An Extensional Resource λ-calculus and its Categorical Semantics

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Motivation: Taylor Expansion and Game Semantics

Taylor Expansion

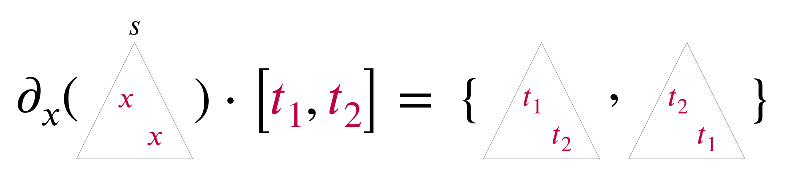
Simply-typed λ-calculus Λ

Terms: $M, N ::= x \mid \lambda x.M \mid M\,N$
Types: $A, B ::= \alpha \mid A \to B$

Resource calculus

$\Lambda_{\text{res}}: \; s, t ::= x \mid \lambda x.s \mid s\,t$
$!\Lambda: \; \mathbf{s}, \mathbf{t} ::= [] \mid s :: \mathbf{t}$

Substitution:



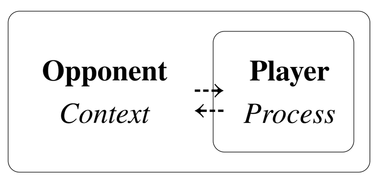
Taylor Expansion

$$\mathcal{T}(x) = \{x\}$$
$$\mathcal{T}(\lambda x.M) = \lambda x.\mathcal{T}(M)$$
$$\mathcal{T}(M\,N) = \mathcal{T}(M)\,\mathcal{T}(N)^!$$

Example:

$$\mathcal{T}(\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f\,x) = \{\ \lambda f.\lambda x.f[],\ \lambda f.\lambda x.f[x],\ \lambda f.\lambda x.f[x,x],\ \lambda f.\lambda x.f[x,x,x], \dots\}$$

Game Semantics



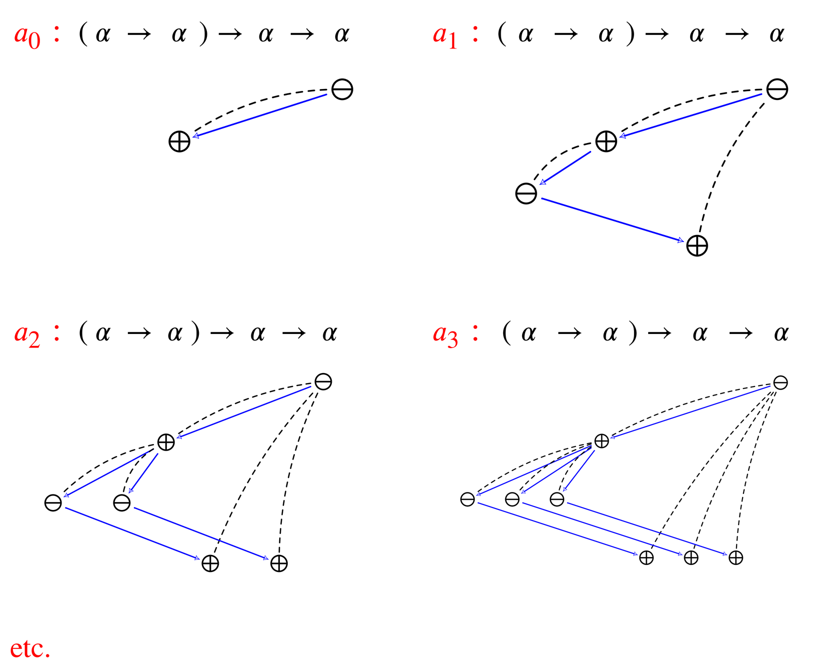
Arenas $\leftrightarrow$ Types
Plays $\leftrightarrow$ Executions
Strategies $\leftrightarrow$ Programs

Interpretation:

$$[\![-]\!]_{\mathcal{G}}: \quad A \mapsto \textbf{Arena}\ [\![A]\!]_{\mathcal{G}}$$
$$\Lambda(A) \to \textbf{Strat}(A) \subseteq \mathcal{P}(\textbf{Aug}(A)).$$

Example:

$[\![\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f\,x]\!]_{\mathcal{G}}$ contains:



Is Taylor Expansion compatible with GS?

Simply-typed case

Example:

$$\lambda f^{\alpha\to\alpha}.f : (\alpha\to\alpha)\to(\alpha\to\alpha).$$

Then the **Taylor expansion** is:

$$\mathcal{T}(\lambda f^{\alpha\to\alpha}.f) = \{\lambda f^{\alpha\to\alpha}.f\}.$$

But on the **game semantics** side, we have

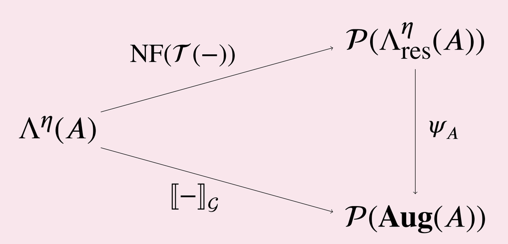
$$[\![\lambda f^{\alpha\to\alpha}.f]\!]_{\mathcal{G}} = [\![\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f\,x]\!]_{\mathcal{G}}.$$

Theorem [Tsukada Ong, 2016]

For any simple-type A, there is

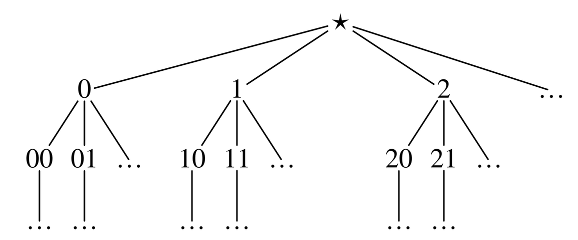
$$\psi_A : \Lambda^{\eta}_{\text{res}}(A) \cong \mathsf{Aug}([\![A]\!]_{\mathcal{G}})$$

s.t. the following diagram commutes.



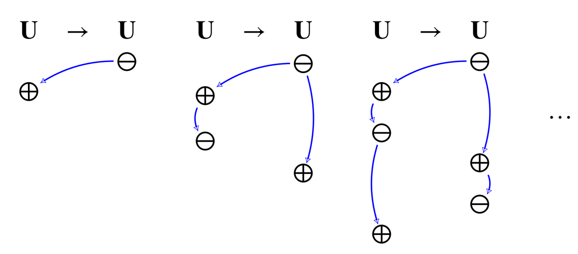
Pure case

We play in the arena $\mathbf{U}$ such that $\mathbf{U} \cong \mathbf{U} \to \mathbf{U}$:

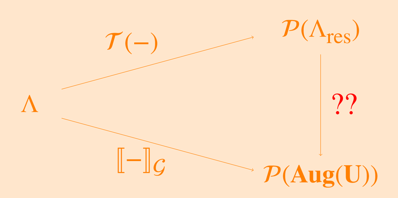


Example:

$[\![\lambda x.x]\!]_{\mathcal{G}}$ (the copycat strategy on $\mathbf{U}$) contains:



For pure λ-calculus, is there a Taylor expansion compatible with game semantics?



Extensional Resource λ-calculus

Extensional λ-calculus

Terms of extensional resource calculus

$$\frac{i \in \mathbb{N}}{\Gamma + x \vdash_{\text{Var}} x_i} \qquad \frac{\Gamma + x \vdash_{\text{App}} t}{\Gamma \vdash_{\text{Val}} \lambda x.t}$$

$$\frac{\Gamma \vdash_{\text{Val}} t \qquad \Gamma \vdash_{\text{Seq}} \vec{u}}{\Gamma \vdash_{\text{App}} t\,\vec{u}}$$

$$\frac{\Gamma \vdash_{\text{Var}} t \qquad \Gamma \vdash_{\text{Seq}} \vec{u}}{\Gamma \vdash_{\text{App}} t\,\vec{u}}$$

$$\frac{(\Gamma \vdash_{\text{Val}} t_j)_{1\le j\le n} \qquad n \in \mathbb{N}}{\Gamma \vdash_{\text{Bag}} [t_1, \dots, t_n]}$$

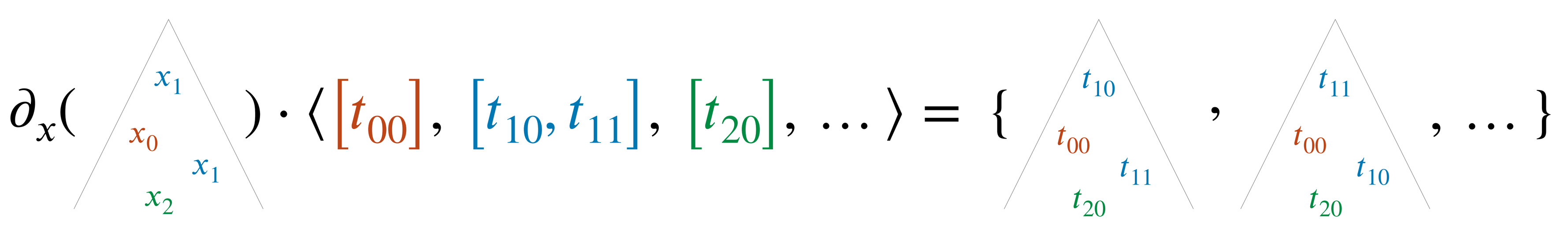
$$\frac{(\Gamma \vdash_{\text{Bag}} t_i)_{i\in\mathbb{N}} \qquad \{i \mid t_i \neq []\} \subseteq_f \mathbb{N}}{\Gamma \vdash_{\text{Seq}} \langle t_i \mid i \in \mathbb{N}\rangle}$$

We define **substitution** by induction on terms:

$$\partial_x(x_k)\cdot\vec{v} = u \ \text{ if } \vec{v} = [u]@k, \quad 0 \text{ otherwise.}$$
$$\partial_x(y_k)\cdot\vec{v} = y_k \ \text{ if } \vec{v} = \vec{[]}, \quad 0 \text{ otherwise.}$$
$$\partial_x(\lambda y.t)\cdot\vec{v} = \lambda y.\partial_x(t)\cdot\vec{v}$$
$$\partial_x(t\,\vec{u})\cdot\vec{v} = \sum_{\vec{v}=\vec{v_1}+\vec{v_2}} (\partial_x(t)\cdot\vec{v_1})(\partial_x(\vec{u})\cdot\vec{v_2})$$
$$\partial_x([t_i \mid i \in I])\cdot\vec{v} = \sum_{\vec{v}=\sum_{i\in I}\vec{v_i}} [\partial_x(t_i)\cdot\vec{v_i} \mid i \in I]$$
$$\partial_x(\langle \mathbf{u}_i \mid i \in \mathbb{N}\rangle)\cdot\vec{v} = \sum_{\vec{v}=\sum_{i\in\mathbb{N}}\vec{v_i}} \langle \partial_x(\mathbf{u}_i)\cdot\vec{v_i} \mid i \in \mathbb{N}\rangle$$

Reduction

$$(\lambda x.t)\vec{u} \to_{\beta} \partial_x(t)\cdot\vec{u}$$

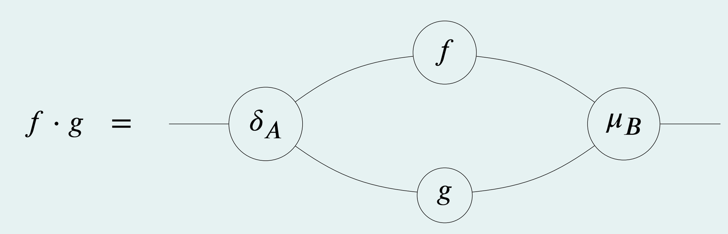


Resource Category

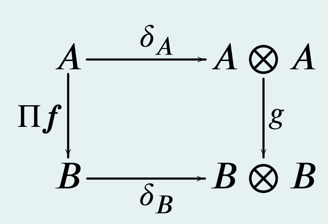
Resource Category

A **resource category** is a symmetric monoidal closed category $\mathcal{C}$ enriched over commutative monoids $(0, +)$ with:

- $\forall A \in \mathcal{C}$, there exists a bialgebra structure $(A, \delta_A, \epsilon_A, \mu_A, \eta_A)$.
- $\forall f, g : A \to B$,



- $\forall A \in \mathcal{C}$, there exists a chosen idempotent $\text{id}^{\bullet}_A : A \to A$ the **pointed identity**. For any $f : A \to B$, f is **pointed**, written $f \in \mathcal{C}_{\bullet}(A, B)$, iff $\text{id}^{\bullet}_B \circ f = f$.
- For any $\boldsymbol{f} \in \mathscr{M}_f(\mathcal{C}_{\bullet}(A, B))$, the following diagram commutes:



where $g = \sum_{\boldsymbol{f}=\boldsymbol{f_1}+\boldsymbol{f_2}} \Pi \boldsymbol{f_1} \otimes \Pi \boldsymbol{f_2}$.

- For any $\boldsymbol{f} \in \mathscr{M}_f(\mathcal{C}_{\bullet}(A, B))$, $\epsilon_B \circ \Pi\boldsymbol{f} = 1$ if $\boldsymbol{f}$ is empty, 0 otherwise.

(and several other coherence axioms...)

Extensional Objects:
Consider $\mathcal{C}$ a resource category with $o \in \mathcal{C}$ fixed.
An **extensional situation** in $\mathcal{C}$ is given by two objects $\mathbf{U}^{\mathbb{N}}, \mathbf{U} \in \mathcal{C}$ along with isos:

$$\mathbf{U} \cong \mathbf{U}^{\mathbb{N}} \multimap o \qquad \mathbf{U}^{\mathbb{N}} \cong \mathbf{U} \otimes \mathbf{U}^{\mathbb{N}}$$

Interpretation:

$$[\![-]\!]_{\text{res}}: \quad \Gamma \longmapsto (\mathbf{U}^{\mathbb{N}})^{\Gamma}$$
$$\text{Var}(\Gamma), \text{Val}(\Gamma) \longrightarrow \mathcal{C}_{\bullet}(\Gamma, \mathbf{U})$$
$$\text{App}(\Gamma) \longrightarrow \mathcal{C}_{\bullet}(\Gamma, o)$$

Substitution:

$$\partial_x([\![t]\!]_{\text{res}})\cdot[\![\vec{u}]\!]_{\text{res}} := [\![t]\!]_{\text{res}} \circ_x [\![\vec{u}]\!]_{\text{res}}$$

Soundness

Consider $\vec{u} \in \text{Seq}(\Gamma)$, then for any $t \in \Lambda_{\text{res}}(\Gamma + x)$, we have:

$$[\![\partial_x(t)\cdot\vec{u}]\!]_{\text{res}} = \partial_x([\![t]\!]_{\text{res}})\cdot[\![\vec{u}]\!]_{\text{res}}$$

WIP: Games and Rel are resource categories.

Extensional Taylor Expansion

Taylor Expansion

$$\mathcal{T}(a) = \lambda y.a\,y^!$$
$$\mathcal{T}(M\,N) = \lambda y.\mathcal{T}(M)(\mathcal{T}(N)^! \cdot y^!)$$
$$\mathcal{T}(\lambda a.M) = \lambda a.\mathcal{T}(M)$$

Notations:

$$y^! = \bigcup_{\mu\in\mathscr{M}_f(\mathbb{N})} \langle \mathcal{T}_i(y)^{\mu(i)} \mid i \in \mathbb{N}\rangle$$
$$\mathcal{T}_i(y) = \lambda z.y_i\,z^!$$

Example:

$$\mathcal{T}(\lambda b.\lambda a.b\,a) = \{\lambda\,bax.(\lambda y.b\,\vec{[]})\,\vec{[]},\ \lambda\,bax.(\lambda y.b\,\vec{[]})\,\langle[\lambda z.a\,\vec{[]}], \dots\rangle,\ \lambda\,bax.(\lambda y.b\,\vec{[]})\,\langle[\lambda z.a\,\vec{[]}, \lambda z.x_1\,\vec{[]}], \dots\rangle,\ \dots\}$$

Compatibility (WIP)

We have:

$$\mathcal{T}(-) : \Lambda \to \Sigma\text{Val}$$
$$\text{NF}(-) : \Sigma\text{Val} \to \Sigma\text{Val}_{\text{NF}}$$
$$[\![-]\!]_{\mathcal{G}} : \Lambda \to \mathcal{P}(\textbf{Aug}(\mathbf{U})).$$

We also have an isomorphism:

$$\psi : \Sigma\text{Val}_{\text{NF}} \cong \mathcal{P}(\textbf{Aug}(\mathbf{U})).$$

And the following diagram commutes.