Positional Injectivity for Innocent Strategies

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Introduction

Programs

Game Semantics —Pos→ **Relational Semantics**

Dynamic: Represent programs with temporal information.
Strategies

Static: Represent states.
Positions

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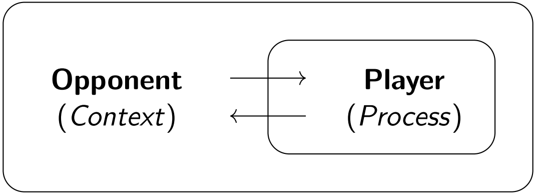
Theorem: Positional Injectivity (for Hyland Ong games)

Pos is injective for total finite innocent strategies.

Outline

Introduction to Game Semantics (Hyland-Ong games)



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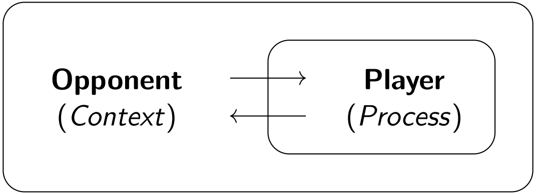
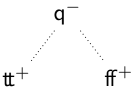


Arenas (*Types*):	The game with its rules
Plays (*Executions*):	A game between two players
Strategies (*Programs*):	Guideline for Player

Arenas (Types)

An **arena** A is a tuple $\langle |A|, \leq_A, \mathrm{pol}_A \rangle$ such that:

- $|A|$ is a set of *events*;
- $\mathrm{pol}_A : |A| \to \{-, +\}$ is a labelling function;
- $\leq_A$ defines a *negative* and *alternating* tree.



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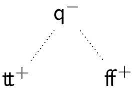

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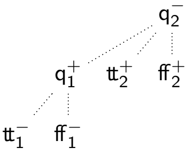

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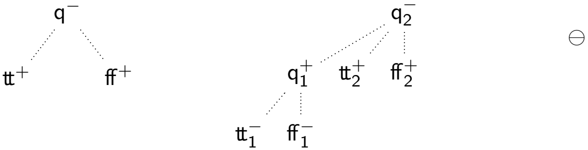

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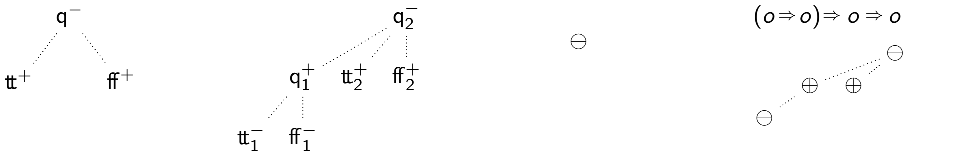


Arena bool

Arena $\mathrm{bool}_1 \Rightarrow \mathrm{bool}_2$

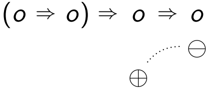
Arena o

Arena $(o \Rightarrow o) \Rightarrow o \Rightarrow o$

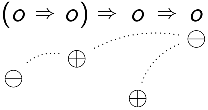
Plays (Executions)

A **play** on arena A is a pointing string $s = s_1 \dots s_n \in |A|^*$ such that:

- The pointers respect $\multimap_A$;
- *alternating*: $\forall 1 \leq i < n$, $\mathrm{pol}_A(s_i) \neq \mathrm{pol}_A(s_{i+1})$;
- *legal*: $\forall 1 \leq i \leq n$, either $s_i = \min(A)$ or s_i has a pointer.



Typical play for $\lambda f^{o \to o}. \lambda x^o. x$



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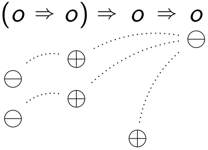

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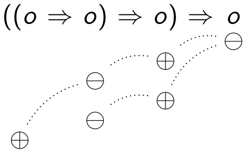
Strategies (Programs)

A **strategy** $\sigma : A$ is a non-empty set $\sigma \subseteq \mathrm{Plays}^+(A)$ satisfying:

$$\begin{array}{rl} \textit{prefix-closed:} & \forall s \in \sigma, \forall t \sqsubseteq^+ s, t \in \sigma\,, \\ \textit{deterministic:} & \forall s \in \sigma,\ sab, sab' \in \sigma \implies sab = sab'\,. \end{array}$$



$K_x : \lambda f^{(o \to o) \to o}.\, f\,(\lambda x^o.\, f\,(\lambda y^o.\, x))$

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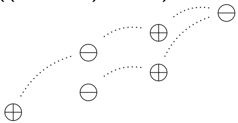
$((o \Rightarrow o) \Rightarrow o) \Rightarrow o$



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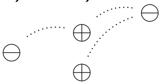

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Innocence

A **P-view** is a play where Opponent moves point to the previous Player move.

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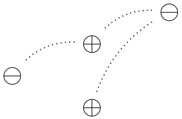

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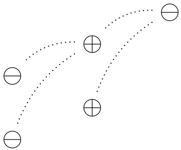

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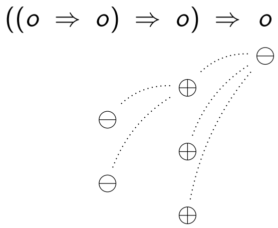
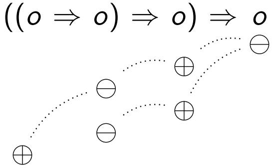


A strategy is **innocent** if Player reacts the same way to every duplication of Opponent moves.

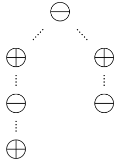
Outline

Positions

The **position** of a play s, noted $\llparenthesis s \rrparenthesis$, is the desequentialization of s (it is s without its temporal order).



A play s of K_x



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Positionality

A strategy $\sigma : A$ is **positional** if for all $sab, t \in \sigma$, $ta' \in \mathrm{Plays}(A)$,

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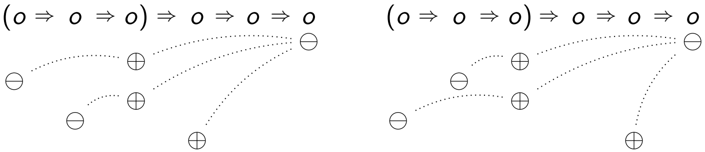

Two maximal P-views for $\lambda f^{o \to o \to o}. \lambda x^o. \lambda y^o. f\,(f\,x\,x)\,(f\,y\,y)$

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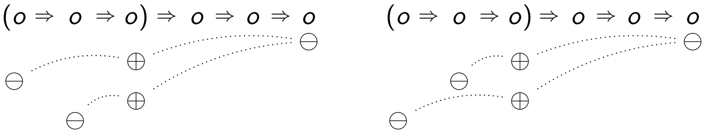

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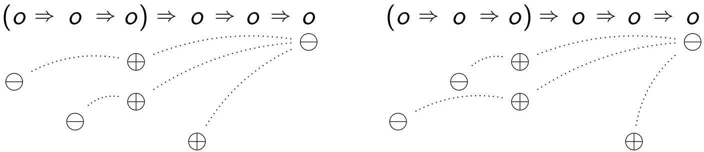

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The **positions** of a strategy $\sigma : A$ are $\llparenthesis \sigma \rrparenthesis = \{ \llparenthesis s \rrparenthesis \mid s \in \sigma \}$.

Question (Positional Injectivity)

For any $\sigma, \tau : A$, do we have $\llparenthesis \sigma \rrparenthesis = \llparenthesis \tau \rrparenthesis \;\Rightarrow\; \sigma = \tau$?

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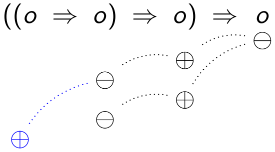

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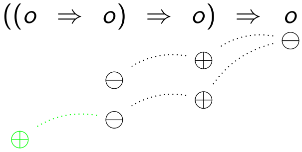

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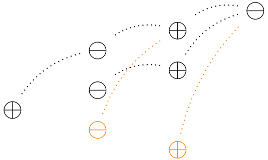

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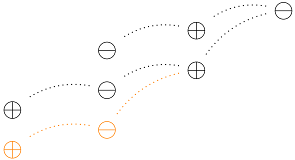

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Positional Injectivity

A strategy $\sigma : A$ is **total** iff for all $s \in \sigma$,

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A strategy $\sigma : A$ is **finite** iff the set of P-views of σ is finite.

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Theorem (Positional Injectivity)

For any $\sigma, \tau : A$ innocent total finite, $\sigma = \tau$ iff $(\!|\sigma|\!) = (\!|\tau|\!)$.

Outline

Augmentations

Augmentations = Plays without Opponent's Scheduling

Augmentations

$$\begin{aligned}\text{Augmentations} &= \text{Plays without Opponent's Scheduling}\\ &= \text{Positions augmented with causal order}\end{aligned}$$

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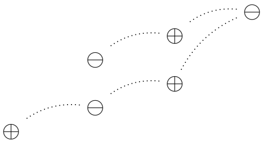

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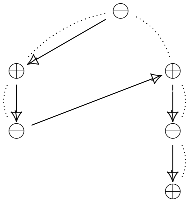

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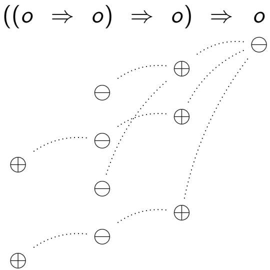

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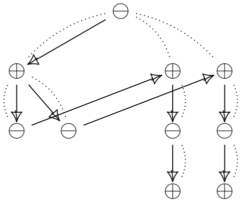

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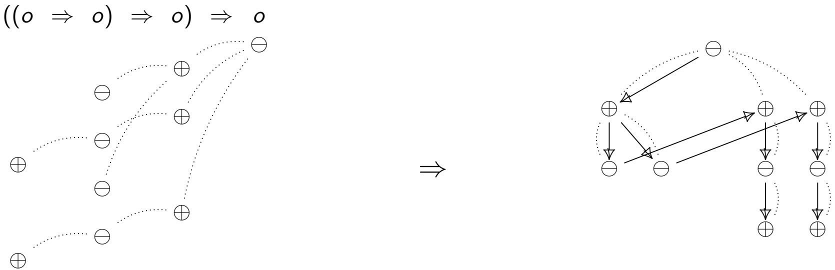
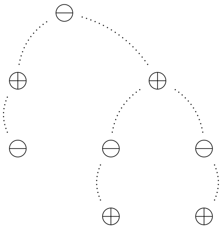


Are **causal games** positionally injective?

Using Duplications of Opponent Moves



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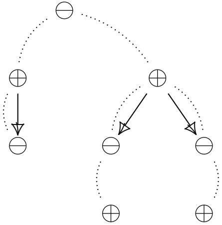
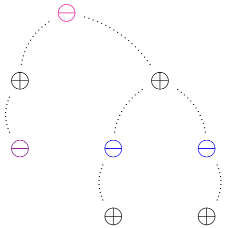
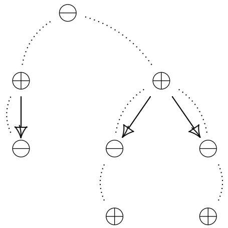


Fork $= \{ \ominus$'s with the same parent and arena image $\}$



$\sharp F_1 = 1 \quad \sharp F_2 = 1 \quad \sharp F_3 = 2$



Using Duplications of Opponent Moves

Fork = { $\ominus$'s with the same parent and arena image }
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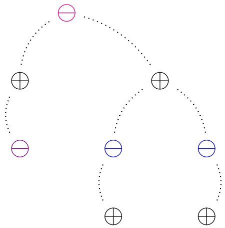

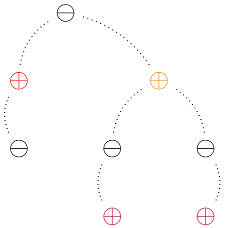

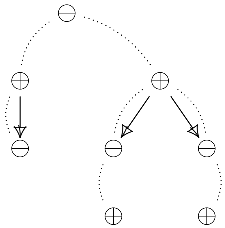

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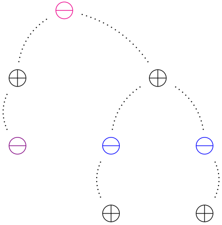

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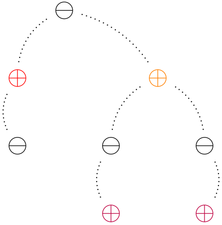

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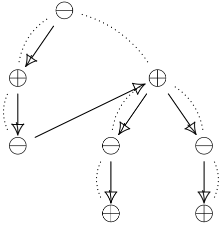
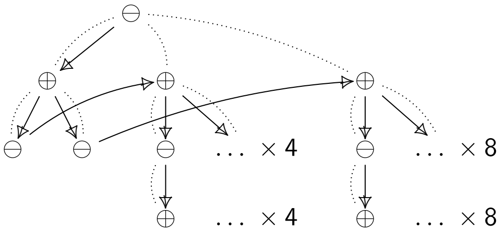


Characteristic Augmentations

A **characteristic augmentation** is an augmentation extracted from the maximal P-views of a strategy, such that each fork has for cardinality a unique power of 2.



A characteristic augmentation of K_y

Positional Injectivity

Theorem: Positional Injectivity (for causal games)

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Theorem: Positional Injectivity (for Hyland Ong games)

For any σ, τ total finite innocent strategies,

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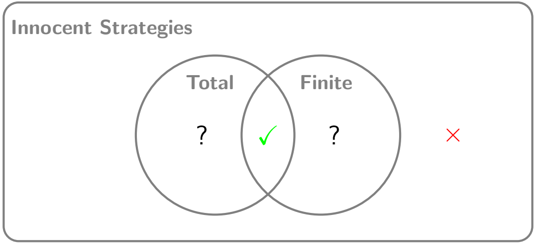
Conclusion

Theorem: Positional Injectivity

Total finite innocent strategies are positionally injective.



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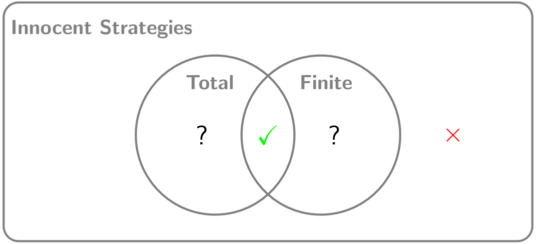
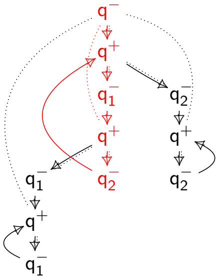
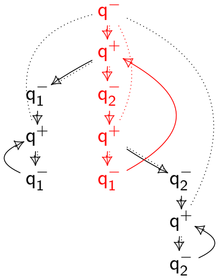


Thank you!

$T_1 = f\ T_2\ R \qquad T_2 = f\ L\ T_1 \qquad L = f\ L\ \bot \qquad R = f\ \bot\ R$



$\lambda f^{o \to o \to o}.\ T_1$



$\lambda f^{o \to o \to o}.\ T_2$