Positional Injectivity for Innocent Strategies

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Introduction

Programs

Game Semantics —Pos→ **Relational Semantics**

Dynamic: Represent programs with temporal information.
Strategies

Static: Represent states.
Positions

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Game Semantics —Pos→ **Relational Semantics**

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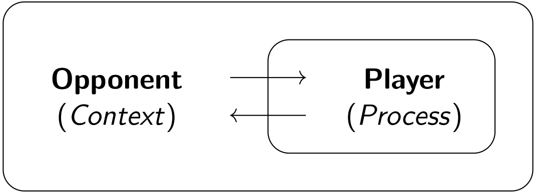
Theorem: Positional Injectivity (for Hyland Ong games)

Pos is injective for total finite innocent strategies.

Outline

Introduction to Game Semantics (Hyland-Ong games)



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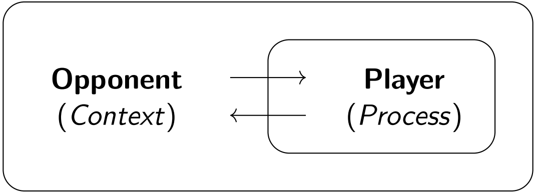


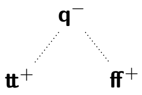
Arenas (*Types*): The game with its rules
Plays (*Executions*): A game between two players
Strategies (*Programs*): Guideline for Player

Arenas (Types)

Definition (Arena)

An **arena** A is a tuple $\langle |A|, \leq_A, \mathrm{pol}_A \rangle$ such that:

- $|A|$ is a set of *events*;
- $\mathrm{pol}_A : |A| \to \{-, +\}$ is a labelling function;
- $\leq_A$ defines a *negative* and *alternating* tree.



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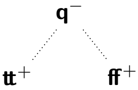

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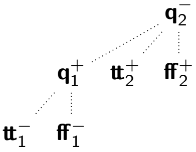

Arena $\mathbf{bool}_1 \Rightarrow \mathbf{bool}_2$

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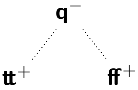

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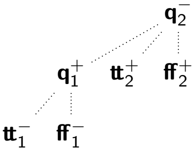

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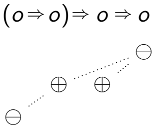


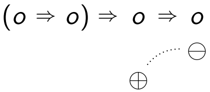
Arena $(o \Rightarrow o) \Rightarrow o \Rightarrow o$

Plays (Executions)

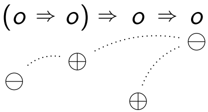
Definition (Play)

A **play** on arena A is a pointing string $s = s_1 \dots s_n \in |A|^*$ such that:

- If s_i points to s_j, then $s_j \rightarrow_A s_i$;
- $\forall 1 \leq i < n$, $\mathrm{pol}_A(s_i) \neq \mathrm{pol}_A(s_{i+1})$;
- $\forall 1 \leq i \leq n$, either $s_i = \min(A)$ or s_i has a pointer.



Typical play for $\lambda f^{o \to o}. \lambda x^o. x$



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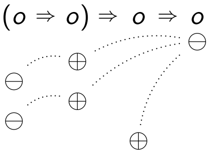

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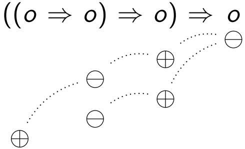
Strategies (Programs)

Definition (Strategy)

A **strategy** $\sigma : A$ is a non-empty set $\sigma \subseteq \mathrm{Plays}^+(A)$ satisfying:

$$\begin{aligned} \textit{prefix-closed:} \quad & \forall s \in \sigma, \forall t \sqsubseteq^+ s, t \in \sigma\,, \\ \textit{deterministic:} \quad & \forall s \in \sigma,\ sab, sab' \in \sigma \implies sab = sab'\,. \end{aligned}$$



$K_x : \lambda f^{(o \to o) \to o}.\, f\,(\lambda x^o.\, f\,(\lambda y^o.\, x))$

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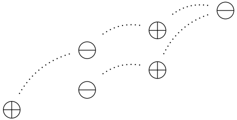

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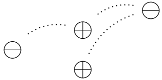

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Innocence

Definition (P-view)

A **P-view** is a play where Opponent moves point to the previous Player move.

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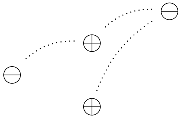

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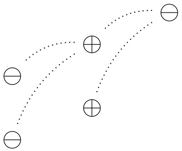

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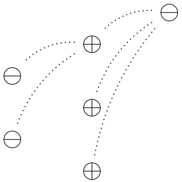

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A **P-view** is a play where Opponent moves point to the previous Player move.

Definition (Innocent Strategy)

A strategy is **innocent** if Player reacts the same way to every duplication of Opponent moves.

Outline

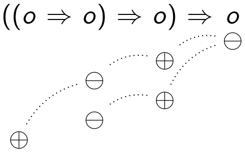
1. Introduction to Game Semantics
2. Positional Injectivity
    - Configurations
    - Positions
    - Positional Injectivity
3. Proof Method

Configurations

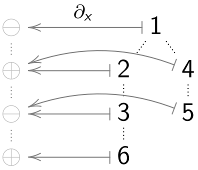
Definition

A **configuration** $x \in \mathscr{C}(A)$ is $x = \langle |x|, \leq_x, \partial_x \rangle$ such that:

- $\langle |x|, \leq_x \rangle$ is a finite tree and $\partial_x : |x| \to |A|$ is the **display map**,
- for all $a \in |x|$, a is $\leq_x$-minimal iff $\partial_x(a)$ is $\leq_A$-minimal,
- for all $a_1, a_2 \in |x|$, if $a_1 \rightarrowtriangle_x a_2$ then $\partial_x(a_1) \rightarrowtriangle_A \partial_x(a_2)$.



K_x



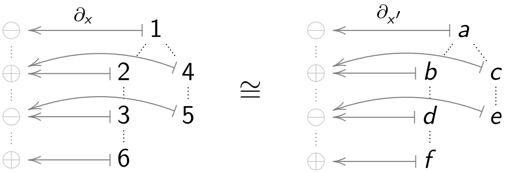
Desequentialization of K_x

Positions

Definition (Isomorphism of configurations)

An **isomorphism** $\varphi : x \cong y$ is a bijection $\varphi : |x| \cong |y|$ such that:

- for all $a \in |x|$, $\partial_y(\varphi(a)) = \partial_x(a)$,
- for all $a_1, a_2 \in |x|$, we have $a_1 \rightarrowtriangle_x a_2$ iff $\varphi(a_1) \rightarrowtriangle_y \varphi(a_2)$.



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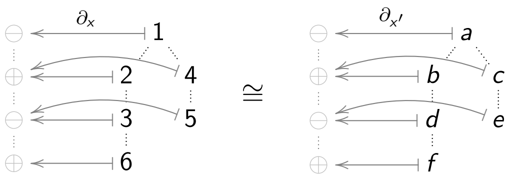


Definition (Position)

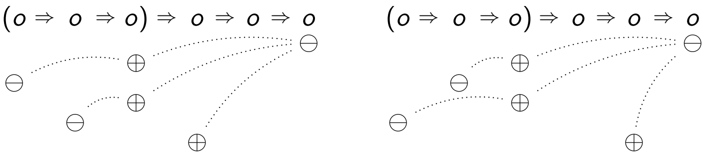
A **position** of A is the isomorphism class of a configuration on A.

Positionality

Definition (Positionality)

A strategy $\sigma : A$ is **positional** if for all $sab, t \in \sigma$, $ta' \in \mathrm{Plays}(A)$,

$$\llparenthesis sa \rrparenthesis = \llparenthesis ta' \rrparenthesis \Rightarrow \exists ta'b \in \sigma, \llparenthesis sab \rrparenthesis = \llparenthesis ta'b \rrparenthesis \,.$$



Two maximal P-views for $\lambda f^{o \to o \to o}.\, \lambda x^o.\, \lambda y^o.\, f\,(f\,x\,x)\,(f\,y\,y)$

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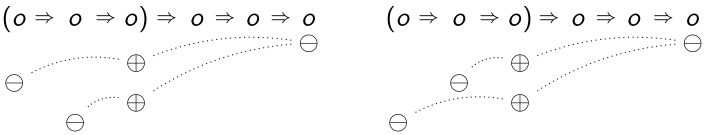

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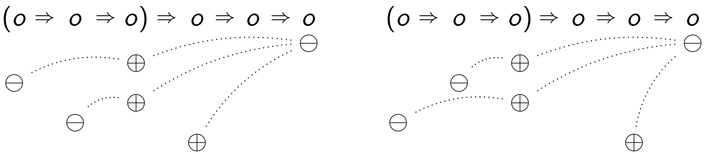

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The **positions** of a strategy $\sigma : A$ are $(\!|\sigma|\!) = \{(\!|s|\!) \mid s \in \sigma\}$.

Question (Positional Injectivity)

For any $\sigma, \tau : A$, do we have $(\!|\sigma|\!) = (\!|\tau|\!) \Rightarrow \sigma = \tau$?

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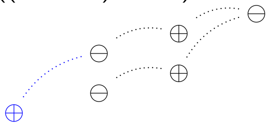

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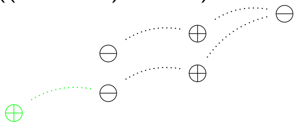

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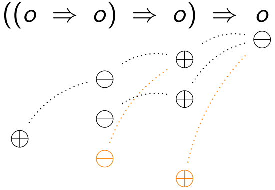

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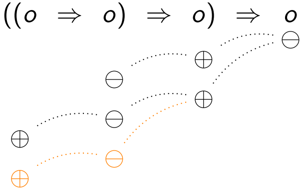

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Positional Injectivity

Definition (Totality)

A strategy $\sigma : A$ is **total** iff for all $s \in \sigma$,

$$sa \in \mathsf{Plays}(A) \;\Rightarrow\; \exists b \text{ such that } sab \in \sigma\,.$$

Definition (Finiteness)

A strategy $\sigma : A$ is **finite** iff the set of P-views of σ is finite.

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Theorem (Positional Injectivity)

For any $\sigma, \tau : A$ innocent total finite, $\sigma = \tau$ iff $\llparenthesis \sigma \rrparenthesis = \llparenthesis \tau \rrparenthesis$.

Outline

Augmentations

Augmentations = Plays without Opponent's Scheduling

Augmentations

$$
\begin{aligned}
\text{Augmentations} &= \text{Plays without Opponent's Scheduling} \\
&= \text{Positions augmented with causal order}
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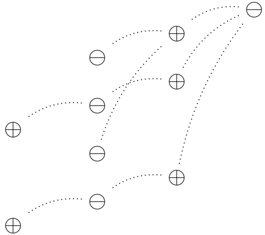

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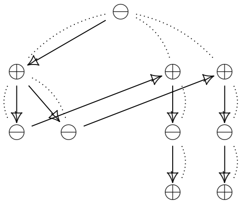

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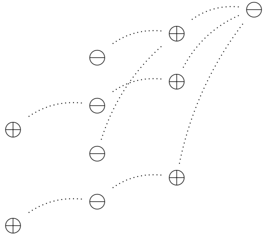

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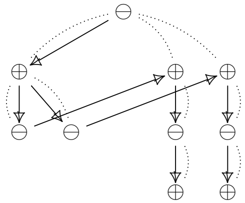
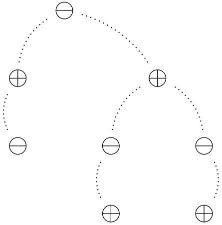


Are **causal games** positionally injective?

Using Duplications of Opponent Moves



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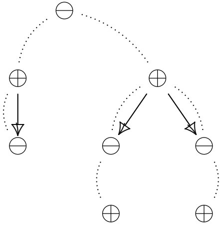
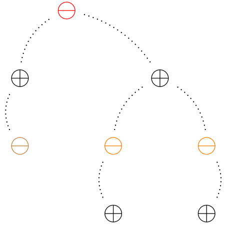
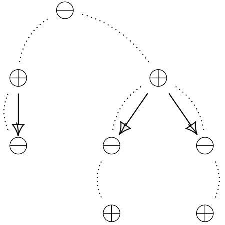


Using Duplications of Opponent Moves

Fork $= \{$ $\ominus$'s with the same parent and arena image $\}$

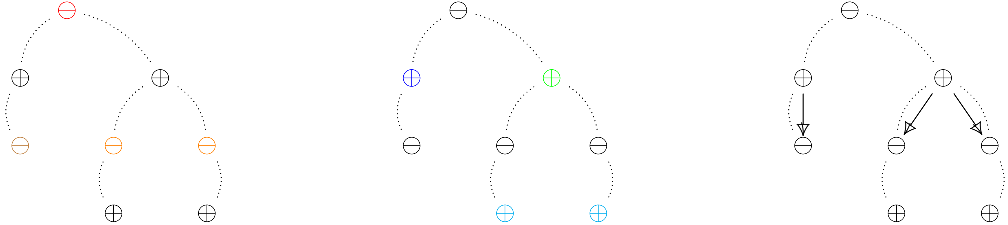


$\sharp F_1 = 1$ $\sharp F_2 = 1$ $\sharp F_3 = 2$



Using Duplications of Opponent Moves

Fork = { ⊖'s with the same parent and arena image }
Clone class = { "similar" ⊕'s }



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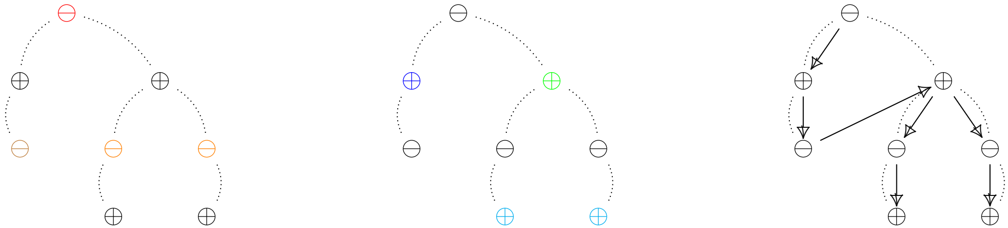

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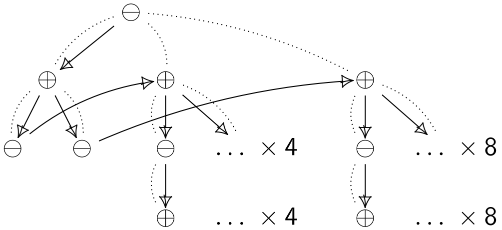
Characteristic Augmentations

Definition (Characteristic Augmentations)

A **characteristic augmentation** is an augmentation extracted from the maximal P-views of a strategy, such that each fork has for cardinality a unique power of 2.



A characteristic augmentation of K_y

Positional Injectivity

Theorem: Positional Injectivity (for causal games)

For any σ, τ finite causal strategies,

$$\llparenthesis \sigma \rrparenthesis = \llparenthesis \tau \rrparenthesis \iff \sigma = \tau \, .$$

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Theorem: Positional Injectivity (for Hyland Ong games)

For any σ, τ total finite innocent strategies,

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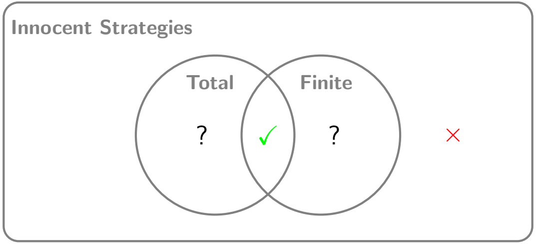
Conclusion

Theorem: Positional Injectivity

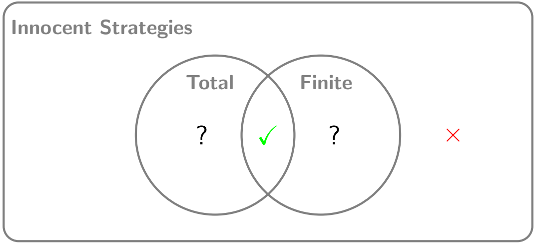
Total finite innocent strategies are positionally injective.



Conclusion

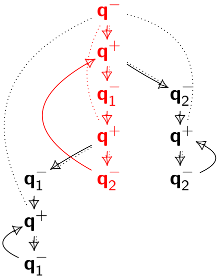
Theorem: Positional Injectivity

Total finite innocent strategies are positionally injective.

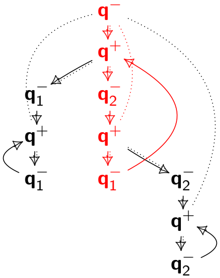


Thank you!

$$T_1 = f\ T_2\ R \qquad T_2 = f\ L\ T_1 \qquad L = f\ L\ \bot \qquad R = f\ \bot\ R$$



$\lambda f^{o \to o \to o}.\ T_1$



$\lambda f^{o \to o \to o}.\ T_2$