

# ULD-Lattices and $\Delta$ -Bonds

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## Abstract

We provide a characterization of upper locally distributive lattices (ULD-lattices) in terms of edge colorings of their cover graphs. In many instances where a set of combinatorial objects carries the order structure of a lattice this characterization yields a slick proof of distributivity or UL-distributivity. This is exemplified by proving a distributive lattice structure on  $\Delta$ -bonds with invariant circular flow-difference. This instance generalizes several previously studied lattice structures, in particular,  $c$ -orientations (Propp),  $\alpha$ -orientations of planar graphs (Felsner, resp. de Mendez) and planar flows (Khuller, Naor and Klein). The characterization also applies to other instances, e.g. to chip-firing games.

## 1 Introduction

The concept of upper locally distributive lattices (ULD) and its dual (lower locally distributive lattices (LLD)) has appeared under several different names, e.g. locally distributive lattices (Dilworth [10]), meet-distributive lattices (Jamison [15, 16], Edelman [11], Björner and Ziegler [7]), locally free lattices (Nakamura [21]). Following Avann [2], Monjardet [20], Stern [26] and others we call them ULDs. The reason for the frequent reappearance of the concept is that there are many instances of ULDs, i.e sets of combinatorial objects that can be *naturally* ordered to form an ULD.

ULDs were first investigated by Dilworth [9] and many different lattice theoretical characterizations of ULDs are known. For a survey on the work up to 1990 we refer to Monjardet [20]. We use the original definition of Dilworth:

**Definition 1** Let  $(P, \leq)$  be a poset.  $P$  is an **upper locally distributive lattice (ULD)** if  $P$  is a lattice and each element has a unique minimal representation as meet of meet-irreducibles, i.e., there is a mapping  $M : P \rightarrow \mathcal{P}(\{m \in P : m \text{ is meet-irreducible}\})$  with the properties

- $x = \bigwedge M_x$  (representation)
- $x = \bigwedge A$  implies  $M_x \subseteq A$  (minimal).

upper locally distributive lattice (ULD)

Let  $D = (V, A)$  be a directed graph. An arc coloring  $c$  of  $D$  is an **U-coloring** if for every  $u, v, w \in V$  with  $u \neq w$  and  $(v, u), (v, w) \in A$  it holds:

U-coloring

(U<sub>1</sub>)  $c(v, u) \neq c(v, w)$ .

(U<sub>2</sub>) There is a  $z \in V$  and arcs  $(u, z), (w, z)$  such that  $c(v, u) = c(w, z)$  and  $c(v, w) = c(u, z)$ .  
(see Figure 1)

**Definition 2** A finite poset  $(P, \leq)$  is called **U-poset** if the arcs of the cover graph  $D_P$  of  $P$  admit a U-coloring.

U-poset

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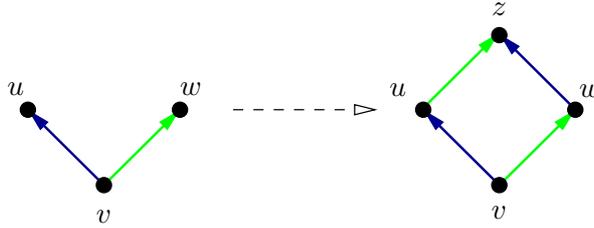


Figure 1: The completion property of U-colorings.

The characterization of ULDs in Section 2 has two parts.

**Theorem 1** (a) If  $D$  is a finite, acyclic digraph admitting a U-coloring, then  $D$  is a cover graph, hence, the transitive closure of  $D$  is a U-poset.

(b) Upper locally distributive lattices are exactly the U-posets with a global minimum.

The duals, in the sense of order reversal, to U-coloring, U-poset and ULD are L-coloring, L-poset and LLD, respectively. The characterization of LLDs dual to Theorem 1 allows easy proofs that the inclusion orders on the following combinatorial structures are lower locally distributive lattices:

- Subtrees of a tree (Boulaye [8]).
- Convex subsets of posets (Birkhoff and Bennett [3]).
- Convex subgraphs of acyclic digraphs, here a set  $C$  is convex if  $x, y \in C$  implies that all directed  $(x, y)$ -paths are in  $C$  (Pfaltz [22]).

These combinatorial structures can also be seen as *convex sets of an abstract convex geometry*. This is no coincidence as every LLD is isomorphic to the inclusion order on the convex sets of an abstract convex geometry (Edelman [11]). The L-colorings of the cover-relations of LLDs are EL-shellings in the language of Björner [4]. They give rise to what is called compatible orderings of the ground set (see [12]). In a more recent work Ardila et al. [1] provide yet another characterisation of LLDs.

Dual to the representations of LLDs as convex sets, every ULD is isomorphic to the inclusion order on the independent sets of an antimatroid (see [19]). In the present paper we represent ULDs as inclusion orders on *multisets* (of colors). In a sense this corresponds to the concepts of *antimatroids with repetition* [7] and *locally free, permutable, left-hereditary languages* [6]. There are in fact objects like chip-firing games that naturally lead to ULD representations by multisets and not by sets, see Subsection 4.5.

In Section 3 we deal with distributive lattices arising from orientations of graphs. To prove distributivity we use the following well known characterization: *Distributive lattices are exactly those lattices that are both ULD and LLD* (Theorem 5).

Let  $D = (V, A)$  be a connected directed graph with upper and lower integral edge capacities  $c_u, c_l : A \rightarrow \mathbb{Z}$ . We are interested in maps  $x : A \rightarrow \mathbb{Z}$  such that  $c_l(a) \leq x(a) \leq c_u(a)$  for all  $a \in A$ . The **circular flow-difference** of  $x$  on a cycle  $C$  with a prescribed direction is

$$\delta(C, x) := \sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a).$$

circular flow-difference

Note that the circular flow-differences  $\delta(C, x)$  on the cycles of a basis of the cycle space uniquely determines the flow-difference of  $x$  on all cycles of the graph.

For a given  $\Delta \in \mathbb{Z}^c$  we consider the set

$$\mathcal{B}_\Delta(D, c_l, c_u) := \{c_l \leq x \leq c_u \mid \delta(C, x) = \Delta_C \text{ for all } C \in \mathcal{C}\}$$

and call it the set of  $\Delta$ -bonds on  $(D, c_l, c_u)$ . The name *bonds* comes from the fact that  $\mathcal{B}_0(D, c_l, c_u)$  is orthogonal to the space of flows of  $D$ . We introduce an order on  $\Delta$ -bonds with prescribed circular flow-difference, i.e., on the elements of  $\mathcal{B}_\Delta(D, c_l, c_u)$  such that:  $\Delta$ -bonds

**Theorem 2**  $\mathcal{B}_\Delta(D, c_l, c_u)$  carries the structure of a distributive lattice.

Theorem 2 is restated in Section 3 in a more precise version as Theorem 6. The power of this result is exemplified in Section 4 where we show that several previously studied distributive lattices can be recognized as special cases of  $\Delta$ -bonds. This is shown in the following cases:

- Lattice of  $c$ -orientations of graphs (Propp [23]).
- Lattice of flow in planar graphs (Khuller, Naor and Klein [17]).
- Lattice of  $\alpha$ -orientations of planar graphs (Felsner [13]).

In Subsection 4.5 we discuss the **chip-firing game** on directed graphs. Important properties of this game can be proved in the context of U-posets and ULD lattices. chip-firing game

## 2 A Characterization of Upper Locally Distributive Lattices

In this section we prove Theorem 1. This new characterization of ULD lattices clearly can be used to show equivalence to any of the many known characterizations of ULDs, see [20]. In our proof we will establish the equivalence to the original definition given by Dilworth [9]. At the end we add a proof of the known fact that a lattice which is both ULD and LLD is actually distributive.

The following lemma is the main tool for the proof.

**Lemma 1** Let  $D = (V, A)$  be a digraph with a U-coloring  $c$ . If  $(x, y)$  is an arc and  $p = x_0, \dots, x_k$  a directed path from  $x = x_0$  to  $z = x_k$ , then there is a sequence  $y_0, \dots, y_\ell$  such that  $(x_i, y_i) \in A$  and  $c(x_i, y_i) = c(x, y)$  for  $i = 1, \dots, \ell$  and either  $\ell = k$  or  $\ell < k$  and  $y_\ell = x_{\ell+1}$ . Figure 2 illustrates the two cases (a) and (b). Case (b) happens iff there is an edge  $(x_\ell, x_{\ell+1})$  on  $p$  with  $c(x, y) = c(x_\ell, x_{\ell+1})$ .

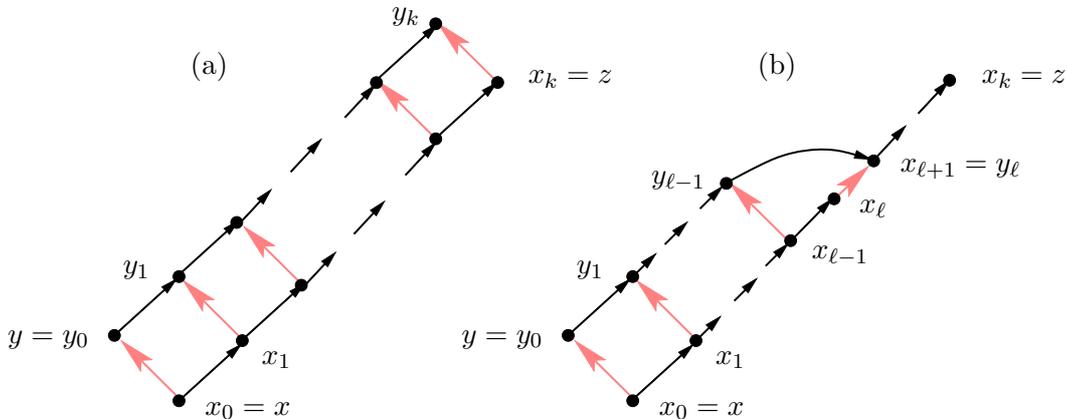


Figure 2: Illustration for Lemma 1.

*Proof.* Recursively applying rule  $U_2$  to edges  $(x_i, y_i)$  and  $(x_i, x_{i+1})$  to define a vertex  $y_{i+1}$  with edges  $(y_i, y_{i+1})$  and  $(x_{i+1}, y_{i+1})$  such that  $c(x_i, y_i) = c(x_{i+1}, y_{i+1})$ . The iteration either ends if  $i = k$  (case (a)), or if the two edges needed for the next application of the rule are the same, i.e.,  $y_i = x_{i+1}$  (case (b)). In this case  $c(x_i, x_{i+1}) = c(x_{i-1}, y_{i-1}) = c(x, y)$ , i.e., there is an edge on the path  $p$  whose color equals the color of edge  $(x, y)$ . Rule  $U_1$  implies that case (b) occurs whenever there is an edge on  $p$  whose color equals the color of edge  $(x, y)$ .  $\square$

**Remark 1** The proof does not imply that  $y_i \neq x_j$  in all cases, as is suggested by Figure 2. An example is given Figure 3. From the analysis below it follows that in all *bad* cases  $D$  is infinite or not acyclic.

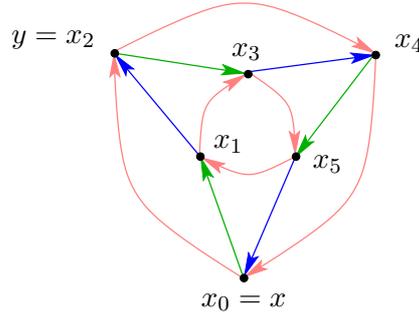


Figure 3: A digraph with a U-coloring. Choosing  $p = x_0, \dots, x_5$  and  $y = x_2$  we get  $y_i = x_{i+2 \pmod{6}}$  for  $i = 0, \dots, 6$ .

From now on we assume that  $D = (V, A)$  is a finite, connected and acyclic digraph with a U-coloring  $c$ . The assumptions imply that the transitive closure of  $D$  is a finite poset  $P_D$ . From the next two propositions it will follow that  $D$  is transitively reduced, i.e., the cover graph of  $P_D$ . Hence,  $P_D$  is a U-poset.

**Proposition 1** There is a unique sink in  $D$ .

*Proof.* Since  $D$  is acyclic and finite it has a sink. Suppose that there are two sinks  $s_0$  and  $s_1$ . Let  $p$  be a shortest  $(s_0, s_1)$ -path in the underlying undirected graph. In  $D$  the first and the last edge of  $p$  are oriented towards the sources. Hence, there is a last vertex  $x$  on  $p$  which is a source on  $p$ , i.e., the final part of  $p$  is  $y \leftarrow x \rightarrow x_1 \rightarrow \dots \rightarrow s_1$ . With  $z = s_1$  this is the precondition of Lemma 1 but both outcomes yield a contradiction: (a) is impossible because  $s_1$  is a sink, (b) is impossible because it implies that there is a shorter  $(s_0, s_1)$ -path than  $p$ .  $\square$

We define the **colorset**  $c(p)$  of a directed path  $p$  as the multi-set of colors used on edges of  $p$ .

**Proposition 2** If  $D$  has a unique sink  $s$  and  $p, p'$  are directed  $(x, z)$ -paths, then  $c(p) = c(p')$ . In particular this allows to define the **colorset**  $c(x)$  of a vertex  $x$  as the colorset of any  $(x, s)$ -path.

*Proof.* First assume that  $z = s$ . In a top down induction we show that the colorsets of all  $(x, s)$ -paths are equal.

Assume that we know that  $c(v)$  is well-defined for all vertices  $v$  that are accessible from  $x$  via directed paths in  $D$ . Let  $p$  and  $p'$  be  $(x, s)$ -paths with different initial edges. Let  $y$  be the successor of  $x$  on  $p'$  and  $a$  be the color of  $(x, y)$ . The edge  $(x, y)$  and the path  $p$  form the precondition for Lemma 1. Since  $z = s$  is the sink the situation has to be as in (b). Let  $p''$  be the path  $y \rightarrow y_1 \rightarrow \dots \rightarrow y_\ell \rightarrow \dots \rightarrow s$ . The assumption for  $y$  yields  $c(p') = a \oplus c(y) =$

colorset  $c(p)$   
of a directed  
path  
colorset  $c(x)$   
of a vertex

$a \oplus c(p'')$ . The coloring rule  $U_2$  implies that  $c(x_i, x_{i+1}) = c(y_i, y_{i+1})$  for all  $i = 0, \dots, \ell - 1$  and  $a = c(x, y) = c(x_\ell, x_{\ell+1})$ , therefore,  $a \oplus c(p'') = c(p)$ . This shows that  $c(x) = c(p') = c(p)$  is well-defined.

If  $p$  is a directed  $(x, z)$ -path, then the concatenation  $p \circ q$  with a  $(z, s)$ -path  $q$  yields a  $(x, s)$ -path, hence,  $c(x) = c(p \circ q) = c(p) \oplus c(q) = c(p) \oplus c(z)$ . This shows that  $c(p) = c(x) \ominus c(z)$  only depends on the end-vertices.  $\square$

Since the colorset of a directed  $(x, z)$ -path in  $D$  only depends on the end-vertices we also know that all  $(x, z)$ -paths have the same length. This implies that  $D$  is transitively reduced. We have thus shown the following which is slightly stronger than statement (a) of Theorem 1.

**Corollary 1** If  $D = (V, A)$  is a finite, connected and acyclic digraph with a U-coloring  $c$ , then  $D$  is a cover graph and its transitive closure is a U-poset  $P_D$  with a rank function and a 1.

Let  $P$  be a U-poset with a global minimum  $\mathbf{0}$ . Define a mapping  $\gamma : P \rightarrow \mathbb{N}^k$ , where  $k$  is the number of colors of the U-coloring. Assuming that the colors used by  $c$  are  $1, \dots, k$  the  $i$ -th component of  $\gamma(x)$  is the multiplicity of color  $i$  on any  $(\mathbf{0}, x)$ -path in the cover graph  $D_P$ . Identifying vectors in  $\mathbb{N}^k$  with multisets of colors we have  $\gamma(x) = c(\mathbf{0}) - c(x)$ .

**Lemma 2** Let  $P$  be a U-poset with a global minimum  $\mathbf{0}$ . The mapping  $\gamma : P \rightarrow \mathbb{N}^k$ , is an order preserving embedding of  $P$  into the dominance order on  $\mathbb{N}^k$ .

*Proof.* The implication from  $y \leq_P z$  to  $\gamma(y) \leq \gamma(z)$  follows from the fact that extending a path requires more colors. In particular the number of edges of color  $i$  on a  $(\mathbf{0}, z)$ -path is at least as large as on a  $(\mathbf{0}, y)$ -path, i.e.,  $\gamma_i(y) \leq \gamma_i(z)$ . In fact  $y <_P z$  implies  $\gamma(y) \neq \gamma(z)$ .

For the converse suppose  $\gamma(y) \leq \gamma(z)$  but  $y \not\leq_P z$ . From the first part and  $y \neq z$  we know  $z \not\leq_P y$ . Let  $x$  be maximal with the property  $x \leq_P z$  and  $x \leq_P y$ . Consider the first edge  $(x, y')$  on a  $(x, y)$ -path in  $D_P$  and let  $p$  be a  $(x, z)$ -path. This is a situation for Lemma 1. Since the color of edge  $(x, y')$  also occurs on  $p$  we are in case (b). This case, however, is impossible because  $y' \leq_P z$ ,  $y' \leq_P y$  and  $y' \geq x$  contradicts the choice of  $x$ .  $\square$

It is easy to see that an embedding  $\gamma : P \rightarrow \mathbb{N}^k$  yields to an embedding  $\gamma' : P \rightarrow \{0, 1\}^{k'}$ . Since generally  $k' > k$  embedding into the grid instead of the cube might be useful.

**Lemma 3** Let  $P$  and  $\gamma$  be as above. For all  $z, y \in P$  there is a  $w \in P$  with  $\gamma(w) = \gamma(y) \vee \gamma(z)$  where  $\vee$  is the componentwise maximum.

*Proof.* For any fixed  $y$  we proceed with top down induction. Given  $z$  consider a maximal  $x$  with the property  $x \leq_P z$  and  $x \leq_P y$ . Let  $(x, y')$  be the first edge on a  $(x, y)$ -path in  $D_P$  and let  $p$  be a  $(x, z)$ -path. Case (b) of Lemma 1 is impossible because  $y'$  would have prevented us from choosing  $x$ . Hence, we are in case (a) and there is a  $z'$  covering  $z$  such that the edges  $(z, z')$  and  $(x, y')$  have the same color  $i$  and moreover, the path from  $x$  to  $z$  has no edge of color  $i$ . Induction implies that there is a  $w'$  such that  $\gamma(w') = \gamma(z') \vee \gamma(y)$ . Since  $\gamma(z') = \gamma(z) + \mathbf{e}_i$  and for the  $i$ -th component  $\gamma_i(z) = \gamma_i(x) < \gamma_i(y)$  holds we can conclude  $\gamma(z') \vee \gamma(y) = \gamma(z) \vee \gamma(y)$ , i.e.,  $w'$  may also serve as  $w$ .  $\square$

**Proposition 3** If a U-poset has a global minimum  $\mathbf{0}$ , then it is a lattice.

*Proof.* The mapping  $\gamma$  is an order embedding of  $P$  into  $\mathbb{N}^k$  (Lemma 2) and the image  $\gamma(P)$  is join-closed (Lemma 3). Together this implies that  $P$  has unique least upper covers (joins exist). Since  $P$  has a  $\mathbf{0}$  there is a lower cover for every pair of elements. Uniqueness for greatest lower covers (meets) follows from the existence of unique joins.  $\square$

The next goal is to show that every element of  $P$  has a unique minimal representation as

a meet of meet-irreducibles. Let  $C(x)$  be the set of colors of the edges emanating from  $x$  in the cover graph. With the next lemma we associate a meet irreducible element with every color  $i \in C(x)$ .

**Lemma 4** Let  $P$  be a U-poset with a  $\mathbf{0}$  and let  $x \neq \mathbf{1}$  be an element of  $P$ . For every  $i \in C(x)$  there is a unique maximal element  $y_i$  such that  $\bullet y_i \geq x$  and  $\bullet \gamma_i(y_i) = \gamma_i(x)$ . The element  $y_i$  is meet-irreducible and  $\gamma_j(y_i) > \gamma_j(x)$  for all  $j \in C(x) \setminus \{i\}$ .

*Proof.* Let  $i \in C(x)$  and consider the set  $S_i(x)$  of all  $y \geq x$  with  $\gamma_i(y) = \gamma_i(x)$ . The set  $S_i(x)$  contains  $x$ , hence, it is non-empty and by Lemma 3 it contains a unique maximal element  $y_i$ . The element  $y_i$  is meet-irreducible, otherwise we could find a successor of  $y_i$  in  $S_i(x)$ . For every  $j \in C(x)$  there is an element  $x_j$  with  $\gamma(x_j) = \gamma(x) + \mathbf{e}_j$ . For  $j \neq i$  the element  $x_j$  is in  $S_i(x)$ , hence,  $\gamma(x_j) \leq \gamma(y_i)$  and  $\gamma_j(x) < \gamma_j(x_j) \leq \gamma_j(y_i)$ .  $\square$

**Proposition 4** A U-poset with a global minimum  $\mathbf{0}$  is an upper locally distributive lattice.

*Proof.* We claim that  $M_x = \{y_i : i \in C(x)\}$  is the unique minimal set of meet-irreducibles with  $x = \bigwedge M_x$ .

Let  $z$  be any lower bound for  $M_x$ , i.e., an element with  $z \leq y_i$  for all  $i \in C(x)$ . Since  $\gamma$  is order preserving and  $\mathbb{N}^k$  is closed under taking meets we have  $\gamma(z) \leq \bigwedge \{\gamma(y_i) : i \in C(x)\}$ . From  $\gamma_i(y_i) = \gamma_i(x)$  it follows that  $\bigwedge \{\gamma(y_i) : i \in C(x)\} = \gamma(x)$ . Since  $\gamma$  is order preserving this implies  $z \leq x$ , i.e.,  $x$  is the unique maximal lower bound for  $M_x$  and the notation  $x = \bigwedge M_x$  is justified.

It remains to show that the representation  $x = \bigwedge M_x$  is the unique minimal representation of  $x$  as meet of meet-irreducibles. Let  $i \in C(x)$  and consider a set  $M$  of meet-irreducibles with  $y_i \notin M$ . It is enough to show that  $x \neq \bigwedge M$ . If  $M$  contains a  $y$  with  $x \not\leq y$ , then  $x \neq \bigwedge M$  is obvious. Consider the set  $S_i(x)$  from the proof of Lemma 4, every element  $y \neq y_i$  in this set is contained in a  $(x, y_i)$ -path  $p$  that contains no  $i$ -colored edge. Lemma 1 implies that there is an  $i$ -colored edge leaving  $y$  together with the edge leaving  $y$  on  $p$  this implies that  $y$  is not meet-irreducible. Hence  $M \cap S_i(x) = \emptyset$ . All  $y > x$  with  $y \notin S_i(x)$  satisfy  $y \geq x_i$  with our assumptions on  $M$  this implies that  $x_i$  with  $\gamma(x_i) = \gamma(x) + \mathbf{e}_i$  is a lower bound on  $M$ , i.e.,  $x \neq \bigwedge M$ .  $\square$

From what we have shown so far we obtain the following criterion for ULD lattices.

**Theorem 3** If  $D$  is a finite, acyclic digraph with a unique source  $\mathbf{0}$  and there is a U-coloring of the arcs of  $D$ , then the transitive closure of  $D$  is an upper locally distributive lattice.

To complete the proof of Theorem 1 it remains to show that every ULD has a representation as U-poset, i.e., we have to present a U-coloring of its cover graph.

**Theorem 4** The cover graph of every finite ULD lattice admits a U-coloring.

Consider the mapping  $\overline{M}$  that takes an element  $x$  of  $P$  to the set  $\overline{M}_x$  of all meet-irreducible elements that are at least as large as  $x$ . The definition of meet irreducible implies that  $x = \bigwedge \overline{M}_x$  for all  $x$ , i.e., the set  $\overline{M}_x$  uniquely characterizes  $x$ . Moreover,  $x \leq y$  iff  $\overline{M}_x \supseteq \overline{M}_y$ .

On the basis of the mappings  $M$  and  $\overline{M}$  we will define a U-coloring of the cover relations of  $P$ . As colors we use the meet-irreducible elements of  $P$ .

**Lemma 5** Let  $P$  be a ULD lattice. A comparability  $x < y$  is a cover iff  $|\overline{M}_x \setminus \overline{M}_y| = 1$ .

*Proof.* An element  $z$  with  $x < z < y$  satisfies  $\overline{M}_y \subsetneq \overline{M}_z \subsetneq \overline{M}_x$  which implies  $|\overline{M}_x \setminus \overline{M}_y| \geq 2$ .

Let  $x < y$  and suppose that  $|\overline{M}_x \setminus \overline{M}_y| \geq 2$ . Since  $\bigwedge \overline{M}_x < \bigwedge \overline{M}_y$  there has to be some  $m \in \overline{M}_x \setminus \overline{M}_y$ . Let  $z = \bigwedge (\overline{M}_x - m)$ , we claim that  $x < y < z$ . From  $\overline{M}_x - m \subset \overline{M}_x$  it

follows that  $z \geq x$ , and since  $m \in M_x$  and  $m \notin M_z$  Definition 1 implies that  $z < x$ . Since  $(\overline{M_x} - m) \supseteq \overline{M_y}$  we have  $z \leq y$ . Let  $m'$  be an element with  $m \neq m' \in \overline{M_x} \setminus \overline{M_y}$ , it follows that  $m' \in \overline{M_z}$  and  $m' \notin \overline{M_y}$ . Therefore  $z \neq y$  and we have shown that  $x < z < y$ , i.e., the pair  $x, y$  is not in a cover relation.  $\square$

To a cover relation  $x \prec y$  we assign the unique meet-irreducible in  $\overline{M_x} \setminus \overline{M_y}$  as its color. Note that this meet-irreducible is a member of  $M_x$ . To verify that this is a U-coloring we have to check the two properties  $U_1$  and  $U_2$ .

**Claim 1.** The coloring satisfies  $U_1$ .

*Proof.* Let  $x \prec y_1$  and  $x \prec y_2$  be two cover relations. Since  $x$  is the meet of  $y_1$  and  $y_2$  we have the representation  $x = \bigwedge (\overline{M_{y_1}} \cup \overline{M_{y_2}})$  of  $x$  as meet of irreducibles, hence,  $M_x \subseteq \overline{M_{y_1}} \cup \overline{M_{y_2}}$ . If both covers had the same color  $m$ , then  $m \in M_x$  but  $m \notin \overline{M_{y_1}} \cup \overline{M_{y_2}}$ , a contradiction.  $\square$

**Claim 2.** The coloring satisfies  $U_2$ .

*Proof.* Let  $x \prec y_1$  and  $x \prec y_2$  be two cover relations such that  $x \prec y_1$  has color  $m_1$  and  $x \prec y_2$  has color  $m_2$ , i.e.,  $\overline{M_{y_i}} = \overline{M_x} - m_i$ . Consider  $z = \bigwedge (\overline{M_x} - m_1 - m_2)$ . Since  $z$  is representable as meet of elements from  $\overline{M_{y_i}}$  we know  $z \geq y_i$  for  $i = 1, 2$ . Since  $y_1$  and  $y_2$  both cover  $x$  it follows that  $z \neq y_1, y_2$ , hence  $z > y_i$  for  $i = 1, 2$ . From  $\overline{M_x} - m_1 - m_2 \subseteq \overline{M_z} \subseteq \overline{M_{y_i}} = \overline{M_x} - m_i$  it follows that  $|\overline{M_{y_i}} \setminus \overline{M_z}| = 1$ . Lemma 5 implies that  $z$  covers each  $y_i$  and the labels of these covers are as required.  $\square$

In many applications of the characterization of ULDs the lattice in question is actually distributive. Such a situation is the topic of the next section. To make the paper self contained we prove the following folklore result.

**Theorem 5** If a finite, acyclic and connected digraph  $D$  admits a U- and a L-coloring then  $D$  is the cover graph of a distributive lattice  $P_D$ . Moreover, the colorings yield an explicit cover-preserving embedding  $P_D \hookrightarrow \mathbb{N}^k$ , where  $k$  is the number of colors.

*Proof.* Corollary 1 and its dual imply that  $D$  is the cover graph of a poset  $P_D$  with  $\mathbf{0}$  and  $\mathbf{1}$ . Hence, with Proposition 3  $P_D$  is a lattice.

Let  $c_U$  and  $c_L$  be a U- and a L-coloring of  $D$ . Consider the coloring  $c = c_U \times c_L$ . The claim is that  $c$  is both a U- and a L-coloring of  $D$ . The rule  $U_1$  and its dual  $L_1$  are immediately inherited from the corresponding rules for  $c_U$  and  $c_L$ . Whenever there is a diamond  $x \prec y_1$ ,  $x \prec y_2$ ,  $y_1 \prec z$ ,  $y_2 \prec z$  the colors of a pair of  $(x, y_i)$  and  $(y_j, z)$  in  $c_U$  and  $c_L$  coincides. This implies rules  $U_2$  and  $L_2$  for  $c$ .

Consider the order embedding  $\gamma : P_D \rightarrow \mathbb{N}^k$  that is based on the coloring  $c$ . By Lemma 3  $\gamma$  is compatible with joins, the dual implies that  $\gamma$  is compatible with meets. Therefore  $\gamma$  is a lattice embedding, i.e.,  $P$  a sublattice of the distributive lattice  $\mathbb{N}^k$ , hence,  $P$  is itself distributive.  $\square$

**Remark 2** Let  $D$  be a digraph with a U-coloring. We need acyclicity, connectedness, finiteness and the unique source to conclude that  $D$  corresponds to a finite ULD lattice. We feel that among these conditions the *unique source* has a somewhat artificial flavor. Abstaining on this condition it can be shown (along the lines of our proof) that the corresponding poset  $P$  is a join-semilattice with the property that for all  $x \in P$  there is a unique minimal set  $M_x$  of meet-irreducibles such that  $x$  is a maximal lower bound for  $M_x$ . Figure 4 shows a small example, in this case  $M_s = M_t = \{u, v\}$ .

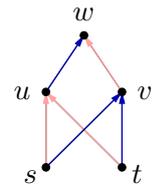


Figure 4:

### 3 The Lattice of $\Delta$ -Bonds

Recall the setting from the introduction: The data are a directed multi-graph  $D = (V, A)$  with upper and lower integral edge capacities  $c_u, c_l : A \rightarrow \mathbb{Z}$  and a number  $\Delta_C$  for each cycle  $C$  of  $D$ . We are interested in the set  $\mathcal{B}_\Delta(D, c_l, c_u)$  of  $\Delta$ -bonds for this data, i.e., the set of  $\Delta$ -bond maps  $x : A \rightarrow \mathbb{Z}$  such that

$$(D_1) \quad c_l(a) \leq x(a) \leq c_u(a) \text{ for all } a \in A. \quad (\text{capacity constraints})$$

$$(D_2) \quad \Delta_C = \sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a) = \delta(C, x) \text{ for all } C. \quad (\text{circular flow conditions})$$

Throughout the discussion we shall assume that the data  $(D, c_l, c_u, \Delta)$  are such that the set of corresponding  $\Delta$ -bonds is non-empty. Moreover, we want to simplify matters by concentrating on connected graphs and getting rid of **rigid** edges, these are edges  $a \in A$  with  $x(a) = y(a)$  for all pairs  $x, y$  of  $\Delta$ -bonds.

Let  $a$  be a rigid edge of  $D$  and let  $D/a$  be obtained from  $D$  by contracting edge  $a$ . Since we allow multiple edges and loops the cycles in  $D/a$  and in  $D$  are in bijection. Let  $C/a$  be the cycle in  $D/a$  corresponding to  $C$  in  $D$ . Define  $\Delta'_{C/a} = \Delta_C$  if  $a \notin C$  and  $\Delta'_{C/a} = \Delta_C - x(a)$  if  $a \in C^+$  and  $\Delta'_{C/a} = \Delta_C + x(a)$  if  $a \in C^-$ . These settings yield the bijection that proves

**Lemma 6**  $\mathcal{B}_\Delta(D, c_l, c_u) \cong \mathcal{B}_{\Delta'}(D/a, c_l, c_u)$ .

Given data  $(D^1, c_l^1, c_u^1, \Delta^1)$  and  $(D^2, c_l^2, c_u^2, \Delta^2)$  there is an obvious extension to a union structure  $(D, c_l, c_u, \Delta)$  where  $D$  is the union of graphs and the  $c_l, c_u, \Delta$  are concatenations of vectors. Since  $\Delta$ -bonds factor into a  $\Delta_1$ - and a  $\Delta_2$ -bond we have:

**Lemma 7**  $\mathcal{B}_\Delta(D, c_l, c_u) \cong \mathcal{B}_{\Delta^1}(D^1, c_l^1, c_u^1) \times \mathcal{B}_{\Delta^2}(D^2, c_l^2, c_u^2)$ .

The data  $(D, c_l, c_u, \Delta)$  are **reduced** if  $D$  is connected and there is no rigid edge. Henceforth we will assume that any given set of data is reduced.

With a partition  $(U, \bar{U})$  of the vertices  $V$  of  $D$  we consider the cut  $S = S[U] \subset A$ . The **forward edges**  $S^+$  of  $S$  are those  $a \in A$  directed from  $U$  to  $\bar{U}$ , **backward edges**  $S^-$  of  $S$  are directed from  $\bar{U}$  to  $U$ .

For  $x : A \rightarrow \mathbb{Z}$  and a subset  $U \subset V$  we define  $y = \text{push}_U(x)$  such that  $y(a) = x(a) + 1$  for all  $a \in S^+[U]$ ,  $y(a) = x(a) - 1$  for all  $a \in S^-[U]$  and  $y(a) = x(a)$  for all  $a \notin S[U]$ . We say that  $y = \text{push}_U(x)$  is obtained by **pushing**  $U$  in  $x$ .

Fix an arbitrary vertex  $v_0$  in  $D$  as the **forbidden vertex**. For  $x, y \in \mathcal{B}_\Delta(D, c_l, c_u)$  define  $x \leq y$  if  $y$  can be reached from  $x$  via a sequence of pushes at sets  $U_i$ , such that  $v_0 \notin U_i$  for all  $i$  and all intermediate states in the sequence are in  $\mathcal{B}_\Delta(D, c_l, c_u)$ .

Below (Corollary 2) we show that the relation “ $\leq$ ” makes  $\mathcal{B}_\Delta(D, c_l, c_u)$  into a partial order  $\mathcal{P}_\Delta(D, c_l, c_u)$ .

The main result of this section is Theorem 2 which can now be stated more precisely:

**Theorem 6** The order  $\mathcal{P}_\Delta(D, c_l, c_u)$  on  $\Delta$ -bonds is a distributive lattice.

It would have been more precise to write  $\mathcal{P}_\Delta(D, c_l, c_u, v_0)$  in the theorem because the actual lattice depends on the choice of the forbidden vertex. Different forbidden vertices yield different lattices on the same ground set. The result of Lemma 7 carries over to the lattices, i.e., if  $D^1$  and  $D^2$  are connected, then  $\mathcal{P}_\Delta(D, c_l, c_u, \{v_1, v_2\}) \cong \mathcal{P}_{\Delta^1}(D^1, c_l^1, c_u^1, v_1) \times \mathcal{P}_{\Delta^2}(D^2, c_l^2, c_u^2, v_2)$ .

The next lemma describes the condition for a legal push, i.e., a push that transforms a  $\Delta$ -bond into a  $\Delta$ -bond.

**Lemma 8** If  $x$  is in  $\mathcal{B}_\Delta(D, c_l, c_u)$  and  $S = S[U]$  is a cut such that  $x(a) < c_u(a)$  for all  $a \in S^+$  and  $x(a) > c_l(a)$  for all  $a \in S^-$ , then  $y = \text{push}_U(x)$  is also in  $\mathcal{B}_\Delta(D, c_l, c_u)$ .

*Proof.* The assumption on  $c_u(a)$  and  $c_l(a)$  implies that  $y = \text{push}_U(x)$  respects the capacity constraints. From the orthogonality of the cycle space and the bond space of  $D$  it follows that  $\delta(C, x) = \delta(C, y)$  for all cycles  $C$ , i.e.,  $y$  satisfies the circular flow conditions.  $\square$

**Lemma 9** If  $x \leq y$ , then  $y$  can be obtained from  $x$  by a sequence of **vertex pushes**, i.e., at cuts  $S[v]$  with  $v \neq v_0$ .

*Proof.* It is enough to show that a single push of a set  $U$  can be replaced by a sequence of vertex pushes. Let  $y = \text{push}_U(x)$  we show that there is a vertex  $w \in U$  such that  $x' = \text{push}_w(x)$  is a  $\Delta$ -bond and  $y = \text{push}_{U \setminus w}(x')$ . This implies the result via induction on the size of  $U$ .

Choose  $w \in U$  arbitrarily. If pushing  $w$  is legal, i.e.,  $x(a) < c_u(a)$  for all  $a \in S[w]^+$  and  $x(a) > c_l(a)$  for all  $a \in S[w]^-$ , then let  $x' = \text{push}_w(x)$ . Note that pushing  $U \setminus w$  in  $x'$  is legal: Indeed for  $a \in S[U \setminus w] \cap S[U]$  we have  $x'(a) = x(a)$  and all  $a \in S[U \setminus w] \setminus S[U]$  are incident to  $w$ , if such an  $a$  is in  $S[U \setminus w]^+$ , then it is in  $S[w]^-$  and  $x'(a) = x(a) - 1 \leq c_u(a) - 1$ , hence,  $x'(a) < c_u(a)$ . The case for  $a \in S[U \setminus w]^-$  is symmetric.

It remains to show that there is a  $w \in U$  such that pushing  $w$  is legal. Choose  $w_1 \in U$  arbitrarily if  $S[w_1]$  is not legal, then there is an incident edge  $a_1 \in S[w_1]^+$  with  $x(a_1) = c_u(a_1)$  or  $a_1 \in S[w_1]^-$  with  $x(a_1) = c_l(a_1)$ . Let  $w_2$  be the second vertex of  $a_1$  and note that  $w_2 \in U$ . If pushing  $w_2$  is not legal, then there is an incident edge  $a_2$  obstructing the push and so on. This yields a sequence  $w_1, a_1, w_2, a_2, \dots$ , either the sequence ends in a vertex  $w_j$  which is legal for pushing or it closes into a cycle. Assume that there is a cycle  $C$  such that  $x(a) = c_u(a)$  for all  $a \in C^+$  and  $x(a) = c_l(a)$  for all  $a \in C^-$ . The condition implies  $\Delta_C = \delta(C, x) = \sum_{a \in C^+} c_u(a) - \sum_{a \in C^-} c_l(a)$ . It follows that every  $\Delta$ -bond  $y$  has  $y(a) = x(a)$  for all  $a \in C$ , i.e., the edges in  $C$  are rigid. However, the data  $(D, c_u, c_l, \Delta)$  are assumed to be reduced, i.e., there are no rigid edges. Hence, there must be a  $w \in U$  that is legal for pushing.  $\square$

In the case of a general set of data  $(D, c_u, c_l, \Delta)$  with rigid edges we would have to allow the pushing of  $U$  iff  $U$  is the vertex set of a connected component of rigid edges.

**Corollary 2** The relation  $\leq$  on  $\mathcal{B}_\Delta(D, c_l, c_u)$  is acyclic, i.e., it is an order relation.

*Proof.* Otherwise we could linearly combine vertex cuts  $S[v]$ , with  $v \neq v_0$ , to zero. But these vertex cuts are a basis of the bond space.  $\square$

Pushes at vertices correspond to the cover relations, i.e., edges of the cover graph, of  $\mathcal{P}_\Delta(D, c_l, c_u)$  (Lemma 9). A coloring of the edges of the cover graph of  $\mathcal{P}_\Delta = \mathcal{P}_\Delta(D, c_l, c_u)$  with colors from  $V \setminus \{v_0\}$  is naturally given by coloring a cover with the vertex of the corresponding vertex cut.

**Lemma 10** The coloring of the edges of the cover graph of  $\mathcal{P}_\Delta$  with colors in  $V \setminus \{v_0\}$  is a U-coloring.

*Proof.* Let  $x \in \mathcal{B}_\Delta(D, c_l, c_u)$  and suppose that pushing  $v$  in  $x$  is legal, i.e., there is a covering colored  $v$  leaving  $x$ . The other element  $x'$  of the covering pair is completely determined by  $x$  and  $v$ . This shows property  $U_1$ .

For  $U_2$  assume that  $u$  and  $v$  can both be pushed in  $x$ . We have to show that they can be pushed in either order. This clearly holds if the vertex cuts of  $u$  and  $v$  are disjoint. Suppose

$(u, v) \in A$  and note that  $c_l(a) < x(a) < c_u$ . Now after pushing  $u$  we still have  $c_l(a) < x(a)$  thus we can still push at  $v$ . Conversely pushing  $v$  preserves  $x(a) < c_u$ , i.e., the push of  $u$  remains legal.  $\square$

A completely symmetric argument shows that the coloring is also a U-coloring for the reversed order, i.e., a L-coloring. Theorem 5 implies that every connected component of  $\mathcal{P}_\Delta$  is a distributive lattice. To complete the proof of Theorem 2 it only remains to show that  $\mathcal{P}_\Delta$  is connected. This is shown in the last lemma of this section.

**Lemma 11** The order  $\mathcal{P}_\Delta$  is connected.

*Proof.* With  $\Delta$ -bonds  $x$  and  $y$  consider  $z = x - y$ . Note that  $z$  is a  $\mathbf{0}$ -bond because  $\delta(C, z) = \delta(C, x) - \delta(C, y) = 0$  for all cycles  $C$ . Since  $\mathbf{0}$ -bonds are just bonds there is a unique expression of  $z$  as a linear combination of vertex cuts  $S[v]$  with  $v \neq v_0$ , we write this as  $z = \sum_v \lambda_v S[v]$ . based on the coefficients  $\lambda_v$  define  $T = \{v \in V : \lambda_v > 0\}$ . Since  $x$  and  $y$  are different and could as well be exchanged we may assume that  $T \neq \emptyset$ .

We claim that pushing  $T$  in  $y$  is legal, i.e.,  $y' = \text{push}_T(y)$  is a  $\Delta$ -bond. First note that  $v_0 \notin T$ . Now let  $a = (v, w)$  be an arc with  $v \in T$  and  $w \in V \setminus T$ , i.e.,  $a \in S[T]^+$ , from  $(x - y)(a) = \lambda_v - \lambda_w > 0$  we obtain  $x(a) > y(a)$ , hence,  $y(a) < c_u(a)$ . For  $a \in S[T]^-$  we obtain  $x(a) < y(a)$ , hence,  $y(a) > c_l(a)$ . Lemma 8 implies that  $y'$  is a  $\Delta$ -bond.

Assuming that  $\mathcal{P}_\Delta$  has several components we may choose  $x$  and  $y$  from different components such that  $\sum_v \lambda_v$  is minimal, where  $x - y = \sum_v \lambda_v S[v]$ . Since  $y'$  is obtained from  $y$  by pushing the set  $T$  it is in the same component as  $y$ . Note that  $x - y' = \sum_{v \in V \setminus T} \lambda_v S[v] + \sum_{v \in T} (\lambda_v - 1) S[v]$  since  $\sum_v \lambda_v - |T| < \sum_v \lambda_v$  this contradicts the choice of  $x$  and  $y$ .  $\square$

## 4 More Applications of the ULD Characterization

In the first three parts of this section we deal with special cases of Theorem 6. As a result we reprove known instances of distributive lattices from graphs. Subsection 4.4 connects from  $\Delta$ -bonds, i.e., special edge weightings, to potentials, i.e., vertex weightings. In Subsection 4.5 we discuss the chip-firing game on directed graphs. A central and previously known result is that the states of this game carry the structure of an ULD lattice. We obtain this as a direct application of our characterization.

### 4.1 The lattice of $c$ -orientations – Propp [23]

Given an orientation  $O$  of a graph  $G = (V, E)$ . We regard a cycle of an undirected graph as an edge-sequence, rather than an edge-set, i.e. a cycle comes with the direction of its traversal. If  $C$  is a cycle in  $G$  then denote by  $c_O(C) := |C_O^+| - |C_O^-|$  the **circular flow-difference** of  $O$  around  $C$ , where  $C_O^+$  is the set of forward arcs of  $C$  in  $O$  and  $C_O^-$  is the set of backward arcs. circular flow-difference

Given a vector  $c$ , which assigns to every cycle  $C$  of  $G$  an integer  $c(C)$ , we call an orientation  $O$  of  $G$  with  $c(C) = c_O(C)$  a  **$c$ -orientation**.  $c$ -orientation

The main result in Propp's article [23] is:

**Theorem 7** Let  $G = (V, A)$  be a graph and  $c \in \mathbb{Z}^C$ . The set of  $c$ -orientations of  $G$  carries the structure of a distributive lattice.

*Proof.* Let  $D = (V, A)$  be any orientation of  $G$ . Define  $\Delta := \frac{1}{2}(c_D - c)$ . We interpret  $x \in \mathcal{B}_\Delta(D, \mathbf{0}, \mathbf{1})$  as the orientation  $O(x)$  of  $G$  which arises from  $D$  by changing the orientation of  $a \in A$  if  $x(a) = 1$ . For an arc set  $A' \subseteq A$  denote by  $x(A') := \sum_{a \in A'} x(a)$ . We calculate

$$\begin{aligned}
c_{O(x)}(C) &= |C_{O(x)}^+| - |C_{O(x)}^-| \\
&= |C_D^+| - x(C_D^+) + x(C_D^-) - (|C_D^-| - x(C_D^-) + x(C_D^+)) \\
&= |C_D^+| - |C_D^-| - 2(x(C_D^+) - x(C_D^-)) \\
&= c_D(C) - 2\delta(x, C) = c_D(C) - 2\Delta_C \\
&= c
\end{aligned}$$

This shows that  $c$ -orientations of  $G$  correspond bijectively to  $\Delta$ -bonds in  $\mathcal{B}_\Delta(D, \mathbf{0}, \mathbf{1})$ . By Theorem 6 we obtain a distributive lattice structure on the set of  $c$ -orientations of  $G$ .  $\square$

From duality of planar graphs and the above theorem Propp derives the following two corollaries:

- The set of  $d$ -factors of a plane bipartite graph can be enhanced with a distributive lattice structure.
- The set of spanning trees of a plane graph can be enhanced with a distributive lattice structure.

## 4.2 The lattice of flow in planar graphs – Khuller, Naor and Klein [17]

Consider a planar digraph  $D = (V, A)$ , with each arc  $a$  having an integer **lower** and **upper** bound on its capacity, denoted  $c_\ell(a)$  and  $c_u(a)$ . For a function  $f : A \rightarrow \mathbb{Z}$  call  $\omega(v, f) := \sum_{a \in \text{in}(v)} f(a) - \sum_{a \in \text{out}(v)} f(a)$  the **excess** at  $v$ . Given a vector  $\Omega \in \mathbb{N}^V$  call  $f$  a  $\Omega$ -**flow** if  $c_\ell(a) \leq f(a) \leq c_u(a)$  for all  $a$  and  $\Omega_v = \omega(v, f)$  for all  $v \in V$ . Denote by  $\mathcal{F}_\Omega(D, c_\ell, c_u)$  the set of  $\Omega$ -flows. excess  
 $\Omega$ -flow

**Theorem 8** If  $D$  is a planar digraph then  $\mathcal{F}_\Omega(D, c_\ell, c_u)$  carries the structure of a distributive lattice.

*Proof.* Given a crossing-free plane embedding of  $D$  we look at the planar dual digraph  $D^*$ . It is an orientation of the planar dual  $G^*$  of the underlying undirected graph  $G$  of  $D$ . Let  $v$  be a vertex of  $G^*$  corresponding to a facial cycle  $C$  of the embedding of  $D$ . Orient an edge incident to  $v$  as outgoing arc of  $v$  if the dual arc is forward when traversing  $C$  in clockwise direction. Given values on the arcs of  $D$  we simply transfer them to the corresponding arcs of  $D^*$ .

Since the excess at a vertex of  $D$  dualizes to the circular flow difference around the corresponding facial cycle of  $D^*$ , we have a correspondence between  $\mathcal{F}_\Omega(D, c_\ell, c_u)$  and  $\mathcal{B}_\Omega(D^*, c_\ell, c_u)$ . This yields the distributive lattice structure on  $\Omega$ -flows of planar graphs.  $\square$

Analogous to the case of  $\Delta$ -bonds we can assume the data  $(D, c_\ell, c_u, \Omega)$  to be reduced. Now the dual operation to vertex pushes is to augment the flow around facial cycles. A natural candidate for the **forbidden facial cycle** is the unbounded face of the planar embedding. forbidden  
facial cycle By flow-augmentation at the remaining facial cycles we can construct the cover graph of a distributive lattice on  $\mathcal{F}_\Omega(D, c_\ell, c_u)$ .

Khuller, Naor and Klein [17] actually only consider the special case of Theorem 8 where  $\Omega = \mathbf{0}$ , these flows without excess are called **circulations**. We restate their result: circulation

**Theorem 9** Let  $D$  be a planar digraph with upper and lower arc capacities  $c_l$  and  $c_u$ . The set of circulations of  $D$  within  $c_l$  and  $c_u$  carries the structure of a distributive lattice.

### 4.3 The lattice of $\alpha$ -orientations in planar graphs – Felsner [13]

Consider a plane graph  $G = (V, E)$ . Given a map  $\alpha : V \rightarrow \mathbb{N}$  an orientation  $X$  of the edges of  $G$  is called an  **$\alpha$ -orientation** if  $\alpha$  records the out-degrees of all vertices, i.e.,  $\text{outdeg}_X(v) = \alpha(v)$  for all  $v \in V$ .  $\alpha$ -orientation

The main result in [13] is:

**Theorem 10** Given a planar graph and a mapping  $\alpha : V \rightarrow \mathbb{N}$  the set of  $\alpha$ -orientations of  $G$  carries the structure of a distributive lattice.

*Proof.* Analogously to the proof of Theorem 7, where c-orientations were interpreted as elements of  $\mathcal{B}_\Delta(C, \mathbf{0}, \mathbf{1})$  we can view  $\alpha$ -orientations as elements of  $\mathcal{F}_\Omega(D, \mathbf{0}, \mathbf{1})$  for some orientation  $D$  of  $G$  and  $\Omega$  depending on  $\alpha$ . Application of Theorem 8 yields the distributive lattice structure on the  $\alpha$ -orientations. □

In view of the present paper  $\alpha$ -orientations appear as a special case of the preceding constructions. Nevertheless they are quite general objects. Special instances of  $\alpha$ -orientations on plane graphs yield lattice structures on

- Eulerian orientations of a plane graph.
- Spanning trees and  $d$ -factors of a plane graph.
- Schnyder woods of a 3-connected plane graph.

It is natural to ask for the structure of  $\alpha$ -orientations of graphs embedded on some surface. Propp [23] comments that to move between the  $d$ -factors in toroidal graphs it is necessary to operate on non-contractible cycles. The second author's diploma thesis [18] investigates  $\alpha$ -orientations on surfaces and generalizations of  $\alpha$ -orientations.

### 4.4 Potentials

For a last and fundamental example set  $\Delta = \mathbf{0}$ . Fix  $v_0 \in V$  and denote by  $x(A') := \sum_{a \in A'} x(a)$  where  $A' \subseteq A$ . Now  $x \in \mathcal{B}_\mathbf{0}(D, c_\ell, c_u)$  means that the potential mapping  $\pi_x(v) := x(p)$  for a  $(v_0, v)$ -path  $p$  is well defined, i.e., independent of the choice of  $p$ . In particular for every  $v_0 \in V$  the set  $\mathcal{B}_\mathbf{0}(D, c_\ell, c_u)$  is in bijection with the set of **feasible vertex-potentials**: feasible  
vertex-  
potentials

$$\Pi_{v_0}(D, c_\ell, c_u) := \{\pi \in \mathbb{N}^V \mid \pi(v_0) = 0 \text{ and } c_\ell(a) \leq \pi(w) - \pi(w') \leq c_u(a) \text{ for all } a \in A\}.$$

Theorem 6 yields that  $\Pi_{v_0}(D, c_\ell, c_u)$  carries the structure of a distributive lattice. Indeed this can be obtained easier since  $\Pi_{v_0}(D, c_\ell, c_u)$  is a suborder of the distributive lattice  $\mathbb{N}^V$  where the order is given by dominance and the lattice operations are max and min. The simple observation that for  $\pi, \pi' \in \Pi_{v_0}(D, c_\ell, c_u)$  also their componentwise minimum and maximum are in  $\Pi_{v_0}(D, c_\ell, c_u)$  yields the distributive lattice structure on feasible vertex potentials.

The proof given in [13, 17, 23] for the respective lattice structures are all based on the construction of potentials corresponding to the objects they investigate. In a forthcoming paper [14] we exploit the potential approach. In this paper will deal with a class of *distributive polytopes*, **D-polytopes** for short and we generalize the notion of  $\Delta$ -bonds to *generalized bonds*. These are in a certain sense the most general structure on graphs that form a distributive lattice. D-polytopes

## 4.5 Chip-Firing Games

ULDs can be represented as *antimatroids with repetition* [7] or *locally free, permutable, left-hereditary languages* [6]. Theorem 1 gives an easy way to obtain such representation. In the following we apply Theorem 1 to the class of *chip-firing games*. As in [5] we consider chip-firing games on directed graphs. For the undirected case see [6]. In general the ULDs coming from chip-firing games are not distributive.

Let  $D = (V, A)$  be a directed graph with a map  $\sigma_0 : V \rightarrow \mathbb{N}$  called a **chip-arrangement**. The number  $\sigma_0(v)$  records the *number of chips* on vertex  $v$  in  $\sigma_0$ . Given a chip-arrangement  $\sigma$  a vertex  $v$  can be **fired** if it contains more chips than its out-degree, i.e.  $\sigma(v) \geq \text{outdeg}(v)$ . Firing  $v$  consists of sending a chip along each of the out-going arcs of  $v$  to their respective end-vertices. The new chip-arrangement is called  $\sigma^v$ . Define a directed graph  $\text{CFG}(D, \sigma_0)$  on the set of chip-arrangements which can be obtained from  $\sigma_0$  by a firing-sequence. The arc  $(\sigma, \sigma^v)$  is naturally colored with  $v \in V$ . By definition of  $\text{CFG}(D, \sigma_0)$  every vertex  $\sigma$  lies on a directed  $(\sigma_0, \sigma)$ -path. If  $\text{CFG}(D, \sigma_0)$  is also acyclic the chip-firing game is called **finite**. We obtain the following well known result: chip-arrangement fire finite

**Proposition 1** The states of a finite chip-firing game carry the structure of a ULD.

*Proof.* Since  $\text{CFG}(D, \sigma_0)$  is acyclic and every vertex lies on a directed  $(\sigma_0, \sigma)$ -path  $\text{CFG}(D, \sigma_0)$  is connected and has a unique sink  $\sigma_0$ . It only remains to show that the natural coloring of  $\text{CFG}(D, \sigma_0)$  is a U-coloring.

Clearly we have property  $U_1$ . To prove property  $U_2$  let  $\sigma \in \text{CFG}(D, \sigma_0)$  and  $v, w \in V$  ready for firing in  $\sigma$ . Since the firing of  $v$  can only increase the number of chips on  $w$ , after firing  $v$  vertex  $w$  can still be fired. This means that coloring the arcs of  $\text{CFG}(D, \sigma_0)$  by the vertices that have been fired leads to a U-coloring. Hence, by Theorem 1, the digraph  $\text{CFG}(D, \sigma_0)$  is the cover graph of a ULD. □

The ULD-properties imply that in a finite game there is a unique chip-arrangement  $\sigma^*$  where the game starting in  $\sigma_0$  ends and that all firing sequences from a given chip-arrangement  $\sigma$  to the maximum  $\sigma^*$  fire the same multiset of vertices.

Observe that the property which makes a U-poset a ULD, namely the existence of a global minimum, is rather artificially achieved in the case of chip-firing games. Instead of considering only the arrangements that can be reached from a starting chip-arrangement  $\sigma_0$  by *upwards* transformations, we can endow the structure by the inverse operation of firing and call it *co-firing*. Co-firing a vertex  $v$  means sending one chip along all the arcs  $(v, w)$  from  $w$  to  $v$ . Note that this requires that every out-neighbor of  $v$  owns a chip. We define the **complete chip-firing game**  $\text{CCFG}(D, \sigma_0)$  to be the digraph on all the chip-arrangements that can be reached from  $\sigma_0$  by firing and co-firing vertices. Under the assumption that  $\text{CFG}(D, \sigma_0)$  is acyclic the obvious coloring of the edges makes  $\text{CCFG}(D, \sigma_0)$  a U-poset. This U-poset is not complete chip-firing game

a lattice, but all its principal upsets(filters) correspond to ULDs obtained from chip-firing games in the original sense, e.g. Figure 5.

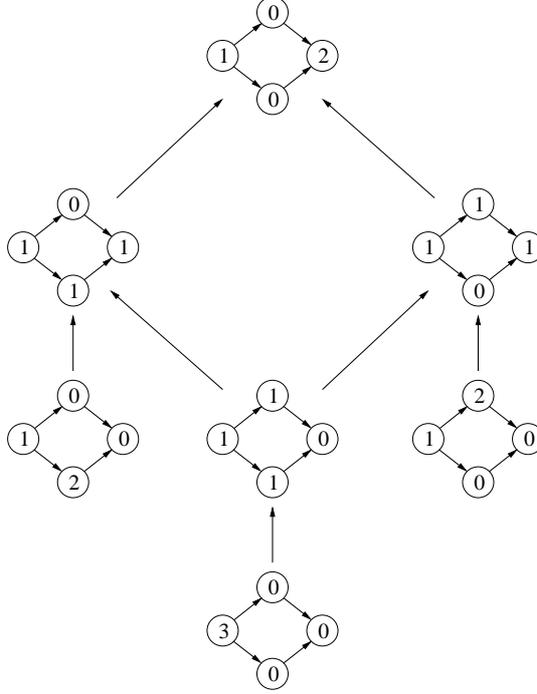


Figure 5: A U-poset induced by a complete chip-firing game.

It could be worthwhile to investigate the U-posets coming from complete chip-firing games. As a first step we provide the following proposition related to Remark 2. Note that the property stated in the proposition does not hold for the U-poset of Figure 4.

**Proposition 2** If  $(P, \leq)$  is a U-poset induced by a CCFG, then for every  $s \in P$  there is a unique inclusion minimal set  $M_s$  of meet-irreducibles such that  $\bigwedge M_s = s$

*Proof.* By Remark 2. we know that for every  $s \in P$  there is a unique inclusion minimal set  $M_s$  of meet-irreducibles such that  $s \in \bigwedge M_s$ . So suppose there are distinct  $s, t \in \bigwedge M_s$ , i.e.  $M_s = M_t$ . Partition  $M_s = U \cup V$  and let  $u = \bigwedge U$  and  $v = \bigwedge V$  and  $w = u \vee v$ . The relation between these elements is as in Figure 4.

For two states  $i < j$  in  $\text{CCFG}(D, \sigma_0)$  let  $p_{i,j}$  be a directed  $(i, j)$ -path and  $c(i, j)$  be the colorset of  $p_{i,j}$ . Recall from Section 2 that  $c(i, j)$  is indeed independent of the choice of  $p_{i,j}$ . Since  $s = u \wedge v$  and  $w = u \vee v$  we have  $c(s, u) = c(v, w)$  and  $c(s, v) = c(u, w)$ . Similarly for  $t$  we obtain  $c(t, u) = c(v, w)$  and  $c(t, v) = c(u, w)$ . This yields  $c(s, u) = c(t, u)$  and  $c(s, v) = c(t, v)$ . But if a chip-configuration  $u \in \text{CCFG}(D, \sigma_0)$  can be obtained from  $s$  and  $t$  by firing the same multiset of vertices then  $s = t$ . This is  $\bigwedge M_s = s$ .  $\square$

## 5 Concluding Remarks

Our characterization of ULDs originates from a characterization of matrices whose *flip-flop* poset generate a distributive lattice in [18]. It turned out that this tool yields handy proofs for

the known distributive lattices from graphs. The extraction of the data that were necessary for the proof led to  $\Delta$ -bonds. While preparing the present paper we observed that every  $\Delta$ -bond lattice is isomorphic to a  $\mathbf{0}$ -bond lattice, hence, the lattice structure can nicely be proved via potentials. This lead us to the notion of a D-polytope, i.e., of a polytope  $P$  such that with points  $x$  and  $y$  also  $\max(x, y)$  and  $\min(x, y)$  are in  $P$ . We can characterize the bounding hyperplanes of D-polytoes. This allows to associate a weighted digraph with every D-polytope such that the *generalized bonds* of the weighted digraph form a distributive lattice. This is the topic of the forthcoming paper [14].

There are two types of problems closely related to this article where we would like to see progress:

- Lattices of  $\Delta$ -bonds depend on the choice of a forbidden vertex  $v_0 \in V$ . Choosing another forbidden vertex  $v_1$  yields a different lattice on the same set of objects. Is there an easy description of the transformation  $\mathcal{P}_\Delta(v_0) \rightarrow \mathcal{P}_\Delta(v_1)$ ?
- The generation of a random element from a distributive lattice is a nice application for *coupling from the past* (c.f. Propp and Wilson [25]). The challenge is to find good estimates for the mixing time, see Propp [24]. What if the lattice is a  $\Delta$ -bond lattice?

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