Positional Injectivity for Innocent Strategies

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Introduction

**Game Semantics**

- **Dynamic:** Represent programs with temporal information.
- **Strategies**

**Relational Semantics**

- **Static:** Represent states.
- **Positions**

**Programs**

**Theorem:** Positional Injectivity (for Hyland Ong games)

Pos is injective for total finite innocent strategies.
Introduction

Programs

Game Semantics

Dynamic: Represent programs with temporal information. Strategies

Relational Semantics

Static: Represent states. Positions

Theorem: Positional Injectivity (for Hyland Ong games)

\( \text{Pos} \) is injective for total finite innocent strategies.
Outline

1 Introduction to Game Semantics
   - Arenas
   - Plays
   - Innocent Strategies

2 Positional Injectivity

3 Proof Method
Introduction to Game Semantics (Hyland-Ong games)

Opponent (Context) \rightarrow Player (Process) \leftarrow

Arenas (Types): The game with its rules
Plays (Executions): A game between two players
Strategies (Programs): Guideline for Player
Introduction to Game Semantics (Hyland-Ong games)

Arenas (Types): The game with its rules
Plays (Executions): A game between two players
Strategies (Programs): Guideline for Player
Arenas (Types)

Definition (Arena)

An arena \( A \) is a tuple \( \langle |A|, \leq_A, \text{pol}_A \rangle \) such that:

- \( |A| \) is a set of events;
- \( \text{pol}_A : |A| \to \{-, +\} \) is a labelling function;
- \( \leq_A \) defines a negative and alternating tree.

Arena \( \text{bool} \)
Arenas (Types)

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\[q^-\]
\[tt^+ \quad ff^+\]

\[q_1^+ \quad tt_2^+ \quad ff_2^+\]

\[tt_1^- \quad ff_1^-\]

$\text{Arena } \mathtt{bool}$

$\text{Arena } \mathtt{bool}_1 \Rightarrow \mathtt{bool}_2$

$\text{Arena } (o \Rightarrow o) \Rightarrow o \Rightarrow o$
Plays (Executions)

Definition (Play)

A **play** on arena $A$ is a pointing string $s = s_1 \ldots s_n \in |A|^*$ such that:

- If $s_i$ points to $s_j$, then $s_j \rightarrow_A s_i$;
- $\forall 1 \leq i < n$, $\text{pol}_A(s_i) \neq \text{pol}_A(s_{i+1})$;
- $\forall 1 \leq i \leq n$, either $s_i = \min(A)$ or $s_i$ has a pointer.

**Typical play for** $\lambda f^{o \rightarrow o}. \lambda x^o. x$

**Typical play for** $\lambda f^{o \rightarrow o}. \lambda x^o. f \, x$

**Typical play for** $\lambda f^{o \rightarrow o}. \lambda x^o. f \, f \, x$
Strategies (Programs)

Definition (Strategy)

A **strategy** $\sigma : A$ is a non-empty set $\sigma \subseteq \text{Plays}^+(A)$ satisfying:

- **prefix-closed**: $\forall s \in \sigma, \forall t \sqsubseteq^+ s, t \in \sigma$,
- **deterministic**: $\forall s \in \sigma, \text{sab}, \text{sab}' \in \sigma \implies \text{sab} = \text{sab}'$.

$$((o \Rightarrow o) \Rightarrow o) \Rightarrow o$$

$$K_x : \lambda f^{(o \Rightarrow o)} \cdot f \left( \lambda x^o . f \left( \lambda y^o . x \right) \right)$$
Strategies (Programs)

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\[
((o \Rightarrow o) \Rightarrow o) \Rightarrow o
\]

\[
K_x : \lambda f^{(o \Rightarrow o) \Rightarrow o}. f (\lambda x^o. f (\lambda y^o. x))
\]

\[
\lambda f^{(o \Rightarrow o) \Rightarrow o}. f (\lambda x^o. f (\lambda y^o. x))
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\lambda f^{(o \Rightarrow o) \Rightarrow o}. f (\lambda x^o. f (\lambda y^o. x))
\]
Innocence

Definition (P-view)

A **P-view** is a play where Opponent moves point to the previous Player move.

\( ((o \Rightarrow o) \Rightarrow o) \Rightarrow o \)
Innocence

Definition (P-view)

A **P-view** is a play where Opponent moves point to the previous Player move.

\[
((o \Rightarrow o) \Rightarrow o) \Rightarrow o
\]
Definition (P-view)
A **P-view** is a play where Opponent moves point to the previous Player move.

Definition (Innocent Strategy)
A strategy is **innocent** if Player reacts the same way to every duplication of Opponent moves.
Outline

1. Introduction to Game Semantics

2. Positional Injectivity
   - Configurations
   - Positions
   - Positional Injectivity

3. Proof Method
Configurations

**Definition**

A *configuration* $x \in C(A)$ is $x = \langle |x|, \leq_x, \partial_x \rangle$ such that:

- $\langle |x|, \leq_x \rangle$ is a finite tree and $\partial_x : |x| \to |A|$ is the **display map**,
- for all $a \in |x|$, $a$ is $\leq_x$-minimal iff $\partial_x(a)$ is $\leq_A$-minimal,
- for all $a_1, a_2 \in |x|$, if $a_1 \xrightarrow{x} a_2$ then $\partial_x(a_1) \xrightarrow{A} \partial_x(a_2)$.

$$(((o \Rightarrow o) \Rightarrow o) \Rightarrow o)$$

$$K_x$$

**Desequentialization of $K_x$**
Definition (Isomorphism of configurations)

An **isomorphism** $\varphi : x \cong y$ is a bijection $\varphi : |x| \cong |y|$ such that:

- for all $a \in |x|$, $\partial_y(\varphi(a)) = \partial_x(a)$,
- for all $a_1, a_2 \in |x|$, we have $a_1 \xrightarrow{x} a_2$ iff $\varphi(a_1) \xrightarrow{y} \varphi(a_2)$. 

\[
\begin{align*}
\begin{array}{ccc}
\Rightarrow & \partial_x & 1 \\
\Rightarrow & 2 & 4 \\
\Rightarrow & 3 & 5 \\
\Rightarrow & 6 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
\Rightarrow & \partial_x' & a \\
\Rightarrow & b & c \\
\Rightarrow & d & e \\
\Rightarrow & f \\
\end{array}
\end{align*}
\]
Definition (Isomorphism of configurations)

An **isomorphism** $\varphi : x \cong y$ is a bijection $\varphi : |x| \cong |y|$ such that:

- for all $a \in |x|$, $\partial_y(\varphi(a)) = \partial_x(a)$,
- for all $a_1, a_2 \in |x|$, we have $a_1 \rightarrow^*_x a_2$ iff $\varphi(a_1) \rightarrow^*_y \varphi(a_2)$.

Definition (Position)

A **position** of $A$ is the isomorphism class of a configuration on $A$. 
A strategy $\sigma : A$ is **positional** if for all $sab, t \in \sigma, ta' \in \text{Plays}(A)$,

$$(sa) = (ta') \Rightarrow \exists ta'b \in \sigma, (sab) = (ta'b).$$

Two maximal P-views for $\lambda f^{o \rightarrow o} \cdot \lambda x^o \cdot \lambda y^o \cdot f (x x) (f y y)$.
Definition (Positionality)

A strategy $\sigma : A$ is **positional** if for all $sab, t \in \sigma, ta' \in \text{Plays}(A)$,

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Two maximal P-views for $\lambda f^{o \to o \to o}. \lambda x^{o}. \lambda y^{o}. f (f x x) (f y y)$.
Positionality

Definition (Positionality)

A strategy $\sigma : A$ is **positional** if for all $sab, t \in \sigma$, $ta' \in \text{Plays}(A)$,

\[
(sa) = (ta') \Rightarrow \exists ta'b \in \sigma, (sab) = (ta'b).
\]

Two maximal P-views for $\lambda f^\circ \to \circ. \lambda x^\circ. \lambda y^\circ. f (f x x) (f y y)$.
Positional Injectivity

Definition (Positions)
The positions of a strategy $\sigma : A$ are $\{\sigma\} = \{ (s) \mid s \in \sigma \}$.

Question (Positional Injectivity)
For any $\sigma, \tau : A$, do we have $\{\sigma\} = \{\tau\} \Rightarrow \sigma = \tau$?
Positional Injectivity

Definition (Positions)
The **positions** of a strategy $\sigma : A$ are $\langle \sigma \rangle = \{ (s) \mid s \in \sigma \}$.

Question (Positional Injectivity)
For any $\sigma, \tau : A$, do we have $\langle \sigma \rangle = \langle \tau \rangle \Rightarrow \sigma = \tau$?

$$
\begin{align*}
((o \Rightarrow o) \Rightarrow o) & \Rightarrow o \\
K_x & : \lambda f^{(o \Rightarrow o) \Rightarrow o} . f (\lambda x^o . f (\lambda y^o . x))
\end{align*}
$$

$$
\begin{align*}
((o \Rightarrow o) \Rightarrow o) & \Rightarrow o \\
K_y & : \lambda f^{(o \Rightarrow o) \Rightarrow o} . f (\lambda x^o . f (\lambda y^o . y))
\end{align*}
$$
Definition (Positions)

The **positions** of a strategy $\sigma : A$ are $\{s\} = \{(s) \mid s \in \sigma\}$.

Question (Positional Injectivity)

For any $\sigma, \tau : A$, do we have $\{\sigma\} = \{\tau\} \Rightarrow \sigma = \tau$?

$$((o \Rightarrow o) \Rightarrow o) \Rightarrow o$$

$$K_x : \lambda f : (o \Rightarrow o) \Rightarrow o. f (\lambda x^o. f (\lambda y^o. x))$$

$$((o \Rightarrow o) \Rightarrow o) \Rightarrow o$$

$$K_y : \lambda f : (o \Rightarrow o) \Rightarrow o. f (\lambda x^o. f (\lambda y^o. y))$$
Positional Injectivity

**Definition (Totality)**

A strategy $\sigma : A$ is **total** iff for all $s \in \sigma$,

$$sa \in \text{Plays}(A) \Rightarrow \exists b \text{ such that } sab \in \sigma.$$ 

**Definition (Finiteness)**

A strategy $\sigma : A$ is **finite** iff the set of P-views of $\sigma$ is finite.

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Positional Injectivity

Definition (Totality)
A strategy $\sigma : A$ is **total** iff for all $s \in \sigma$,

$$sa \in \text{Plays}(A) \Rightarrow \exists b \text{ such that } sab \in \sigma.$$  

Definition (Finiteness)
A strategy $\sigma : A$ is **finite** iff the set of P-views of $\sigma$ is finite.

Theorem (Positional Injectivity)
For any $\sigma, \tau : A$ innocent total finite, $\sigma = \tau$ iff $\|\sigma\| = \|\tau\|$. 

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Outline

1. Introduction to Game Semantics
2. Positional Injectivity
3. Proof Method
   - Augmentations
   - Characteristic Augmentations
   - Positional Injectivity
Augmentations

Augmentations = Plays without Opponent’s Scheduling
Augmentations

Augmentations = Plays without Opponent’s Scheduling
= Positions augmented with causal order
Augmentations

Augmentations = Plays without Opponent’s Scheduling
= Positions augmented with causal order

\(((o \Rightarrow o) \Rightarrow o) \Rightarrow o\)
Augmentations

Augmentations = Plays without Opponent’s Scheduling

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\[ ((o \Rightarrow o) \Rightarrow o) \Rightarrow o \]
Augmentations

Augmentations = Plays without Opponent’s Scheduling
= Positions augmented with causal order

$$(((o \Rightarrow o) \Rightarrow o) \Rightarrow o)$$

Are causal games positionally injective?
Using Duplications of Opponent Moves
Using Duplications of Opponent Moves
Using Duplications of Opponent Moves

**Fork** = \{ θ’s with the same parent and arena image \}

\[ \#F_1 = 1 \quad \#F_2 = 1 \quad \#F_3 = 2 \]
Proof Method

Characteristic Augmentations

Using Duplications of Opponent Moves

**Fork** = \{ \bigoplus’s with the same parent and arena image \}

**Clone class** = \{ “similar” \bigoplus’s \}

\[
\begin{align*}
\#F_1 &= 1 \\
\#F_2 &= 1 \\
\#F_3 &= 2 \\
\#C_1 &= 1 \\
\#C_2 &= 1 \\
\#C_3 &= 2
\end{align*}
\]
Using Duplications of Opponent Moves

**Fork** = \{ \ominus’s with the same parent and arena image \}

**Clone class** = \{ “similar” \oplus’s \}

\[
\begin{align*}
\# F_1 &= 1 & \# F_2 &= 1 & \# F_3 &= 2 \\
\# C_1 &= 1 & \# C_2 &= 1 & \# C_3 &= 2
\end{align*}
\]
Characteristic Augmentations

Definition (Characteristic Augmentations)

A characteristic augmentation is an augmentation extracted from the maximal P-views of a strategy, such that each fork has for cardinality a unique power of 2.

A characteristic augmentation of $K_y$
Theorem: Positional Injectivity (for causal games)

For any $\sigma$, $\tau$ finite causal strategies,

$$\mathcal{L}(\sigma) = \mathcal{L}(\tau) \iff \sigma = \tau.$$
Theorem: Positional Injectivity (for causal games)
For any $\sigma$, $\tau$ finite causal strategies,

$$\langle \sigma \rangle = \langle \tau \rangle \iff \sigma = \tau.$$ 

Theorem: Positional Injectivity (for Hyland Ong games)
For any $\sigma$, $\tau$ total finite innocent strategies,

$$\langle \sigma \rangle = \langle \tau \rangle \iff \sigma = \tau.$$
Conclusion

Theorem: Positional Injectivity

Total finite innocent strategies are positionally injective.
Conclusion

**Theorem: Positional Injectivity**

Total finite innocent strategies are positionally injective.

Thank you!
\( T_1 = f \ T_2 \ R \quad T_2 = f \ L \ T_1 \quad L = f \ L \perp \quad R = f \perp \ R \)

\[
\begin{array}{c}
\lambda f^{o\rightarrow o\rightarrow o}. \ T_1 \\
\lambda f^{o\rightarrow o\rightarrow o}. \ T_2
\end{array}
\]