Labeled sample compression schemes for complexes of oriented matroids

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Abstract We show that the topes of a complex of oriented matroids (abbreviated COM) of VC-dimension d admit a proper labeled sample compression scheme of size d. This considerably extends results of Moran and Warmuth and the authors and is a step towards the sample compression conjecture – one of the oldest open in computational learning theory. On the one hand, our approach exploits the rich combinatorial cell structure of COMs via oriented matroid theory. On the other hand viewing tope graphs of COMs as partial cubes creates a fruitful link to metric graph theory.

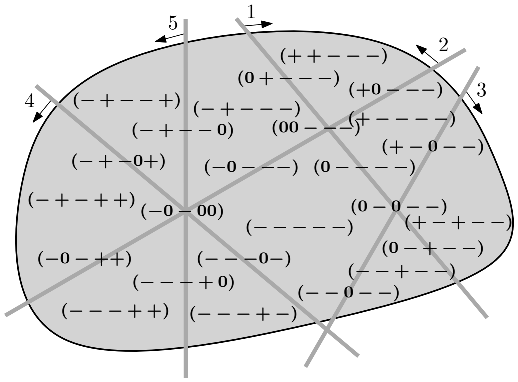
1. Introduction

Littlestone and Warmuth [41] introduced sample compression schemes as an abstraction of the underlying structure of learning algorithms. The idea is that a sample compression scheme of size k allows to compress any list of labeled examples to a list of size k, while still being able to reconstruct any element of the list. Usually, the set of possible lists of examples is coded as a set system $\mathcal{C}$, called *concept class*. There are two types of sample compression schemes: labeled [41, 30] and unlabeled [29, 6, 39]. A labeled compression scheme of size k compresses every sample of $\mathcal{C}$ to a labeled subsample of size $\leq k$ and an unlabeled compression scheme of size k compresses every sample of $\mathcal{C}$ to a subset of size $\leq k$ of the domain of the sample (see the end of the introduction for precise definitions). The Vapnik-Chervonenkis dimension (*VC-dimension*) of a set system, was introduced by Vapnik and Chervonenkis [51] as a complexity measure of set systems. VC-dimension is central in PAC-learning and plays an important role in combinatorics, algorithmics, discrete geometry, and combinatorial optimization. In particular, it coincides with the rank in the theory of (complexes of) oriented matroids. Furthermore, within machine learning and closely tied to the topic of this paper, the importance of the VC-dimension is illustrated by the fact that a sample compression scheme of a concept class of VC-dimension d needs to be of size at least d. Conversely, the *sample compression conjecture* of [30, 41] states that *any set family of VC-dimension d has a sample compression scheme of size $O(d)$*. This question remains one of the oldest open problems in computational learning theory. The best-known general bound is due to Moran and Yehudayoff [45] and shows that labeled compression schemes of size $O(2^d)$ exist. From below, Pálvölgyi and Tardos [49] recently exhibited a concept class of VC-dimension 2 with no unlabeled compression scheme of size 2.

For more structured concept classes better upper bounds are known. Ben-David and Litman [6] proved a compactness lemma, which reduces the existence of labeled or unlabeled compression schemes for arbitrary concept classes to finite concept classes. They also obtained unlabeled compression schemes for regions in arrangements of affine hyperplanes (which correspond to realizable affine oriented matroids in our language). Finally, they obtained sample compression schemes for concept classes by embedding them into concept classes for which such schemes were known. Helmbold, Sloan, and Warmuth [34] constructed unlabeled compression schemes of size d for intersection-closed concept classes of VC-dimension d. They compress each sample to a minimal generating set and show that the size of this set is upper bounded by the VC-dimension. An important class for which positive results are available is given by ample set systems [23] (originally introduced as lopsided sets [40]). They capture an important variety of combinatorial objects, e.g., (conditional) antimatroids [25], diagrams of (upper locally) distributive lattices, median graphs or CAT(0) cube complexes [3] and were rediscovered in various disguises, e.g. as *extremal for (reverse) Sauer* [8] and *shattering-extremal* [46]. Moran and Warmuth [44] provide labeled sample compression schemes of size d for ample set systems of VC-dimension d. For maximum concept classes (a subclass of ample set systems) unlabeled sample compression schemes of size d have been designed by Chalopin et al. [9]. They also combinatorially characterized unlabeled compression schemes for ample classes via the existence of *unique sink orientations* of their graphs. However, the existence of such orientations is open.

A structure somewhat opposed to ample classes are Oriented Matroids (OMs) [7]. These objects provide a framework for the analysis of combinatorial properties of geometric configurations occurring in discrete geometry and in machine learning. Point and vector configurations, order types, hyperplane and pseudo-line arrangements, convex polytopes, directed graphs, and linear programming find a common generalization in this language. The Topological Representation Theorem [31] connects the theory of OMs on a deep level to arrangements of pseudo-spheres and distinguishes it from the theory of ordinary matroids.

Complexes of Oriented Matroids (COMs) were introduced not long ago in [4] as a natural common generalization of ample classes and OMs. Indeed, ample classes can be seen as COMs with cubical cells, while OMs are COMs with a single cell. In general COMs the cells are OMs and the resulting cell complex is contractible. In the realizable setting, a COM corresponds to the intersection pattern of a hyperplane arrangement with an open convex, see Figure 1. Recently, realizable COMs have received attention in the theory of neural codes [36]. Other examples of COMs not contained in the class of OMs and ample classes include linear extensions of a poset or acyclic orientations of mixed graphs [4], CAT(0) Coxeter complexes [4, 33], hypercellular and Pasch graphs [15], and Affine Oriented Matroids [5, 20]. Apart from the above, COMs have spiked research and appear as the next natural class to attack in different areas such as combinatorial semigroup theory [43], algebraic combinatorics in relation to the Varchenko determinant [35], with respect to neural codes [36], poset cones [22], as well as sweeping sequences [48]. In particular, relations to COMs have already been established within sample compression, see [9, 16, 17].



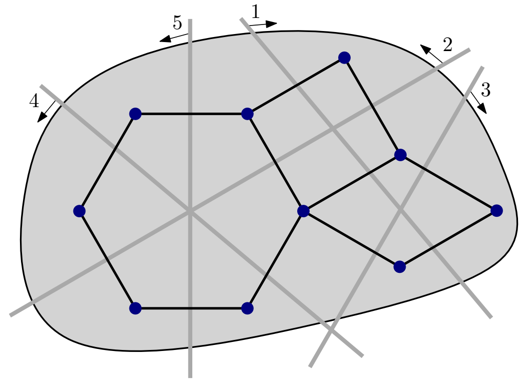


FIGURE 1. A realizable COM $\widetilde{\mathcal{M}}$ and its tope graph $\widetilde{G} = G(\widetilde{\mathcal{M}})$.

A central feature of COMs, is that they can be studied via their tope graphs. Indeed, the recent characterization of their tope graphs [37] establishes an embedding of the theory of COMs into metric graph theory, with theoretical and algorithmic implications. On one hand, tope graphs of COMs form a subclass of the ubiquitous metric graph class of partial cubes, i.e., isometric subgraphs of hypercubes, with applications ranging from interconnection networks [32] and media theory [28] to chemical graph theory [27]. On the other hand, tope graphs of COMs can be recognized in polynomial time [26, 37]. The graph theoretic view has been used in several recent publications, see [38, 42, 14].

As we explain later, COMs can be defined as sets of sign vectors, which is another unifying feature for OMs and ample set systems. This turns out to be beneficial for the present paper, since the language of sign vectors is perfectly suited for defining sample compression schemes formally. The following formulation is due to [10], for classical formulations, see [41, 44, 45]. Let U be a finite set, called the *universe* and $\mathcal{C}$ be a family of subsets of U, called a *concept class*. We view $\mathcal{C}$ as a set of $\{-1,+1\}$-vectors, i.e., $\mathcal{C} \subseteq \{-1,+1\}^U$. We also consider sets of $\{-1,0,+1\}$-vectors, i.e., subsets of $\{-1,0,+1\}^U$ endowed with the product order $\leq$ between sign vectors relative to the ordering $0 \leq -1,+1$. The sign vectors of the set $\downarrow \mathcal{C} = \bigcup_{C\in\mathcal{C}}\{S \in \{-1,0,+1\}^U : S \leq C\}$ are *realizable samples* for $\mathcal{C}$. A *labeled sample compression scheme* of size k for a concept class $\mathcal{C} \subseteq \{-1,+1\}^U$ is a pair (α,β) of mappings, where $\alpha : \downarrow \mathcal{C} \to \{-1,0,+1\}^U$ is called *compression function* and $\beta : \{-1,0,+1\}^U \to \{-1,+1\}^U$ the *reconstruction function* such that for any realizable sample $S \in \downarrow \mathcal{C}$, it holds :

$$\alpha(S) \leq S \leq \beta(\alpha(S)) \text{ and } |\underline{\alpha}(S)| \leq k,$$

where $\underline{\alpha}(S)$ is the support of the sign vector $\alpha(S)$. The condition $S \leq \beta(\alpha(S))$ means that the restriction of $\beta(\alpha(S))$ on the support of S coincides with the input sample S. In particular, if S is a concept of $\mathcal{C}$, then $\beta(\alpha(S)) = S$, i.e., the reconstructor must reconstruct the input concept. A labeled sample compression scheme is *proper* if $\beta(\alpha(S)) \in \mathcal{C}$ for all $S \in \downarrow C$. Notice that the labeling compression schemes of size $O(2^d)$ of [45] are not proper and they use additional information. The compression schemes developed in [10] for balls in graphs are proper but also use additional information. The *unlabeled sample compression schemes* [39] (which are not the subject of this paper) are defined analogously, with the difference that in the unlabeled case $\alpha(S)$ is a subset of size at most k of the support of S.

The definition of labeled compression scheme implies that if $\mathcal{C}$ is an extension of a concept class $\mathcal{C}'$ and (α,β) is a labeled sample compression scheme for $\mathcal{C}$, then (α,β) is a labeled sample compression scheme for $\mathcal{C}'$. However, (α,β) is no longer proper for $\mathcal{C}'$. Still, this yields an approach (suggested in [50] and implicit in [30])to obtain improper schemes. For instance, using the result of Moran and Warmuth [44] that ample set systems of VC-dimension d admit labeled sample compression schemes of size d, one can try to extend a given set system to an ample set system without increasing the VC-dimension too much and then apply their result. Two recent works of the authors go into this direction: In [16] it is shown that partial cubes of VC-dimension 2 can be extended to ample set systems of VC-dimension 2. Furthermore, in [17] it is shown that OMs and complexes of uniform oriented matroids can be extended to ample set systems without increasing the VC-dimension. Thus, in these classes there exist improper labeled sample compression schemes whose size is the VC-dimension.

On the other hand, there exist partial cubes of VC-dimension 3 that cannot be extended to ample set systems of VC-dimension 3, see [17], as well as set systems of VC-dimension 2, that cannot be extended to partial cubes of VC-dimension 2, see [16]. In [17] it is conjectured that every COM of VC-dimension d can be extended to an ample set system of VC-dimension d. This would yield improper labeled sample compression schemes for COMs of size d.

Here, we follow a different strategy to give (stronger) proper labeled sample compression schemes for general COMs. Our work extends the result of [44] substantially from ample set systems to the class of COMs. More precisely, we show that the set system defined by the topes of a COM satisfies the strong form of the sample compression conjecture, i.e., COMs of VC-dimension d admit *proper labeled sample compression schemes of size d.*

2. Preliminaries

2.1. **OMs and COMs.** We recall the basic theory OMs and COMs from [7] and [4], respectively. Let U be a set of size m and let $\mathcal{L}$ be a *system of sign vectors*, i.e., maps from U to $\{-1,0,+1\}$. The elements of $\mathcal{L}$ are referred to as *covectors* and denoted by capital letters X,Y,Z. For $X \in \mathcal{L}$, the subset $\underline{X} = \{e \in U : X_e \neq 0\}$ is the *support* of X and its complement $X^0 = U \setminus \underline{X} = \{e \in U : X_e = 0\}$ is the *zero set* of X. For $X,Y \in \mathcal{L}$, $\mathrm{Sep}(X,Y) = \{e \in U : X_eY_e = -1\}$ is the *separator* of X and Y. The *composition* of X and Y is the sign vector $X \circ Y$, where $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ if $X_e = 0$.

Definition 1. An *oriented matroid* (OM) is a system of sign vectors $\mathcal{M} = (U,\mathcal{L})$ satisfying

(C) (Composition) $X \circ Y \in \mathcal{L}$ for all $X,Y \in \mathcal{L}$.

(SE) (Strong elimination) for each pair $X,Y \in \mathcal{L}$ and for each $e \in \mathrm{Sep}(X,Y)$, there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in U \setminus \mathrm{Sep}(X,Y)$.

(Sym) (Symmetry) $-\mathcal{L} = \{-X : X \in \mathcal{L}\} = \mathcal{L}$, that is, $\mathcal{L}$ is closed under sign reversal.

We only consider *simple* systems of sign-vectors $\mathcal{L}$, i.e., if for each $e \in U$, $\{X_e : X \in \mathcal{L}\} = \{-1,0,+1\}$ and for all $e \neq f$ there exist $X,Y \in \mathcal{L}$ with $\{X_eX_f, Y_eY_f\} = \{+1,-1\}$.

Let $\leq$ be the product ordering on $\{-1,0,+1\}^U$ relative to the ordering $0 \leq -1,+1$. The poset $(\mathcal{L},\leq)$ of an OM $\mathcal{M}$ with an artificial global maximum $\hat{1}$ forms the (graded) *big face lattice* $\mathcal{F}_{\mathrm{big}}(\mathcal{M})$. The length of maximal chains of $\mathcal{F}_{\mathrm{big}}(\mathcal{M})$ minus 1 is the *rank* of $\mathcal{L}$ and denoted $\mathrm{rank}(\mathcal{M})$. The rank of the underlying matroid $\underline{\mathcal{M}}$ equals $\mathrm{rank}(\mathcal{M})$ [7, Thm 4.1.14].

The *topes* $\mathcal{T}$ of $\mathcal{M}$ are the co-atoms of $\mathcal{F}_{\mathrm{big}}(\mathcal{M})$. By simplicity the topes are $\{-1,+1\}$-vectors and $\mathcal{T}$ can be seen as a family of subsets of U. For each $T \in \mathcal{T}$, $e \in U$ belongs to the set if and only if $T_e = +1$. The *tope graph* $G(\mathcal{M})$ of an OM $\mathcal{M}$ is the subgraph of the hypercube $\mathrm{Q}(U)$ induced by the vertices corresponding to $\mathcal{T}$, see Figure 1. It is well-known that tope graphs of OMs are partial cubes and that $\mathcal{M}$ can be recovered up to isomorphism from its tope graph $G(\mathcal{M})$, see e.g. [7]. *Thus, OMs can be treated as tope graphs.*

Another important axiomatization of OMs is in terms of *cocircuits* of $\mathcal{L}$. These are the atoms of $\mathcal{F}_{\mathrm{big}}(\mathcal{L})$. Their collection is denoted by $\mathcal{C}^*$ and axiomatized as follows: a system of sign vectors $(U,\mathcal{C}^*)$ is an *oriented matroid* (OM) if $\mathcal{C}^*$ satisfies (Sym) and the two axioms:

(Inc) (Incomparability) $\underline{X} \subseteq \underline{Y}$ implies $X = \pm Y$ for all $X,Y \in \mathcal{C}^*$.

(E) (Elimination) for each pair $X,Y \in \mathcal{C}^*$ with $X \neq -Y$ and for each $e \in \mathrm{Sep}(X,Y)$, there exists $Z \in \mathcal{C}^*$ such that $Z_e = 0$ and $Z_f \in \{0, X_f, Y_f\}$ for all $f \in U$.

The set $\mathcal{L}$ of covectors can be derived from $\mathcal{C}^*$ by taking the closure of $\mathcal{C}^*$ under composition.

COMs are defined by replacing the global axiom (Sym) with a weaker local axiom:

Definition 2. A *complex of oriented matroids* (COMs) is a system of sign vectors $\mathcal{M} = (U,\mathcal{L})$ satisfying (SE) and the following axiom:

(FS) (Face symmetry) $X \circ -Y \in \mathcal{L}$ for all $X,Y \in \mathcal{L}$.

One can see that OMs are exactly the COMs containing the zero vector $\mathbf{0}$, see [4]. The twist between (Sym) and (FS) allows to keep on using the same concepts, such as topes, tope graphs, the sign-order and the big face (semi)lattice in a completely analogous way. On the other hand, it leads to a combinatorial and geometric structure that is build from OMs as cells but is much richer than OMs. Let $\mathcal{M} = (U,\mathcal{L})$ be a COM and $X \in \mathcal{L}$ a covector. The *face* of X is $\uparrow X := \{Y \in \mathcal{L} : X \leq Y\}$, sometimes denoted $\mathrm{F}(X)$, see [4, 7]. A *facet* is an inclusion maximal proper face. By [4, Lemma 4], each face $\uparrow X$ of a COM $\mathcal{M}$ is an OM, which we denote $\mathcal{M}(X)$. *Ample classes* (called also lopsided [3, 40] or extremal [8, 44]) are exactly the COMs, in which all faces are cubes. Since OMs are COMs, each face of an OM is an OM and the facets correspond to cocircuits. The *topes* $\mathcal{T}$ and the *tope graph* $G_{\mathcal{M}}$ of a COM are defined as for OMs. Again, the topes are $\mathcal{T} \subseteq \{-1,+1\}^U$, $G(\mathcal{M})$

is a partial cube, and the COM $\mathcal{M}$ can be recovered from $G(\mathcal{M})$, see [4, 37]. For $X \in \mathcal{L}$, the topes in $\uparrow X$ induce a subgraph $[X]$ of $G(\mathcal{M})$, which is isomorphic to the tope graph of $\mathcal{M}(X)$.

2.2. **Deletion and duality.** We continue with restrictions and deletions in OMs and COMs. Let $\mathcal{M} = (U, \mathcal{L})$ be a COM and $A \subseteq U$. Given a sign vector $X \in \{\pm 1, 0\}^U$ by $X \setminus A$ (or by $X_{|U \setminus A}$) we refer to the *restriction* of X to $U \setminus A$, that is $X \setminus A \in \{\pm 1, 0\}^{U \setminus A}$ with $(X \setminus A)_e = X_e$ for all $e \in U \setminus A$. The *deletion* of A is defined as $\mathcal{M} \setminus A = (U \setminus A, \mathcal{L} \setminus A)$, where $\mathcal{L} \setminus A := \{X \setminus A : X \in \mathcal{L}\}$. We often consider the following type of deletion. For a covector $X \in \mathcal{L}$, we denote by $\mathcal{M}(X) = (U \setminus \underline{X}, \uparrow X \setminus \underline{X})$ the OM defined by the face $\uparrow X$, i.e., $\mathcal{M}(X) = \mathcal{M} \setminus \underline{X}$. The classes of COMs and OMs are closed under deletion, see [4, Lemma 1]. The cocircuits and the covectors of deletions of OMs are described in the following way:

Lemma 1. [7] *Let $\mathcal{M} = (U, \mathcal{L})$ be an* OM *with the set of cocircuits $\mathcal{C}^*$ and $A \subseteq U$. Then the cocircuits of $\mathcal{M} \setminus A$ are $\mathcal{C}^* \setminus A$ and the covectors of $\mathcal{M} \setminus A$ are $\mathcal{L} \setminus A$.*

We briefly recall the duality of OMs, see [7, Section 3.4]. Two sign-vectors $X, Y \in \{\pm 1, 0\}^U$ are *orthogonal*, denoted $X \perp Y$, if either $\underline{X} \cap \underline{Y} = \varnothing$ or there are $e, f \in \underline{X} \cap \underline{Y}$ such that $X_e Y_e = -X_f Y_f$. Oriented matroids can be defined in terms of their *circuits* $\mathcal{C}$, which can be derived from the cocircuits $\mathcal{C}^*$ using the following result:

Theorem 1. [7, Theorem 3.4.3] *Let $\mathcal{C}^*$ be the set of cocircuits of an OM $\mathcal{M}$. The set $\mathcal{C}$ of circuits of $\mathcal{M}$ consists of all minimal $Y \in \{\pm 1, 0\}^U \setminus \{\boldsymbol{0}\}$ such that $Y \perp X$ for any $X \in \mathcal{C}^*$.*

2.3. **Partial cubes and pc-minors.** Tope graphs of OMs and COMs are partial cubes, which we introduce now. Let $G = (V, E)$ be a finite, connected, simple graph. The *distance* $d(u, v) := d_G(u, v)$ between vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v) := \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$ consists of all vertices on shortest (u, v)-paths. A subgraph H is *convex* if $I(u, v) \subseteq H$ for any $u, v \in H$ and *gated* [24] if for every vertex $x \notin H$ there exists a vertex x' (the *gate* of x) in H such that $x' \in I(x, y)$ for each vertex y of H. It is easy to see that gates are unique and that gated sets are convex. An induced subgraph H of G is *isometric* if the distance between vertices in H is the same as that in G. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all $u, v \in V$. A graph G is a *partial cube* if it admits an isometric embedding into a hypercube Q_m. For an edge uv of G, let $W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$. For an edge uv, the sets $W(u, v)$ and $W(v, u)$ are called *complementary halfspaces* of G.

Theorem 2. [21] *A graph G is a partial cube if and only if G is bipartite and for any edge uv the sets $W(u, v)$ and $W(v, u)$ are convex.*

Djoković [21] introduced the following binary relation Θ on the edges of G: for two edges $e = uv$ and $e' = u'v'$, we set $e \Theta e'$ if $u' \in W(u, v)$ and $v' \in W(v, u)$. If G is a partial cube, then Θ is an equivalence relation and let E_e be a Θ-class. Let $\{G_e^-, G_e^+\}$ be the complementary halfspaces of G defined by setting $G_e^- := G(W(u, v))$ and $G_e^+ := G(W(v, u))$ for an arbitrary edge $uv \in E_e$. An *elementary restriction* consists of taking one of the halfspaces G_e^- and G_e^+. A *restriction* is a convex subgraph of G induced by the intersection of a set of halfspaces of G. Since any convex subgraph of a partial cube G is the intersection of halfspaces [1, 2, 11], the restrictions of G coincide with the convex subgraphs of G. Denote by $\pi_e(G)$ an *elementary contraction*, i.e., the graph obtained from G by contracting the edges in E_e. For a vertex v of G, let $\pi_e(v)$ be the image of v under the contraction. We apply π_e to subsets $S \subseteq V$, by setting $\pi_e(S) := \{\pi_e(v) : v \in S\}$. By [12, Theorem 3], the class of partial cubes is closed under contractions. Since contractions commute, for a set A of Θ-classes, we denote by $\pi_A(G)$ the isometric subgraph of $\mathrm{Q}_{m-|A|}$ obtained from G by contracting the equivalence classes of edges from A. Contractions and restrictions also commute in partial cubes. A *pc-minor* of G is a partial cube obtained from G by restrictions and contractions.

Since tope graphs of COMs and OMs are partial cubes, we can describe deletions and contractions on sign-vectors in terms of partial cubes.

Lemma 2. *Let $\mathcal{M} = (U, \mathcal{L})$ be a COM and $A \subseteq U$. Then $\pi_A(G(\mathcal{M}))$ is the tope graph of $\mathcal{M} \setminus A$. In particular, if $X \in \mathcal{L}$, then the tope graph of $\mathcal{M}(X) = \mathcal{M} \setminus \underline{X}$ is isomorphic to $[X]$.*

The following lemma is implicit in [4] and explicit in [37]:

Lemma 3. *For each covector X of a COM $\mathcal{M}$, the subgraph $[X]$ of $G(\mathcal{M})$ is gated.*

2.4. **VC-dimension.** Let $\mathcal{S}$ be a family of subsets of an m-element set U. A subset X of U is *shattered* by $\mathcal{S}$ if for all $Y \subseteq X$ there exists $S \in \mathcal{S}$ such that $S \cap X = Y$. The *Vapnik-Chervonenkis dimension* (VC-dimension) [51] $\text{VC-dim}(\mathcal{S})$ of $\mathcal{S}$ is the cardinality of the largest subset of U shattered by $\mathcal{S}$. Any set family $\mathcal{S} \subseteq 2^U$ can be viewed as a subset of vertices of the m-dimensional hypercube $\mathrm{Q}_m = \mathrm{Q}(U)$. Denote by $G(\mathcal{S})$ the *1-inclusion graph* of $\mathcal{S}$, i.e., the subgraph of Q_m induced by the vertices of Q_m corresponding to $\mathcal{S}$. A subgraph G of Q_m has VC-dimension d if G is the 1-inclusion graph of a set family of VC-dimension d. For partial cubes, the VC-dimension can be formulated in terms of pc-minors: a partial cube G has VC-dimension $\leq d$ if and only if G does not have Q_{d+1} as a pc-minor. This is well-defined, since the embeddings of partial cubes are unique up to isomorphism, see e.g. [47, Chapter 5].

The *VC-dimension* $\text{VC-dim}(\mathcal{M})$ of a COM $\mathcal{M} = (U, \mathcal{L})$ is the VC-dimension of its tope graph $G(\mathcal{M})$. The *VC-dimension* $\text{VC-dim}(X)$ of a covector $X \in \mathcal{L}$ of $\mathcal{M}$ is the VC-dimension of the OM $\mathcal{M}(X)$, i.e., it is the VC-dimension of the graph $[X]$. The VC-dimension of OMs, COMs, and their covectors can be expressed in the following way:

Lemma 4. [17, Lemma 13] *For a COM $\mathcal{M}$, $\text{VC-dim}(\mathcal{M}) = \max\{\text{VC-dim}(\mathcal{M}(X)) : X \in \mathcal{L}\}$. If $\mathcal{M}$ is an OM and X a cocircuit of $\mathcal{M}$, then $\text{VC-dim}(X) + 1 = \text{VC-dim}(\mathcal{M}) = \text{rank}(\mathcal{M})$.*

That $\text{VC-dim}(X) = \text{VC-dim}(\mathcal{M}) - 1$ for cocircuits X of an OM $\mathcal{M}$ follows from the fact that the cocircuits are atoms of the big face lattice $\mathcal{F}_{\text{big}}(\mathcal{M})$ and this lattice is graded.

3. Auxiliary results

We establish and recall several auxiliary results about OMs and COMs. We also develop a correspondence between realizable samples and convex subgraphs of partial cubes.

3.1. **More about shattering in OMs and COMs.** We continue with new results about shattering in OMs and COMs. Let G be a partial cube, H a convex subgraph and E_e a Θ-class of G. We say that E_e *crosses* H if H contains an edge of E_e. If E_e does not cross H and there exists an edge uv of E_e with $u \in H$ and $v \notin H$, then E_e and H *osculate*. Otherwise, E_e is *disjoint* from H. Denote by $\text{osc}(H)$ the set of all e such that E_e osculates with H and by $\text{cross}(H)$ the set of all e such that E_e crosses H.

Lemma 5. *Let G be a partial cube, H a convex subgraph, and $e \notin \text{osc}(H)$. Then $\pi_e(H)$ is a convex in $\pi_e(G)$ and $\text{osc}(\pi_e(H)) = \text{osc}(H)$, where $\text{osc}(H)$ and $\text{osc}(\pi_e(H))$ are considered in G and $\pi_e(G)$, respectively.*

Proof. Let $H' = \pi_e(H)$. First, since $e \notin \text{osc}(H)$, the fact that H' is a convex subgraph of $\pi_e(G)$ comes from [15, Lemma 5]. Then, the inclusion $\text{osc}(H) \subseteq \text{osc}(H')$ is obvious. If there exists $e' \in \text{osc}(H') \setminus \text{osc}(H)$, then there exists an edge $\pi_e(u)\pi_e(v)$ in $\pi_e(E_{e'})$ with $\pi_e(u) \in V(H')$ and $\pi_e(v) \notin V(H')$. Then $\pi_e(u)\pi_e(v)$ comes from an edge uv of G belonging to $E_{e'}$. Since $e' \notin \text{osc}(H)$, the vertices u and v do not belong to H. This implies that there exists an edge uw of E_e with $w \in V(H)$. Then E_e and H osculate, a contradiction. $\square$

Lemma 6. *Let G be a partial cube and H a gated subgraph of G. If $D \subseteq \text{cross}(H)$ is shattered by G, then D is shattered by H.*

Proof. Pick any Θ-class E_e with $e \in D$ and let v be any vertex of G. If v belongs to the halfspace G_e^- of G, then the gate v' of v in H also belongs to G_e^-. Indeed, since E_e crosses H, there exists a vertex $w \in G_e^- \cap H$. Then $v' \in I(v,w) \subset G_e^-$ by convexity of G_e^- and since v' is the gate of v in H. Analogously, if $v \in G_e^+$, then $v' \in G_e^+$.

Since G shatters D, for any sign vector $X \in \{-1,+1\}^D$, there exists a vertex v_X of G, whose restriction to D coincides with X. This means that for any $e \in D$, the vertex v_X belongs to the halfspace $G_e^{X_e}$. Since the gate v'_X of v_X in H also belongs to $G_e^{X_e}$, the restriction of v'_X to D also coincides with X. This implies that H also shatters D. □

The next lemma shows that the *independent sets* of the underlying matroid $\underline{\mathcal{M}}$ are exactly the sets shattered by an OM $\mathcal{M}$, i.e., the sets not containing supports of circuits of $\mathcal{M}$.

Lemma 7. *Let $\mathcal{M} = (U, \mathcal{L})$ be an* OM *and D be a subset of U. Then D is shattered by $\mathcal{M}$ if and only if D is independent in the underlying matroid $\underline{\mathcal{M}}$.*

Proof. First suppose that D is not shattered by $\mathcal{M}$. We assert that there is a circuit Y of $\mathcal{M}$ with support included in D. Let $|D| = d+1$. We proceed by induction on $|U| + |D|$. Let $\mathcal{M}' := \mathcal{M} \setminus (U \setminus D)$. Since $G(\mathcal{M})$ does not shatter D and $G(\mathcal{M}')$ is a pc-minor of $G(\mathcal{M})$, $G(\mathcal{M}')$ also does not shatter D. Therefore, if D is a proper subset of U, then by induction hypothesis, there exists a circuit Y' of $\mathcal{M}'$ with $\underline{Y'} \subseteq D$. Consider the sign vector $Y \in \{\pm 1, 0\}^U$ defined by setting $Y_e = Y'_e$ if $e \in D$ and $Y_e = 0$ if $e \in U \setminus D$. By Lemma 1, for any cocircuit X of $\mathcal{M}$, $X' = X \setminus (U \setminus D)$ is a cocircuit of $\mathcal{M}'$. Since $X' \perp Y'$, we conclude that $X \perp Y$. By Theorem 1, Y is a circuit of $\mathcal{M}$ and we are done. Thus we can suppose that $U = D$.

If D contains a proper subset D', which is not shattered by $\mathcal{M}$, then we can apply the induction hypothesis and find a circuit Y with $\underline{Y} \subseteq D' \subset D$, and we are done. Thus we can suppose that all proper subsets of D are shattered by $\mathcal{M}$. Since $U = D$, this implies that VC-dim$(\mathcal{M}) = |D| - 1 = d$. By Lemma 4, VC-dim$(X) = d - 1$ for any cocircuit X of $\mathcal{M}$.

If $\mathcal{M}$ contains a cocircuit X such that $\mathcal{M}(X)$ does not shatter the set $X^0 \cap D$, then by induction assumption applied to $\mathcal{M}(X)$ we can find a circuit Y' of $\mathcal{M}(X)$ with $\underline{Y'} \subseteq X^0 \cap D$. Then extending Y' to Y as in the previous case, we obtain a circuit Y of $\mathcal{M}$ whose support is included in D and we are done. Therefore, we can suppose that for any cocircuit X of $\mathcal{M}$, $\mathcal{M}(X)$ shatters the set $X^0 \cap D$. Since VC-dim$(X) = d-1$ and $U = D$, this implies that $|X^0 \cap D| = d - 1$. Hence, the support $\underline{X}$ of each cocircuit X consists of two elements.

Since D is not shattered by $\mathcal{M}$, there exists a sign-vector $Y' \in \{-1,+1\}^D$ such that for any tope T of $\mathcal{M}$, the restriction of T to D is different from Y'. By symmetry, $-Y'$ also is not shattered by $\mathcal{M}$. Consider the sign vector $Y \in \{\pm 1, 0\}^U$ defined by setting $Y_e = Y'_e$ if $e \in D$ and $Y_e = 0$ if $e \in U \setminus D$. Then $Y, -Y \notin \mathcal{L}$. We assert that Y is a circuit of $\mathcal{M}$. By Theorem 1, we have to show that $Y \perp X$ for any cocircuit X of $\mathcal{M}$. We assert that $X_f Y_f = -X_{f'} Y_{f'}$, where $\underline{X} = \{f, f'\}$. Indeed, since Y and $-Y$ do not belong to $\uparrow X$, $\mathrm{Sep}(Y, -Y) = D$, and $X^0 = D \setminus \{f, f'\}$, we must have $X_f \neq Y_f, X_{f'} \neq -Y_{f'}$ or $X_{f'} \neq Y_{f'}, X_f \neq -Y_f$. In the first case we have $X_f Y_f = -1, X_{f'} Y_{f'} = +1$ and in the second case we have $X_f Y_f = +1, X_{f'} Y_{f'} = -1$. This proves that if D is an independent set of $\underline{\mathcal{M}}$, then D is shattered by $\mathcal{M}$.

Conversely, let D be a set of size d shattered by $\mathcal{M}$ and suppose by way of contradiction that D contains a circuit of $\underline{\mathcal{M}}$. Since any subset of D is also shattered by $\mathcal{M}$, we can suppose without loss of generality that for any $e \in D$, $D \setminus \{e\}$ is an independent set of $\underline{\mathcal{M}}$, i.e. that D is a circuit of $\underline{\mathcal{M}}$. By passing from $\mathcal{M}$ to $\mathcal{M}' = \mathcal{M} \setminus (U \setminus D)$, we can also suppose that $U = D$, i.e., that VC-dim$(\mathcal{M}) = d$. Since $\mathcal{M}$ shatters the set $D = U$, any sign vector from $\{\pm 1\}^D$ is a tope of $\mathcal{M}$. Consider the two circuit signatures Y and $-Y$ of the circuit D. They are obviously topes of $\mathcal{M}$. Consider also a cocircuit X of $\mathcal{M}$ such that $-Y \in \uparrow X$ and $Y \notin \uparrow X$ (such a X exists because $\mathcal{M}$ is a simple OM). By Lemma 4 VC-dim$(X) = d - 1$, thus D contains an element f such that X shatters $D \setminus \{f\}$. This implies that $D \setminus \{f\} \subseteq X^0$. Hence, Y and X are not orthogonal, contrary to Theorem 1. This shows that each set D shattered by $\mathcal{M}$ is an independent set of $\underline{\mathcal{M}}$. □

An *antipode* of a vertex v in a partial cube G is a (necessarily unique) vertex $-v$ such that $G = I(v, -v)$. A partial cube G is *antipodal* if all its vertices have antipodes. By (Sym), a tope graph of a COM is the tope graph of an OMs if and only if it is antipodal, see [37].

The next lemma can be seen as dual analogue of Lemma 7. It shows that the VC-dimension of OMs is defined locally at each tope T, by shattering subsets of $\mathrm{osc}(T)$.

Lemma 8. *Let $\mathcal{M} = (U, \mathcal{L})$ be an* OM *of rank d with tope graph $G(\mathcal{M})$. For any tope T of $\mathcal{M}$,* $\mathrm{osc}(T)$ *contains a subset D of size d shattered by $\mathcal{M}$.*

Proof. We proceed by induction on the size of U. If $\mathrm{osc}(T) = U$, then we are obviously done. Thus suppose that there exists $e \notin \mathrm{osc}(T)$. Consider the tope graph $G' = \pi_e(G)$ of the oriented matroid $\mathcal{M}' = \mathcal{M} \setminus e$. Let $T' = \pi_e(T)$. Then $\mathrm{osc}(T') = \mathrm{osc}(T)$ by Lemma 5. If $\mathrm{rank}(\mathcal{M}') = d$, by induction hypothesis the set $\mathrm{osc}(T')$ contains a subset D of size d shattered by G'. Since G' is a pc-minor of G, $D \subset \mathrm{osc}(T)$ is also shattered by G and we are done.

Thus, let $\mathrm{rank}(\mathcal{M}') < \mathrm{rank}(\mathcal{M})$. If the Θ-class E_e of G crosses the faces $\uparrow X$ of all cocircuits $X \in \mathcal{L}$, then $\mathcal{L}$ is not simple. Therefore, there exists a cocircuit $X \in \mathcal{L}$ whose face $\uparrow X$ is not crossed by E_e. However, since when we contract E_e the rank decreases by 1, the resulting OM $\mathcal{M}'$ coincides with $\uparrow X$. Indeed, after contraction the rank of $\uparrow X$ remains the same. Hence, if X would remain a cocircuit, then the global rank would not decrease. Hence, G' is the tope graph of $\mathcal{M}(X)$. Since G is an antipodal partial cube and $G_e^+ = \uparrow X$, we have $G_e^- \cong G_e^+$. This shows that $G \cong G_e^+ \Box K_2 \cong G' \Box K_2$. This implies that E_e osculate with $\{T\}$ in G, contrary to the assumption $e \notin \mathrm{osc}(T)$. $\square$

Next we give a shattering property of COMs. It is in its proof where the axiom (SE) of COMs is crucial. The *distance* $d(A, B)$ between sets A, B of vertices of G is $\min\{d(a, b) : a \in A, b \in B\}$. The set $\mathrm{pr}_B(A) = \{a \in A : d(a, B) = d(A, B)\}$ is the *metric projection* of B on A. For two covectors $X, Y \in \mathcal{L}$ of a COM $\mathcal{M}$, we denote by $\mathrm{pr}_{[X]}([Y])$ the metric projection of $[X]$ on $[Y]$ in $G(\mathcal{M})$. Since $[X]$ and $[Y]$ are gated by Lemma 3, $\mathrm{pr}_{[X]}([Y])$ consists of the gates of vertices of $[X]$ in $[Y]$. Two faces $\uparrow X$ and $\uparrow Y$ of $\mathcal{M}$ are *parallel* if $\mathrm{pr}_{[X]}([Y]) = [Y]$ and $\mathrm{pr}_{[Y]}([X]) = [X]$. A *gallery* between two parallel faces $\uparrow X$ and $\uparrow Y$ of $\mathcal{M}$ is a sequence of faces $(\uparrow X = \uparrow X_0, \uparrow X_1, \ldots, \uparrow X_{k-1}, \uparrow X_k = \uparrow Y)$ such that any two faces of this sequence are parallel and any two consecutive faces $\uparrow X_{i-1}, \uparrow X_i$ are facets of a common face of $\mathcal{L}$. A *geodesic gallery* between $\uparrow X$ and $\uparrow Y$ is a gallery of length $|\mathrm{Sep}(X, Y)|$. Two parallel faces $\uparrow X, \uparrow Y$ are *adjacent* if $|\mathrm{Sep}(X, Y)| = 1$, i.e., $\uparrow X$ and $\uparrow Y$ are opposite facets of a face of $\mathcal{L}$. See Figure 2 and recall the following result:

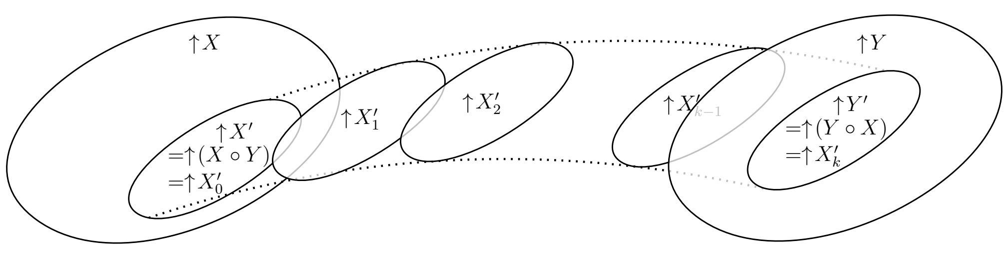


FIGURE 2. Illustration of Lemmas 9 and 10.

Lemma 9. [17, Proposition 8] *Let $\mathcal{M} = (U, \mathcal{L})$ be a COM and $X, Y \in \mathcal{L}$. Then:*

(i) *$d([X], [Y]) = |S(X, Y)|$ and the gates of $[Y]$ in $[X]$ are the vertices of $[X \circ Y] \subseteq [X]$;*
(ii) *$\uparrow X$ and $\uparrow Y$ are parallel if and only if $\underline{X} = \underline{Y}$. If $\uparrow X$ and $\uparrow Y$ are parallel, then they are connected by a geodesic gallery;*
(iii) $\mathrm{pr}_{[Y]}([X]) = [X \circ Y]$, $\mathrm{pr}_{[X]}([Y]) = [Y \circ X]$, *and $\uparrow(X \circ Y)$ and $\uparrow(Y \circ X)$ are parallel.*

A covector $X \in \mathcal{L}$ of a COM $\mathcal{M} = (U, \mathcal{L})$ *maximally shatters* a set $D \subseteq U$ if $[X]$ shatters D but $[X]$ does not shatter any superset of D. We also say that $X \in \mathcal{L}$ *minimally shatters* a set D if $[X]$ shatters D but D is not shattered by $[X']$ for any covector $X' > X$.

Lemma 10. *Let $\mathcal{M} = (U, \mathcal{L})$ be a COM and $X, Y \in \mathcal{L}$. Then:*

(i) *if $[X]$ and $[Y]$ shatter D, then the projections $[X \circ Y]$ and $[Y \circ X]$ also shatter D;*
(ii) *if $[X]$ maximally shatters D and $[Y]$ shatters D, then $[X \circ Y] = [X]$ and $\uparrow X$ is not a facet of $\mathcal{M}$;*
(iii) *if both $[X]$ and $[Y]$ shatter D, then there exist covectors $X' \geq X, Y' \geq Y$ such that $[X']$ and $[Y']$ both maximally shatter D, and $\uparrow X'$ and $\uparrow Y'$ are parallel.*

Proof. **Property (i):** Since $[X]$ and $[Y]$ shatter D, for any sign vector $Z \in \{\pm 1\}^D$ we can find two topes $T' \in [X]$ and $T'' \in [Y]$, such that $T'_{|D} = Z = T''_{|D}$. Since $X < T'$ and $Y < T''$, from $T'_{|D} = Z = T''_{|D}$ we conclude that $(X \circ Y)_{|D} < Z$ and in $[X \circ Y]$ we can find a tope T whose restriction to D coincides with Z. This proves that $[X \circ Y]$ shatters D, establishing (i).

Property (ii): If $[X]$ maximally shatters D, then VC-dim$(X) = |D| =: d$. By property (i), $[X \circ Y]$ also shatters D. If $\uparrow(X \circ Y)$ is a proper face of $\uparrow X$, then we obtain a contradiction with Lemma 4. Thus $\uparrow(X \circ Y) = \uparrow X$, showing that $X = X \circ Y$. This establishes the first assertion. By Lemma 9, the faces $\uparrow X$ and $\uparrow(Y \circ X)$ are parallel and therefore are connected by a geodesic gallery $(\uparrow X = \uparrow X_0, \uparrow X_1, \ldots, \uparrow X_k = \uparrow(Y \circ X))$. Then $\uparrow X$ and $\uparrow X_1$ are facets of a common face of $\mathcal{L}$, thus $\uparrow X$ is not a facet of $\mathcal{M}$. This proves (ii).

Property (iii): Let $d = |D|$. We can suppose that both X and Y minimally shatter the set D. Indeed, if D is shattered by a proper face $\uparrow X'$ of $\uparrow X$, then we can replace the pair X, Y by the pair X', Y so that $[X']$ and $[Y]$ still shatter D. Thus D is not shattered by any proper faces of $\uparrow X$ and $\uparrow Y$. Since by (i), D is shattered by $[X \circ Y]$ and $[Y \circ X]$, we conclude that $X = X \circ Y$ and $Y = Y \circ X$ and thus the faces $\uparrow X$ and $\uparrow Y$ are parallel.

It remains to show that $[X]$ and $[Y]$ maximally shatter D. Suppose by way of contradiction that $[X]$ shatters a larger set $D' := D \cup \{e\}$. Consider the OM $\mathcal{M}' = \uparrow X \setminus (U \setminus D')$. Since $[X]$ shatters D', $\mathcal{M}'$ also shatters D'. Moreover, $\mathcal{M}'$ maximally shatters D', i.e., VC-dim$(\mathcal{M}') = d + 1$. Since $\mathcal{M}'$ is a simple OM, $\mathcal{M}'$ contains two adjacent topes T'_1, T'_2 with $\mathrm{Sep}(T'_1, T'_2) = \{e\}$ and we can find a cocircuit X'' of $\mathcal{M}'$ such that $T'_1 \in [X'']$ and $T'_2 \notin [X'']$. By Lemma 4 applied to $\mathcal{M}'$, we conclude that X'' has VC-dimension d. Hence, X'' must shatter the set D. By Lemma 1, there is a cocircuit X' of $\uparrow X$ such that $X'' = X' \setminus (U \setminus D')$. Since X'' shatters D, X' also shatters D. Since $X < X'$, this contradicts our assumption that X minimally shatters D. This establishes (iii). $\square$

3.2. **Realizable and full samples as convex subgraphs.** We establish a correspondence between realizable samples and convex subgraphs of partial cubes. Let $\mathcal{L} \subset \{-1, 0, +1\}^U$ be a system of sign vectors whose topes $\mathcal{T}$ induce an isometric subgraph G of $\mathrm{Q}(U)$. Recall that $\downarrow \mathcal{L} = \bigcup_{X \in \mathcal{L}} \{S \in \{-1, 0, +1\}^U : S \leq X\}$ is the *set of realizable samples* for $\mathcal{L}$. This construction is called *polar complex* in neural codes [36]. Since for any $X \in \mathcal{L}$ there exists $T \in \mathcal{T}$ such that $X \leq T$, we have $\downarrow\mathcal{L} = \downarrow\mathcal{T}$, see Figure 3.

For a realizable sample $S \in \downarrow \mathcal{L}$, let $\uparrow S = \{X \in \mathcal{L} : S \leq X\}$ and let $[S]$ be the subgraph of G induced by all topes $T \in \mathcal{L}$ from $\uparrow S$. For OMs, the set $\uparrow S$ is called *supertope* in [35]. For COMs, $\uparrow S$ is called the *fiber* of S and it is known that they are COMs [4]. Since for any $S \in \downarrow \mathcal{L}$ there exists $T\mathcal{T}$ such that $S \leq T$, $[S] \neq \emptyset$. Moreover, $[S]$ is the intersection of halfspaces of G of the form G^+_e if $S_e = +1$ and G^-_e if $S_e = -1$. Hence, $[S]$ is a nonempty convex subgraph of G for all $S \in \downarrow \mathcal{L}$. If $\mathcal{M} = (U, \mathcal{L})$ is a COM and $S \in \mathcal{L}$, then $\uparrow S$ is the face of S and $[S]$ is gated by Lemma 3.

Any convex subgraph H of a partial cube G is the intersection of all halfspaces of G containing H [1, 2, 11]. However, H can be represented in different ways as the intersection of halfspaces. Indeed, any representation of H as an intersection of halfspaces of G yields a realizable sample S, where $S_e = \pm 1$ if $G^{\pm}_e$ participates in the representation and $S_e = 0$ otherwise. Notice that the

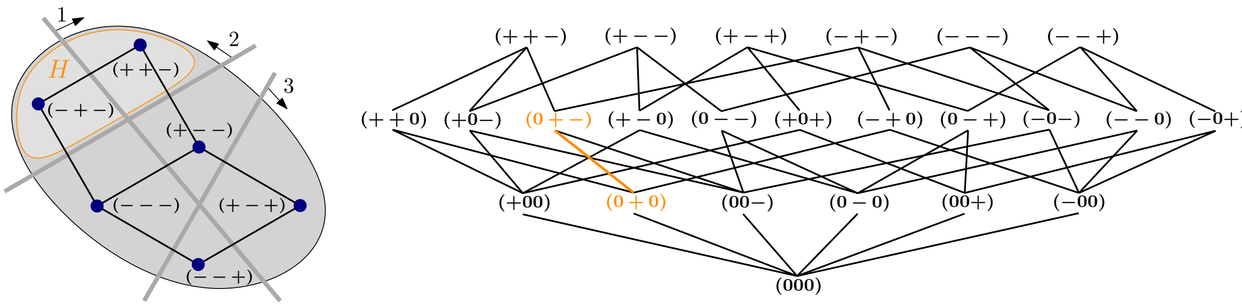


FIGURE 3. Left: the tope graph $\widetilde{G}'$ of the restriction $\widetilde{\mathcal{M}}'$ of $\widetilde{\mathcal{M}}$ to $\{1,2,3\}$ and a convex subgraph H of $\widetilde{G}'$. Right: the realizable samples of $\widetilde{\mathcal{M}}'$ and the interval $I(H)$ (in orange).

Θ-classes osculating with H have to be part of every representation of H and the Θ-classes crossing H take part in no representation of H. This leads to two canonical representations of H, one using only the halfspaces whose Θ-class osculates with H and one using all halfspaces containing H. We define the corresponding realizable samples:

$$(S_\perp)_e = \begin{cases} -1 & \text{if } e \in \operatorname{osc}(H) \text{ and } H \subseteq G_e^-, \\ +1 & \text{if } e \in \operatorname{osc}(H) \text{ and } H \subseteq G_e^+, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad (S^\top)_e = \begin{cases} -1 & \text{if } H \subseteq G_e^-, \\ +1 & \text{if } H \subseteq G_e^+, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(S^\top)^0 = \operatorname{cross}(H)$ and $(S_\perp)^0 = U \setminus \operatorname{osc}(H)$, i.e., $(S_\perp)^0$ consists of all e such that E_e crosses or is disjoint from H. If S is a sample arising from the representation of H as the intersection of halfspaces, then $S_\perp \leq S \leq S^\top$. Moreover, any sample S from the order interval $I(H) := [S_\perp, S^\top]$ arises from a representation of H, i.e., $[S] = [S_\perp] = [S^\top] = H$. Thus, for any convex subgraph H of G the set of all $S \in \downarrow\mathcal{L}$ such that $[S] = H$ is an interval $I(H) = [S_\perp, S^\top]$ of $(\downarrow\mathcal{L}, \leq)$. Note that the intervals $I(H)$ partition $\downarrow\mathcal{L}$. Moreover:

Lemma 11. *If $S, S' \in\downarrow\mathcal{L}$ and $S \leq S'$, then $[S'] \subseteq [S]$.*

Lemma 12. *If $X \in \mathcal{L}, S \in\downarrow\mathcal{L}$ such that $\operatorname{Sep}(X,S) = \varnothing$ and $\widehat{S} = X \circ S$, then $[\widehat{S}] = [X] \cap [S]$.*

Proof. Since $\widehat{S} = X \circ S$, we have $X \leq \widehat{S}$ and by Lemma 11 we have $[\widehat{S}] \subsetneq [X]$. Now we prove that $[\widehat{S}] \subseteq [S]$. Indeed, otherwise there exists a tope T of $\mathcal{L}$ such that $T \in [\widehat{S}] \setminus [S]$. This implies that $\widehat{S} < T$ and there exists an element $e \in U$ such that $T_e \neq S_e \neq 0$. Since $\widehat{S} = X \circ S$, this implies that $X_e = T_e$, which is impossible because $\operatorname{Sep}(X,S) = \varnothing$. This proves that $[\widehat{S}] \subseteq [X] \cap [S]$.

To prove the converse inclusion $[X] \cap [S] \subseteq [\widehat{S}]$ pick any tope T of $\mathcal{L}$ belonging to $[X] \cap [S]$. Then $X < T$ and $S < T$. Suppose by way of contradiction that $T \notin [\widehat{S}]$, i.e., $\widehat{S} \not< T$. Then there exists $e \in U$ such that $\widehat{S}_e \neq 0$ and $\widehat{S}_e \neq T_e$, say $\widehat{S}_e = -1$ and $T_e = +1$. Since $\widehat{S} = X \circ S$, the equality $\widehat{S}_e = -1$ implies that either $X_e = -1$ or that $X_e = 0$ and $S_e = -1$. Since $T_e = +1$, in the first case we get a contradiction with $X < T$ and in the second case we get a contradiction with $S < T$. Hence, $[X] \cap [S] \subseteq [\widehat{S}]$ and we are done. $\square$

We say that a sample $S \in\downarrow \mathcal{L}$ is *full* if the pc-minor $G' = \pi_{S^0}(G)$ has VC-dimension $d = \text{VC-dim}(G)$. Let $\downarrow\mathcal{L}_f$ denote the set of all full samples of $\mathcal{L}$. Note that all topes of $\mathcal{L}$ are full samples since their zero set is empty. A convex subgraph H of G is *full* if $S_\perp(H)$ is full. The image of H in G' is a single vertex v_H and its degree is $|\operatorname{osc}(H)|$. If $D \subset \operatorname{osc}(v_H) = \operatorname{osc}(H)$ of size d is shattered by G', since G' is a pc-minor of G, D is also shattered by G. Hence, a convex set H of G is full if and only if G shatters a subset D of $\operatorname{osc}(H)$ of size $d = \text{VC-dim}(G)$. If H is a full convex subgraph of a COM, not all samples in $I(H)$ have to be full. We show next, that the above problem does not arise in OMs.

Lemma 13. *Let $\mathcal{M} = (U, \mathcal{L})$ be an OM of rank d and let $G = G(\mathcal{M})$ be its tope graph. A sample $S \in \downarrow\mathcal{L}$ is full if and only if the convex subgraph $[S]$ is full.*

Proof. First notice that since in OMs the rank and the VC-dimension are equal, a sample S is full if and only if $\mathrm{rank}(\mathcal{M} \setminus S^0) = d$.

First suppose that the convex subgraph H is full. Then the sample $S_\perp := S_\perp(H)$ is full. Since $\underline{S_\perp} = \mathrm{osc}(H) \subseteq \underline{S}$ for any $S \in I(H) = [S_\perp, S^\top]$, we get $S^0 \subset (S_\perp)^0$, thus $\mathrm{rank}(\mathcal{M} \setminus S^0) \geq \mathrm{rank}(\mathcal{M} \setminus (S_\perp)^0) = d = \mathrm{rank}(\mathcal{M})$. Hence, $\mathrm{rank}(\mathcal{M} \setminus S^0) = d$, i.e., S is a full sample.

Conversely, let S be a full sample and we assert that $H = [S]$ is a full convex subgraph. Let $\mathcal{M}' = \mathcal{M} \setminus \mathrm{cross}(H)$ and let $G' = \pi_{\mathrm{cross}(H)}(G)$ be its tope graph. Since $\mathrm{cross}(H) \subseteq S^0$ and S is full, $\mathrm{rank}(\mathcal{M}') = d$ and hence VC-dim$(G') = d$. The image of H in G' is a single vertex v_H. By Lemma 5, $\mathrm{osc}(v_H) = \mathrm{osc}(H)$. By Lemma 8, $\mathrm{osc}(v_H)$ contains a subset of size d shattered by $\mathcal{M}'$, whence H is full. □

4. The main result

Theorem 3. *The set $\mathcal{T}$ of topes of a complex of oriented matroids $\mathcal{M} = (U, \mathcal{L})$ of VC-dimension d admits a proper labeled sample compression scheme of size d.*

The proof of the theorem is based on the *distinguishing* and the *localization lemma*, provided in the next subsections. Compressor and reconstructor are given in the last subsection and are illustrated by Example 1.

4.1. The distinguishing lemma.

In this subsection, $\mathcal{M} = (U, \mathcal{L})$ is an OM of rank d. The *distinguishing lemma* allows to distinguish full samples of $\mathcal{M}$ by their restriction to subsets of size d. It is constructing a function $f_\mathcal{M}$ that assigns such a subset to each full sample and is used by both compressor and reconstructor.

Lemma 14. *Let $\mathcal{M} = (U, \mathcal{L})$ be an* OM *of VC-dimension d. Then there exists a function $f_\mathcal{M} : \downarrow\mathcal{L}_f \to \binom{U}{d}$ such that for all $S, S' \in \downarrow\mathcal{L}_f$:*

- (i) *if $e \in f_\mathcal{M}(S)$, then $e \in \mathrm{osc}([S])$,*
- (ii) *$f_\mathcal{M}(S)$ is shattered by $\mathcal{M}$,*
- (iii) *if $e \notin \mathrm{osc}([S])$, then $f_\mathcal{M}(S) = f_{\mathcal{M}\setminus e}(S \setminus e)$,*
- (iv) *if $S_{|f_\mathcal{M}(S)} = S'_{|f_\mathcal{M}(S')}$, then $[S] = [S']$.*

Proof. Let $G := G(\mathcal{M})$ be the tope graph of $\mathcal{M}$. We proceed by induction on d. If $d = 1$, then $U = \{e\}$ and G is an edge between the topes $T_1 = (-1)$ and $T_2 = (+1)$. Then T_1 and T_2 are the only full samples of $\mathcal{M}$ (the unique other sample (0) is not full). Defining $f_\mathcal{M}(T_1) = f_\mathcal{M}(T_2) = \{e\}$, we obtain a function satisfying the conditions (i)-(iv).

Before treating the general case $d \geq 2$, we establish the following claim:

Claim 1. If S is a full sample and $e \in \mathrm{osc}([S])$, then there exists a cocircuit X of $\mathcal{M}$ such that $e \in \underline{X}$ and $X \leq S$. Moreover, $S \setminus \underline{X}$ is a full sample of $\mathcal{M}(X)$.

Proof. Since S is a full sample, $\mathcal{M}' = \mathcal{M} \setminus S^0$ has VC-dimension d and thus rank d. Moreover, $[S \setminus S^0]$ is a tope T of $\mathcal{M}'$. By Lemma 5, $e \in \mathrm{osc}([S]) = \mathrm{osc}([T])$ and T is incident to an edge in E_e, i.e., there is a tope T' of $\mathcal{M}'$ such that $\mathrm{Sep}(T, T') = \{e\}$. Let X' be a cocircuit of $\mathcal{M}'$ such that its face $\uparrow X'$ contains T but not T'. This cocircuit X' exists, otherwise all cocircuits Y' of $\mathcal{M}'$ would have $Y'_e = 0$, but $\mathcal{M}'$ is simple because $\mathcal{M}$ is. Now, since $\mathcal{M}'$ has VC-dimension d, $\mathcal{M}'(X') \cong \uparrow X'$ has VC-dim $d - 1$ by Lemma 4. Thus, there exists a covector X of $\mathcal{M}$ such that $X' = X \setminus S^0$ and VC-dim$(X) = d - 1$. If X is not a cocircuit, then $\uparrow X$ is a proper face of $\uparrow Y$ for a cocircuit Y of $\mathcal{M}$. Since the VC-dimension of any proper face is strictly smaller than the VC-dimension of the face itself and since $\mathcal{M}$ has rank d, we obtain a contradiction. Thus X is a cocircuit of $\mathcal{M}$. In particular, $e \in \underline{X}$ and $X \leq S$.

It remains to show that $S \setminus \underline{X}$ is a full sample of $\mathcal{M}(X)$, i.e., that the VC-dimension of $(\uparrow X \setminus \underline{X}) \setminus S^0$ is $d-1$. To see this note that by Lemma 1, $X \setminus S^0 \in \mathcal{C}^*(\mathcal{M} \setminus S^0)$ and since S is full the VC-dimension of $\mathcal{M} \setminus S^0$ is d. Hence, the VC-dimension of $\mathcal{M}(X \setminus S^0)$ is $d-1$ by Lemma 4. But since all sign-vectors in $\uparrow X$ agree on $\underline{X}$, we have $\mathcal{M}(X \setminus S^0) \cong (\uparrow X \setminus \underline{X}) \setminus S^0$. Hence, VC-dim$((\uparrow X \setminus \underline{X}) \setminus S^0) =$ VC-dim$(\mathcal{M}(X \setminus S^0)) =$ VC-dim$(\mathcal{M}) - 1 = d - 1$. □

Let now $d \geq 2$. Fix a linear order on $U = \{1, \dots, m\}$. For a full sample S, define the function $f_{\mathcal{M}}$ recursively by setting $f_{\mathcal{M}}(S) = \{e_S, f_{\mathcal{M}(X)}(S \setminus \underline{X})\}$, where e_S is the smallest element of U such that E_{e_S} osculates with $[S]$ and X is any cocircuit of $\mathcal{M}$ such that $e_S \in \underline{X}$ and $X \leq S$. By Claim 1, X exists and by Lemma 11 $[S] \subseteq [X]$ holds. Now we prove that $f_{\mathcal{M}}$ satisfies the conditions (i)-(iv). By Claim 1, $S \setminus \underline{X}$ is a full sample of $\mathcal{M}(X)$ to which we can apply the induction hypothesis.

Condition (i): If $e \in f_{\mathcal{M}}(S)$, then either $e = e_S$ or $e \in f_{\mathcal{M}(X)}(S \setminus \underline{X})$. In the first case, E_e and $[S]$ osculate by the choice of e_S from osc$([S])$. In the second case, E_e and $[S \setminus \underline{X}]$ osculate in the tope graph of $\mathcal{M}(X)$ by induction hypothesis. Since the tope graph of $\mathcal{M}(X)$ is isomorphic to $[X]$, E_e crosses $[X]$ and thus $e \notin \underline{X}$. Moreover, since $X \leq S$, by Lemma 11, $[S] \subseteq [X]$. Thus $[S \setminus \underline{X}]$ is isomorphic to $[S]$. Hence, E_e and $[S]$ osculate in G.

Condition (ii): Suppose that $f_{\mathcal{M}}(S) = \{e_S, f_{\mathcal{M}(X)}(S \setminus \underline{X})\}$ is not shattered by $\mathcal{M}$. Define $D' = f_{\mathcal{M}(X)}(S \setminus \underline{X})$. By induction hypothesis, D' is shattered by $\mathcal{M}(X)$. Hence, by Lemma 7 there is a circuit Y of $\mathcal{M}$ such that $\underline{Y} \subseteq \{e_S\} \cup D'$ and $e_S \in \underline{Y}$. On the other hand, we have that $D' \subseteq X^0$ and $e_S \in \underline{X}$. Thus, $|\underline{Y} \cap \underline{X}| = 1$, and since X is a cocircuit and Y is a circuit, this contradicts orthogonality of circuits and cocircuits in OMs, see Theorem 1.

Condition (iii): Let $e \notin$ osc$([S])$. Then clearly $e \notin$ osc$([S \setminus \underline{X}])$ in $[X \setminus \underline{X}]$. Thus, $e \notin f_{\mathcal{M}}(S)$. Moreover, contracting a class that does not osculate with $[S]$ cannot yield a new class that osculates with $[S]$ by Lemma 5. Thus, by the definition of $f_{\mathcal{M}}$ and induction hypothesis we have $f_{\mathcal{M}}(S) = \{e_S, f_{\mathcal{M}(X)}(S \setminus \underline{X})\} = \{e_S, f_{\uparrow X \setminus (\underline{X} \cup \{e\})}(S \setminus (\underline{X} \cup \{e\}))\} = f_{\mathcal{M} \setminus e}(S \setminus e)$.

Condition (iv): Let S, S' be two full samples such that $S_{|f_{\mathcal{M}}(S)} = S'_{|f_{\mathcal{M}}(S')}$. In particular, $f_{\mathcal{M}}(S) = \{e_S, f_{\mathcal{M}(X)}(S \setminus \underline{X})\} = \{e_{S'}, f_{\mathcal{M}(X')}(S' \setminus \underline{X'})\} = f_{\mathcal{M}}(S')$. By the minimality in the choice of the elements e_S and $e_{S'}$ both are the smallest elements of the respective sets $f_{\mathcal{M}}(S)$ and $f_{\mathcal{M}}(S')$, whence $e_S = e_{S'} =: e$. This means that $f_{\mathcal{M}(X)}(S \setminus \underline{X}) = f_{\mathcal{M}(X')}(S' \setminus \underline{X'}) =: D'$ and for cocircuits X and X' both faces $\uparrow X \cong \mathcal{M}(X)$ and $\uparrow X' \cong \mathcal{M}(X')$ shatter the same set $D' \subseteq U$. By Lemma 10 this implies that $X = X'$ or $X = -X'$. Indeed, let $X \neq X'$. Since X, X' maximally shatter D', by Lemma 10(ii) $X = X \circ X'$ and $X' = X' \circ X$. By Lemma 10(iii) there exists a geodesic gallery between $\uparrow X$ and $\uparrow X'$. Since X and X' are cocircuits of $\mathcal{M}$, $\uparrow X$ and $\uparrow X'$ are facets of $\mathcal{M}$. Therefore $\uparrow X$ and $\uparrow X'$ must be consecutive in the gallery and the face containing them as facets must coincide with $\mathcal{M}$. Thus, $X = -X'$.

But $X = -X'$ cannot happen, otherwise, since $e \in \underline{X} \cap \underline{X'}$, we have $e \in$ Sep(X, X') and since $S \geq X$ and $S' \geq X'$, we get $S_e = -S'_e$, which contradicts the assumption $S_{|f_{\mathcal{M}}(S)} = S'_{|f_{\mathcal{M}}(S')}$. Hence, $X = X'$. Hence $S, S' \geq X$ and by Lemma 11 we get $[S] \subseteq [X]$ and $[S'] \subseteq [X]$. By induction hypothesis, $[S \setminus \underline{X}] = [S' \setminus \underline{X}]$ in $[X]$. This means that $[S] \cap [X] = [S'] \cap [X]$, but since $[S] \subseteq [X]$ and $[S'] \subseteq [X]$, we conclude that $[S] = [S']$. □

4.2. **The localization lemma.** The *localization lemma* designates for any realizable sample of a COM the set of all potential covectors whose faces may contain topes which can be used by the reconstructor.

Let $\mathcal{M} = (U, \mathcal{L})$ be a COM of VC-dimension d and let $S \in \downarrow\mathcal{L}$ be a realizable sample. Consider the tope $S' = S \setminus S^0$ of the COM $\mathcal{M}' := \mathcal{M} \setminus S^0$ and let X' be a minimal covector of $\mathcal{M}'$ such that $S' \geq X'$. Note that if $\mathcal{M}'$ is an OM, then $X' = \mathbf{0}$ and $\uparrow X' = \mathcal{M}'$. By Lemma 4, the OM

$\mathcal{M}'(X') \cong \uparrow X'$ has VC-dimension $\leq d$. Let

$$\mathcal{H}_{S,X'} := \{X \in \mathcal{L} : X \setminus S^0 = X' \text{ and } \mathcal{M}(X) \text{ has the same VC-dimension as } \mathcal{M}'(X')\}.$$

Let $D \subseteq U \setminus S^0$ be a set of size $d' = \text{VC-dim}(X')$ shattered by the OM $\mathcal{M}'(X')$. Let also

$$\mathcal{H}_D := \{X \in \mathcal{L} : \mathcal{M}(X) \text{ maximally shatters } D\}.$$

Lemma 15. *Let $S \in \downarrow\mathcal{L}$, X' be a minimal covector of $\mathcal{M}' = \mathcal{M} \setminus S^0$ such that $S \setminus S^0 = S' \geq X'$, and let $D \subseteq U$ be shattered by $\mathcal{M}'(X') = \uparrow X'$. Then $\varnothing \neq \mathcal{H}_{S,X'} = \mathcal{H}_D$.*

Proof. By Lemma 1 there must be a covector $X \in \mathcal{L}$ such that $X \setminus S^0 = X'$. Moreover, if $\mathcal{M}'(X') = \uparrow X'$ shatters D, then $\mathcal{M}(X)$ also shatters D because the tope graph of $\mathcal{M}'(X')$ is a pc-minor of the tope graph of $\mathcal{M}(X)$. Suppose that $\mathcal{M}(X)$ shatters a superset of D. Then there is a covector $Y \geq X$ of $\mathcal{M}$ such that $\mathcal{M}(Y)$ shatters D. Hence, $Y \setminus S^0 \geq X \setminus S^0 = X'$, but $\mathcal{M}'(Y \setminus S^0)$ and $\mathcal{M}'(X')$ have the same VC-dimension, so by Lemma 4 $Y \setminus S^0 = X'$. Hence $Y \in \mathcal{H}_{S,X'}$. In particular, we have shown that any element of $\mathcal{H}_{S,X'}$ shatters D and $|D|$ is its VC-dimension so it maximally shatters D. Hence, $\mathcal{H}_{S,X'} \subseteq \mathcal{H}_D$.

It remains to show $\mathcal{H}_D \subseteq \mathcal{H}_{S,X'}$. Let $Y \in \mathcal{H}_D \setminus \mathcal{H}_{S,X'}$ and set $Y' = Y \setminus S^0$. By assumption we have $X' \neq Y'$ and since $\mathcal{M}(Y)$ maximally shatters D and $D \subseteq \underline{S}$, also $\mathcal{M}'(Y')$ maximally shatters D. In particular, $D \subseteq X'^0 \cap Y'^0 = (X' \circ Y')^0$. By Lemma 9 the gates of $[Y']$ in $[X']$ are the topes of $\uparrow(X' \circ Y') \subseteq \uparrow X'$. Thus, $[X' \circ Y']$ is a gated subgraph of $[X']$, and $[X' \circ Y']$ is crossed by D, and D is shattered by $[X']$. By Lemma 6, the VC-dimension of $\mathcal{M}'(X' \circ Y')$ is at least $|D|$, which is the VC-dimension of $\mathcal{M}'(X')$. Lemma 4 yields $X' \circ Y' = X'$.

If $\text{Sep}(X', Y') = \varnothing$, then $\uparrow X' = \uparrow(X' \circ Y') \subseteq \uparrow Y'$. Since $\uparrow X'$ is a maximal face of $\mathcal{M}'$, we get $X' = Y'$. If $\text{Sep}(X', Y') \neq \varnothing$, then by Lemma 10 there exists a geodesic gallery $(\uparrow X' = \uparrow X_0, \uparrow X_1, \ldots, \uparrow X_k = \uparrow Y')$ from $\uparrow X'$ to $\uparrow Y'$ in $\mathcal{M}'$. By the definition of a gallery, the union of $\uparrow X'$ and $\uparrow X_1$ is a face $\uparrow Z \supsetneq \uparrow X'$ of $\mathcal{M}'$. Thus, $\uparrow X'$ is not a maximal face of $\mathcal{M}'$ and this contradicts the assumption that X' is a minimal covector of $\mathcal{M}'$. $\square$

4.3. **The labeled compression scheme.** Now, we describe the compression and the reconstruction and prove their correctness. The compression map generalizes the compression map for ample classes of [44]. However, the reconstruction map is much more involved than the reconstruction map for ample classes.

Compression. Let $\mathcal{M} = (U, \mathcal{L})$ be a COM of VC-dimension d. For a sample $S \in \downarrow\mathcal{L}$ of $\mathcal{M}$, consider the tope $S' = S \setminus S^0$ of $\mathcal{M} \setminus S^0 =: \mathcal{M}'$ and let X' be a minimal covector of $\mathcal{M}'$ such that $S' \geq X'$. Denote by $\mathcal{M}'(X') = \uparrow X' \setminus \underline{X'}$ the OM defined by the face $\uparrow X'$ of $\mathcal{M}'$.Define

$$\alpha(S)_e = \begin{cases} S_e & \text{if } e \in f_{\mathcal{M}'(X')}(S'), \\ 0 & \text{otherwise.} \end{cases}$$

The map α is well-defined since S' is a tope of $\mathcal{M}'(X')$ and hence the sample S' is full in $\mathcal{M}'$. Moreover, by definition we have $\alpha(S) \leq S$, whence $\alpha(S) \in \downarrow\mathcal{L}$. Finally, by Lemma 4 the OM $\mathcal{M}'(X')$ has VC-dimension at most d and thus, by Lemma 14 $\alpha(S)$ has support of size $\leq d$.

Reconstruction. To define the reconstruction map β, pick $C \in \{\pm 1, 0\}^U$ in the image of α and let $D := \underline{C}$. Let X be any covector from $\mathcal{H}_D$, i.e., X is a covector of $\mathcal{L}$ that maximally shatters D. By Lemma 15, such X exists. Now, let $\widetilde{S} \in \downarrow\mathcal{L}$ be a sample satisfying:

(1) $\widetilde{S} \geq X$;
(2) $\text{Sep}(\widetilde{S}, C) = \varnothing$;
(3) $\widetilde{S}$ is full in $\mathcal{M}(X)$;
(4) $f_{\mathcal{M}(X)}(\widetilde{S}) = D$.

Finally, set $\beta(C)$ to be any tope T of $\mathcal{M}$ with $T \geq \widetilde{S}$.

To show that β is well-defined we give a canonical sample $\widehat{S}$ satisfying (1)-(4). For $S \in \downarrow\mathcal{L}$, let $C = \alpha(S)$, $D = \underline{C}$, and $X \in \mathcal{H}_D$. By Lemma 15, X satisfies $X \setminus S^0 = X'$, where X' is the minimal covector of $\mathcal{M}' = \mathcal{M} \setminus S^0$ chosen in the definition of $\alpha(S)$. Set $\widehat{S} := X \circ S \geq X$.

Claim 2. The sample $\widehat{S}$ satisfies the conditions (1)-(4) of the definition of β. Moreover, $[\widehat{S}] = [X] \cap [S]$ and $\widehat{T} \geq S$ for any tope $\widehat{T} \in [\widehat{S}]$.

Proof. Let $C = \alpha(S)$ for $S \in \downarrow \mathcal{L}$. Since $X \setminus S^0 = X' \leq S' = S \setminus S^0$, we have $\mathrm{Sep}(X, S) = \varnothing$. By Lemma 12 $[\widehat{S}] = [X] \cap [S]$ is a proper convex subgraph of $[X]$. Since $X \setminus \widehat{S}^0 = X'$ and both $\mathcal{M}(X), \mathcal{M}'(X')$ have the same VC-dimension $|D|$, the sample $\widehat{S}$ is full in $\mathcal{M}(X)$.

Finally, since X' and S' can be obtained from X and $\widehat{S}$ by deletion of non-osculating elements of S^0, by Lemma 14(iii) we get $f_{\mathcal{M}(X)}(\widehat{S}) = f_{\mathcal{M}'(X')}(S') = D$. Since $[\widehat{S}] = [X] \cap [S] \neq \varnothing$, this intersection contains at least one tope and therefore $\beta(C)$ is well-defined. Moreover, for any tope $\widehat{T} \in [\widehat{S}]$ we have $\widehat{T} \geq S$ because $[\widehat{S}] = [X] \cap [S]$. $\square$

Correction. We prove that (α, β) defines a proper labeled sample compression scheme.

Claim 3. For all samples $S \in \downarrow \mathcal{L}$, $\beta(\alpha(S)) \geq S$.

Proof. We show that for any choice of $\widetilde{S}$ satisfying the conditions (1)-(4) in the definition of β and for any choice of a tope $\widetilde{T} \in [\widetilde{S}]$, we have $\widetilde{T} \geq S$. To prove this, we show that $[\widetilde{S}] = [\widehat{S}]$, where $\widehat{S}$ is the canonical sample defined in Claim 2. Since this implies that $\widetilde{T} \in [\widehat{S}]$, by the second assertion of Claim 2 we get $\widetilde{T} \geq S$.

So, let $\widetilde{S}, \widetilde{S}'$ be two full samples of $\mathcal{M}(X)$ such that $\widetilde{S}, \widetilde{S}' \geq X$, $\mathrm{Sep}(\widetilde{S}, C) = \mathrm{Sep}(\widetilde{S}', C) = \varnothing$, and $f_{\mathcal{M}(X)}(\widetilde{S}) = f_{\mathcal{M}(X)}(\widetilde{S}') = D$. By Lemma 14 applied to $\mathcal{M}(X)$, we have $[\widetilde{S}] = [\widetilde{S}']$. This proves that all samples $\widetilde{S}$ available to β yield the same (nonempty) convex subgraph $[\widetilde{S}] = [\widehat{S}] = [X] \cap [S]$. Since $\widehat{T} \geq S$ for any tope $\widehat{T}$ from $[\widehat{S}]$, we conclude that for any choice of $\widetilde{T} \in [\widetilde{S}]$ as $\beta(\alpha(S))$ we have $\widetilde{T} \geq S$. Hence, $\beta(\alpha(S)) \geq S$. $\square$

Example 1. Consider the tope graph G of a COM $\mathcal{M}$ of VC-dimension 3 and a realizable sample $S = (++-0-0+0)$ in Figure 4. $[S]$ is induced by 7 topes drawn as white vertices of G. Contracting the 3 dashed Θ-classes corresponding to $\{4,6,8\} = S^0$, yields the tope graph G' of $\mathcal{M}' = \mathcal{M} \setminus S^0$. Then $S' = S \setminus S^0 = (++--+)$. The compressor picks a minimal covector $X' = (+0--+)$ of $\mathcal{M}'$ with $S' \geq X'$; X' corresponds to the thick red edge in G'. The compressor returns $\alpha(S) = (0+000000)$ and $D = \{2\}$. The reconstructor receives $C = (0+000000) = \alpha(S)$, defines $D = \underline{C} = \{2\}$ and constructs the set $\mathcal{H}_D$. There are six covectors of $\mathcal{M}$ belonging to $\mathcal{H}_D$ corresponding to the thick red edges in G. By the localization lemma, they are the covectors which have the same VC-dimension as X' and agree with X' on $\{1,2,3,5,7\} = \underline{S}$. The reconstructor picks $X = (+0----+-) \in \mathcal{H}_D$. The OM $\mathcal{M}(X)$ is composed of the covectors X and the ends T and T' of the corresponding red edge. Among T and T', only the tope $T = (++-----+-)$ satisfies the conditions (1)-(4) in the definition of β. Thus, $\beta(\alpha(S))$ is set to T, which is a white node of G.

5. Conclusion

We have presented proper labeled compression schemes of size d for COMs of VC-dimension d. Even though we made strong use of the structure of COMs, it is tempting to extend our approach to other classes, e.g., bouquets of oriented matroids [19], strong elimination systems [4], or CW left-regular-bands [43]. Our treatment of realizable samples as convex subgraphs suggests an angle at general partial cubes.

To achieve *improper* labeled compression schemes our results provide a new approach, extending the one of [16, 17] presented in the introduction. Is it possible to extend a given set system or a partial cube to a COM without increasing the VC-dimension too much?

In unlabeled sample compression schemes, the compressor α is less expressive since its image is in 2^U and has to satisfy $\alpha(S) \subseteq \underline{S}$. Unlabeled compression schemes exist for realizable affine oriented

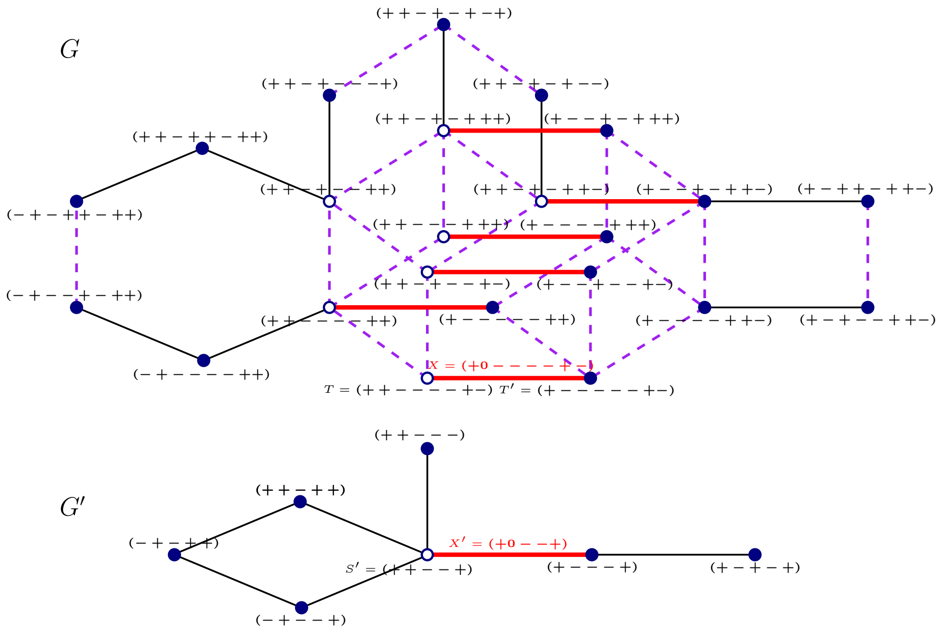


Figure 4. An illustration of Example 1.

matroids [6] and ample set systems with corner peelings [9, 39]. The notion of corner peeling has been extended to COMs [38], but it is open if they yield unlabeled compression schemes for COMs.

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