Enumerating minimal dominating sets in the incomparability graphs of bounded dimension posets

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Typical question:

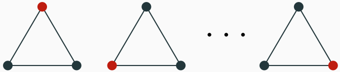
Given input I, list all objects of type X in I.

Examples:

- cycles, cliques, stable sets, dominating sets of a graph
- transversals of a hypergraph
- antichains of a partial order
- variable assignments satisfying a formula
- answers to a query
- trains to Paris leaving tomorrow before 10:00
- ...

Remark: possibly many objects!

$3^{n/3} \approx 1.4422^n$



Two perspectives about complexity

Input-sensitive: in terms of input size

Theorem (Fomin, Grandoni, Pyatkin, and Stepanov, 2008)

There is an $O(1.7159^n)$-time algorithm enumerating all minimal dominating sets in n-vertex graphs.

$\rightarrow$ *basically upper-bounds the number of objects*

Output-sensitive: in terms of input+output size

Theorem (Fredman and Khachiyan, 1996)

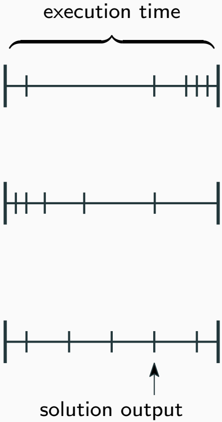
There is a $N^{o(\log N)}$-time algorithm enumerating all minimal dominating sets in n-vertex graphs, where $N = n + |\mathcal{D}(G)|$.

$\rightarrow$ *many techniques (reverse search, flashlight search, ordered generation, proximity search, etc.)*

“Fast” output-sensitive algorithms

Let n be input size, e.g., number of vertices of a graph G

Let d be output size, e.g., number of maximal cliques in G



output-polynomial

algo. stops in poly$(n + d)$*-time*

incremental-polynomial

outputs i^{th} *solution in* poly$(n + i)$*-time*

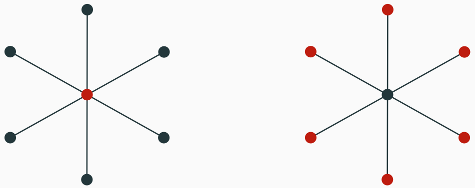
polynomial-delay

poly(n)*-time between two cons. outputs*

Minimal dominating sets

- $N(v)$: neighborhood of vertex v, $N[v] = N(v) \cup \{v\}\}$
- dominating set (DS): $D \subseteq V(G)$ s.t. $V(G) = D \cup N(D)$

  "D can see everybody else"
- minimal dominating set: inclusion-wise minimal DS



Private neighbors & Irredundant sets

- $N(S) = \bigcup_{v \in S} N(v) \setminus S$: neighborhood of vertex set S
- dominating set (DS): $D \subseteq V(G)$ s.t. $V(G) = D \cup N(D)$

  "D can see everybody else"
- minimal dominating set: inclusion-wise minimal DS
- private neighbors $\mathrm{Priv}(D, v)$ of $v \in D$:

  vertices that are $\begin{cases} \text{dominated by } v, \text{ and} \\ \text{not dominated by } D \setminus \{v\} \end{cases}$ (possibly v)
- irredundant set: $S \subseteq V(G)$ s.t. every $x \in S$ has a priv. neighbor

Observation ★

A DS is minimal if and only if it is irredundant.

if all its vertices have a private neighbor.

if $\mathrm{Priv}(D, v) \neq \emptyset$ *for all* $v \in D$

Minimal DS Enumeration (Dom-Enum)

input: a n-vertex graph G.

output: the set $\mathcal{D}(G)$ of minimal DS of G.

Dream goal: an output-poly. poly(N) algorithm, $N = n + |\mathcal{D}(G)|$

General case: open, best is quasi-polynomial $N^{o(\log N)}$

Hard case: co-bipartite graphs

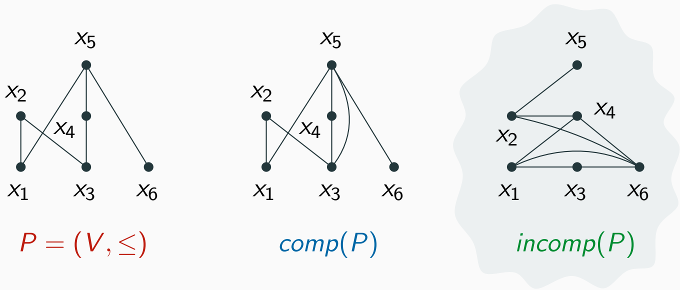
Known cases:

- **output poly.:** log(n)-degenerate graphs, Kt-free graphs
- **incr. poly.:** chordal bipartite graphs, bounded conformality graphs
- **poly. delay:** degenerate, line, and chordal graphs
- **linear delay:** permutation and interval graphs, etc.

- poset $P = (V, \leq)$: refl., trans., antisymmetric relation $\leq$ on V

  $x \leq x \qquad x \leq y \wedge y \leq x \implies x = y$
- comp(P): graph on V s.t. $uv \in E$ if $u \leq v$ or $v \leq u$
- incomp(P): complementary of comp(P)

  i.e., an edge for every incomparable pair of elements



Minimal DS Enumeration (Dom-Enum)

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General case: open, best is quasi-polynomial $N^{o(\log N)}$

Hard case: co-bipartite graphs, hence incomparability graphs

Known cases: ⚠

- **output poly.:** log(n)-degenerate graphs, Kt-free graphs
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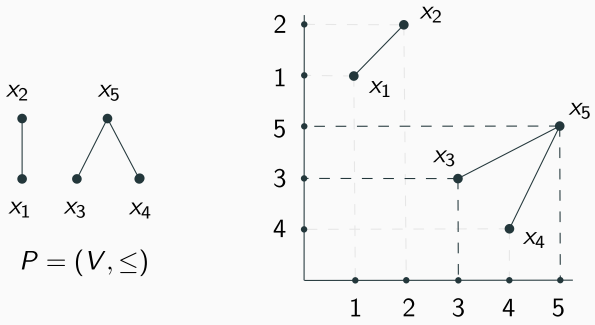
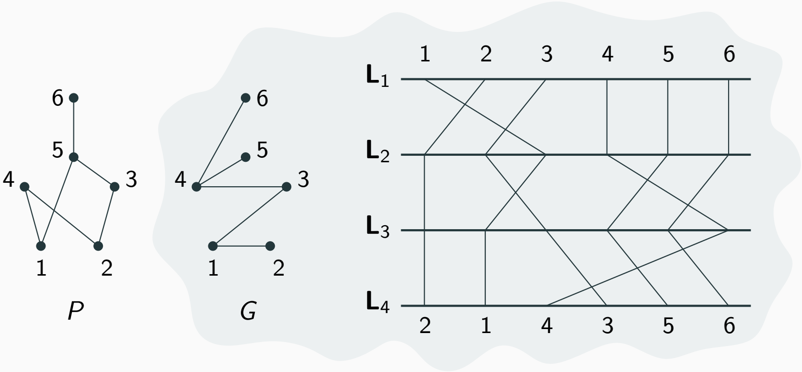
Poset dimension of $P = (V, \leq)$:

Least integer d such that elements of P can be embedded into $\mathbb{R}^d$ in such a way that $x \leq y$ in P if and only if the point of x is below the point of y with respect to the product order of $\mathbb{R}^d$



Theorem (Golumbic, Rotem, and Urrutia, 1983)

A graph G is the incomparability graph of a poset of dimension d if and only if it is the intersection graph of the concatenation of d permutation diagrams.



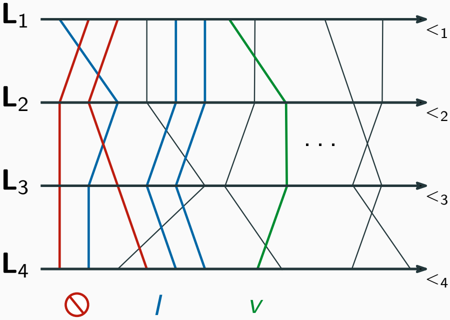
Observation ★

A DS is minimal if and only if it is irredundant.

if all its vertices have a private neighbor.

- make grow irredundant sets to minimal dominating sets
- ensure that each constructed partial set leads to a solution

→ from left to right!



Definition of "right":

- $\mathsf{R}(I) = \{v \in V \setminus I : \exists j,\ \forall u \in I,\ u <_j v\}$

Right-Extension Problem:

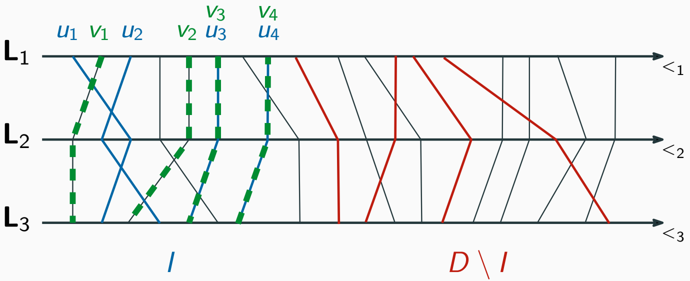
Given $I \subseteq V(G)$, decide whether I can be *extended to the right into a min DS*, i.e., whether $\exists D \in \mathcal{D}(G)$ s.t. $I \subseteq D$ and $D \setminus I \subseteq \mathrm{R}(I)$

Observation ♢

Set $I = \{u_1, \dots, u_p\}$ *can be extended to the right iff*

$$\exists v_1, \dots, v_p \in \mathrm{Priv}(I, u_1) \times \dots \times \mathrm{Priv}(I, u_p)$$

$$s.t.\ R(I) \setminus N[v_1, \dots, v_p] \text{ dominates } G - N[I]$$

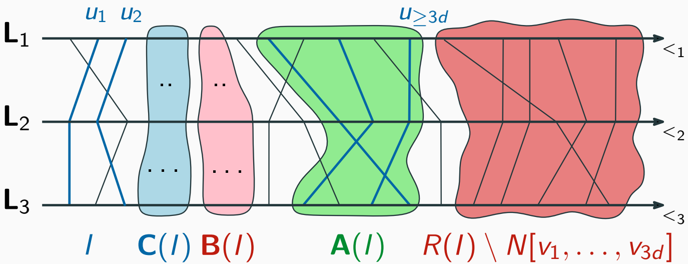


Extension problem: the unbounded case $I \geq 3d$

- 1st layer of I: $\mathbf{A}(I) = (a_1, \ldots, a_d)$ so that $a_1 = \mathrm{Max}_{<_1}(I)$, and $\forall i \in \{2, \ldots, d\}$, $a_i = \mathrm{Max}_{<_i}(I \setminus \{a_1, \ldots, a_{i-1}\})$
- $\mathbf{B}(I) = \mathbf{A}(I \setminus \mathbf{A}(I))$
- $\mathbf{C}(I) = \mathbf{A}(I \setminus (\mathbf{A}(I) \cup \mathbf{B}(I)))$

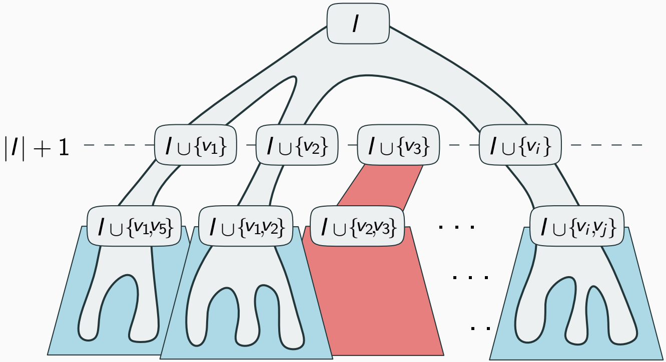
Theorem ♢

Set I, $|I| \geq 3d$ with $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} = \{u_1, \ldots, u_{3d}\}$ can be ext. iff
$\exists v_1, \ldots, v_{3d} \in \mathrm{Priv}(I, u_1) \times \cdots \times \mathrm{Priv}(I, u_{3d})$
s.t. $R(I) \setminus N[v_1, \ldots, v_{3d}]$ *dominates* $G - N[I]$



The algorithm

- start with $I = \emptyset$
- for every $v \in \mathsf{R}(I)$, check if $I' = I \cup \{v\}$ *extends to the right into a min DS*, i.e., whether $\exists D \in \mathcal{D}(G)$ s.t. $I' \subseteq D$ and $D \setminus I' \subseteq \mathsf{R}(I')$
- explore if it is the case, AND, if $I = \mathrm{Parent}(I') = I' \setminus \mathrm{Max}_{<_1}(I')$



Main theorem and future work

Theorem (Bonamy, D., Micek, and Nourine)

The set $\mathcal{D}(G)$ of minimal DS of incomp. graphs of posets of dimension d can be enumerated in time $O(n^{3d+4})$ and poly. space.

- complexity improvements? can we get $f(d) \cdot n^{O(1)}$ delay?
- what about comparability graphs of bounded dimension?

Theorem (Bonamy, D., Micek, and Nourine)

Minimal DS of comp. graphs of posets of dimension d can be enumerated in incremental-polynomial time and poly. space.

- remaining important cases:

  C_4-free ? ✗

  (general) comparability graphs? ✗

  unit disk? ✗