Strategies as Resource Terms, and their Categorical Semantics

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Strategies as **Resource Terms**

λ-calculus

$$\Lambda : \; M, N, \ldots = x \mid \lambda x.M \mid MN$$

Reduction:

$$(\lambda x.M)\, N \rightarrow_\beta M[N/x]$$

Example:

$$(\lambda x.xx)y \rightarrow_\beta yy$$

$$(\lambda x.xx)(\lambda x.xx) \rightarrow_\beta (\lambda x.xx)(\lambda x.xx)$$

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Resource λ-calculus

$$\Delta : \quad s, t, \ldots = x \mid \lambda x.s \mid \langle s \rangle \bar{t}$$

$$\Delta^! : \quad \bar{s}, \bar{t}, \ldots = [s_1, \ldots, s_n]$$

Reduction:

$$\langle \lambda x.s \rangle \bar{t} \to_\beta \sum_{\sigma \in S} s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(n)}/x_n]$$

Example:

$$\langle \lambda x.\langle x \rangle [x] \rangle [\lambda x.\langle x \rangle [x], \lambda x.\langle x \rangle [x]]$$

$$\to_\beta \; 2 \cdot (\langle \lambda x.\langle x \rangle [x] \rangle [\lambda x.\langle x \rangle [x]]) \;\to_\beta\; 0$$

Strategies as **Resource Terms**

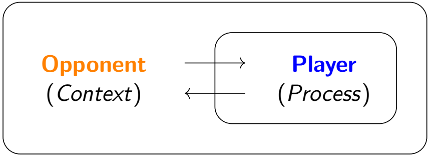
Taylor Expansion $\mathcal{T} : \Lambda \to \mathbb{R}^{+\Delta}$

$$
\begin{aligned}
\mathcal{T}(x) &\stackrel{\text{def}}{=} x \\
\mathcal{T}(\lambda x.M) &\stackrel{\text{def}}{=} \lambda x.\mathcal{T}(M) \\
\mathcal{T}(MN) &\stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}} \frac{1}{n!}\mathcal{T}(M)\,\mathcal{T}(N)^n
\end{aligned}
$$

where $\mathcal{T}(N)^n \stackrel{\text{def}}{=} [\underbrace{\mathcal{T}(N),\ldots,\mathcal{T}(N)}_{n \text{ copies}}]$.

Strategies as Resource Terms

Game Semantics:



Example:

$x + y = ?$

$x = ?$

$x = 2$

$y = ?$

$y = 3$

$x + y = 5$

Strategies as Resource Terms

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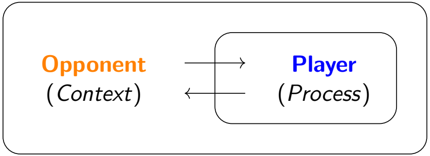

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Types	$\leftrightarrow$	**Arenas**	:	"The game with its rules"
Executions	$\leftrightarrow$	**Plays**	:	"A game between two players"
Programs	$\leftrightarrow$	**Strategies**	:	"Guideline for Player"

Strategies **as** Resource Terms

Tsukada & Ong, 2016

Resource Terms (in β-normal, η-expanded form) $\simeq$ HO **Plays**$_{/\sim}$

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This paper:

- More direct proof
- Pointer Concurrent Games
- Dynamic aspect
- Resource Categories

$$\begin{array}{ccc} s & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(s) \\ {\scriptstyle [\![-]\!]}\downarrow & & \wr\wr \\ [\![s]\!] & = & [\![\mathcal{N}(s)]\!] \end{array}$$

Part I – Static Isomorphism

Resource λ-calculus (1)

Terms of the calculus

$$\Delta \ni s, t, u, \ldots ::= x \mid \lambda x.s \mid s\,[t_1, \ldots, t_n]\ .$$

Substitution:

$$y\langle \bar{t}/x\rangle \overset{\text{def}}{=} \begin{cases} t & \text{if } y = x \text{ and } \bar{t} = [t] \\ y & \text{if } y \neq x \text{ and } \bar{t} = [] \\ 0 & \text{otherwise} \end{cases}$$

$$(\lambda z.s)\langle \bar{t}/x\rangle \overset{\text{def}}{=} \lambda z.s\langle \bar{t}/x\rangle$$

$$(s\,\bar{u})\langle \bar{t}/x\rangle \overset{\text{def}}{=} \sum_{\bar{t} \lhd \bar{t}_1 * \bar{t}_2} (s\langle \bar{t}_1/x\rangle)\,(\bar{u}\langle \bar{t}_2/x\rangle)$$

$$[s_1, \ldots, s_n]\langle \bar{t}/x\rangle \overset{\text{def}}{=} \sum_{\bar{t} \lhd \bar{t}_1 * \cdots * \bar{t}_n} [s_1\langle \bar{t}_1/x\rangle, \ldots, s_n\langle \bar{t}_n/x\rangle]$$

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$$(\lambda z.s)\langle \bar{t}/x\rangle \overset{\text{def}}{=} \lambda z.s\langle \bar{t}/x\rangle$$

Reduction:

$$\frac{}{(\lambda x.s)\,\bar{t} \to s\langle \bar{t}/x\rangle} \qquad \frac{s \to S'}{\lambda x.s \to \lambda x.S'} \qquad \frac{s \to S'}{s\,\bar{t} \to S'\,\bar{t}} \qquad \frac{s \to S'}{[s] * \bar{t} \to [S'] * \bar{t}} \qquad \frac{\bar{t} \to \bar{T}'}{s\,\bar{t} \to s\,\bar{T}'}$$

Resource λ-calculus (2)

Types

$$F, G, H, \ldots ::= o \mid F \to G$$

If $\vec{F} = \langle F_1, \ldots, F_n \rangle$, we write $\vec{F} \to G \stackrel{\text{def}}{=} F_1 \to \cdots \to F_n \to G$.

Typing rules:

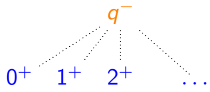
$$\frac{\Gamma, \vec{x} : \vec{F} \vdash_{\text{Base}} s : o}{\Gamma \vdash_{\text{Val}} \lambda \vec{x}.s : \vec{F} \to o}\ \text{abs} \qquad \frac{\Gamma \vdash_{\text{Val}} s : \vec{F} \to o \qquad \Gamma \vdash_{\text{Seq}} \vec{t} : \vec{F}}{\Gamma \vdash_{\text{Base}} s\,\vec{t} : o}\ \text{hr} \qquad \frac{\Gamma \vdash_{\text{Var}} x : \vec{F} \to o \qquad \Gamma \vdash_{\text{Seq}} \vec{t} : \vec{F}}{\Gamma \vdash_{\text{Base}} x\,\vec{t} : o}\ \text{hv}$$

$$\frac{}{\Gamma, x : F \vdash_{\text{Var}} x : F}\ \text{id} \qquad \frac{\Gamma \vdash_{\text{Val}} s_1 : F \quad \cdots \quad \Gamma \vdash_{\text{Val}} s_n : F}{\Gamma \vdash_{\text{Bag}} [s_1, \ldots, s_n] : F}\ \text{bag} \qquad \frac{\Gamma \vdash_{\text{Bag}} \bar{s}_1 : F_1 \quad \cdots \quad \Gamma \vdash_{\text{Bag}} \bar{s}_n : F_n}{\Gamma \vdash_{\text{Seq}} \langle \bar{s}_1, \ldots, \bar{s}_n \rangle : \langle F_1, \ldots, F_n \rangle}\ \text{seq}$$

Hyland-Ong Games

Arena $A = \langle |A|, \leq_A, \mathrm{pol}_A \rangle$

- $|A|$ set of *events*
- $\mathrm{pol}_A : |A| \to \{-, +\}$ labelling function
- $\leq_A$ defines an *alternating* tree



Arena **nat**

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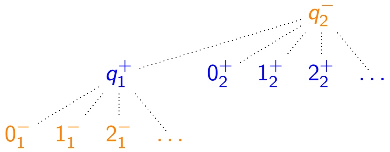

Arena $\mathbf{nat}_1 \to \mathbf{nat}_2$

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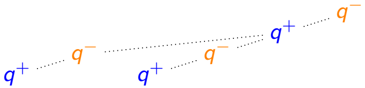
$((o \to o) \to (o \to o) \to o) \to o$



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Play s on an arena A

A pointing string $s = s_1 \dots s_n \in |A|^*$ s.t.:

- If s_i points to s_j, then $s_j \rightarrowtriangle_A s_i$
- $\forall 1 \leq i < n$, $\mathrm{pol}_A(s_i) \neq \mathrm{pol}_A(s_{i+1})$
- $\forall 1 \leq i \leq n$, $s_i \in \min(A)$ or s_i has a pointer

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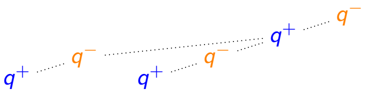

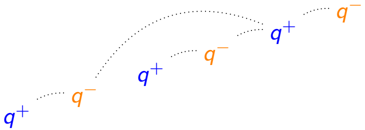

$$\lambda f^{(o \to o) \to (o \to o) \to o}. f\, [\lambda x^o . x]\, [\lambda y^o . y]$$

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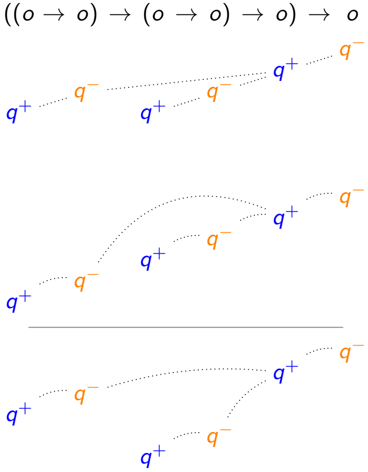


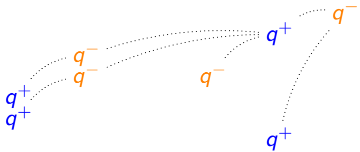
Pointer Concurrent Games

Configuration on A

$x = \langle |x|, \leq_x, \partial_x \rangle \in \mathcal{C}(A)$ with:

- $\langle |x|, \leq_x \rangle$ finite forest
- $\partial_x : |x| \to |A|$ preserves minimality and causality

$((o \to o) \to (o \to o) \to o) \to o$



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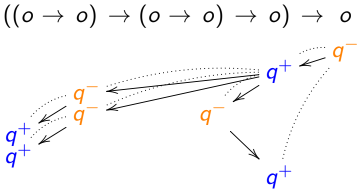


Augmentation on A

$\mathsf{p} = \langle |\mathsf{p}|, \leq_{(\!|\mathsf{p}|\!)}, \leq_{\mathsf{p}}, \partial_{\mathsf{p}} \rangle \in \mathrm{Aug}(A)$ with:

- $(\!|\mathsf{p}|\!) = \langle |\mathsf{p}|, \leq_{(\!|\mathsf{p}|\!)}, \partial_{\mathsf{p}} \rangle \in \mathcal{C}(A)$
- $\langle |\mathsf{p}|, \leq_{\mathsf{p}} \rangle$ is a forest s.t.
  - *rule-abiding*
  - *courteous*
  - *deterministic*
  - *$+$-covered*
  - *negative*

$\lambda f^{(o \to o) \to (o \to o) \to o}. f\,[\lambda x^o.x,\ \lambda y^o.y]\,[\lambda z^o.f\,[\,]\,[\,]\,]$

Augmentations as Normal Resource Terms

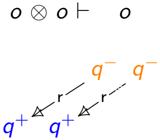
Theorem

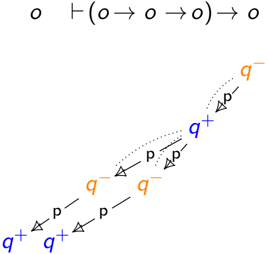
$$\begin{array}{rcrcl} \| - \| & : & \mathsf{Val}_{\mathsf{nf}}(\Gamma; F) & \simeq & \mathsf{Aug}_{\bullet}(\llbracket \Gamma \rrbracket \vdash \llbracket F \rrbracket) \\ \| - \| & : & \mathsf{Base}_{\mathsf{nf}}(\Gamma) & \simeq & \mathsf{Aug}_{\bullet}(\llbracket \Gamma \rrbracket \vdash o) \\ \| - \| & : & \mathsf{Bag}_{\mathsf{nf}}(\Gamma; F) & \simeq & \mathsf{Aug}(\llbracket \Gamma \rrbracket \vdash \llbracket F \rrbracket) \\ \| - \| & : & \mathsf{Seq}_{\mathsf{nf}}(\Gamma; \vec{F}) & \simeq & \mathsf{Aug}(\llbracket \Gamma \rrbracket \vdash \llbracket \vec{F} \rrbracket) \end{array}$$

Part II – Dynamics

Composition of augmentations (1)

How do we compose $\mathsf{r} \in \mathrm{Aug}((o \otimes o) \vdash o)$ and $\mathsf{p} \in \mathrm{Aug}(o \vdash (o \to o \to o) \to o)$?





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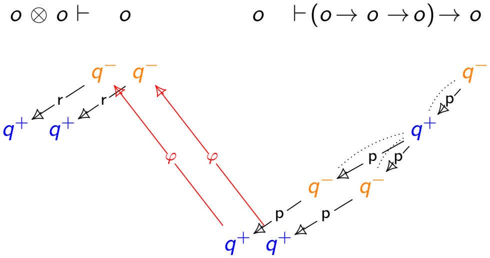

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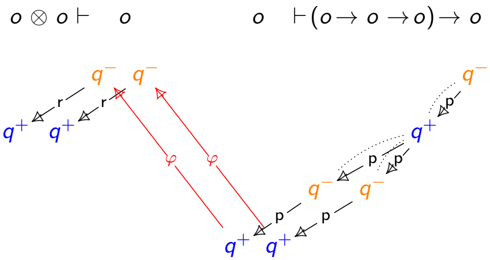

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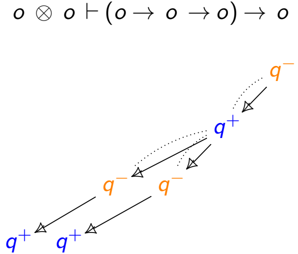

$\mathsf{p} \odot_{\varphi} \mathsf{r}$

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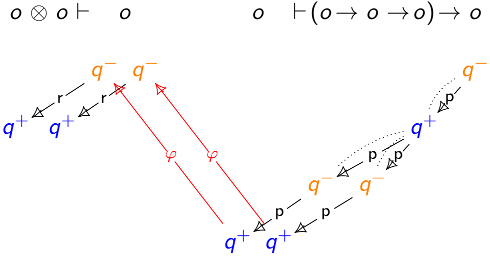

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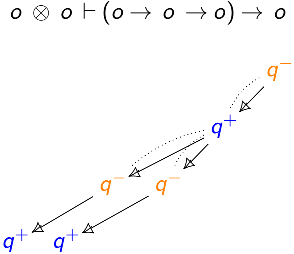

$\mathsf{p} \odot_{\varphi} \mathsf{r}$

The composition is defined *according to an isomorphism* $\varphi : \mathsf{r}_{rhs} \cong \mathsf{p}_{lhs}$!

Composition of augmentations (2)

Composition of augmentations:

$$\mathsf{p} \odot \mathsf{r} \quad \stackrel{\text{def}}{=} \quad \sum_{\varphi : \mathsf{r}_{rhs} \cong \mathsf{p}_{lhs}} \mathsf{p} \odot_{\varphi} \mathsf{r}$$

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Strategy $\sigma : A$

A **strategy** on arena A is a function $\sigma : \mathrm{Aug}(A) \to \overline{\mathbb{R}}_+$.

$$\sigma = \sum_{\mathsf{p} \in \mathrm{Aug}(A)} \sigma(\mathsf{p}) \cdot \mathsf{p}$$

Composition of augmentations (2)

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Composition of strategies $\sigma : A \vdash B$ and $\tau : B \vdash C$:

$$\tau \odot \sigma \quad \overset{\text{def}}{=} \quad \sum_{\mathsf{r} \in \mathrm{Aug}(A \vdash B)} \; \sum_{\mathsf{p} \in \mathrm{Aug}(B \vdash C)} \sigma(\mathsf{r})\tau(\mathsf{p}) \cdot (\mathsf{p} \odot \mathsf{r})$$

PCG as a category

PCG

- Objects: Arenas
- Morphisms $\mathbf{PCG}(A, B)$: Strategies $\{\sigma : A \vdash B\}$

But what is its categorical structure?

Resource Categories (1)

Resource Category $\mathcal{C}$

An additive symmetric monoidal category $\mathcal{C}$ s.t., for each $A \in \mathcal{C}$:

- A has a bialgebra structure $(A, \delta_A, \epsilon_A, \mu_A, \eta_A)$
- A has a pointed identity $\mathrm{id}_A^{\bullet}$

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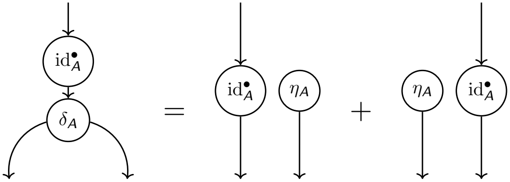
Pointed Identity $\mathrm{id}_A^\bullet$:

An idempotent $\mathrm{id}_A^\bullet \in \mathcal{C}(A, A)$ s.t.

- $\epsilon_A \circ \mathrm{id}_A^\bullet = 0$
- $\mathrm{id}_A^\bullet \circ \eta_A = 0$
- "non-duplicable"
- "non-duplicative"



Resource Categories (2)

Pointed morphism $f \in \mathcal{C}_\bullet(A, B)$

A morphism $f \in \mathcal{C}(A, B)$ s.t.

$$\mathrm{id}_B^\bullet \circ f = f$$

Resource Categories (2)

Pointed morphism $f \in \mathcal{C}_\bullet(A, B)$

A morphism $f \in \mathcal{C}(A, B)$ s.t.

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Key Lemma

For any bag of pointed morphisms $\bar{f} \in \mathscr{B}(\mathcal{C}_\bullet(A, B))$,

(a) the diagram on the right commutes

(b) $\epsilon_B \circ \Pi\bar{f} = \begin{cases} 1 & \text{if } \bar{f} \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$

$$\begin{array}{ccc} A & \xrightarrow{\delta_A} & A \otimes A \\ {\scriptstyle \Pi\bar{f}}\downarrow & & \downarrow{\scriptstyle \sum_{\bar{f} \lhd \bar{f}_1 * \bar{f}_2} \Pi\bar{f}_1 \otimes \Pi\bar{f}_2} \\ B & \xrightarrow[\delta_B]{} & B \otimes B \end{array}$$

Strategies as Resource Terms

Interpretation of resource terms.
For $\Gamma, A \in \mathcal{C}$ and $\vec{A} = \langle A_1, \ldots, A_n \rangle$, we define

$$\begin{aligned} \mathsf{Val}_{\mathcal{C}}(\Gamma; A) &\stackrel{\text{def}}{=} \mathcal{C}_\bullet(\Gamma, A) & \qquad \mathsf{Seq}_{\mathcal{C}}(\Gamma; \vec{A}) &\stackrel{\text{def}}{=} \Pi_{1 \le i \le n} \mathsf{Bag}_{\mathcal{C}}(\Gamma; A_i) \\ \mathsf{Base}_{\mathcal{C}}(\Gamma) &\stackrel{\text{def}}{=} \mathcal{C}_\bullet(\Gamma, o) & \qquad \mathsf{Bag}_{\mathcal{C}}(\Gamma; A) &\stackrel{\text{def}}{=} \mathscr{B}(\mathsf{Val}_{\mathcal{C}}(\Gamma; A))\,. \end{aligned}$$

Compatibility with normal forms.

If $S \in \Sigma\mathsf{Val}(\Gamma; F)$ and $S \Rightarrow S'$, then $[\![S]\!] = [\![S']\!]$.
If $s \in \mathsf{Val}(\Gamma; F)$ with normal form $s \Rightarrow^* \sum_{i \in I} s_i$, then $[\![s]\!] = \sum_{i \in I} [\![s_i]\!]$.

Theorem

PCG is a closed resource category.

Conclusion

Theorem

$$\mathsf{Val}_{\mathsf{nf}}(\Gamma; F) \;\simeq\; \mathbf{PCG}_\bullet([\![\Gamma]\!], [\![F]\!])$$

$$\begin{array}{ccc} s & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(s) \\ {\scriptstyle [\![-]\!]}\big\downarrow & & \wr\| \\ [\![s]\!] & = & [\![\mathcal{N}(s)]\!] \end{array}$$

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Future work:

-

$$\begin{array}{ccccc} M & \xrightarrow{\mathcal{T}(-)} & \mathcal{T}(M) & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(\mathcal{T}(M)) \\ {\scriptstyle [\![-]\!]}\big\downarrow & & {\scriptstyle [\![-]\!]}\big\downarrow & & \wr\| \\ [\![M]\!] & = & [\![\mathcal{T}(M)]\!] & = & [\![\mathcal{N}(\mathcal{T}(M))]\!] \end{array}$$

Conclusion

Theorem

$$\mathsf{Val}_{\mathsf{nf}}(\Gamma; F) \;\simeq\; \mathbf{PCG}_{\bullet}(\llbracket \Gamma \rrbracket, \llbracket F \rrbracket)$$

$$\begin{array}{ccc} s & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(s) \\ {\scriptstyle \llbracket - \rrbracket}\downarrow & & \wr\!\mid \\ \llbracket s \rrbracket & = & \llbracket \mathcal{N}(s) \rrbracket \end{array}$$

Future work:

- $$\begin{array}{ccccc} M & \xrightarrow{\mathcal{T}(-)} & \mathcal{T}(M) & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(\mathcal{T}(M)) \\ {\scriptstyle \llbracket - \rrbracket}\downarrow & & {\scriptstyle \llbracket - \rrbracket}\downarrow & & \wr\!\mid \\ \llbracket M \rrbracket & = & \llbracket \mathcal{T}(M) \rrbracket & = & \llbracket \mathcal{N}(\mathcal{T}(M)) \rrbracket \end{array}$$
- What about pure λ-calculus?

Conclusion

Theorem

$$\mathrm{Val}_{\mathsf{nf}}(\Gamma; F) \;\simeq\; \mathbf{PCG}_{\bullet}(\llbracket \Gamma \rrbracket, \llbracket F \rrbracket)$$

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s & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(s) \\
{\scriptstyle \llbracket - \rrbracket}\downarrow & & \wr \\
\llbracket s \rrbracket & = & \llbracket \mathcal{N}(s) \rrbracket
\end{array}$$

Future work:

- $$\begin{array}{ccccc}
M & \xrightarrow{\mathcal{T}(-)} & \mathcal{T}(M) & \xrightarrow{\mathcal{N}(-)} & \mathcal{N}(\mathcal{T}(M)) \\
{\scriptstyle \llbracket - \rrbracket}\downarrow & & {\scriptstyle \llbracket - \rrbracket}\downarrow & & \wr \\
\llbracket M \rrbracket & = & \llbracket \mathcal{T}(M) \rrbracket & = & \llbracket \mathcal{N}(\mathcal{T}(M)) \rrbracket
\end{array}$$
- What about pure λ-calculus?

Thank you!

Augmentations as Normal Resource Terms

Propositions

$$
\begin{array}{rrcl}
\langle -, \ldots, - \rangle : & \prod_{i=1}^{n} \mathsf{Aug}(\Gamma \vdash A_i) & \simeq & \mathsf{Aug}(\Gamma \vdash \otimes_{1 \leq i \leq n} A_i) \\
- * \cdots * - : & \mathscr{B}(\mathsf{Aug}_\bullet(\Gamma \vdash A)) & \simeq & \mathsf{Aug}(\Gamma \vdash A) \\
\Lambda_{\Gamma, A, B} : & \mathsf{Aug}(\Gamma \otimes A \vdash B) & \simeq & \mathsf{Aug}(\Gamma \vdash A \Rightarrow B) \\
\square : & \sum_{1 \leq i \leq n} \mathsf{Aug}(\Gamma \vdash \vec{B}_i^{\otimes}) & \simeq & \mathsf{Aug}_\bullet(\Gamma \vdash o)
\end{array}
$$

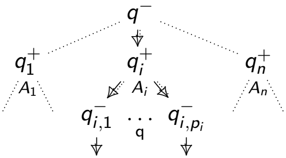

Illustration of $\square_i(\mathsf{q})$

Theorem

$$
\begin{array}{rcrcl}
\| - \| & : & \mathsf{Val}_{\mathsf{nf}}(\Gamma; F) & \simeq & \mathsf{Aug}_\bullet(\llbracket \Gamma \rrbracket \vdash \llbracket F \rrbracket) \\
\| - \| & : & \mathsf{Base}_{\mathsf{nf}}(\Gamma) & \simeq & \mathsf{Aug}_\bullet(\llbracket \Gamma \rrbracket \vdash o) \\
\| - \| & : & \mathsf{Bag}_{\mathsf{nf}}(\Gamma; F) & \simeq & \mathsf{Aug}(\llbracket \Gamma \rrbracket \vdash \llbracket F \rrbracket) \\
\| - \| & : & \mathsf{Seq}_{\mathsf{nf}}(\Gamma; \vec{F}) & \simeq & \mathsf{Aug}(\llbracket \Gamma \rrbracket \vdash \llbracket \vec{F} \rrbracket)
\end{array}
$$