Enumerating minimal dominating sets in K_t-free graphs and variants

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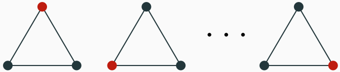
Typical question:

Given input I, list all objects of type X in I.

Examples:

- cycles, cliques, stable sets, dominating sets of a graph
- transversals of a hypergraph
- antichains of a partial order
- variable assignments satisfying a formula
- answers to a query
- trains to Paris leaving tomorrow before 10:00
- ...

Remark: possibly many objects!
$3^{n/3} \approx 1.4422^n$



Two perspectives about complexity

Input-sensitive: in terms of input size

Theorem (Fomin, Grandoni, Pyatkin, and Stepanov, 2008)

There is an $O(1.7159^n)$-time algorithm enumerating all minimal dominating sets in n-vertex graphs.

$\rightarrow$ *basically upper-bounds the number of objects*

Output-sensitive: in terms of input+output size

Theorem (Fredman and Khachiyan, 1996)

There is a $N^{o(\log N)}$-time algorithm enumerating all minimal dominating sets in n-vertex graphs, where $N = n + |\mathcal{D}(G)|$.

$\rightarrow$ *many techniques (reverse search, backtrack search, ordered generation, proximity search, etc.)*

“Fast” output-sensitive algorithms

Let n be input size, e.g., number of vertices of a graph G

Let d be output size, e.g., number of maximal cliques in G

output-polynomial

algo. stops in poly$(n + d)$*-time*

incremental-polynomial

outputs i^{th} *solution in* poly$(n + i)$*-time*

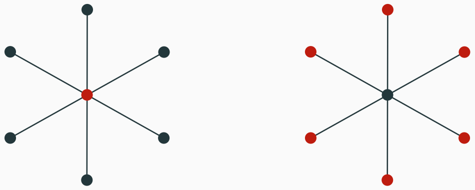
polynomial-delay

poly(n)*-time between two cons. outputs*

Minimal dominating sets

- $N(v)$: neighborhood of vertex v
- dominating set (DS): $D \subseteq V(G)$ s.t. $V(G) = D \cup N(D)$

  "D can see everybody else"
- minimal dominating set: inclusion-wise minimal DS



Private neighbors and irredundant sets

- $N(v)$: neighborhood of vertex v
- dominating set (DS): $D \subseteq V(G)$ s.t. $V(G) = D \cup N(D)$

  "D can see everybody else"
- minimal dominating set: inclusion-wise minimal DS
- private neighbor of $v \in D$:

  $$\text{vertex that is} \begin{cases} \text{dominated by } v, \text{ and} \\ \text{not dominated by } D \setminus \{v\} \end{cases} \quad \text{(possibly } v\text{)}$$
- irredundant set: $S \subseteq V(G)$ s.t. every $x \in S$ has a priv. neighbor

Observation

A DS is minimal if and only if it is irredundant.

if all its vertices have a private neighbor.

Minimal DS Enumeration (Dom-Enum)

input: a n-vertex graph G.

output: the set $\mathcal{D}(G)$ of minimal DS of G.

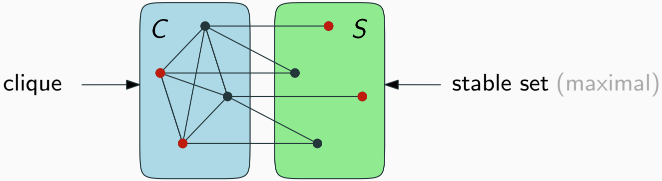
Dream goal: an output-poly. poly(N) algorithm, $N = n + |\mathcal{D}(G)|$

General case: open, best is quasi-polynomial $N^{o(\log N)}$

Known cases:

- **output poly.:** log(n)-degenerate graphs
- **incr. poly.:** chordal bipartite graphs, bounded conformality graphs
- **poly. delay:** degenerate, line, and chordal graphs
- **linear delay:** permutation and interval graphs, graphs with bounded clique-width, split and P_7-free chordal graphs

Dom-Enum in split graphs (Kanté et al., 2014)



Proposition (Kanté, Limouzy, Mary, and Nourine, 2014)

A set $D \subseteq V(G)$ is a minimal DS of G iff D dominates S and every $v \in D$ has a private neighbor in S.

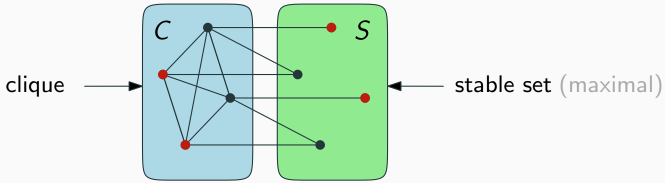
Then: $D \cap S = \{$all vertices not dominated by $D \cap C\}$

Enumeration: complete every irredundant set $X \subseteq C$ in S

→ the family of such X's is an independence set system

→ can be enumerated with linear delay

Dom-Enum in split graphs (Kanté et al., 2014)



Theorem (Kanté, Limouzy, Mary, and Nourine, 2014)

There is a linear-delay (and poly. space) algorithm enumerating minimal dominating sets in split graphs.

Then: $D \cap S = \{$all vertices not dominated by $D \cap C\}$

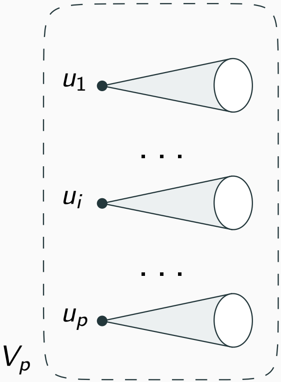
Enumeration: complete every irredundant set $X \subseteq C$ in S

$\rightarrow$ the family of such X's is an independence set system

$\rightarrow$ can be enumerated with linear delay

Peeling graphs

Goal: enumerating minimal DS one neighborhood at a time

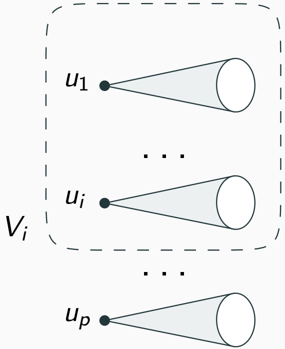


Peeling: sequence $(V_0, \dots, V_p)$ s.t.

1. $V_p = V(G)$
2. for $i \in \{1, \dots, p\}$:
   $V_{i-1} = V_i \setminus \{u_i\} \setminus N(u_i)$
3. $V_0 = \emptyset$

Extending partial solutions

Goal: enumerating minimal DS one neighborhood at a time



Dominating set (DS) of $\mathbf{V_i}$:

- $D \subseteq V(G)$ s.t. $V_i \subseteq D \cup N(D)$

  "D can see everybody else in V_i"

Plan:

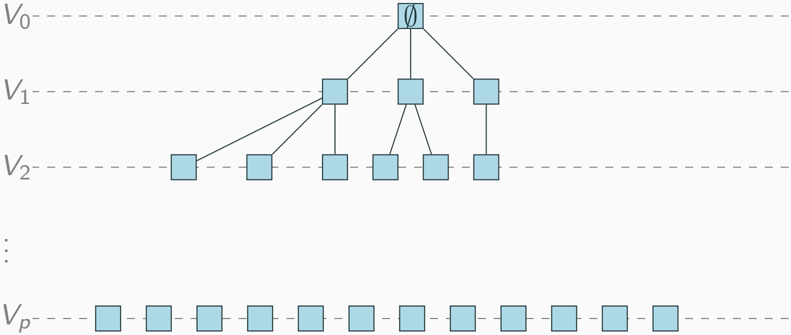
1. given minimal DS of V_i

   allowing vertices of $G - V_i$

2. enumerate those of V_{i+1}

   allowing vertices of $G - V_{i+1}$

The algorithm

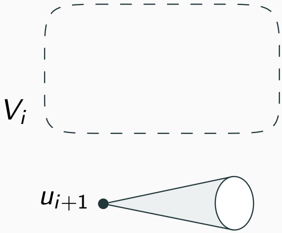


Important wanted properties:

- no cycle (using a *parent relation*: lexicographical order)
- no leaf before level p (no *exponential blowup*)

Minimal DS of V_{i+1} from those of V_i

Goal: extend each minimal DS D of V_i to a minimal DS of V_{i+1}



Observation:

- possibly D **minimally** dominates V_{i+1}
- if not then $D \cup \{u_{i+1}\}$ does

extension is always possible, hence

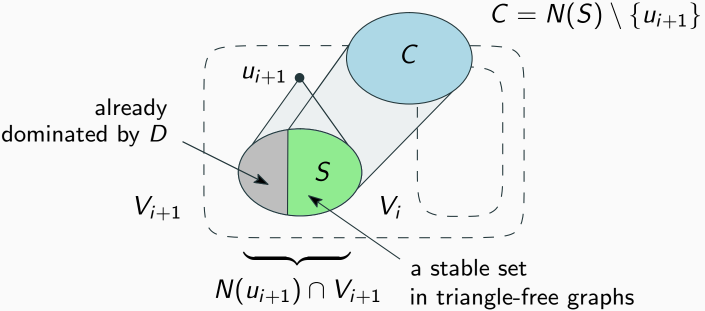
$$|\text{minimal DS of } V_i| \leq |\text{minimal DS of } V_{i+1}| \leq |\text{minimal DS of } G|$$

candidate extension of D: minimal set X s.t. $D \cup X$ dominates V_{i+1}

Lemma

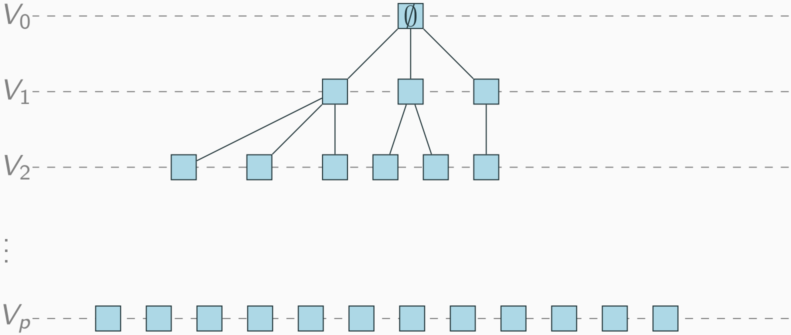
|candidate extensions of D| $\leq$ |minimal DS of G|

Which are the candidate extensions? The triangle-free case



- if u_{i+1} dom. by D: they only have to dominate S
  $\rightarrow$ exactly the minimal DS of Split(C, S)
- if u_{i+1} not dom. by D: they should also dom. u_{i+1}
  irredundant $\{t\} \cup Q$ s.t. $\begin{cases} t \in N(u_{i+1}) \\ Q \text{ minimal DS of Split}(C, S) \end{cases}$

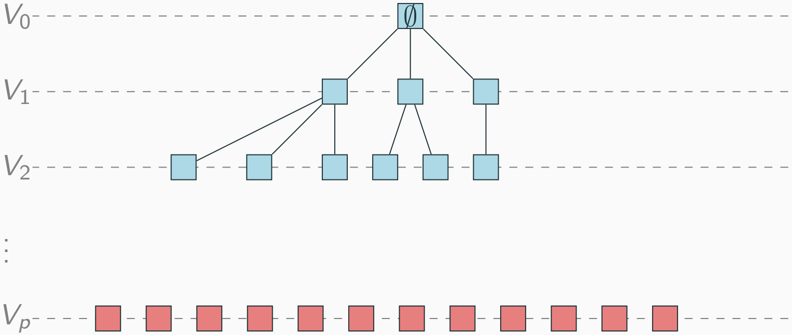
Complexity: the triangle-free case



For each minimal DS of V_i:

- compute all candidate extensions; in time $O(\text{poly}(n) \cdot |\mathcal{D}(G)|)$
- only keep the $X \cup D$'s that are minimal and children of D

Complexity: the triangle-free case



Theorem (Bonamy, D., Heinrich, and Raymond, 2019)

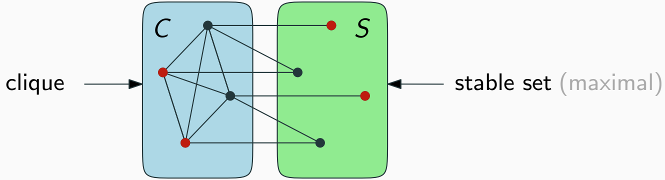
The set $\mathcal{D}(G)$ of minimal DS of any triangle-free graph G can be enumerated in time $O(\text{poly}(n) \cdot |\mathcal{D}(G)|^2)$ and polynomial space.

Bicolored graph

Bicolored graph: graph G with a bipartition A, B of $V(G)$

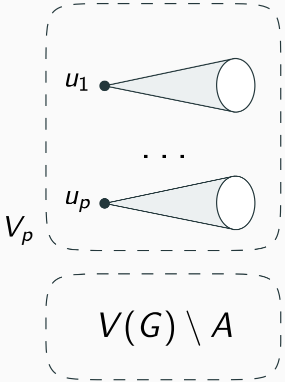
- $G(A)$: bicolored graph of bipartition $(A, V(G) \setminus A)$
- dominating set (DS) of $G(A)$: $D \subseteq V(G)$ s.t. $A \subseteq D \cup N(D)$

  "D can see everybody else in A"
- $\mathcal{D}(G, A)$: inclusion-wise minimal DS of $G(A)$



On split graphs of (maximal) stable set S, $\mathcal{D}(G) = \mathcal{D}(G, S)$

Goal: enum. minimal DS of $G(A)$ one neighborhood at a time



Peeling: sequence $(V_0, \ldots, V_{p+1})$ s.t.

1. $V_{p+1} = V(G)$
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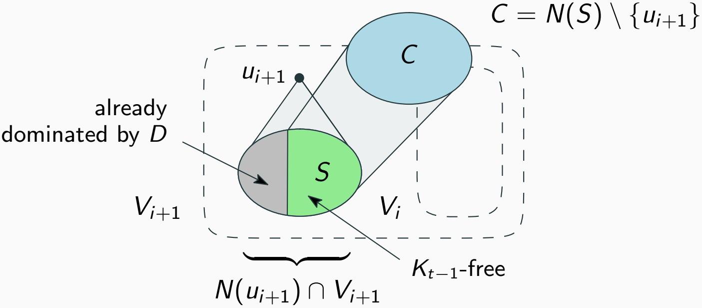

- if u_{i+1} dom. by D: they only have to dominate S
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- if u_{i+1} not dom. by D: they should also dom. u_{i+1}

irredundant $\underbrace{\{t\} \cup Q}_{\triangle!}$ s.t. $\begin{cases} t \in N(u_{i+1}) \\ Q \text{ minimal DS of } G(S \setminus \{t\} \setminus N(t)) \end{cases}$

Theorem (Bonamy, D., Heinrich, Pilipczuk, and Raymond)

The set $\mathcal{D}(G)$ of minimal DS of any graph G can be enumerated in time $O(n^{2^{t+1}} \cdot |\mathcal{D}(G)|^{2^t})$ and poly. space where $t = \omega(G) + 1$.

Future work:

- complexity improvements? delay is still open for bipartite graphs

Theorem (Bonamy, D., Heinrich, and Raymond, 2019)

Deciding if a vertex set S can be extended into a minimal DS is NP-complete in bipartite graphs.

- extensions to other classes?

  $K_t + K_2$-free, paw-free, diamond-free ✓

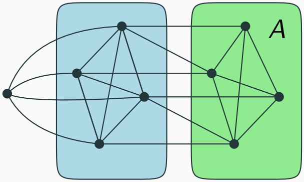
  C_4-free ? ✗

  comparability and unit disk ? ✗

Limitations of bicolored minimal DS enumeration

Observation

Enumerating the minimal DS of $G(S)$ is harder than Dom-Enum whenever A can contain an arbitrary large clique.



Dom-Enum is as hard in co-bipartite graphs as in general graphs (Kanté et al., 2014)