Finite Coxeter lattices and lattices of finite closure systems: some (lower) bounded lattices

Nathalie Caspard

LACL, University Paris 12 Val-de-Marne, France
and CAMS, EHESS, Paris, France

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**Sketch of the talk**

1. **(Lower) bounded lattices and the doubling operation**

2. **Finite Coxeter lattices**
   - Coxeter lattices
   - The class $\mathcal{HH}$ of lattices
   - All lattices of $\mathcal{HH}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{HH}$

3. **The lattice of finite closure systems**
Outline

1. (Lower) bounded lattices and the doubling operation

2. Finite Coxeter lattices
   - Coxeter lattices
   - The class $\mathcal{HH}$ of lattices
   - All lattices of $\mathcal{HH}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{HH}$

3. The lattice of finite closure systems
(Lower) bounded lattices and the doubling operation

Finite Coxeter lattices
The lattice of finite closure systems

(Lower) BOUNDED LATTICES

Definition (McKenzie [10], 1972)

A homomorphism $\alpha : L \rightarrow L'$ is called lower bounded if the inverse image of each element of $L'$ is either empty or has a minimum.

A lattice is lower bounded if it is the lower bounded homomorphic image of a free lattice.

An upper bounded lattice is defined dually and a lattice is bounded if it is lower and upper bounded.
THE DOUBLING CONSTRUCTION, DAY [6], 1970

(Lower) bounded lattices and the doubling operation

Finite Coxeter lattices

The lattice of finite closure syst

\begin{tikzpicture}
    \node (A) at (0,0) {a};
    \node (B) at (1,1) {b};
    \node (C) at (1,2) {c};
    \node (D) at (2,1) {d};
    \node (E) at (1,0) {e};
    \node (F) at (2,0) {f};
    \node (G) at (3,0) {g};
    \node (H) at (4,0) {h};
    \node (I) at (1,3) {i};
    \node (J) at (2,3) {j};
    \node (K) at (3,3) {k};

    \draw (A) -- (B) -- (C) -- (D) -- (E) -- (A);
    \draw (A) -- (I);
    \draw (B) -- (I);
    \draw (C) -- (I);
    \draw (D) -- (I);
    \draw (A) -- (J);
    \draw (B) -- (J);
    \draw (C) -- (J);
    \draw (D) -- (J);
    \draw (A) -- (K);
    \draw (B) -- (K);
    \draw (C) -- (K);
    \draw (D) -- (K);
    \draw (E) -- (F);
    \draw (F) -- (G);
    \draw (G) -- (H);
    \draw (H) -- (E);
\end{tikzpicture}
The doubling construction, Day [6], 1970

(Lower) bounded lattices and the doubling operation

Finite Coxeter lattices The lattice of finite closure syst
(Lower) bounded lattices and the doubling operation

The doubling construction, Day [6], 1970
The doubling construction, Day [6], 1970
The doubling construction, Day [6], 1970

(Lower) bounded lattices and the doubling operation
GENERALISATION TO LOWER PSEUDO-INTERVALS
GENERALISATION TO LOWER PSEUDO-INTERVALS
GENERALISATION TO LOWER PSEUDO-INTERVALS
Generalisation to convex sets
CHARACTERISATION OF BOUNDED LATTICES

Theorem (Day [7], 1979)

Let $L$ be a lattice. The following are equivalent:

- $L$ is bounded,
- it can be constructed starting from $2$ by a finite sequence of interval doublings.
Theorem (Day [7], 1979)

Let $L$ be a lattice. The following are equivalent:

- $L$ is lower bounded,
- it can be constructed starting from $2$ by a finite sequence of lower pseudo-intervals.
THEOREM (DAY [7], 1979)

Let $L$ be a lattice. The following are equivalent:

- $L$ is upper bounded,
- it can be constructed starting from $2$ by a finite sequence of upper pseudo-intervals.
AN EXAMPLE OF BOUNDED LATTICE
An example of bounded lattice
An example of bounded lattice
Perm(3) is bounded
Permutohedron on 4 elements:
Permutohedron on 4 elements: bounded too
### Permutohedron on 5 Elements:

<table>
<thead>
<tr>
<th>1, 2, 4, 5, 3</th>
<th>3, 1, 4, 2, 5</th>
<th>3, 1, 5, 2, 4</th>
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<th>3, 1, 2, 4, 5</th>
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<tbody>
<tr>
<td>4, 1, 2, 3, 5</td>
<td>1, 4, 5, 2, 3</td>
<td>1, 5, 2, 4, 3</td>
<td>5, 1, 2, 3, 4</td>
<td>2, 1, 5, 3, 4</td>
</tr>
</tbody>
</table>
PERMUTOHEDRON ON 5 ELEMENTS: BOUNDED AGAIN
In fact...

Permutohedron is bounded
In fact...

Permutohedron is bounded

And in fact...

All finite Coxeter lattices are bounded
Outline

1. (Lower) bounded lattices and the doubling operation

2. Finite Coxeter lattices
   - Coxeter lattices
   - The class $\mathcal{H}$ of lattices
   - All lattices of $\mathcal{H}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{H}$

3. The lattice of finite closure systems
What is a Coxeter group?

Definition

A group $W$ is a \textit{Coxeter group} if $W$ has a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = e$$

where $m(s, s) = 1$ for any $s$ in $S$ (all generators have order 2), and $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in $S$. 
List of all finite irreducible Coxeter groups

The four infinite families:

- $A_n$ (symmetric groups),
- $B_n$,
- $D_n$,
- and $I_n$ (dihedral groups).
List of all finite irreducible Coxeter groups

1. The four infinite families:
   - $A_n$ (symmetric groups),
   - $B_n$,
   - $D_n$,
   - and $I_n$ (dihedral groups).

2. and the six isolated groups: $E_6$, $E_7$, $E_8$, $F_4$, $H_3$ and $H_4$. 
Coxeter lattices

**Coxeter graph of finite irreducible Coxeter groups**

- $A_n$
- $B_n (n \geq 2)$
- $D_n (n \geq 4)$
- $I_n (n \geq 5)$
- $E_6$
- $E_7$
- $E_8$
- $F_4$
- $H_3$
- $H_4$
The lattice structure of Coxeter groups
The lattice structure of Coxeter groups

Cayley graph of a group ordered by the (right) weak order

If \( \ell(w) < \ell(ws) \).
If $\ell(w) < \ell(ws)$.

**Theorem (Björner, 1984)**

The weak order on any finite Coxeter group is a (autodual) lattice.
Sketch of the proof:

Finite Coxeter lattices are bounded.
Finite Coxeter lattices are bounded

Sketch of the proof:

1. Defining a new class of lattices: $\mathcal{H}$,
Finite Coxeter lattices are bounded

Sketch of the proof:
1. Defining a new class of lattices: $\mathcal{H}$,
2. Showing that lattices of $\mathcal{H}$ are bounded,
Finite Coxeter lattices are bounded

Sketch of the proof:

1. Defining a new class of lattices: $\mathcal{HH}$,
2. Showing that lattices of $\mathcal{HH}$ are bounded,
3. Showing that finite Coxeter lattices are in $\mathcal{HH}$. 
Finite Coxeter lattices are bounded

Sketch of the proof:

1. Defining a new class of lattices: $\mathcal{H}$,
**Hat, AntiHat and 2-facet**

**Definition**

- A *Hat* \((y, x, z)^\wedge\) :

  ![Hat Diagram](image)

- An *AntiHat* \((y, x, z)^\vee\) :

  ![AntiHat Diagram](image)
The class $\mathcal{HH}$ of lattices

**Hat, antihat and 2-facet**

**Definition**

- A **Hat** $(y, x, z)^\land$:

```
       x
      / \    /
     y   z  y
```

- An **antihat** $(y, x, z)^\lor$:

```
       x
      / \    /
     y   z  y
```
**Definition**

- **a Hat** $(y, x, z)^\wedge$ :

- **an antiHat** $(y, x, z)^\lor$ :

- **a 2-facet** $F_{y,x,z}$ :

\[
\begin{align*}
\text{a Hat } & (y, x, z)^\wedge : \\
& \quad \text{Diagram of a Hat} \\
\text{an antiHat } & (y, x, z)^\lor : \\
& \quad \text{Diagram of an antiHat} \\
\text{a 2-facet } & F_{y,x,z} : \\
& \quad \text{Diagram of a 2-facet}
\end{align*}
\]
The class $\mathcal{H}$ of lattices

**Definition of a 2-facet labelling**

*Fig.: Example of a 2-facet labelling*
The class $\mathcal{HH}$ of lattices

**Definition of a 2-facet labelling**

**Fig.:** Example of a 2-facet labelling
DEFINITION OF A 2-FACET LABELLING

Fig.: Example of a 2-facet labelling
**Definition of a 2-facet labelling**

![Diagram showing two 2-facet labellings of lattices]

**Fig.: Another example of a 2-facet labelling**
The class $\mathcal{H}$ of lattices

**Definition of a 2-facet labelling**

**Fig.: Another example of a 2-facet labelling**
The class $\mathcal{HH}$ of lattices

**2-FACET RANK FUNCTION ON A 2-FACET LABELLING**

**Definition**

![Diagram of 2-facet rank function on a 2-facet labelling]
The class $\mathcal{H}$ of lattices

**2-FACET RANK FUNCTION ON A 2-FACET LABELLING**

**Definition**

This is a function $r$ from $T = \{t_1, \ldots, t_i, \ldots t_p\}$ to $\mathbb{R}$.
2-FACET RANK FUNCTION ON A 2-FACET LABELLING

Definition

This is a function $r$ from $T = \{t_1, \ldots, t_i, \ldots t_p\}$ to $\mathbb{R}$ such that:
2-FACET RANK FUNCTION ON A 2-FACET LABELLING

**Definition**

This is a function \( r \) from \( T = \{ t_1, ..., t_i, ... t_p \} \) to \( \mathbb{R} \) such that:

So: \( r(t_1) < r(t_2) < r(t_3) \)

and \( r(t_6) < r(t_5) < r(t_4) \)

and \( r(t_1), r(t_6) < r(t_7) \)
2-FACET RANK FUNCTION ON A 2-FACET LABELLING
Here \( r(t_1) < r(t_5), r(t_3) \)
The class $\mathcal{HH}$ of lattices

2-FACET RANK FUNCTION ON A 2-FACET LABELLING

Here $r(t_1) < r(t_5), r(t_3)$ and $r(t_2) < r(t_6), r(t_3)$
Definition

A lattice is *semidistributive* if, for all \( x, y, z \in L \):

- \( x \land y = x \land z \) implies \( x \land y = x \land (y \lor z) \)
- \( x \lor y = x \lor z \) implies \( x \lor y = x \lor (y \land z) \)
Definition

A lattice is semidistributive if, for all $x, y, z \in L$:

- $x \land y = x \land z$ implies $x \land y = x \land (y \lor z)$
- $x \lor y = x \lor z$ implies $x \lor y = x \lor (y \land z)$

Proposition (Day, Nation, Tschchantz [8], 1989)

Bounded lattices are semidistributive.
The class $HH$ of lattices

**The class $HH$ of lattices**

Definition

A finite lattice $L$ is in the class $HH$ if it satisfies:

1. $L$ is semidistributive,
2. to every hat $(y, x, z)$ of $L$ is associated an anti-hat $(y', y \land z, z')$ of $L$ such that $[y \land z, x]$ is a 2-facet,
3. to every anti-hat $(y, x, z)$ of $L$ is associated a hat $(y', y \lor z, z')$ of $L$ such that $[x, y \lor z]$ is a 2-facet,
4. there exists a 2-facet labelling $T$ on the (covering) edges of $L$ and a 2-facet rank function $r$ on $T$. 


### The Class $\mathcal{HH}$ of Lattices

**Definition**

A finite lattice $L$ is in the class $\mathcal{HH}$ if it satisfies:

1. $L$ is **semidistributive**,  
2. to every hat $(y, x, z)^\wedge$ of $L$ is associated an anti-hat $(y', y \wedge z, z')^\vee$ of $L$ such that $[y \wedge z, x]$ is a 2-facet,  
3. to every anti-hat $(y, x, z)^\vee$ of $L$ is associated a hat $(y', y \vee z, z')^\wedge$ of $L$ such that $[x, y \vee z]$ is a 2-facet,  
4. there exists a **2-facet labelling** $T$ on the (covering) edges of $L$ and a **2-facet rank function** $r$ on $T$. 

---
All lattices of $\mathcal{H}$ are bounded

First part of the theorem

All lattices of $\mathcal{H}$ are bounded

How do we prove this?
All lattices of $\mathcal{H}$ are bounded

**RECALLING ARROW RELATION RELATIONS...**

$$j \downarrow m : j \land m = j^-$$

$$j \uparrow m : j \lor m = m^+$$
Characterising semidistributivity with arrow relations

Proposition (Day [7], 1979)

A lattice $L$ is semidistributive if and only if the relation $\uparrow$ on $J \times M$ induces a bijection between $J$ and $M$.
All lattices of $\mathcal{H}$ are bounded

**Characterising semidistributivity with arrow relations**

**Proposition (Day [7], 1979)**

A lattice $L$ is semidistributive if and only if the relation $\uparrow$ on $J \times M$ induces a bijection between $J$ and $M$.

**Notation**

In any semidistributive lattice $L$, we can denote by $(j, m_j)$ – or by $(j_m, m)$ – the elements of $J_L \times M_L$ which are bijective for the relation $\uparrow$. 
All lattices of $\mathcal{H}$ are bounded

**Relations on the edges of the lattices of $\mathcal{H}$**

We write: $bd \preceq t_2 gi$ and $ab \preceq t_4 ce$ and $ac \preceq t_1 be \preceq t_1 hi$ and so: $ac \leq t_1 hi$. 

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All lattices of $\mathcal{H}$ are bounded

**Relations on the edges of the lattices of $\mathcal{H}$**

We write: $bd \prec_{t_2} gi$
All lattices of $\mathcal{H}$ are bounded.

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All lattices of $\mathcal{HH}$ are bounded

**Relations on the edges of the lattices of $\mathcal{HH}$**

We write: $bd \prec_{t_2} gi$

and $ab \prec_{t_4} ce$

and $ac \prec_{t_1} be \prec_{t_1} hi$
All lattices of $\mathcal{HH}$ are bounded

**Relations on the edges of the lattices of $\mathcal{HH}$**

We write: $bd \prec_{t_2} gi$

and $ab \prec_{t_4} ce$

and $ac \prec_{t_1} be \prec_{t_1} hi$

and so: $ac \leq_{t_1} hi$. 
All lattices of $\mathcal{HH}$ are bounded

**Using the $\leq_t$ Relations**

**Theorem**

Let $m$ be meet-irreducible in $L \in \mathcal{HH}$ and let $(m, m^+)$ be labelled by $t$.

The set $E_m = \{(x, y) : (x, y) \leq_t (m, m^+)\}$ is not empty and has a least element $(u, v)$.

Moreover $v$ is a join-irreducible, $v^- = u$ and $v \uparrow m$. 
All lattices of $\mathcal{H}$ are bounded
All lattices of $\mathcal{H}$ are bounded.
Finite Coxeter lattices The lattice of finite closure systems

All lattices of $\mathcal{H}$ are bounded
Lemma

Let $L \in \mathcal{H}$ and $T$ a 2-facet labelling of $L$. There exists a label $t \in T$ such that for any hat $(y, x, z)^\wedge$ whose arc $(y, x)$ or $(z, x)$ is labelled by $t$, $F(y, x, z)$ is a diamond.
Lemma

Let $L \in \mathcal{H}$ and $T$ a 2-facet labelling of $L$. There exists a label $t \in T$ such that for any hat $(y, x, z)^{\wedge}$ whose arc $(y, x)$ or $(z, x)$ is labelled by $t$, $F(y, x, z)$ is a diamond.

More precisely:

If

and if $r(t)$ is maximum in $r(T)$
Lemma

Let $L \in \mathcal{HH}$ and $T$ a 2-facet labelling of $L$. There exists a label $t \in T$ such that for any hat $(y, x, z)^\land$ whose arc $(y, x)$ or $(z, x)$ is labelled by $t$, $F^{(y, x, z)}$ is a diamond.

More precisely:

If

and if $r(t)$ is maximum in $r(T)$ then:
Definition

Let $L$ be a lattice and $I \subseteq L$ an interval of $L$. We say that $I$ is \textit{contractible} (in $L$) if $L$ can be obtained from a lattice $L_0$ by the doubling of an interval $I_0 \subseteq L_0$ (with $I = I_0 \times 2$).
All lattices of $\mathcal{HH}$ are bounded

"Disconstructing" an interval to construct a second lemma

Definition

Let $L$ be a lattice and $I \subseteq L$ an interval of $L$. We say that $I$ is **contractible** (in $L$) if $L$ can be obtained from a lattice $L_0$ by the doubling of an interval $I_0 \subseteq L_0$ (with $I = I_0 \times 2$).

Lemma

Let $L \in \mathcal{HH}$, $j \in J_L$ and $t$ the label of the arcs $(j^-, j)$ and $(m_j, m_j^+)$. Assume all 2-facets contained in $[j^-, m_j^+]$ and which have one edge labelled by $t$ are isomorphic with diamonds.
All lattices of \( \mathcal{H} \) are bounded

"Disconstructing" an interval to construct a second lemma

**Definition**

Let \( L \) be a lattice and \( I \subseteq L \) an interval of \( L \). We say that \( I \) is **contractible** (in \( L \)) if \( L \) can be obtained from a lattice \( L_0 \) by the doubling of an interval \( I_0 \subseteq L_0 \) (with \( I = I_0 \times 2 \)).

**Lemma**

Let \( L \in \mathcal{H} \), \( j \in J_L \) and \( t \) the label of the arcs \( (j^-, j) \) and \( (m_j, m^-_j) \).

Assume all 2-facets contained in \([j^-, m^-_j]\) and which have one edge labelled by \( t \) are isomorphic with diamonds.

Then the interval \( I_{j,m_j} = [j^-, m^-_j] \) is **contractible**.
All lattices of $\mathcal{H}$ are bounded

**ILLUSTRATION OF THE LEMMA**
All lattices of \( \mathcal{H} \) are bounded

**ILLUSTRATION OF THE LEMMA**
(Lower) bounded lattices and the doubling operation

Finite Coxeter lattices  The lattice of finite closure syst

All lattices of $\mathcal{HH}$ are bounded

**ILLUSTRATION OF THE LEMMA**
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**ILLUSTRATION OF THE LEMMA**
All lattices of $\mathcal{H}$ are bounded

**At last...**

**Theorem**

The class $\mathcal{H}$ of lattices is closed for the contraction of a contractible interval w.r.t. a label whose 2-facet rank function is maximal.

**Hence the result:** lattices of $\mathcal{H}$ are bounded!
All lattices of $\mathcal{H}$ are bounded

**At last...**

**Theorem**

*The class $\mathcal{H}$ of lattices is closed for the contraction of a contractible interval w.r.t. a label whose 2-facet rank function is maximal.*

**Hence the result**: *Lattices of $\mathcal{H}$ are bounded!*
All lattices of $\mathcal{H}$ are bounded

**NOT ALL BOUNDED LATTICES ARE IN $\mathcal{H}$**
All lattices of $\mathcal{H}$ are bounded

**NOT ALL BOUNDED LATTICES ARE IN $\mathcal{H}$**

A bounded lattice that does not belong to $\mathcal{H}$

WHY ???
Finite Coxeter lattices are in $\mathcal{H}$

Second part of the theorem

*Finite Coxeter lattices are in $\mathcal{H}$*

How do we prove this?
Finite Coxeter lattices are in $\mathcal{H}$

**A STRONG RESULT**

**Proposition (L.C.d.P.-B., 1994)**

*Finite Coxeter lattices are semidistributive.*
Finite Coxeter lattices are in $\mathcal{H}$

**A STRONG RESULT**

**Proposition (L.C.d.P.-B., 1994)**

*Finite Coxeter lattices are semidistributive.*

**Proposition (Duquenne and Cherfouh, 1994)**

*Permutohedron is semidistributive.*
Finite Coxeter lattices are in $\mathcal{H}$

**Reflections as Elements and Edge Labels**

**Definition**

$$T_W = \{ t \in W : \exists s \in S, \exists w \in W \text{ such that } t = wsw^{-1} \}$$

is the set of the *reflections* of the Coxeter group $W$. 
Finite Coxeter lattices are in $\mathcal{H}$

**Reflections as elements and edge labels**

**Definition**

$$T_W = \{ t \in W : \exists s \in S, \exists w \in W \text{ such that } t = wsw^{-1} \}$$

is the set of the *reflections* of the Coxeter group $W$.

**Two labellings of the edges**: the $g$-labelling

![Diagram showing two labellings of the edges](image)
Finite Coxeter lattices are in \( \mathcal{H} \)

**Reflections as elements and edge labels**

**Definition**

\[ T_W = \{ t \in W : \exists s \in S, \exists w \in W \text{ such that } t = wsw^{-1} \} \]

is the set of the *reflections* of the Coxeter group \( W \).

**Two labellings of the edges: the \( g \)-labelling and the \( r \)-labelling**

\[ t = wsw^{-1} \]
Finite Coxeter lattices are in $\mathcal{H}$

**PROPERTIES OF THE REFLECTIONS**

**Proposition (L.C.d.P.-B.)**

*Two "opposite" edges of a 2-facet of a Coxeter lattice are labelled by the same reflection.*
Finite Coxeter lattices are in $\mathcal{H}$. 

**PROPERTIES OF THE REFLECTIONS**

**Proposition (L.C.d.P.-B.)**

*Two "opposite" edges of a 2-facet of a Coxeter lattice are labelled by the same reflection.*

Diagram:

```
   t_1
   /   \
 t_4 /     \ t_2
     /       \
   t_3
     \
   t_4
```

Corollary

The $r$-labelling on the edges of any finite Coxeter lattice is a 2-facet labelling.
Finite Coxeter lattices are in \( \mathcal{H} \)

**Properties of the Reflections**

**Proposition (L.C.d.P.-B.)**

Two "opposite" edges of a 2-facet of a Coxeter lattice are labelled by the same reflection.

**Corollary**

The \( r \)-labelling on the edges of any finite Coxeter lattice is a 2-facet labelling.
Finite Coxeter lattices are in $\mathcal{H}$

**Properties of the Length Function**

**Theorem (L.C.d.P.-B.)**

The length function $\ell$ on every Coxeter lattice $L_W$ is a 2-facet rank function when defined on the $r$-labelling of the edges of $L_W$. 
Finite Coxeter lattices are in $\mathcal{HH}$

**Properties of the Length Function**

**Theorem (L.C.d.P.-B.)**

The length function $\ell$ on every Coxeter lattice $L_W$ is a 2-facet rank function when defined on the $r$-labelling of the edges of $L_W$.

So:

**Theorem**

Every Coxeter lattice is in the class $\mathcal{HH}$ and therefore is bounded.
Finite Coxeter lattices are in \( \mathcal{H} \)

**TWO ADDITIONAL RESULTS**

**Theorem**

Let \( L_W \) be a Coxeter lattice and \( W_H \) a parabolic subgroup of \( W \). There exists a series of interval contractions that leads from \( L_W \) to the lattice \( L_{W_H} \) of its parabolic subgroup \( W_H \).
Finite Coxeter lattices are in $\mathcal{H}$

**TWO ADDITIONAL RESULTS**

**Theorem**

Let $L_W$ be a Coxeter lattice and $W_H$ a parabolic subgroup of $W$. There exists a series of interval contractions that leads from $L_W$ to the lattice $L_{W_H}$ of its parabolic subgroup $W_H$.

**Proposition**

There exists a particular interval doubling series from a given Coxeter lattice generated by $n$ generators to the Coxeter lattice of the same family, generated by $n + 1$ generators.
Outline

1. (Lower) bounded lattices and the doubling operation

2. Finite Coxeter lattices
   - Coxeter lattices
   - The class $\mathcal{H}$ of lattices
   - All lattices of $\mathcal{H}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{H}$

3. The lattice of finite closure systems
**Definition**

A *closure system* $\mathcal{C}$ on $S$: a subset of $2^S$ which contains $S$ and is closed under set intersection.

**Example** ($S = \{1, 2, 3, 4\}$)
The lattice \((\mathcal{M}_n, \subseteq)\) of closure systems on a finite set \(S\)

Example \((n=2)\)

\[
\begin{align*}
\{\emptyset, 1, S\} & \quad \{\emptyset, 2, S\} \\
\{\emptyset, S\} & \quad \{1, S\} \quad \{2, S\} \\
\{S\} & \quad 2^S
\end{align*}
\]
Structures cryptomorphic with:

- closure operators,
- finite lattices,
- full implicational systems (or full systems of dependencies).
Structures cryptomorphic with:
- closure operators,
- finite lattices,
- full implicational systems (or full systems of dependencies).

**Theorem**

The lattice $(\mathbb{M}_n, \subseteq)$ of closure systems is lower bounded.

**How do we prove this?**
Two dependence relations on the join-irreducibles of $M_n$

- The dependence relation $\delta$ (Monjardet [11], 1990),
- The strong dependence relation $\delta_d$ (Day [7], 1979).

**Definition**

1. $j \delta j'$ if $j = j'$ or if $\exists x \in L$ with $j < j' \lor x$, $j \not\leq x$ and $j' \not\leq x$.
2. $j \delta_d j'$ if $j = j'$ or if $\exists x \in L$ with $j < j' \lor x$ and $j \not\leq j' - \lor x$. 
Two dependence relations on the join-irreducibles of $\mathbb{M}_n$

- The dependence relation $\delta$ (Monjardet [11], 1990),
- The strong dependence relation $\delta_d$ (Day [7], 1979).

**Definition**

1. $j \delta j'$ if $j = j'$ or if $\exists x \in L$ with $j < j' \lor x$, $j \not\leq x$ and $j' \not\leq x$.
2. $j \delta_d j'$ if $j = j'$ or if $\exists x \in L$ with $j < j' \lor x$ and $j \not\leq j' \neg \lor x$.

In particular, we have $\delta_d \subseteq \delta$. 
**Characterising $\delta$ and $\delta_d$ with the arrow relations**

<table>
<thead>
<tr>
<th>Proposition</th>
<th>Expression</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$j \delta j' \iff \exists m \in M : j \uparrow m \text{ and } j' \not\leq m$.</td>
</tr>
<tr>
<td>2</td>
<td>$j \delta_d j' \iff \exists m \in M : j \uparrow m \text{ and } j' \downarrow m$.</td>
</tr>
</tbody>
</table>
Some results

Proposition

In any lattice $L$, the following are equivalent:

1. $L$ is atomistic,
2. $\forall j \in J, \forall m \in M, j \nleq m$ implies $j \downarrow m$,
3. $\delta_d = \delta$. 

Some results

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Moreover:

Proposition (Day [7], 1979)

A lattice $L$ is lower bounded if and only if $\delta_d$ has no circuit.
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The join-irreducibles of $\mathbb{M}_n$

For $A \subset S$, we set $\mathcal{C}_A = \{A, S\}$. 
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Proposition

The lattice $\mathcal{M}_n$ is atomistic and $\delta_d = \delta$. 
What about the meet-irreducibles of $\mathbb{M}_n$?

**Proposition**

Let $\mathcal{C}$ be a closure system of $\mathbb{M}_n$. The following holds:

$$\mathcal{C} \in M_{\mathbb{M}_n} \iff \mathcal{C} = \mathcal{C}_{A,i} = \{X \subseteq S : A \not\subseteq X \text{ or } i \in X\}.$$
Finally...

**Proposition**

Let $C_A$ and $C_B$ be two join-irreducible elements of $\mathbb{M}_n$.

$$C_A \delta C_B \iff A \subseteq B \subseteq S$$

So $\delta$ is an order relation.

Theorem

The lattice $\mathbb{M}_n$ of closure systems is lower bounded. It is not bounded since it is not semidistributive.
Finally...

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Let \( C_A \) and \( C_B \) be two join-irreducible elements of \( \mathbb{M}_n \).  

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\]

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The lattice \( \mathbb{M}_n \) of closure systems is lower bounded.

It is *not* bounded since it is *not* semidistributive.
MY COLLEAGUES (AND FRIENDS!)

Fig.: C. le Conte de Poly-Barbut, CAMS, EHESS, Paris.
MY COLLEAGUES (AND FRIENDS!)

**Fig.:** B. Leclerc and B. Monjardet, CAMS, EHESS, Paris.
THE FINAL WORD.
No QUESTIONS..?


Recollectingarrowrelations...

\[ j \downarrow m : j \land m = j^- \]

\[ j \uparrow m : j \lor m = m^+ \]
AND THE $A$-CONTEXT OF A LATTICE
... AND THE $A$-CONTEXT OF A LATTICE

\[
\begin{array}{c|ccccc}
 & a & z & t & v & w \\
\hline
y & \uparrow & \times & \times & \times & \times \\
z & \times & \downarrow & \times & \downarrow \\
t & \downarrow & \times & \uparrow & \times \\
u & \downarrow & \uparrow & \downarrow & \times & \times \\
\end{array}
\]
... And the $A$-context of a lattice

Any lattice has $(|J|! \times |M|!)$ tableaux to describe its $A$-context.
ON SEMIDISTRIBUTIVITY

Definition

A lattice is **meet-semidistributive** if, for all \( x, y, z \in L \),

\[ x \land y = x \land z \text{ implies } x \land y = x \land (y \lor z). \]

**Join-semidistributive** lattices are defined dually and a lattice is **semidistributive** if it is meet- and join-semidistributive.
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**Proposition (Day, Nation, Tschantz [8], 1989)**

*Bounded lattices are semidistributive.*
RESULTS

Proposition (Duquenne and Cherrouh, L.C.d.P.-B., 1994)
Permutohedron is semidistributive.
Results

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A lattice $L$ is semidistributive if and only if the relation $\uparrow$ on $J \times M$ induces a bijection between $J$ and $M$. 
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Permutohedron is semidistributive.

Proposition (Day [7], 1979)

A lattice $L$ is semidistributive if and only if the relation $\updownarrow$ on $J \times M$ induces a bijection between $J$ and $M$.

Hence:

Given any total order on $J_{\text{Perm}(n)}$, there exists a unique total order on $M_{\text{Perm}(n)}$—say $L^*_M$—such that $T = (A_L, L_J, L^*_M)$ has all $\updownarrow$ on the principal diagonal.
A SIMPLE IDEA FROM A STRONG RESULT

Definition

Let $L$ be a semidistributive lattice. A tableau $T = (A_L, L_J, L_M)$ of the $A$-context of $L$ is a $B$-tableau if the following hold:

1. The $|J|$ arrows $\uparrow$ of $T$ are on the principal diagonal of $T$,
2. All arrows $\uparrow$ are below this diagonal and all arrows $\downarrow$ are above.

Proposition (Geyer [9], 1994)

A lattice is bounded if and only if its $A$-context admits a $B$-tableau.
### A $B$-Tableau of $\text{Perm}(4)$

<table>
<thead>
<tr>
<th>$J \setminus M$</th>
<th>3421</th>
<th>4231</th>
<th>3241</th>
<th>2431</th>
<th>4312</th>
<th>4213</th>
<th>3214</th>
<th>2413</th>
<th>4132</th>
<th>3142</th>
<th>1432</th>
</tr>
</thead>
<tbody>
<tr>
<td>1243</td>
<td>↓</td>
<td>×</td>
<td>↓</td>
<td>×</td>
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<td></td>
</tr>
</tbody>
</table>

Here: $L_J$ is equal to $Lex(J)$.

In fact:

**Theorem**

The tableau $T = (A_{Perm(n)}, Lex_J, L_M^*)$ of the $A$-context of the lattice $Perm(n)$ is a $B$-tableau.
**Recalls**

**Definition**

- $A(\alpha)$: the set of *agreements* of $\alpha$,
- $D(\alpha)$: the set of *disagreements* of $\alpha$.  

*Example* $\alpha = 3241 \in \text{Perm}(4)$.

- $A(3241) = \{24, 34\}$ and $D(3241) = \{32, 31, 21, 41\}$.  

The weak order defined on $\text{Perm}(n)$ is characterised by:

$\alpha \leq \beta \iff A(\beta) \subseteq A(\alpha) \iff D(\alpha) \subseteq D(\beta)$. 

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**RECALLS**

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**The weak order defined on $\text{Perm}(n)$ is characterised by**:
- $\alpha \leq \beta \iff A(\beta) \subseteq A(\alpha) \iff D(\alpha) \subseteq D(\beta)$.
RESULT

\( \alpha \in J_{Perm}(n) \) if and only if there exists a unique ordered pair \( vu \) of adjacent elements in \( \alpha \) such that \( u < v \).
**Expression of the elements of** \( J_{Perm(n)} \)

**Result**

\[ \alpha \in J_{Perm(n)} \text{ if and only if there exists a unique ordered pair } vu \text{ of adjacent elements in } \alpha \text{ such that } u < v. \]

**Example**

\[ 4123, 1324 \in J_{Perm(4)} \]
**Expression of the elements of $J_{Perm(n)}$**

<table>
<thead>
<tr>
<th><strong>Result</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \in J_{Perm(n)}$ if and only if there exists a unique ordered pair $v u$ of adjacent elements in $\alpha$ such that $u &lt; v$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Example</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$4123, 1324 \in J_{Perm(4)}$ but $1432, 4213 \notin J_{Perm(4)}$.</td>
</tr>
</tbody>
</table>
Expression of the elements of $J_{\text{Perm}}(n)$

Result

$\alpha \in J_{\text{Perm}}(n)$ if and only if there exists a unique ordered pair $vu$ of adjacent elements in $\alpha$ such that $u < v$.

Example

$4123, 1324 \in J_{\text{Perm}}(4)$ but $1432, 4213 \notin J_{\text{Perm}}(4)$.

So, in other words:

$$\alpha \in J_{\text{Perm}}(n) \iff \alpha = A|\overline{A} = Bv|u\overline{B}$$

with $u < v$ and $A = Bv$ and $\overline{A} = u\overline{B}$ the two maximal linear suborders of $\alpha$ compatible with $0_{\text{Perm}}(n) = 1...i...n$. 
Expression of the elements of $M_{Perm(n)}$

**Result**

$\alpha \in M_{Perm(n)}$ if and only if there exists a unique ordered pair $l p$ of adjacent elements in $\alpha$ such that $l < p$.

**Example**

$4213, 1432 \in M_{Perm(4)}$ but $1342, 4231 \notin M_{Perm(4)}$. 
### Expression of the Elements of $M_{Perm(n)}$

<table>
<thead>
<tr>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \in M_{Perm(n)}$ if and only if there exists a unique ordered pair $lp$ of adjacent elements in $\alpha$ such that $l &lt; p$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example</th>
</tr>
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<tbody>
<tr>
<td>$4213, 1432 \in M_{Perm(4)}$ but $1342, 4231 \notin M_{Perm(4)}$.</td>
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<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \in M_{Perm(n)} \iff \alpha = C</td>
</tr>
</tbody>
</table>
**Characterising the $A$-context of $\text{Perm}(n)$**

**Lemma**

Let $\gamma = Bv|u\bar{B} \in J_{\text{Perm}(n)}$ and $\mu = Cl|p\bar{C} \in M_{\text{Perm}(n)}$. 
CHARACTERISING THE $A$-CONTEXT OF $\text{Perm}(n)$

**Lemma**

Let $\gamma = B v | u \overline{B} \in J_{\text{Perm}(n)}$ and $\mu = \text{Cl}|p \overline{C} \in M_{\text{Perm}(n)}$.

1. $\gamma \leq \mu \iff D(\gamma) \subseteq D(\mu) \iff A(\mu) \subseteq A(\gamma)$.
2. $\gamma \uparrow \mu \iff pl \in D(\gamma)$ and $D(\gamma) \subseteq D(\mu^+)$.
3. $\gamma \downarrow \mu \iff uv \in A(\mu)$ and $A(\mu) \subseteq A(\gamma^-)$.
4. $\gamma \updownarrow \mu \iff pl \in D(\gamma), uv \in A(\mu), D(\gamma) \subseteq D(\mu^+)$ and $A(\mu) \subseteq A(\gamma^-)$. 

(Lower) bounded lattices and the doubling operation  Finite Coxeter lattices  The lattice of finite closure syst
Characterising the bijection between \( J \) and \( M \) induced by \( \uparrow \)

**Proposition**

1. Let \( \gamma = Bu|vB \) be a join-irreducible and \( \mu \) a meet-irreducible of \( \text{Perm}(n) \).

\[
\gamma \downarrow \mu \iff \mu = Cu|vC \quad \text{with} \quad \begin{cases} C = \{x \in B : u < x\} \cup \{x \in B : v < x\}, > \\ \overline{C} = \{x \in B : x < u\} \cup \{x \in B : x < v\}, > \end{cases}
\]

2. Let \( \mu = Cl|p\overline{C} \) be a meet-irreducible and \( \gamma \) a join-irreducible of \( \text{Perm}(n) \).

\[
\gamma \downarrow \mu \iff \gamma = Bp|l\overline{B} \quad \text{with} \quad \begin{cases} B = \{x \in C : x < p\} \cup \{x \in \overline{C} : x < l\}, < \\ \overline{B} = \{x \in C : p < x\} \cup \{x \in \overline{C} : l < x\}, < \end{cases}
\]
Let $L_J$ be a linear order on $J_{Perm(n)}$ and $L^*_M$ the "associated" linear order on $M_{Perm(n)}$. The following are equivalent:

1. $T = (A_{Perm(n)}, L_J, L^*_M)$ is a $B$-tableau of $Perm(n)$.
2. $L_J$ is a linear extension of $(J, \leq_{Perm(n)})$ and $L^*_M$ a linear extension of $(M, \geq_{Perm(n)})$.
An additional result

Theorem

Let $L_J$ be a linear order on $J_{\text{Perm}(n)}$ and $L_M^*$ the "associated" linear order on $M_{\text{Perm}(n)}$. The following are equivalent:

1. $T = (A_{\text{Perm}(n)}, L_J, L_M^*)$ is a $B$-tableau of $\text{Perm}(n)$,
2. $L_J$ is a linear extension of $(J, \leq_{\text{Perm}(n)})$ and $L_M^*$ a linear extension of $(M, \geq_{\text{Perm}(n)})$. 
Not all tableaux of \( \text{Perm}(n) \) are \( B \)-tableaux

Proof:
1324
2134
1243
2341
3412
4123
1423
1342
2413
3124
2314
**Proof:**

Not all tableaux of $\text{Perm}(n)$ are $B$-tableaux.

Proof:

```
2314  2341  3124  3412  2413  4123
1234  1342  3142  2341  1324  1243
```

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**Not all tableaux of Perm(n) are B-tableaux**

Proof:

![Diagram](image_url)

**Fig:** A linear extension $L_J$ of $(J, \leq_{Perm(4)})$ for which $L^*_M$ on $M$ is not a linear extension of $(M, \geq_{Perm(4)})$. 
Not all tableaux of \( \text{Perm}(n) \) are \( B \)-tableaux.

Proof:

\[
\begin{array}{cccc}
2341 & 3412 & 4123 \\
9 & 10 & 11 \\
2314 & 3124 & 2413 \\
4 & 5 & 6 \\
2134 & 1324 & 1243 \\
1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
4312 & 4231 & 3421 \\
1 & 2 & 3 \\
4132 & 4213 & 3142 \\
5 & 4 & 10 \\
1432 & 2143 & 3214 \\
11 & 6 & 9 \\
\end{array}
\]

\[
\begin{array}{cccc}
1342 & 1423 \\
7 & 8 \\
3142 & 2314 \\
9 & 10 \\
3241 \\
7 \\
\end{array}
\]
**Definition**

A *closure system* \( \mathcal{C} \) on \( S \): a subset of \( 2^S \) which contains \( S \) and is closed under set intersection.

**Example** \((S = \{1, 2, 3, 4\})\)
Proposition

The set of all the lattices that can be obtained from \( L \in \mathcal{H} \) by a series of interval contractions is a distributive lattice when ordered by the following natural order relation: \( L < L' \) if \( L \) can be obtained from \( L' \) by a series of interval contractions.