' : A$ means that a and a' are equal morphisms of $\mathbb{C}$. Suppose we have a term $c : \Gamma \vdash \mathrm{I}_A(a,a')$. For the first rule, note that $\mathrm{I}_A(a,a')(\mathrm{id}_\Gamma) \circ c = \mathrm{id}_\Gamma$, because c is a term. But id_Γ factors through $\mathrm{I}_A(a,a')(\mathrm{id}_\Gamma)$. Thus it equalizes a and a', and it follows that $a = a'$. For the second rule, note that the first rule implies that $a = a'$. Thus id_Γ equalizes a and a' and there is a unique factorisation of id_Γ through $\mathrm{I}_A(a,a')(\mathrm{id}_\Gamma)$. Since c and $\mathrm{r}_{A,a}$ are both such factorisations $c = \mathrm{r}_{A,a}$.

Proposition 42. *Let $\mathbb{C}$ be a finitely complete category, then $(\mathbb{C}, T_\mathbb{C})$ supports identity types.*

***Π*-types.** If $\mathbb{C}$ is a lccc, then the cwf $H(\mathbb{C})$ supports Π-types. Let $\overrightarrow{A}$ be a functorial family over Γ and $\overrightarrow{B}$ over $\Gamma{\cdot}\overrightarrow{A}$. Then the value of the family $\Pi(\overrightarrow{A},\overrightarrow{B})$ at substitution $\delta : \Delta \to \Gamma$ is defined by $\Pi_{\overrightarrow{A}(\delta)}(\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))$, where Π_f is the right adjoint of f^* obtained by the lcc structure. If $\alpha : \Omega \to \Delta$ and $\delta : \Delta \to \Gamma$, we have to define a morphism $\Pi(\overrightarrow{A},\overrightarrow{B})(\delta,\alpha)$ yielding a pullback diagram. For this purpose, first consider the following chain of isomorphisms in $\mathbb{C}/\Omega$:

$$\begin{aligned}
\Pi_{\overrightarrow{A}(\delta\alpha)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta\alpha)) &= \Pi_{\overrightarrow{A}(\delta\alpha)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)\overrightarrow{A}(\delta,\alpha)) \\
&\cong \Pi_{\overrightarrow{A}(\delta\alpha)}(\overrightarrow{A}(\delta,\alpha))^*(\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta))) \\
&\cong \alpha^*(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))
\end{aligned}$$

The first isomorphism is by uniqueness of the pullback of $\overrightarrow{B}(\mathrm{id},\delta)$ along $\overrightarrow{A}(\delta,\alpha)$, while the second is by the Beck-Chevalley condition applied to the pullback square of $\overrightarrow{A}(\delta,\alpha)$. Let us call ϕ this isomorphism. The action of α^* also gives a canonical morphism $h : \mathrm{dom}(\alpha^*(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))) \to \mathrm{dom}(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))$, thus we define:

$$\Pi(\overrightarrow{A},\overrightarrow{B})(\delta,\alpha) = h\phi : \mathrm{dom}(\Pi(\overrightarrow{A},\overrightarrow{B})(\delta)) \to \mathrm{dom}(\Pi(\overrightarrow{A},\overrightarrow{B})(\delta\alpha))$$

As needed this defines a pullback square since it is obtained as an isomorphism and a pullback, hence the definition of the functorial family $\Pi(\overrightarrow{A},\overrightarrow{B})$ is now complete, since the equations come from the universal property of the pullback. The fact that $\Pi(\overrightarrow{A},\overrightarrow{B})[\delta]$ and $\Pi(\overrightarrow{A}[\delta],\overrightarrow{B}[\langle\delta\mathrm{p},\mathrm{q}\rangle])$ coincide on objects (of $\mathbb{C}/\Gamma$) is a straightforward calculation, from which the fact that they coincide on morphisms can be directly deduced.

The combinators λ and ap come from natural applications of the adjunction $(\overrightarrow{A}(\mathrm{id}))^* \dashv \Pi_{\overrightarrow{A}(\mathrm{id})}$, and the computation rules follow from the properties of adjunctions. Behaviour of the combinators λ and ap under substitution require to rework the proof of the Beck-Chevalley conditions for lcccs. As in [5], we will not give the details.

Proposition 43. *Let $\mathbb{C}$ be a lccc, then $(\mathbb{C}, T_{\mathbb{C}})$ supports Π-types.*

D.2 Image of 1-cells

Lemma 44. *Let $(F, \sigma) : (\mathbb{C}, T) \to (\mathbb{D}, T')$ be a pseudo cwf-morphism with families of isomorphisms θ and ρ. Then for any $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and type $A \in \mathrm{Type}(\Gamma)$, we have:*

$$F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) = \rho^{-1}_{\Gamma,A} \langle F(\delta)\mathrm{p}, \mathrm{q}\rangle \theta^{-1}_{A,\delta} \rho_{\Delta, A[\delta]}$$

Proof. Direct calculation.

$$\begin{aligned}
F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) &= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \{\theta^{-1}_{A,\delta \mathrm{p}}\}(\sigma^{A[\delta \mathrm{p}]}_{\Delta A[\delta]}(\mathrm{q}))\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \{\theta^{-1}_{A,\delta \mathrm{p}}\}(\{\theta_{A[\delta],\mathrm{p}}\}(\mathrm{q}[\rho_{\Delta,A[\delta]}]))\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\}(q[\rho_{\Delta,A[\delta]}])\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \mathrm{q}[\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\langle id, q[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\delta}\langle (F\mathrm{p})\mathrm{p}, \mathrm{q}\rangle]\rangle\langle id, q[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \mathrm{q}[\theta^{-1}_{A,\delta}\langle F\mathrm{p}, \mathrm{q}[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta \mathrm{p}), \mathrm{q}[\theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}]\rangle \\
&= \rho^{-1}_{\Gamma,A} \langle F(\delta)\mathrm{p}, \mathrm{q}\rangle \theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}
\end{aligned}$$

Using preservation of substitution extension and q, then coherence of θ and manipulation of cwf combinators.

From any functor $F : \mathbb{C} \to \mathbb{D}$ preserving finite limits, its extension to $(F, \sigma_F) : (\mathbb{C}, T_{\mathbb{C}}) \to (\mathbb{D}, T_{\mathbb{D}})$ relies heavily on the following lemma.

Lemma 45 (Generation of isomorphisms). *Let $(\mathbb{C}, T)$ and $(\mathbb{D}, T')$ be two cwfs, $F : \mathbb{C} \to \mathbb{D}$ a functor preserving finite limits, a family of functions $\sigma_\Gamma : \mathrm{Type}(\Gamma) \to \mathrm{Type}'(F\Gamma)$ and a family of isomorphisms $\rho_{\Gamma,A} : F(\Gamma \cdot A) \to F\Gamma \cdot \sigma_\Gamma(A)$ such that $\mathrm{p}\rho_{\Gamma,A} = F\mathrm{p}$. Then there exists an unique choice of functions σ^A_Γ on terms and of isomorphisms $\theta_{A,\delta}$ such that (F, σ) is a weak cwf-morphism.*

Proof. By Lemma 28, the unique way of extending σ to terms is by exploiting the redundancy between terms and sections and set $\sigma^A_\Gamma(a) = \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]$. To generate θ, we exploit the two squares below:

$$\begin{array}{ccc}
F\Delta \cdot \sigma_\Gamma(A)[F\delta] & \xrightarrow{\langle (F\delta)\mathrm{p}\mathrm{q}\rangle} & F\Gamma \cdot \sigma_\Gamma(A) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{p}} \\
F\Delta & \xrightarrow[F\delta]{} & F\Gamma
\end{array}
\qquad
\begin{array}{ccc}
F\Delta \cdot \sigma_\Delta(A[\delta]) & \xrightarrow{\rho_{\Gamma,A} F(\langle \delta \mathrm{p}, \mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}} & F\Gamma \cdot \sigma_\Gamma(A) \\
{\scriptstyle \mathrm{p}}\downarrow & & \downarrow{\scriptstyle \mathrm{p}} \\
F\Delta & \xrightarrow[F\delta]{} & F\Gamma
\end{array}$$

The first square is a standard substitution pullback. The second is a pullback because F preserves finite limits and $\rho_{\Gamma,A}$ and $\rho_{\Delta,A[\delta]}$ are isomorphisms. The isomorphism $\theta_{A,\delta}$ is then defined as the unique mediating morphism from the first to the second. There is no other possible choice for $\theta_{A,\delta}$: indeed, in an arbitrary pseudo cwf-morphism $\theta_{A,\delta}$ necessarily already commutes with the projections of these pullback diagrams (easy consequence of Lemma 44), therefore whenever (F,σ) is such that F preserves finite limits, $\theta_{A,\delta}$ must coincide with this mediating arrow above.

We must now check that θ, defined as above, satisfies the necessary coherence and naturality conditions. Clearly, θ defined as above commutes with the projections and satisfies $\theta_{A,\mathrm{id}} = \mathrm{id}$. We must now check that θ, defined as above, satisfies the necessary coherence and naturality conditions, which will be a consequence of the universal property of the pullback above. For the coherence condition, consider now that we have $\alpha : \Omega \to \Delta$ and $\delta : \Delta \to \Gamma$. The morphism $\theta_{A[\delta],\alpha}\boldsymbol{T'}(F\alpha)(\theta_{A,\delta})$ is an isomorphism between $\Omega\cdot\sigma_\Gamma(A)[F(\delta\alpha)]$ and $\Omega\cdot\sigma_\Omega(A[\delta\alpha])$, thus we just have to prove that it preserves the projections (of the pullback corresponding to these expressions, as in the diagram above) to conclude by uniqueness of such an isomorphism. The two equations to prove are therefore the following.

$$\mathrm{p}\theta_{A[\delta],\alpha}\boldsymbol{T'}(F\alpha)(\theta_{A,\delta}) = \mathrm{p} \tag{3}$$

$$\rho_{\Gamma,A}F(\langle\delta\alpha\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Omega,A[\delta\alpha]}\theta_{A[\delta],\alpha}\boldsymbol{T'}(F\alpha)(\theta_{A,\delta}) = \langle(F(\delta\alpha))\mathrm{p},\mathrm{q}\rangle \tag{4}$$

Equation (3) is clear by property of $\theta_{A[\delta],\alpha}$ and construction of $(F\alpha)^*\theta_{A,\delta}$, while equation (4) is a consequence of Lemma 44.

It remains to prove that θ satisfies the naturality condition. Let $f : A \to B$ be a morphism in $\boldsymbol{T}(\Gamma)$. We need to establish the following equality:

$$\boldsymbol{\sigma}_\Delta(\boldsymbol{T}(\delta)(f))\theta_{A,\delta} = \theta_{B,\delta}\boldsymbol{T'}(F\delta)(\boldsymbol{\sigma}_\Gamma(f))$$

It follows from the fact that both sides of this equation make the two triangles commute in the following diagram, which is a pullback diagram because F preserves finite limits.

$$\begin{tikzcd}
F\Delta\cdot\sigma_\Gamma(A)[F\delta] \arrow[drr, bend left, "\boldsymbol{\sigma}_\Gamma(f)\langle(F\delta)\mathrm{p},\mathrm{q}\rangle"] \arrow[dr, dotted] \arrow[ddr, bend right, "\mathrm{p}"'] & & \\
& F\Delta\cdot\sigma_\Delta(B[\delta]) \arrow[r, "\rho_{\Gamma,B}F(\langle\delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,B[\delta]}"] \arrow[d, "\mathrm{p}"] & F\Gamma\cdot\sigma_\Gamma(B) \arrow[d, "\mathrm{p}"] \\
& F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}$$

The proof that the two triangles commute only involves the definition of $\theta_{A,\delta}$ and $\theta_{B,\delta}$, along with manipulation of cwf combinators. This ends the proof that (F,σ) is a weak cwf-morphism.

Proposition 46. *If $F : \mathbb{C} \to \mathbb{D}$ preserves finite limits, then σ_F preserves democracy.*

Proof. The functor F preserves finite limits, thus it preserves in particular the terminal object: let us denote by $\iota : [] \to F[]$ the inverse to the terminal projection. Let us note now that since the two involved cwfs have been built with Hofmann's construction, their democratic structure is trivial; we have $[] \cdot \overline{\Gamma} = \Gamma$ and $\gamma_\Gamma = \mathrm{id}$. In particular, we have $F([] \cdot \overline{\Gamma}) = F(\Gamma) = [] \cdot \overline{F\Gamma}$. Thus to get preservation of the democratic structure, it is natural to choose:

$$d_\Gamma = \langle \iota, \mathrm{q} \rangle \rho^{-1}_{[],\overline{\Gamma}} : [] \cdot \sigma_{[]}(\overline{\Gamma}) \to [] \cdot \overline{F\Gamma}[\langle\rangle]$$

which makes the coherence condition essentially trivial.

Proposition 47. *If $(F, \sigma) : (\mathbb{C}, T) \to (\mathbb{D}, T')$ such that $(\mathbb{C}, T)$ and $(\mathbb{D}, T')$ supports identity types and F preserves finite limits, then σ preserves identity types.*

Proof. Let $A \in \mathit{Type}(\Gamma)$, and $a, a' \in \Gamma \vdash A$, then $\Gamma \cdot \mathrm{I}_A(a, a')$ along with its projection to Γ is an equalizer of $\langle \mathrm{id}, a \rangle$ and $\langle \mathrm{id}, a' \rangle$. Indeed if $\delta : \Delta \to \Gamma$ such that $\langle \mathrm{id}, a \rangle \delta = \langle \mathrm{id}, a' \rangle \delta$, it is straightforward to see that the morphism $h = \langle \delta, \mathrm{r}_{A[\delta], a[\delta]} \rangle$ typechecks and satisfies $\mathrm{p}h = \delta$. It is also the unique such morphism because of the uniqueness of identity proofs. But F is left exact and in particular preserves equalizers, hence the pair $(F(\Gamma \cdot \mathrm{I}_A(a, a')), F(\mathrm{p}))$ defines an equalizer of $F(\langle \mathrm{id}, a \rangle) = \rho^{-1}_{\Gamma,A} \langle \mathrm{id}, \sigma^A_\Gamma(a) \rangle$ and $F(\langle \mathrm{id}, a' \rangle) = \rho^{-1}_{\Gamma,A} \langle \mathrm{id}, \sigma^A_\Gamma(a') \rangle$. From this it is obvious that the pair $(F\Gamma \cdot \sigma_\Gamma(A), \mathrm{p})$ is an equalizer of $\langle \mathrm{id}, \sigma^A_\Gamma(a) \rangle$ and $\langle \mathrm{id}, \sigma^A_\Gamma(a') \rangle$. But for the same reason as in the beginning of the proof, the pair $(F\Gamma \cdot \mathrm{I}_{\sigma_\Gamma(A)}(\sigma^A_\Gamma(a), \sigma^A_\Gamma(a')), \mathrm{p})$ is already such an equalizer, therefore they must be isomorphic and (F, σ) preserves identity types.

We will now address the corresponding proposition for preservation of Π-types by (F, σ), provided F preserves lcc structure. The proof will make use of thr following notion.

Definition 48. *If $(\mathbb{C}, T)$ is a cwf (not necessarily supporting Π-types), Γ a context in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, let us call* a Π-object *of A and B any type $\Pi(A, B)$ such that for all term $c : \Gamma \cdot A \vdash B$ there is $\lambda(c) : \Gamma \vdash \Pi(A, B)$, for all $c : \Gamma \vdash \Pi(A, B)$ and $a : \Gamma \vdash A$ there is $\mathrm{ap}(c, a) : \Gamma \vdash B[\langle \mathrm{id}, a \rangle]$ satisfying the computation rules for Π-types (but no requirements w.r.t. substitution). Then it is straightforward to check that just as exponentials $A \Rightarrow B$ of A and B are unique up to isomorphism, Π-objects of A and B are unique up to type isomorphism.*

This notion extends to categories by relating them to the cwf built with Hofmann's construction: if $\mathbb{C}$ is any category with a terminal object, if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathbb{C}$, we will call a Π-object of f and g any morphism $\Pi(f, g) : D \to C$ such that $\widehat{\Pi(f, g)}$ is a Π-object of $\widehat{f}$ and $\widehat{g}$ in $(\mathbb{C}, T_\mathbb{C})$.

Lemma 49. *The two notions of Π-objects coincide: if $(\mathbb{C}, T)$ is a cwf, Γ a context in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, then a type C is a Π-object of A and B if and only if p_C is a Π-object of p_A and p_B.*

Proof. Obvious by the correspondence between terms of type A and sections of p_A.

Lemma 50. *If $\mathbb{C}$ is any category with a terminal object, if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathbb{C}$, then there is only one Π-object of f and g up to isomorphism in $\mathbb{C}/C$.*

Proof. The proof exactly mimics the proof of uniqueness of exponential objects.

Lemma 51. *If $\mathbb{C}$ and $\mathbb{C}'$ are lcccs and $F : \mathbb{C} \to \mathbb{C}'$ is a functor preserving finite limits, then if F preserves the lcc structure, then it preserves Π-objects.*

Proof. Recall that if $\mathbb{C}$ is locally cartesian closed and if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathbb{C}$, then there is a morphism $\Pi_f(g) : - \to \Gamma$, obtained by the following pullback in $\mathbb{C}/C$:

$$\begin{tikzcd}
\Pi_f(g) \arrow[r] \arrow[d] & (gf)^f \arrow[d, "g^f"] \\
1 \arrow[r, "\Lambda(\mathrm{id})"] & f^f
\end{tikzcd}$$

this extends to a functor $\Pi_f : \mathbb{C}/B \to \mathbb{C}/C$, which is right adjoint to the pullback functor f^*. Exploiting this adjunction, it is straightforward to prove that $\Pi_f(f)$ is a Π-object of f and g in $\mathbb{C}$. By uniqueness, it is *the* Π-object of f and g, up to isomorphism. But since F preserves lcc structure it preserves pullbacks and local exponentiation, thus it maps (up to isomorphism) this pullback diagram into the following pullback:

$$\begin{tikzcd}
F(\Pi_f(g)) \arrow[r] \arrow[d] & (F(g)F(f))^{F(f)} \arrow[d, "F(g)^{F(f)}"] \\
1 \arrow[r, "\Lambda(\mathrm{id})"] & F(f)^{F(f)}
\end{tikzcd}$$

So $F(\Pi_f(g))$ is as required a Π-object of $F(f)$ and $F(g)$.

Proposition 52. *If $(F, \sigma) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ such that $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ supports Π-types and F preserves lccc structure, then σ preserves Π-types.*

Proof. Let Γ be a context of $\mathbb{C}$, $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, obviously $\Pi(A, B)$ is a Π-object of A and B. Hence, $\mathrm{p}_{\Pi(A,B)}$ is a Π-object of p_A and p_B by Lemma 49. But F preserves Π-objects by Lemma 51, so $F(\mathrm{p}_{\Pi(A,B)})$ is a Π-object of $F(\mathrm{p}_A)$ and $F(\mathrm{p}_B)$. But $F(\mathrm{p}_A)$ is isomorphic to $\mathrm{p}_{\sigma_\Gamma(A)}$ (the isomorphism being $\rho_{\Gamma,A}$) and $F(\mathrm{p}_B)$ is isomorphic to $\mathrm{p}_{\sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}]}$ (the isomorphism being $\langle \rho_{\Gamma,A}^{-1}\mathrm{p}, q\rangle \rho_{\Gamma\cdot A,B}^{-1}$), therefore $F(\mathrm{p}_{\Pi(A,B)})$ is a Π-object of $\mathrm{p}_{\sigma_\Gamma(A)}$ and $\mathrm{p}_{\sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}]}$, hence it must be isomorphic to $\mathrm{p}_{\Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}])}$ by Lemma 50, so we have the required isomorphism $F(\Gamma \cdot \Pi(A, B)) \to F\Gamma \cdot \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}])$.

D.3 Image of 2-cells

Lemma 53 (Completion of pseudo cwf-transformations). *Suppose (F, σ) and (G, τ) are pseudo cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T)$ such that F and G preserve finite limits and $\phi : F \overset{\bullet}{\to} G$ is a natural transformation, then there exists a family of morphisms $(\psi_\phi)_{\Gamma,A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ such that (ϕ, ψ_ϕ) is a pseudo cwf-transformation from (F, σ) to (G, τ).*

Proof. We set $\psi_{\Gamma,A} = \langle \mathrm{p}, \mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle : F\Gamma\cdot\sigma_\Gamma A \to F\Gamma\cdot\tau_\Gamma(A)[\phi_\Gamma]$. To check the coherence law, consider the following composition of pullback squares.

$$\begin{tikzcd}
F\Delta\cdot\tau_\Delta(A[\delta])[\phi_\Delta] \arrow[r, "\langle \phi_\Delta \mathrm{p}, \mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"] & G\Delta\cdot\tau_\Delta(A[\delta]) \arrow[r, "\rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle)(\rho'_{\Delta,A[\delta]})^{-1}"] \arrow[d, "\mathrm{p}"] & G\Gamma\cdot\tau_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "\phi_\Delta"'] & G\Delta \arrow[r, "G\delta"'] & G\Gamma
\end{tikzcd}$$

The two paths $\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A})$ and $\psi_{\Delta,A[\delta]}\theta_{A,\delta}$ of the coherence diagram behave in the same way with respect to this pullback. Here is the calculation for the first path of the coherence diagram:

$$\begin{aligned}
&\rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle)(\rho'_{\Delta,A[\delta]})^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
&= \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
&= \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
&= \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta p, q[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
&= \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\mathrm{p}\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle, q[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}]\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
&= \langle(G\delta)\mathrm{p},\mathrm{q}\rangle\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
&= \langle(G\delta)\mathrm{p},\mathrm{q}\rangle\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\psi_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
&= \langle(G\delta)\phi_\Delta\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
&= \langle\phi_\Gamma(F\delta)\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
&= \langle\phi_\Gamma\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle\langle(F\delta)\mathrm{p},\mathrm{q}\rangle \\
&= \rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle
\end{aligned}$$

where we use Lemma 44, then only the definition of $\psi_{\Gamma,A}$, naturality of ϕ and manipulation of cwf combinators. The calculation for the other path follows:

$$
\begin{aligned}
&\rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \phi_\Delta \mathrm{p}, \mathrm{q}\rangle \psi_{\Delta,A[\delta]} \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \phi_\Delta \mathrm{p}, \mathrm{q}\rangle \langle \mathrm{p}, \mathrm{q}[\rho'_{\Delta,A[\delta]} \phi_{\Delta A[\delta]} \rho^{-1}_{\Delta,A[\delta]}]\rangle \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \phi_\Delta \mathrm{p}, \mathrm{q}[\rho'_{\Delta,A[\delta]} \phi_{\Delta A[\delta]} \rho^{-1}_{\Delta,A[\delta]}]\rangle \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \mathrm{p} \rho'_{\Delta,A[\delta]} \phi_{\Delta A[\delta]} \rho^{-1}_{\Delta,A[\delta]}, \mathrm{q}[\rho'_{\Delta,A[\delta]} \phi_{\Delta A[\delta]} \rho^{-1}_{\Delta,A[\delta]}]\rangle \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \rho'_{\Delta,A[\delta]} \phi_{\Delta A[\delta]} \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) \phi_{\Delta A[\delta]} \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} \phi_{\Gamma A} F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} \phi_{\Gamma A} \rho^{-1}_{\Gamma,A} \rho_{\Gamma,A} F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= {} & \rho'_{\Gamma,A} \phi_{\Gamma A} \rho^{-1}_{\Gamma,A} \langle (F\delta)\mathrm{p}, \mathrm{q}\rangle
\end{aligned}
$$

We have used naturality of ϕ, preservation of the first projection by (F,σ) and (G,τ) and manipulations on cwf combinators.

Lemma 54. *Completion of pseudo cwf-transformations commutes with both notions of composition,* i.e. *if $\phi : F \xrightarrow{\bullet} G$ and $\phi' : G \xrightarrow{\bullet} H$, then*

$$
\begin{aligned}
(\phi', \psi_{\phi'})(\phi, \psi_\phi) &= (\phi'\phi, \psi_{\phi'\phi}) \\
(\phi, \psi_\phi) \star 1 &= (\phi \star 1, \psi_{\phi \star 1}) \\
1 \star (\phi, \psi_\phi) &= (1 \star \phi, \psi_{1 \star \phi}) \\
(\phi', \psi_{\phi'}) \star (\phi, \psi_\phi) &= (\phi' \star \phi, \psi_{\phi' \star \phi})
\end{aligned}
$$

whenever these expressions typecheck.

Proof. The first equality is just a straightforward verification, and the second and third are trivial given the definition of ψ_1. The fourth though, requires a more involved calculation with arguments really similar to those used to prove Lemma 53. We only detail the third case. Imagine we have the following situation:

$$
\begin{tikzcd}
(\mathbb{C}, T) \arrow[r, shift left, "{(F,\sigma)}"] \arrow[r, shift right, "{(G,\tau)}"'] & (\mathbb{C}', T') \arrow[r, shift left, "{(F',\sigma')}"] \arrow[r, shift right, "{(G',\tau')}"'] & (\mathbb{C}'', T'')
\end{tikzcd}
$$

Let us call θ and ρ the components of (F,σ), θ' and ρ' the components of (F',σ'), t and r the components of (G,τ) and t' and r' the components of (G',τ'). Let us also consider natural transformations $\phi : F \xrightarrow{\bullet} G$ and $\phi' : F' \xrightarrow{\bullet} G'$. Let us recall that the vertical composition of pseudo cwf-transformations follow those of 2-cells in the 2-category of indexed categories over arbitrary bases, which means $(\phi, \psi_\phi) \star (\phi', \psi_{\phi'}) = (\phi \star \phi', m)$, where $m_{\Gamma,A}$ is obtained by:

$$
\sigma'_{F\Gamma}(\sigma_\Gamma A) \xrightarrow{\boldsymbol{\sigma'}_{F\Gamma}((\psi_\phi)_{\Gamma,A})} \sigma'_{F\Gamma}(\tau_\Gamma A[\phi_\Gamma]) \xrightarrow{{\theta'_{\tau_\Gamma A, \phi_\Gamma}}^{-1}} \sigma'_{G\Gamma}(\tau_\Gamma A)[F'\phi_\Gamma] \xrightarrow{\boldsymbol{T''}(F'\phi_\Gamma)((\psi_{\phi'})_{G\Gamma, \tau_\Gamma A})} \tau'_{G\Gamma}(\tau_\Gamma A)[\phi'_{G\Gamma} F'(\phi_\Gamma)]
$$

which the following calculation relates to $(\psi_{\phi\star\phi'})_{\Gamma,A}$:

$$
\begin{aligned}
m_{\Gamma,A} &= \boldsymbol{T''}(F'\phi_\Gamma)((\psi_{\phi'})_{G\Gamma,\tau_\Gamma A})\theta'_{\tau_\Gamma A,\phi_\Gamma}{}^{-1}\boldsymbol{\sigma'}_{F\Gamma}((\psi_\phi)_{\Gamma,A}) \\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\langle (F'\phi_\Gamma)\mathrm{p}, \mathrm{q}\rangle]\rangle\theta'_{\tau_\Gamma A,\phi_\Gamma}{}^{-1}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1} \\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\langle (F'\phi_\Gamma)\mathrm{p}, \mathrm{q}\rangle\theta'_{\tau_\Gamma A,\phi_\Gamma}{}^{-1}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}{}^{-1}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}{}^{-1}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle r_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(r_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}G'(r_{\Gamma,A})\phi'_{G(\Gamma\cdot A)}F'(\phi_{\Gamma\cdot A})F'(\rho^{-1}_{\Gamma,A})\rho'_{F\Gamma,\sigma_\Gamma A}{}^{-1}]\rangle \\
&= \langle \mathrm{p}, \mathrm{q}[\rho^{G'G}_{\Gamma,A}(\phi\star\phi')_{\Gamma\cdot A}\rho^{F'F}_{\Gamma,A}{}^{-1}]\rangle \\
&= (\psi_{\phi\star\phi'})_{\Gamma,A}
\end{aligned}
$$

We have first unfolded the action of $\boldsymbol{T''}$ and $\boldsymbol{\sigma'}$, then applied Lemma 44, unfolded the definition of $\psi_{\phi'}$ and ψ_ϕ, then used naturality of ϕ' and ϕ. Of course there are a lot of simplification steps, involving the preservation of the first projection by all the present pseudo cwf-morphisms and manipulation of cwf combinators.

Proposition 55. *There are pseudofunctors* $H : \mathbf{FL} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ *and* $H : \mathbf{LCC} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ *defined by:*

$$
\begin{aligned}
H\mathbb{C} &= (\mathbb{C}, T_\mathbb{C}) \\
HF &= (F, \sigma_F) \\
H\phi &= (\phi, \psi_\phi)
\end{aligned}
$$

Proof. First, note that as proved in Lemma 54, H is functorial on 2-cells.

For each $\mathbb{C}$ we need an invertible 2-cell $H_\mathbb{C} : Id_{(\mathbb{C},T_\mathbb{C})} \to H(Id_\mathbb{C})$, this will be the identity 2-cell since we have in fact $H(Id_\mathbb{C}) = (Id_\mathbb{C}, \sigma_{Id_\mathbb{C}}) = id_{\mathbb{C},T_\mathbb{C}}$ by construction of $\sigma_{Id_\mathbb{C}}$.

For each two functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{E}$ we need an isomorphism $H_{F,G} : HG \circ HF \to H(G \circ F)$, natural in F and G. It is given by $H_{F,G} = (1_{GF}, \psi_{1_{GF}})$. The naturality condition amounts to the fact that the following square commutes:

$$
\begin{tikzcd}
(G,\sigma_G)(F,\sigma_F) \arrow[r, "(1_{GF},\psi_{1_{GF}})"] \arrow[d, "(\phi,\psi_\phi)\star(\phi',\psi_{\phi'})"] & (GF,\sigma_{GF}) \arrow[d, "(\phi'\star\phi,\psi_{\phi'\star\phi})"] \\
(G',\sigma_{G'})(F',\sigma_{F'}) \arrow[r, "(1_{GF},\psi_{1_{GF}})"] & (G'F',\sigma_{G'F'})
\end{tikzcd}
$$

which is a direct consequence of Lemma 54. The coherence laws w.r.t. associativity of composition and identities also stems from Lemma 54. In fact, Lemma 54 implies that to check the validity of any equation involving vertical and horizontal compositions of pseudo cwf-transformations built with Lemma 53 and identity pseudo cwf-transformations, it suffices to check the equality of the corresponding base natural transformation, ignoring the modifications.

E Proofs of Section 6

Definition 56 (The pseudo cwf-morphism $\eta_{(\mathbb{C},T)}$). *For each cwf $(\mathbb{C},T)$, context Γ of $\mathbb{C}$ and type $A \in \mathrm{Type}(\Gamma)$. Consider:*

- *The identity functor $Id_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$,*
- *For each context Γ and type $A \in \mathrm{Type}(\Gamma)$, the functorial family $\sigma_\Gamma(A)$ defined by:*

$$\sigma_\Gamma(A)(\delta) = \mathrm{p}_{A[\delta]}$$
$$\sigma_\Gamma(A)(\delta,\gamma) = \langle \gamma \mathrm{p}, \mathrm{q} \rangle$$

- *The isomorphism $\rho_{\Gamma,A} = \mathrm{id}_{\Gamma \cdot A}$.*

Then by Lemma 45, it completes in an unique way to a pseudo cwf-morphism $\eta_{(\mathbb{C},T)} : (\mathbb{C},T) \to (\mathbb{C},T_{\mathbb{C}}) = HU((\mathbb{C},T))$.

Lemma 57 (The pseudonatural transformation η). *The family $\eta_{(\mathbb{C},T)}$: $(\mathbb{C},T) \to HU((\mathbb{C},T))$ is pseudonatural in $(\mathbb{C},T)$.*

Proof. For each pseudo cwf-morphism (F,σ), the pseudonaturality square relates two pseudo cwf-morphisms whose base functor is F. Hence, the necessary invertible pseudo cwf-transformation is obtained using Lemma 53 from the identity natural transformation on F. The coherence conditions are straightforward consequences of Lemma 54.

Definition 58 (The pseudo cwf-morphism $\epsilon_{(\mathbb{C},T)}$). *For each cwf $(\mathbb{C},T)$, context Γ of $\mathbb{C}$ and type $A \in \mathrm{Type}(\Gamma)$. Consider:*

- *The identity functor $Id_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$,*
- *For each context Γ and functorial family $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$, the type $\tau_\Gamma(\overrightarrow{A})$ defined by:*

$$\tau_\Gamma(\overrightarrow{A}) = \mathrm{Inv}(\overrightarrow{A}(\mathrm{id}_\Gamma))$$

- *The isomorphism $\rho_{\Gamma,A} : \mathrm{dom}(\overrightarrow{A}) \to \Gamma \cdot \mathrm{Inv}(\overrightarrow{A}(\mathrm{id}_\Gamma))$ is the isomorphism between $\overrightarrow{A}(\mathrm{id})$ and $\mathrm{p}_{(\mathrm{Inv}\overrightarrow{A}(\mathrm{id}_\Gamma))}$ in $\mathbb{C}/\Gamma$.*

Then by Lemma 45, it completes in an unique way to a pseudo cwf-morphism $\epsilon_{(\mathbb{C},T)} : HU(\mathbb{C},T) = (\mathbb{C},T_{\mathbb{C}}) \to (\mathbb{C},T)$.

Lemma 59 (The pseudonatural transformation ϵ). *The family* $\epsilon_{(\mathbb{C},T)}$: $HU(\mathbb{C},T) \to (\mathbb{C},T)$ *is pseudonatural in* $(\mathbb{C},T)$.

Proof. Exactly the same reasoning as for η.

Theorem 60. *We have the following biequivalences of* 2*-categories.*

$$\mathbf{FL} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}} \qquad\qquad \mathbf{LCC} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$$

Proof. We need to define invertible modifications $m : \eta\epsilon \to 1$ and $m' : \epsilon\eta \to 1$. Taken at each cwf $(\mathbb{C},T)$, $m_{(\mathbb{C},T)}$ and $m'_{(\mathbb{C},T)}$ are both generated from the identity natural transformation using Lemma 53. It is obvious by Lemma 54 that they satisfy the required coherence law.