Taylor Expansion is Game Semantics

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Introduction

Game Semantics

STRATEGIES

↓ *def*

PLAYS

Introduction

Game Semantics

STRATEGIES

↓ *def*

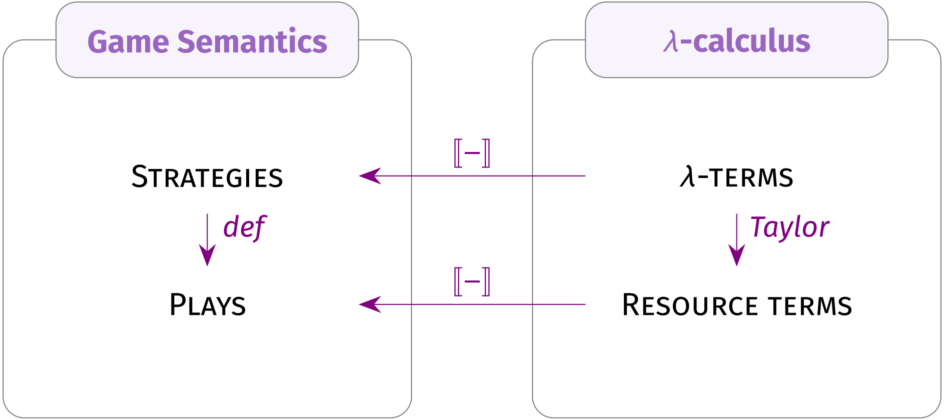
PLAYS

λ-calculus

λ-TERMS

↓ *Taylor*

RESOURCE TERMS


Game Semantics
λ-calculus
Strategies
λ-terms
⟦−⟧
def
Taylor
Plays
Resource terms
⟦−⟧

Resource Calculus

λ-calculus Λ

$$M, N, \ldots = x \mid \lambda x.M \mid MN$$

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$$(\lambda x.M)\, N \rightarrow_\beta M[N/x]$$

Examples:

$$(\lambda x.xx)y \rightarrow_\beta yy$$

$$(\lambda x.xx)(\lambda x.xx) \rightarrow_\beta (\lambda x.xx)(\lambda x.xx)$$

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Resource-calculus Δ

$$s, t, \ldots = x \mid \lambda x.s \mid \langle s\rangle[t_1, \ldots, t_n]$$

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$s, t, \ldots = x \mid \lambda x.s \mid \langle s \rangle [t_1, \ldots, t_n]$

β-reduction:

$\langle \lambda x.s \rangle \bar{t} \rightarrow_\beta \sum_{\sigma \in S} s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(n)}/x_n]$

Example:

$\langle \lambda x.\langle x \rangle [x] \rangle [\lambda x.\langle x \rangle [x], \lambda x.\langle x \rangle [x]]$

$\rightarrow_\beta 2 \cdot (\langle \lambda x.\langle x \rangle [x] \rangle [\lambda x.\langle x \rangle [x]]) \rightarrow_\beta 0$

Taylor Expansion

Taylor Expansion $\mathcal{T} : \Lambda \longrightarrow \mathbb{R}^{+\Delta}$

$$
\begin{array}{rcl}
\mathcal{T}(x) & = & x \\
\mathcal{T}(\lambda x.M) & = & \lambda x.\mathcal{T}(M) \\
\mathcal{T}(MN) & = & \sum_{n\in\mathbb{N}} \frac{1}{n!}\mathcal{T}(M)\ \mathcal{T}(N)^n
\end{array}
$$

where $\mathcal{T}(N)^n = [\,\underbrace{\mathcal{T}(N),\dots,\mathcal{T}(N)}_{n \text{ copies}}\,]$.

Resource Terms as Augmentations

[Tsukada & Ong, 2016]

Resource Terms (in β-normal, η-expanded form) $\simeq$ HO **Plays**$_{/\sim}$

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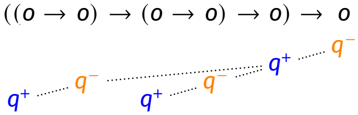
[B., Clairambault & Vaux Auclair, 2023]

Resource Terms (in β-normal, η-expanded form) $\simeq$ PCG **Aug**

and the isomorphism is compatible with β-reduction.

Arena $A = \langle |A|, \leq_A, \mathrm{pol}_A \rangle$

- $|A|$ set of *events*
- $\mathrm{pol}_A : |A| \to \{-, +\}$ labelling function
- $\leq_A$ defines an *alternating* tree



Innocence in Hyland-Ong Games

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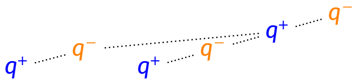
Play s **on an arena** A

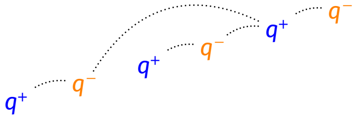
A pointing string $s = s_1 \ldots s_n \in |A|^*$ s.t.:

- If s_i points to s_j, then $s_j \rightarrowtriangle_A s_i$
- $\forall 1 \leq i < n$, $\mathrm{pol}_A(s_i) \neq \mathrm{pol}_A(s_{i+1})$
- $\forall 1 \leq i \leq n$, $s_i \in \min(A)$ or s_i has a pointer

$$\lambda f^{(o \to o) \to (o \to o) \to o}.f\ [\lambda x^o.x]\ [\lambda y^o.y]$$

$((o \to o) \to (o \to o) \to o) \to o$





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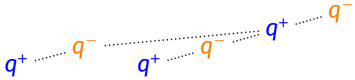
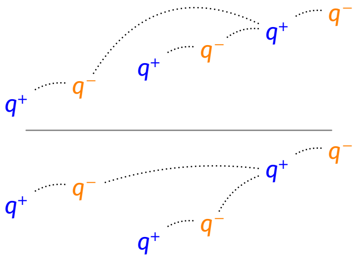


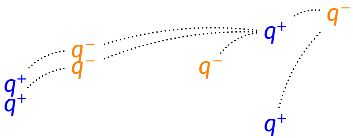


Configuration on A

$x = \langle |x|, \leq_x, \partial_x \rangle \in \mathcal{C}(A)$ with:

- $\langle |x|, \leq_x \rangle$ finite forest
- $\partial_x : |x| \to |A|$ preserves minimality and causality

$((o \to o) \to (o \to o) \to o) \to o$



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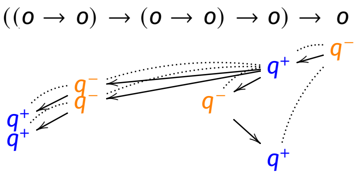
Augmentation on A

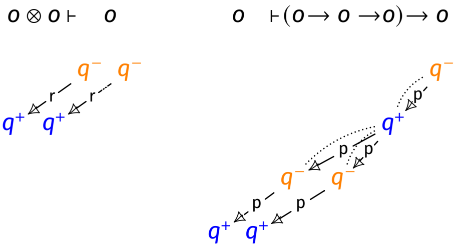
$\mathsf{p} = \langle |\mathsf{p}|, \leq_{(\!|\mathsf{p}|\!)}, \leq_{\mathsf{p}}, \partial_{\mathsf{p}} \rangle \in \mathrm{Aug}(A)$ with:

- $(\!|\mathsf{p}|\!) = \langle |\mathsf{p}|, \leq_{(\!|\mathsf{p}|\!)}, \partial_{\mathsf{p}} \rangle \in \mathscr{C}(A)$
- $\langle |\mathsf{p}|, \leq_{\mathsf{p}} \rangle$ is a forest s.t.
  - *rule-abiding*
  - *courteous*
  - *deterministic*
  - *+-covered*
  - *negative*



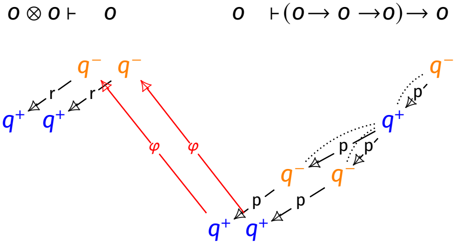
$$\lambda f^{(o\to o)\to(o\to o)\to o}.f\ [\lambda x^o.x,\ \lambda y^o.y]\ [\lambda z^o.f\ [\,]\ [\,]\,]$$

How do we compose r $\in$ Aug$((o \otimes o) \vdash o)$ and p $\in$ Aug$(o \vdash (o \to o \to o) \to o)$?



Composition of augmentations

How do we compose $\mathsf{r} \in \mathrm{Aug}((o \otimes o) \vdash o)$ and $\mathsf{p} \in \mathrm{Aug}(o \vdash (o \to o \to o) \to o)$?



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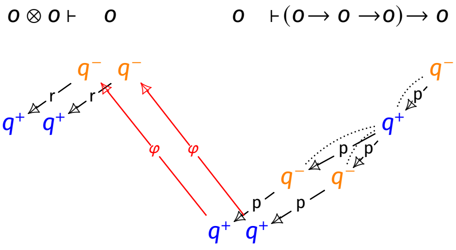

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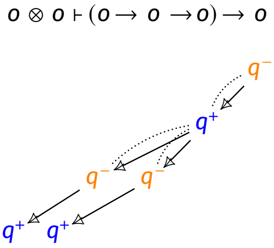

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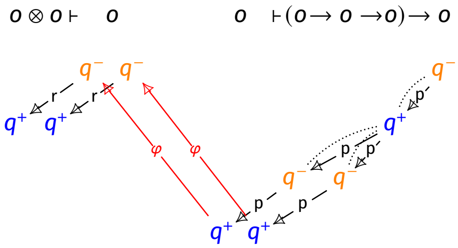
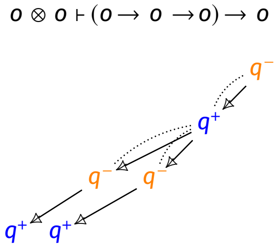


$p \circledast_{\varphi} r$



$p \odot_{\varphi} r$

The composition is defined *according to an isomorphism* $\varphi : r_{rhs} \cong p_{lhs}$!

Composition of augmentations

Consider $\mathsf{r} \in \mathrm{Aug}(A \vdash B)$, $\mathsf{p} \in \mathrm{Aug}(B \vdash C)$.

$$\mathsf{p} \odot \mathsf{r} \quad \stackrel{\mathrm{def}}{=} \quad \sum_{\varphi : \mathsf{r}_{rhs} \cong \mathsf{p}_{lhs}} \mathsf{p} \odot_{\varphi} \mathsf{r}$$

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Reminder:

$$\langle \lambda x.s \rangle \bar{t} \rightarrow_{\beta} \sum_{\sigma \in S} s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(n)}/x_n]$$

Composition and Strategies

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Strategy $\sigma : A$

A **strategy** σ on an arena A is a function $\sigma : \mathrm{Aug}(A) \to \mathbb{R}^+$.

$$\sigma = \sum_{\mathsf{p} \in \mathrm{Aug}(A)} \sigma(\mathsf{p}) \cdot \mathsf{p}$$

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Composition of strategies $\sigma : A \vdash B$ and $\tau : B \vdash C$:

$$\tau \odot \sigma \stackrel{\text{def}}{=} \sum_{\mathrm{r} \in \mathrm{Aug}(A \vdash B)} \sum_{\mathrm{p} \in \mathrm{Aug}(B \vdash C)} \sigma(\mathrm{r})\tau(\mathrm{p}) \cdot (\mathrm{p} \odot \mathrm{r})$$

Resource Categories

Composition of strategies $\sigma : A \vdash B$ and $\tau : B \vdash C$:

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PCG is a category with:

Objects: Arenas

Morphisms $\mathbf{PCG}(A, B)$: Strategies $\{\sigma : A \vdash B\}$

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PCG is a closed *resource category* with:

Objects: Arenas

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Theorem [BCVA23]

$$\Delta_{\mathrm{NF}}(\Gamma \vdash F) \simeq \mathbf{PCG}_{\bullet}(\llbracket \Gamma \rrbracket, \llbracket F \rrbracket)$$

$$\begin{array}{ccc}
s & \xrightarrow{\mathcal{NF}(-)} & \mathcal{NF}(s) \\
\llbracket - \rrbracket \Big\downarrow & & \wr \\
\llbracket s \rrbracket & = & \llbracket \mathcal{NF}(s) \rrbracket
\end{array}$$

Theorem [BCVA23]

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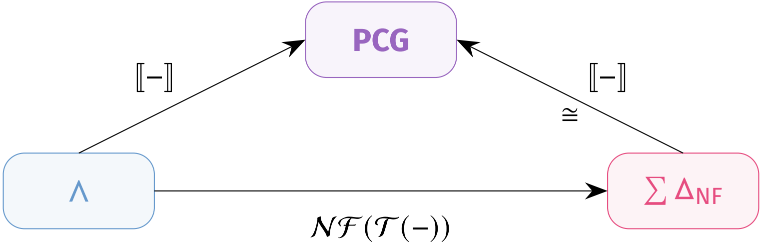
Remark: **PCG** restricted to its co-monoid morphisms is a *CCC*.

Theorem (WIP)

$$\begin{array}{ccccc} M & \xrightarrow{\mathcal{T}(-)} & \mathcal{T}(M) & \xrightarrow{\mathcal{NF}(-)} & \mathcal{NF}(\mathcal{T}(M)) \\ \llbracket - \rrbracket \downarrow & & \llbracket - \rrbracket \downarrow & & \wr \\ \llbracket M \rrbracket & = & \llbracket \mathcal{T}(M) \rrbracket & = & \llbracket \mathcal{NF}(\mathcal{T}(M)) \rrbracket \end{array}$$

Conclusion



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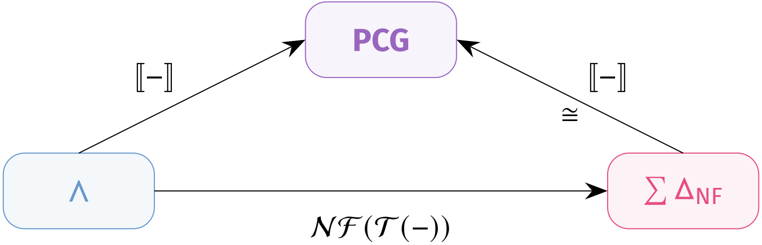

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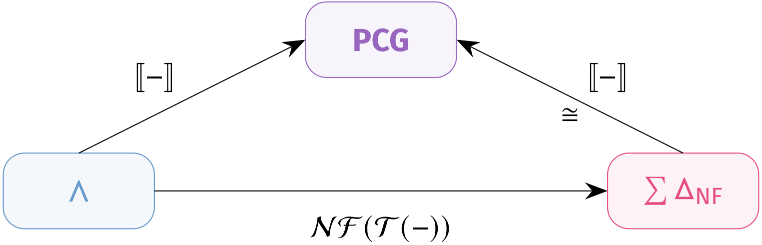

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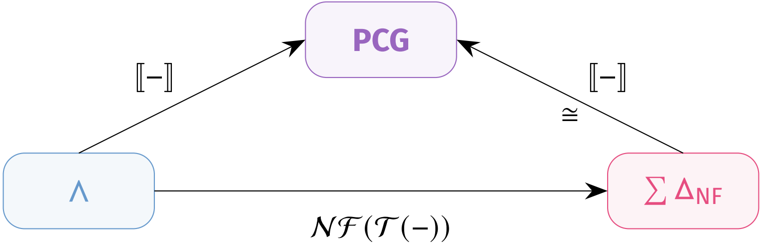

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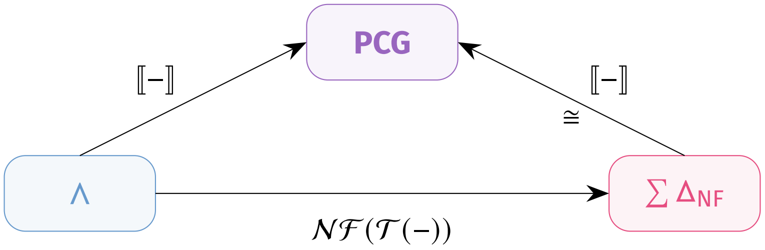

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