Avoidable paths in graphs

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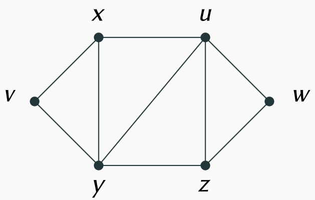
- A graph G is chordal if every induced cycle in G is a triangle.
- A vertex $v \in V(G)$ is simplicial if its neighborhood is a clique.

Theorem (Dirac, 1961)

Every chordal graph has a simplicial vertex.



A chordal graph and a simplicial vertex v.

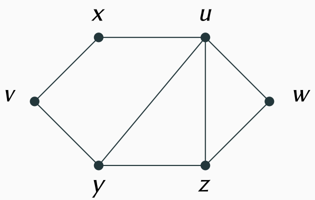
Vertex u is not simplicial.

Generalization: avoidable vertices

- A vertex $v \in V(G)$ is avoidable if every induced path on three vertices with middle vertex v is contained in an induced cycle in G.

Theorem (Ohtsuki, Cheung, and Fujisawa, 1976)

Every graph has an avoidable vertex.



A (non-chordal) graph and an avoidable vertex v.

Vertex u is not avoidable (xuw is not in an induced cycle).

- A vertex $v \in V(G)$ is avoidable if every induced path on three vertices with middle vertex v is contained in an induced cycle in G.

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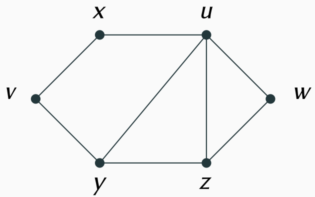
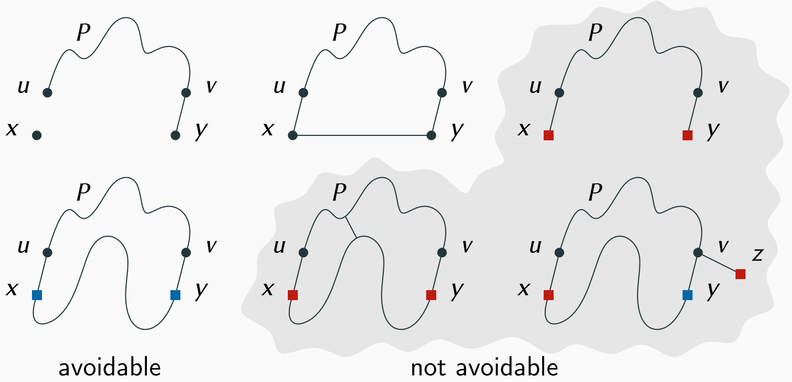


simplicial $\implies$ avoidable
in chordal graphs:
simplicial $\iff$ avoidable

A (non-chordal) graph and an avoidable vertex v.
Vertex u is not avoidable (xuw is not in an induced cycle).

- An extension of an induced path P in G is an induced path xPy in G for some vertices $x, y \in V(G)$.
- A path is failing if it is not contained in an induced cycle of G.
- A path is avoidable if it is induced and has no failing extension.



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Conjecture A (Beisegel et al., 2019)

For every positive integer k, every graph either is P_k-free or contains an avoidable P_k.

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Conjecture A (Beisegel et al., 2019)

For every positive integer k, every graph either is P_k-free or contains an avoidable P_k.

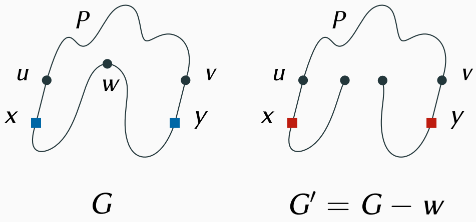
Theorem (Chvátal et al., 2002)

For every positive integer k, every $C_{\geq k+3}$-free graph either is P_k-free or contains an avoidable P_k.

Avoidable paths in subgraphs

- An extension of an induced path P in G is an induced path xPy in G for some vertices $x, y \in V(G)$.
- A path is failing if it is not contained in an induced cycle of G.
- A path is avoidable if it is induced and has no failing extension.

→ Given a subgraph G' of G, we say that P is an avoidable path of G in G' if it is avoidable in G and $V(P) \subseteq V(G')$.



A stronger induction hypothesis

Basic property H_B

Given a positive integer k and a graph G, the property $H_B(G,k)$ holds if either G is P_k-free or there is an avoidable P_k in G.

Refined property H_R

Given a positive integer k, a graph G and a vertex $u \in V(G)$, the property $H_R(G,k,u)$ holds if either $G - N[u]$ is P_k-free or there is an avoidable P_k of G in $G - N[u]$.

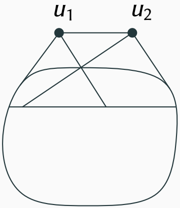
Given a positive integer k and a graph G, the property $H_R(G,k)$ holds if $H_R(G,k,u)$ holds for every $u \in V(G)$.

- The conjecture reads as: $H_B(G,k)$ holds for every G and k.

⚠ Property $H_R(G,k)$ does not directly imply property $H_B(G,k)$.

Lemma B

Let k be a positive integer, G a graph and $u_1 u_2$ an edge of G.
Let G' be the graph obtained from G by merging the two vertices u_1 and u_2 into one vertex u. If $G' - N[u]$ contains a P_k, then $H_R(G', k, u)$ implies $H_R(G, k, u_1)$.



G

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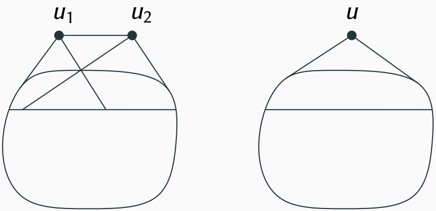

G $\qquad$ $G' = G_{|u_1 u_2 \rightsquigarrow u}$

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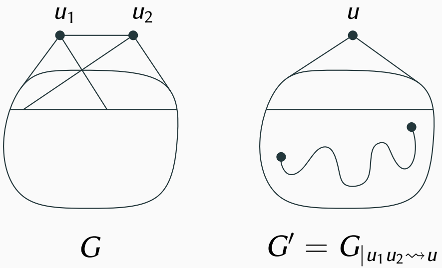

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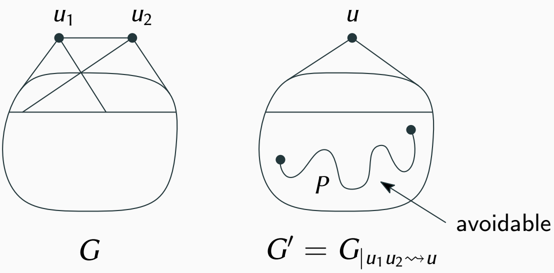

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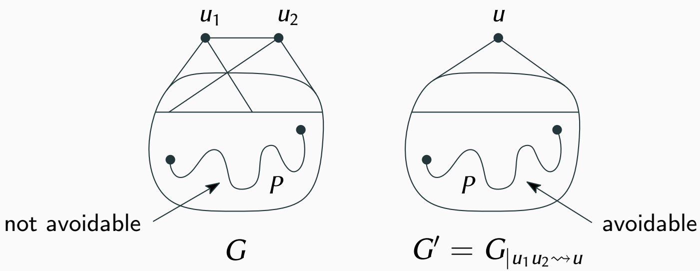

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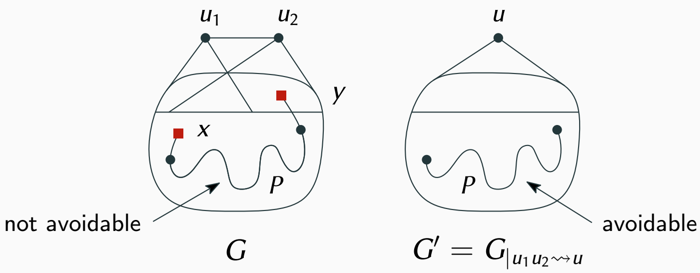

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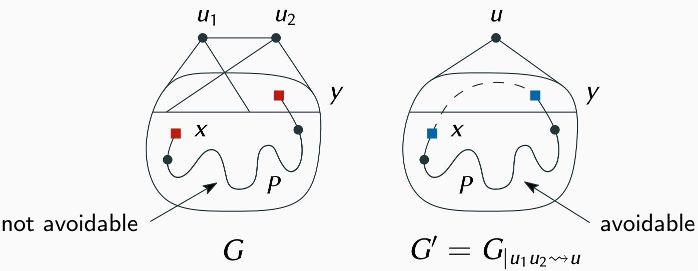

□

Proof of Conjecture A

Theorem (Bonamy, D., Hatzel, and Thiebaut, 2019)

For every positive integer k and every graph G, both properties $\mathcal{H}_B(G, k)$ and $\mathcal{H}_R(G, k)$ hold.

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- Consider a counterexample G minimum with respect to $|V(G)|$.

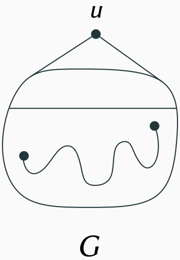
→ We show that $H_R(G,k)$ and $H_B(G,k)$ hold for every k to obtain a contradiction.

Proof of Conjecture A: H_R property

Lemma

The property $H_R(G, k)$ holds for every k.

- By contradiction: suppose there exists u and a P_k in $G - N[u]$, and every P_k in $G - N[u]$ has a failing extension in G.



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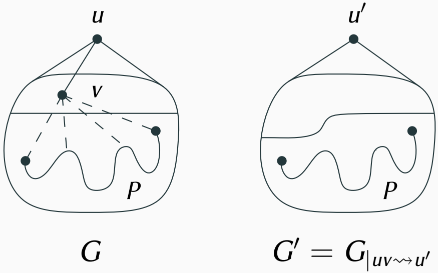

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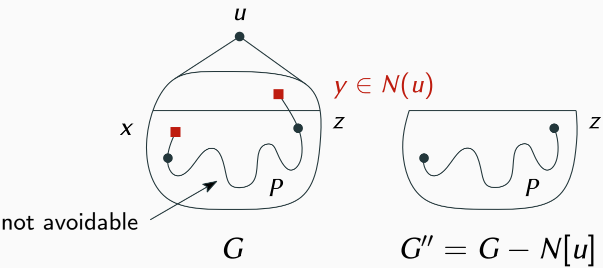

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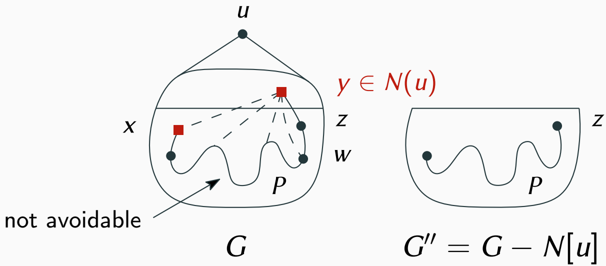
- Consider an induced path P and a failing extension xPy in G:

→ $xP - z$ does not dominate y!



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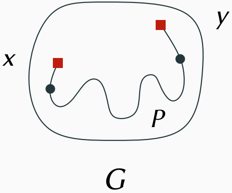

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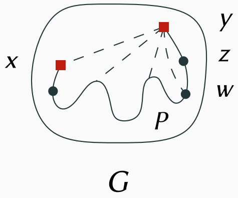

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An algorithmic proof

Theorem (Bonamy, D., Hatzel, Thiebaut, 2019)

For every positive integer k, every graph either is P_k-free or contains an avoidable P_k.

Algorithm 1 finds an avoidable path of given length in a given graph, if any.

procedure FindAvoidablePathRefined(G, k, u)
    for all $v \in N(u)$ **do**
        if InducedPath$(G - N[\{u, v\}], k) \neq$ `null` **then**
            $G' \leftarrow G$ with u and v merged into u'
            return FindAvoidablePathRefined(G', k, u')
    return FindAvoidablePath$(G - N[u], k)$

procedure FindAvoidablePath(G, k)
    for all $u \in V(G)$ **do**
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            return FindAvoidablePathRefined(G, k, u)
    return InducedPath(G, k)

Further (1)

Corollary 1

For every positive integer k, graph G and subset $X \subseteq V(G)$ such that $G[X]$ is connected, either $G - N[X]$ is P_k-free or there is an avoidable P_k of G in $G - N[X]$.

Further (1)

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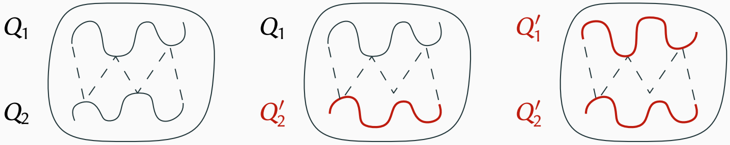
Corollary 2

For every positive integer k and graph G, either G does not contain two non-adjacent P_k, or it contains two non-adjacent avoidable P_k.



Further (2)

Question

For every positive integer k, does every graph G either not contain two disjoint P_k, or contain two disjoint avoidable P_k?

- Yes for $k = 1, 2$ [Beisegel et al. 2019].

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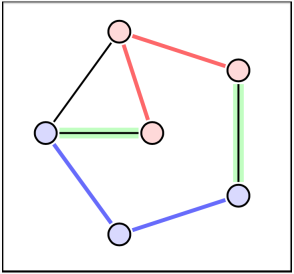

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