Combinatorics from Concurrency: the Nice Labelling Problem for Event Structures*

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Abstract

An event structures is a mathematical model of a concurrent process. It consists of a set of local events ordered by a causality relation and separated by a conflict relation. A global state, or configuration, is an order ideal whose elements are pairwise not in conflict. Configurations, ordered by subset inclusion, form a poset whose Hasse diagram codes the state-transition graph of the process. By labelling edges, the state-transition graph can be enriched with the structure of a deterministic concurrent automaton on some alphabet. The nice labelling problem *asks to minimize the cardinality of the alphabet.*

In this talk we shall first introduce the nice labelling problem and show how it translates to a graph coloring problem. We shall then survey on the problem and present the results that are available to us nowadays. The survey will also be the occasion to point out new perspectives, open problems, and research paths.

Keywords: *Event structures, graph-coloring, poset-covering by chains.*

1. Introduction

A principal aim of this note is to introduce the reader to the combinatorics that arises from modelling concurrency. This is a rich, attractive, and challenging field, providing a wide space for interactions to the computer scientist and to the mathematician working in the theory of relations. We shall achieve our goal by focusing on the *nice labelling problem for event structures*.

Let us mention first that ordered sets have been widely used to model concurrent computation [16, 18]. The resulting theory did have a clear impact on the problem of efficiently exploring the state space of concurrent processes [12, 15] and gave birth to a number of verification tools [14, 8] based on ordered sets. Yet this theory is far from being complete, the exploration of state spaces is still object of improvements and discoveries [7] and problems with a deep theoretical content, such as Thiagarajan's conjecture on regular event structures [3, 23, 17], are open. The nice labelling problem for event structures was posed in [20, 19, 2]. We shall survey on the problem and present the results that are available to us nowadays. Thus we shall recall some recent advances [21, 22] that come to integrate what by now might be called standard knowledge [6, 2]. This survey will also be the occasion to point out new perspectives, open problems, and research paths.

Event structures, introduced in [16], are a widely recognized model of true concurrent computation. This model has found many uses since then, within concurrency theory, in semantics [24, 25, 4] and verification [12], and even outside the scope of concurrency, in logic [9, 13] The fortune of this model is possibly due to its level of mathematical

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abstraction which has made possible a formal comparison of the multitude of models within concurrency [25]. The nice labelling problem, by asking to represent an event structure, a mathematical object whose main ingredients are posets and relations, as an automaton over some language, contributes to this comparison. It tightens up two traditional approaches to theory of concurrent computation: the first, domain theory [18, 1], is based on the theory of ordered sets, the second, trace theory [5], counts its origins in automata and language theory.

Let us describe the problem with more precisions. An event structure is made up of a set of local events E which is ordered by a causality relation $\leq$. Moreover, a conflict relation $\smile$, that may only relate causally independent events, is given. A configuration, that is, a global state of the computation comprehensive of its history, is modeled as a subset of events, lower closed w.r.t. the causality relation, which also is an independent set w.r.t. the conflict relation. Configurations may be organized into a poset, the domain the an event structure, representing all the concurrent non-deterministic executions. The Hasse diagram of this poset codes the state-transition graph of the event structure as an abstract process. By labeling the transitions of this graph with letters from some alphabet, we can enrich the graph with the structure of a deterministic concurrent automaton. The nice labelling problem asks to find a labelling that uses the minimum number of letters. The problem is actually equivalent to a graph coloring problem: we can associate to an event structure $\mathcal{E}$ a graph $\mathcal{G}(\mathcal{E})$, of which we are asked to compute the chromatic number. It might also be considered as generalization of Dilworth's problem of covering a poset by disjoint chains.

The note is structured as follows. We present in Section 2 the elementary definitions. Most of the Section is then devoted to explaining how – given the domain of an event structure $\mathcal{E}$ – a deterministic concurrent edge labeling of its Hasse diagram translates to a coloring of the graph $\mathcal{G}(\mathcal{E})$. We formalize then the nice labelling problem directly in terms of the graph $\mathcal{G}(\mathcal{E})$, as a coloring problem. In section 3 we present what constitute – up to now – the building blocks of known results. These blocks amount to a fine understanding of the structure $\mathcal{G}(\mathcal{E})$ when restricted to an antichain. What is peculiar of such an analysis is its geometric flavor, suggesting that a proper geometric approach might be taken in the future. Finally, in section 4, we survey on the main available results on the nice labelling problems, taking the opportunity to present them under new perspectives.

Acknowledgement. We are thankful to Rémi Morin for introducing us to the theory of concurrency. We would also like to thank Maurice Pouzet, for participating to us his enthusiasm for this subject, as well as for the many productive fruitful on the combinatorics of event structures.

2. Event Structures and the Nice Labelling Problem

Event structures with binary conflict are a common model of concurrent computation, introduced already in [16]. Let us review their definition.

Definition 2.1. An *event structure* is a triple $\mathcal{E} = \langle E, \leq, \smile \rangle$ where:

- $\langle E, \leq \rangle$ is a finite partially ordered set.The relation $\leq$ is called *causality*.
- $\smile \subseteq E \times E$ is a symmetric and irreflexive binary relation, such that $x \smile y$ and $y \leq z$ implies $x \smile z$. This relation is called *conflict*.

Let us remark that we consider within this paper finite event structures only. It is also possible to consider infinite event structures, in which case, for $x \in E$, the set $\{ y \in E \mid y \leq x \}$ is required to be finite. This requirement reflects the fact the past of a computation is always finite.

We observe next that $x \smile y$ implies that x, y have no common upper bound, in particular they are not comparable w.r.t. $\leq$. Otherwise, if $x, y \leq z$ and $x \smile y$, then $z \smile z$, contradicting $\smile$ being irreflexive.

It is often more convenient to analyze event structures in term of the complement of the conflict relation, and in term of a number of derived relations. Let

$$x \frown\!\!\!\!\!\!\_\,\, y \text{ iff not } x \smile y\,,$$

and call this relation *weak concurrency*. The *concurrency* relation is then defined as follows:

$$x \frown y \text{ iff } x \frown\!\!\!\!\!\!\_\,\, y \text{ and } x, y \text{ are not comparable w.r.t. the order}\,.$$

Finally, pairs in *minimal conflict* are as follows:

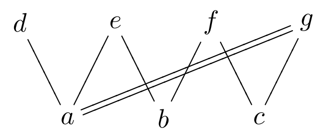
$$x \,\underline{\smile}\, y \text{ iff } (i)\; x \smile y\,,\; (ii)\; x' < x \text{ implies } x' \simeq y\,, \quad \text{and } (iii)\; y' < y \text{ implies } x \simeq y'\,.$$

Let us remark that every pair in conflict is over a pair in minimal conflict: if $x \smile y$ then $x' \leq x$ and $y' \leq y$ for some x', y' such that $x' \,\underline{\smile}\, y'$. This allows to give concise presentations of event structures.

Example 2.2. Let us consider the following example from [2]: let $\mathcal{E} = \langle E, \leq, \smile \rangle$ be the event structure defined by

- $E = \{ a, b, c, d, e, f, g \}$,
- $\leq = \{ (a, d), (a, e), (b, e), (b, f), (c, f), (c, g) \}$,
- $\smile = \{ \{a, g\}, \{d, g\}, \{e, g\} \}$.

As the pairs in conflict are determined by those in minimal conflict, we can limit us to present the Hasse diagram of the causality relation together with the pairs in minimal conflict, double dashed in the following diagram:



Let us now understand in which sense an event structure models concurrent computation. Let

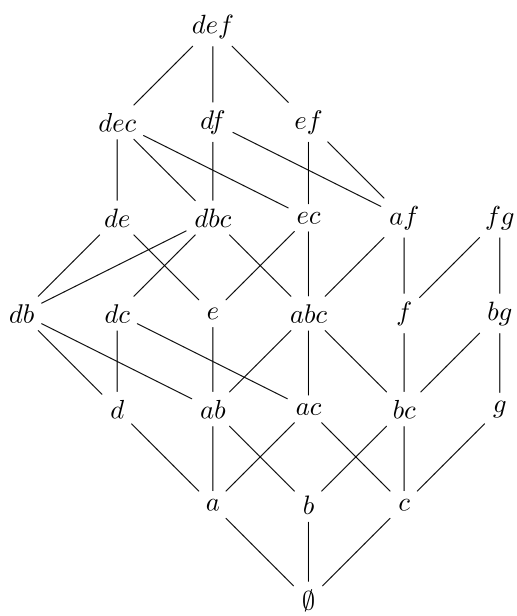
$$\mathcal{I} = \{ I \subseteq E \mid y \leq x \in I \text{ implies } y \in I \}$$

be the collection of order ideals of $\langle E, \leq \rangle$ and let

$$\mathcal{D} = \{ I \in \mathcal{I} \mid I \text{ is a clique w.r.t. } \simeq \}\,.$$

An element of $\mathcal{D}$ is called a *configuration* de $\mathcal{E}$. Now, $\mathcal{D}(\mathcal{E}) = \langle \mathcal{D}, \subseteq \rangle$ is itself an ordered set, named the *domain associated to* $\mathcal{E}$.

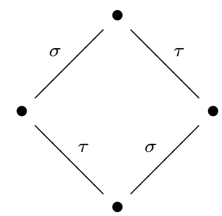
A configuration of an event structure represents a state (comprehensive of its history) of a global computation: it is a collection of local events which pairwise are either one in the past of the other (that is, comparable w.r.t. the causality order) or part of a common computation (that is, they are related by the concurrency relation). For $I, J \in \mathcal{D}$, $I \subseteq J$ means that J may occur after I in time. The Hasse diagram of $\mathcal{E}$ represents therefore the state-transition graph of $\mathcal{E}$ as a process. We draw next such a diagram for the event structure introduced in Example 2.2, with each configuration labeled by its maximal events.



We are now in possession of all the ingredients required to introduce the nice labelling problem. We obtain a representation of the process $\mathcal{E}$ as an automaton if we color the edges of the Hasse diagram by letters of some alphabet. It is quite natural, however, to ask this coloring to satisfy the following conditions.

Determinism: transitions outgoing from the same state have different colors.

Concurrency: every square of the diagram has to be colored according to the following pattern, suggesting that actions σ, τ may take place in parallel:



By pushing down colors along such squares, we see that a concurrent edge-coloring is determined by the configurations having a unique incoming transition and by the color of this transition. In turn, these configurations are in bijection with elements of $\mathcal{E}$. Hence:

Lemma 2.3. *There is a bijection between concurrent edge-colorings of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ and colorings of elements of $\mathcal{E}$.*

An edge (I, J) of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ is such that $J = I \cup \{ x \}$ for some $x \in E \setminus I$. The bijection constructs an edge-coloring λ' of the Hasse diagram from a coloring of λ of E by letting $\lambda'(I, I \cup \{ x \}) = \lambda(x)$.

We are left to analyze how the condition on determism of a concurrent edge-coloring transfers to a coloring of E. To this goal, let us introduce the *orthogonality* relation between events:

$$x \mathrel{\overline{\frown}} y \text{ iff } x \mathrel{\underline{\smile}} y \text{ or } x \frown y \,,$$

and define the graph of $\mathcal{E}$:

$$\mathcal{G}(\mathcal{E}) = \langle E, \overline{\frown} \rangle \,.$$

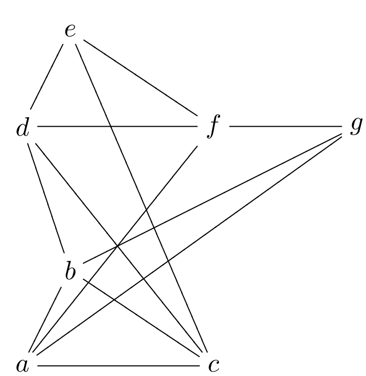
The following Lemma is the key to understand the role of the orthogonality relation.

Lemma 2.4. *A set $\{ x_1, \ldots, x_n \}$ is a clique of $\mathcal{G}(\mathcal{E})$ if and only if there exists a configuration $I \in \mathcal{D}$ s.t. $(I, I \cup \{ x_i \})$, $i = 1, \ldots, n$, are distinct transitions of the Hasse diagram of $\mathcal{D}(\mathcal{E})$.*

Therefore, the degree of $\mathcal{E}$, that is, the maximum outdegree of configurations in $\mathcal{D}$, coincides with the clique number of $\mathcal{G}(\mathcal{E})$. Letting $n = 2$ in the statement of Lemma 2.4, we obtain the following Proposition:

Proposition 2.5. *There is a bijection between concurrent deterministic edge-colorings of the Hasse diagram of $\mathcal{D}(\mathcal{E})$ and colorings the graph of $\mathcal{G}(\mathcal{E})$.*

Given the central role of the graph $\mathcal{G}(\mathcal{E})$, we draw next the graph of the event structure introduced in Example 2.2:



We leave the reader to verify that the chromatic number of this graph is 4. We can now define the relevant notions.

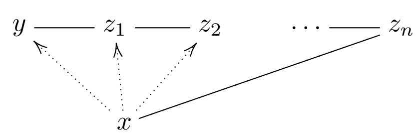
Definition 2.6. A *nice labelling* of the event structure $\mathcal{E}$ is a coloring of $\mathcal{G}(\mathcal{E})$. The *degree* of $\mathcal{E}$, $\omega(\mathcal{E})$, is the clique number of $\mathcal{G}(\mathcal{E})$, i.e. the number $\omega(\mathcal{G}(\mathcal{E}))$. The *index* of $\mathcal{E}$, $\chi(\mathcal{E})$, is the chromatic number of $\mathcal{G}(\mathcal{E})$, i.e. the number $\chi(\mathcal{G}(\mathcal{E}))$.

Relations such as $\omega(\mathcal{E}) \leq \chi(\mathcal{E})$ follow immediately. The *nice labelling problem* asks to compute $\chi(\mathcal{E})$ for a given $\mathcal{E}$. It was shown to be an NP-complete problem in [2]. More generally, given a class $\mathcal{K}$ of event structures, the problem asks whether there is an upper bound to the set $\{ \chi(\mathcal{E}) \mid \mathcal{E} \in \mathcal{K} \}$. Of particular interest are the classes $\mathcal{K}_n$ obtained by bounding the degree, $\mathcal{E} \in \mathcal{K}_n$ if and only if $\omega(\mathcal{E}) \leq n$.

We end this section with some observations. Among the properties of the orthogonality relation, let us remark the following one:

$$x \leq y \text{ and } y \underline{\frown} z \text{ implies } x \leq z \text{ or } x \underline{\frown} z\,. \tag{clock}$$

This property already holds for the concurrency relation and is inherited by the orthogonality relation. To see that the clock property holds of $\frown$, observe that the property is stating that the weak concurrency relation is down-closed, an immediate consequence of being the complement of the up-closed conflict relation. We call this property the *clock property*, since it can be represented as in the diagram below by letting the comparability relation to move as an arrow of a clock, until it finds the border of the clock, the relation $x \underline{\frown} z_n$.



Among all the relations $\underline{\frown}$ on a poset $\langle P, \leq \rangle$ satisfying the clock property, it is possible to characterize those that arise as orthogonality relations. Say that a pair x, y is covered if there exists $x' \geq x$, $y' \geq y$ with $x' > x$ or $y' > y$, such that $x' \underline{\frown} y'$ or x', y' comparable. Then $\underline{\frown}$ is an orthogonality relation if and only if it is closed in the following sense: if $x, y \in P$ are such that (i) $x' < x$ implies that x', y is covered or $x' < y$ and (ii) $y' < y$ implies that x, y' is covered or $y' < x$, then $x \underline{\frown} y$.

A second remark concerns Lemma 2.4. It can be verified that the collection $\mathcal{D}$ of all configurations of $\mathcal{E}$ satisfies the following conditions:

- if $I \in \mathcal{D}$, then I is a lower set of $\mathcal{E}$,
- if $x \in E$, then $\{ y \in E \mid y \leq x \} \in \mathcal{D}$,
- if $J \subseteq I \in \mathcal{D}$, then $J \in \mathcal{D}$.

That is, the domain $\mathcal{D}(\mathcal{E})$ is a sort of ordered simplicial complex, or a chopped lattice in the sense of [10]. As a chopped lattice, $\mathcal{D}(\mathcal{E})$ also satisfies the distributive identity whenever this makes sense. It has already been proposed in [25] to generalize the definition of an event structure to that of a triple $\langle E, \leq, \mathcal{D} \rangle$, where $\mathcal{D}$ satisfies the above conditions. The proposed generalization is quite natural. As a matter of fact, three events might be pairwise not in conflict, but yet have no reason to give rise to a common parallel computation. This might happen for example if a semaphore of capacity two is part of the computation. The nice labelling problem for such a generalized event structure might still be asked. In this new context, it is still possible to reduce a concurrent deterministic edge-coloring of the Hasse diagram to a coloring of some graph $\mathcal{G}(\mathcal{E})$ having E as set of vertices. Yet Lemma 2.4 fails and the outdegree of the Hasse diagram of $\langle \mathcal{D}, \subseteq \rangle$ is simply a lower bound for the clique number of $\mathcal{G}(\mathcal{E})$. It is an open problem to establish whether some stronger relation holds between these two parameters.

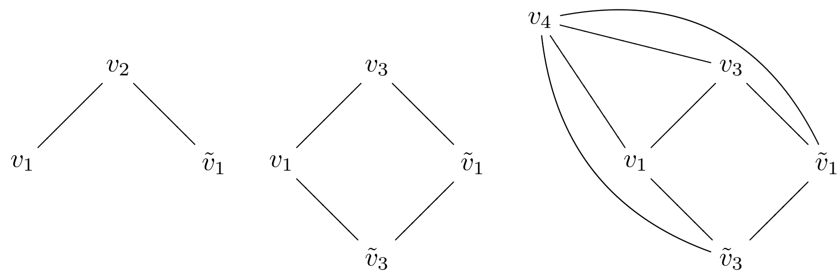


Figure 1. The graphs G_2, G_3, G_4.

3. Geometry on Antichains

The following results, appeared in [2, 21], are the building blocks for the results on the nice labelling problem [2, 22]. They concern the structure of the graph $\mathcal{G}(\mathcal{E})$ restricted to an antichain. What is remarkable is their geometric flavor.

For a graph $G = \langle V, E \rangle$, let $v * G = \langle V \uplus \{ v \}, E \cup E' \rangle$ where $E' = \{ \{v, v'\} \mid v' \in V \}$. This operation amounts to *adding a cone* to G.The *suspension* of G consists of adding two cones to G. Denoted by $(v, \tilde{v}) * G$, it is defined similarly: $(v, \tilde{v}) * G = \langle V \uplus \{ v \} \uplus \{ \tilde{v} \}, E \cup E'$ where $E' = \{ \{v, v'\}, \{\tilde{v}, v'\} \mid v' \in V \}$.

We define next a sequence of graphs G_n, for $n \geq -1$. The graph G_{-1} is the empty graph. If n is even, then $G_n = v_n * G_{n-1}$, and, if n is odd, $G_n = (v_n, \tilde{v}_n) * G_{n-2}$. For $n = 2, 3, 4$, the graphs G_n appear in Figure 1. The following is a graph-theoretic characterization of the graphs G_n. To this goal, let P be the following property: *if x, y are distinct nodes of $G = \langle V, E \rangle$ such that $\{x, y\} \notin E$, then they both form a cone over $V \setminus \{ x, y \}$.*

Proposition 3.1. *The graphs G_n have property P. Moreover, if a graph $\langle V, E \rangle$ has property P, then it contains a copy of G_k as a subgraph, with $k = \mathrm{card}(V) - 1$.*

For the sake of the studying the graph $\mathcal{G}(\mathcal{E})$, we have the following result.

Proposition 3.2. *Let $\mathcal{E}$ be an event structure with $\omega(\mathcal{E}) \leq n$. If $G_n = \langle V, E \rangle$ appears as a subgraph of $\mathcal{G}(\mathcal{E})$, then V contains a pair of comparable elements.*

In other words, an antichain of $\mathcal{E}$ induces a subgraph of $\mathcal{G}(\mathcal{E})$ having G_n as a forbidden subgraph. For $n = 2$, this result was proved in [2, Lemma 2.2]. For $n = 3$, we made extensive use of this result in [22].

The construction of the graphs G_n reminds the construction of disks and spheres in geometry. This is not just an analogy and the similarity may be explained by introducing $\mathcal{C}(G)$, the clique complex of a graph $G = \langle V, E \rangle$. Vertices of $\mathcal{C}(G)$ are those of G, and a subset $S \subseteq V$ is a simplex of $\mathcal{C}(G)$ if and only if it is a clique within G. We have then:

Proposition 3.3. *For each $n \geq 0$, (the geometric realization of) $\mathcal{C}(G_{2n})$ is (homeomorphic to) a disk in dimension n and $\mathcal{C}(G_{2n+1})$ is a sphere in dimension n.*

It might be valuable to explore this geometric point view further. We proved in [21] that all the 1-dimensional spheres are forbidden on antichains of a $\mathcal{G}(\mathcal{E})$, with $\omega(\mathcal{E}) \leq 3$, not just the simple sphere G_3. This result is rephrased by saying that cycles of length equal to or greater than 4 are forbidden. In an attempt to generalize this result to higher degrees and dimensions – but also willing to lay down a bridge to the geometrical perspective on concurrency developed in [11] – we introduced the homology of an event structure.

Let $\mathcal{A}(\mathcal{E})$ be the set of antichains of $\langle E, \leq \rangle$. These antichains may be organized into a poset $\langle \mathcal{A}(\mathcal{E}), \gg \rangle$ by defining $A \gg B$ if for all $a \in A$ there exists $b \in B$ such that $a \geq b$. Now, for $A \in \mathcal{A}(\mathcal{E})$, let $\mathbb{H}_*(A)$ be the sequence of homology groups of the clique complex $\mathcal{C}(\mathcal{G}(\mathcal{E})_{|A})$, $\mathcal{G}(\mathcal{E})_{|A}$ being the subgraph of $\mathcal{G}(\mathcal{E})$ induced by A.

Proposition 3.4. *There is a well defined homology functor $\mathbb{H}_*$ from the poset $\langle \mathcal{A}(\mathcal{E}), \gg \rangle$ to the category of infinite sequences of Abelian groups.*

4. Results on the Nice Labelling Problem

It is time to recall the known results on the nice labelling problem. The first one is the celebrated Dilworth's theorem. We mention it to elucidate the sense for which the nice labelling problem generalizes the problem of covering a poset by chains.

Theorem 4.1 (Dilworth [6])**.** *If the conflict relation of $\mathcal{E}$ is empty, then $\chi(\mathcal{E}) = \omega(\mathcal{E})$.*

As a matter of fact, if the conflict relation is empty, then $x \frown y$ if and only if x, y are not comparable. We remark that, in the general case, an independent set in $\mathcal{G}(\mathcal{E})$ is a forest w.r.t the causality order. Observe next that the conflict relation of $\mathcal{E}$ is empty if and only if there is no pair of events $x, y \in E$ in minimal conflict, i.e. such that $x \smile y$. Dilworth's Theorem, as a statement about event structures with a limited number of minimal conflicts, has the following generalization:

Theorem 4.2 (Assous et al. [2])**.** *For each $n \geq 0$ there exists $m \geq 0$ such that if $\mathcal{E}$ has at most n pairs $(x, y) \in E^2$ such that $x \smile y$, then the width of the poset $\langle E, \leq \rangle$ is bounded by m; consequently, $\chi(\mathcal{E}) \leq m$.*

The next results deal with event structures of fixed the degree.

Theorem 4.3 (Assous et al. [2])**.** *If $\omega(\mathcal{E}) = 2$, then $\chi(\mathcal{E}) = 2$.*

We sketch next an alternative proof of the above theorem, showing that $\mathcal{G}(\mathcal{E})$ contains no odd length simple cycle, hence it is bipartite. To this goal it is enough to prove that $\mathcal{G}(\mathcal{E})$ does not contain chordless simple cycles of odd length; as a matter of fact, an odd length simple cycle is split by a chord into a pair of cycles of shorter length, one of which still has odd length. The following Lemma, however, proves a stronger property, and calls for the problem of characterizing graphs that are of the form $\mathcal{G}(\mathcal{E})$ with $\omega(\mathcal{E}) = 2$.

Lemma 4.4. *If $\omega(\mathcal{E}) = 2$, then all the chordless cycles of $\chi(\mathcal{E})$ have length 4.*

Proof. Let v_i, $0 \leq i < n$, be a chordless simple cycle of length n within $\mathcal{G}(\mathcal{E})$: $v_i \frown v_{i+1}$, $0 \leq i < n$, where the sums on indexes, here and in the rest of the proof, are modulo n.

Since $\omega(\mathcal{E}) = 2$, we cannot have $n = 3$. Observe next that, for $0 \leq i < n$, $v_{i-1} \frown v_i \frown v_{i+1}$ form a G_2 and, by Proposition 3.2, v_{i-1}, v_{i+1} are comparable. We can then exclude the case $n = 5$, since then we would obtain that a cycle of length 5 is the comparability graph of some poset, which cannot be the case.

Let us suppose now that $n \geq 6$. Observe that if $v_i < v_j$ for some j, then $v_i < v_k$ for all k such that $v_i \frown v_k$ does not hold. This is an immediate consequence of the clock property, and of the fact that the cycle $v_0 \ldots v_n$ is chordless and simple. Thus we can split the set $\{ v_i \mid 0 \leq i < n \}$ into the set of its minima, $m = \{ v_i \mid \exists j \text{ s.t. } v_i < v_j \}$, and the set of its maximal elements, $M = \{ v_j \mid \exists i \text{ s.t. } v_i < v_j \}$. Observe that $m \cap M = \emptyset$ and that $m, M \neq \emptyset$. W.l.o.g. we can assume $v_2 \in M$, which implies then $v_0, v_4 \in m$. Since $v_0 \frown v_4$ does not hold, we have $v_0 < v_4 < v_0$, a contradiction. □

Let us move to higher degrees, and recall the following result.

Proposition 4.5 (Assous et al. [2])**.** *There exists a family of event structures $\mathcal{E}_n$, $n \geq 3$, such that $\omega(\mathcal{E}_n) = n$ and $\chi(\mathcal{E}_n) = n + 1$.*

For $n = 3$, the event structure $\mathcal{E}_n$ is the one presented in Example 2.2. While looking for structural properties that are responsible for the inequality $\omega(\mathcal{E}) \neq \chi(\mathcal{E})$, we ended up considering ordered structures that are trees. Thus let us say that an event structure is *tree-like* if its order is a tree. The following was the main result presented in [22]:

Theorem 4.6. *If $\mathcal{E}$ is tree-like and $\omega(\mathcal{E}) \leq 3$, then $\chi(\mathcal{E}) \leq 3$.*

We present next the ideas behind the proof. While the following discussion does not constitute an alternative proof of Theorem 4.6, its goal is to present these ideas by means a simple analogy which might help clarifying the original proof [22, §4]. The analogy is that of some brothers in the process of inheriting the fortune of their unique parent. The eldest – and luckiest – brother inherits all the fortune. Younger brothers need to build up their own fortune, by complying with the rules of the society they are aware of: they should not steel.

For $x \in E$, let us say that p is the *parent* of x if p is the unique lower cover of x. Let us say that y is a *brother*[1] of x if x, y have the same parent. It is easy to see that if x, y are brothers, then $x \frown y$. In particular, taking into account that $\omega(\mathcal{E}) \leq 3$, we can have at most three pairwise brothers. Define the *society* of x as

$$\mathcal{S}^x = \{\, y \in E \mid x \frown y, y \text{ is not above any brother of } x \,\}\,.$$

Let $\lhd$ be a linear order on E respecting the tree-height: $\mathrm{h}(x) < \mathrm{h}(y)$ implies $x \lhd y$. We think the relation $x \lhd y$ as stating that x is older in age than y. Define now

$$O^x_{\lhd} = \{\, y \in E \mid x \frown y, y \lhd x \,\}\,.$$

In constructing a coloring of $\mathcal{G}(\mathcal{E})$ we shall assume that we have already defined a partial coloring λ of $\mathcal{G}(\mathcal{E})$ whose support is the set $\{\, z \in E \mid z \lhd x \,\}$, which uses 3 colors. To extend it to a partial coloring whose support includes x, we need to color x so to be different from all colors appearing in $O^x_{\lhd}$.

Our first remark is that if $y \in O^x_{\lhd}$, then $y \in \mathcal{S}^x$ or y is an older brother of x. In particular, if x is an *eldest brother*, then $O^x_{\lhd} \subseteq \mathcal{S}^x$. Thus, we can extend λ on x, by decreeing that x *inherits the fortune* – that is, the color – *of his parent* p, $\lambda(x) = \lambda(p)$. This is still a partial proper coloring of $\mathcal{G}(\mathcal{E})$: let $y \in O^x_{\lhd}$, then $x \frown y$ implies $p < y$ or $p \frown y$, by the clock property. If $p < y$, considering that also $y \lhd x$ and $\lhd$ respects the height, then y has to be an older brother of x, a contradiction. Thus we have $p \frown y$ and $\lambda(x) = \lambda(p) \neq \lambda(y)$.

Whenever there is more than one brother, then we are left with two questions: who is the eldest brother – that is, how should $\lhd$ be defined on brothers – and, then, how younger brothers enrich themselves without steeling from their society – that is, how should we extend λ to younger brothers. The answers to both questions are as follows, cf. [22, Lemmas 3,4]:

Lemma 4.7. *If x has two brothers, then $\mathcal{S}^x = \emptyset$. If x, y are the only two brothers with the same parent, then $\mathcal{S}^x \subseteq \mathcal{S}^y$ or $\mathcal{S}^y \subseteq \mathcal{S}^x$, and $\mathcal{S}^x \cap \mathcal{S}^y$ is linearly ordered.*

If x, y, z are three distinct brothers, then the Lemma ensures that we can choose the eldest in any arbitrary way: since $O^x_{\lhd} \subseteq \{\, y, z \,\}$, then we shall have no problems extending λ using three colors. If x, y are the only brothers, then we decree that x is the eldest of them if $\mathcal{S}^y \subseteq \mathcal{S}^x$. Let m be the minimum element of $\mathcal{S}^y = \mathcal{S}^x \cap \mathcal{S}^y$, then we define $\lambda(y)$ to be the color different from $\lambda(x)$ and $\lambda(m)$. That this gives rise to a partial coloring of $\mathcal{G}(\mathcal{E})$ is ensured by the following structural property, cf. [22, Lemma 5]:

Lemma 4.8. *If y is a younger brother, $z \in \mathcal{S}^y$, and $z \neq m$, then z is an eldest brother.*

Considering that eldest brothers inherit the color of their parents and that $\mathcal{S}^y$ is a linear order, then we derive $\lambda(z) = \lambda(m)$ for $z \in \mathcal{S}^y$. Consequently, $\lambda(y) \neq \lambda(m) = \lambda(z)$. Lemma 4.8 is easily deduced from Lemma 4.7: if $z \in \mathcal{S}^y$ and $z \neq m$, then x, y are above no brother of z, otherwise $m \leq y$ instead of $m \frown y$. Hence $x, y \in \mathcal{S}^z$, so that if w is an older brother of z, then the relation $\mathcal{S}^z \subseteq \mathcal{S}^w$ implies that $\{\, x, y, z, w \,\}$ is a clique within $\mathcal{G}(\mathcal{E})$, a contradiction.

We end up here this succinct exposition of the ideas behind Theorem 4.6.

Coming back to our search for structural properties of event structures ensuring the equality $\chi(\mathcal{E}) = \omega(\mathcal{E})$, it might be asked whether tree-likeness suffices to this goal, even in higher degree. The answer is already found in [2] toghether with a simple method of coding a graph as a subgraph of some graph of the form $\mathcal{G}(\mathcal{E})$:

Proposition 4.9 (Assous et al. [2])**.** *There is a family $\mathcal{E}_n$, $n \geq 3$, of tree-like event structures such that $\omega(\mathcal{E}_n) = \binom{n}{2} + 2$ and $\chi(\mathcal{E}_n) \geq \binom{n}{2} + \log(n)$.*

[1] We intentionally avoid the word "sibling", which implies the existence of a pre-existing order among sons of the same vertex of a tree.

It might be noticed that in the above family of event structures the ratio $\frac{\chi(\mathcal{E}_n)}{\omega(\mathcal{E}_n)}$ converges for n tending to infinity, thus providing some evidence that the index of an event structure cannot be so far away from its degree. That this is not enough as an evidence is witnessed by the following recent result, obtained in collaboration with Maurice Pouzet, showing that the gap between the index and the the degree may grow quite fast.

Proposition 4.10. *There is a family $\mathcal{E}_n$, $n \geq 1$, of tree-like event structures such that*

$$\frac{\chi(\mathcal{E}_n)}{\omega(\mathcal{E}_n)} \geq (\frac{5}{4})^{n-1} .$$

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