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CYCLING IN HYPERCUBES

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Abstract

We study isometric subgraphs found in hypercubes, called partial cubes. We focus on three aspects: understanding the cycle space of such subgraphs, exploring established subfamilies and properties, and finding symmetric ones.

As we show, convex cycles in partial cubes have many intriguing properties, from spanning a simply connected connected space to forming complex substructures such as intertwinings and traverses. We analyze partial cubes with high girth to obtain results on structure and degree of such graphs. This knowledge is transferred to symmetric partial cubes to obtain a complete classification of cubic, vertex-transitive ones and to find a connection between partial cubes having mirror automorphisms and finite Coxeter groups. We study various subfamilies of partial cubes to expose a connection between (pseudo-) hyperplane arrangements, antipodal subgraphs, oriented matroids, median graphs, and many other structures found in partial cubes. With our main tool, the concept of partial cube minors, we create a map of partial cubes determining the hierarchical structure of subfamilies of partial cubes, and classifying many purely in the language of partial cubes. Lastly, computational and enumerative properties of partial cubes bounded by their isometric dimension are discussed, together with a result showing that finding isomorphisms of graphs is GI-complete already for one of the simplest classes of partial cubes: median graphs.

Contents

1

Introduction

Hypercubes are basic mathematical objects widely known in different disciplines of mathematics, other sciences, and even found their way to popular culture[1] and general discussions. Their attractiveness comes from their high dimensional, sometimes non-intuitive properties, making them a mystical object for general public, but a playground for mathematicians.

This thesis explores substructures found in hypercube graphs. In particular, it focuses on isometric subgraphs called partial cubes. These subgraphs inherit their metric properties from hypercubes and exhibit the richness of hypercube substructures. Besides pure curiosity there are various motivations for exploring such graphs. Numerous subfamilies of partial cubes have naturally and independently emerged from diverse areas of mathematics but also other areas of research, such as chemistry, theory of social choices, genetics, etc [6, 34, 35, 58, 60]. Partial cubes are capable of capturing the complexity of vector arrangements in an Euclidean space having intimate connections with linear algebra and geometry. Moreover, high dimensional properties of hypercubes are reflected in partial cubes making them an essential tool for understanding such phenomena.

In the thesis we present our contributions to the topic, together with some resent developments in the area. The topics chosen to be presented here are the one that are directly connected to our studies, or we find important and interesting to understand. Hence the choice is heavily biased and by no means we claim that it covers all the knowledge about partial cubes. Nevertheless, we like to see this thesis as a survey of the topics in partial cubes that are of interest in the resent years and a good starting point for future research. The thesis is mostly compiled from various published results that we cite on the go rather than reproving them, but includes also many new (unpublished) results for which we provide proofs. To distinguish our contributions to the topic we use bold font for citing our papers. We build the thesis on papers **[22, 61, 65, 66, 67]**. Paper **[22]** was written together with V. Chepoi and K. Knauer, while **[61]** is is a result of cooperation with K. Knauer. We are grateful to the coauthors for their contributions.

The layout of the thesis is the following. In Chapter 2 we define partial cubes and some necessary toolbox for their analysis. We present some basic characterizations of them and focus on the properties of subgraphs useful for the analysis in the further chapters. The most important concept given in this chapter is the definition of a partial cube minor, that

[1] see film Cube, 1997

we introduced in **[22]**. The partial cube minor relation gives a new view of the structure of partial cubes since it allows characterizations of its families and properties through forbidden minor characterizations. It offers a unification of many results on partial cubes and gives algorithmic recognizability of the classes defined by finitely many forbidden minors.

In Chapter 3, we focus on the cycle space of partial cubes. We find the understanding of the cycles one of the crucial insights into understanding the structure of partial cubes. The latter is one of the reasons for the title of this thesis besides suggesting that it includes various results about hypercubes structure. We analyze substructures that cycles can form and use this knowledge on partial cubes with high girth. The topic was motivated as a toolbox to prove results from Chapter 5, but outgrew in a research of its own and found application also in other topics. The main result of this chapter states that there is no finite partial cube with girth greater than six and minimal degree at least three. Moreover, partial cubes with girth greater than six are analyzed to show that such graphs have tree-like structure. Similar results hold if the girth of a partial cube is greater than four, provided that a particular isometric subgraph is not in it.

Chapter 4 is intended to give a layout of families of partial cubes. Besides being one of the motivations for the research of partial cubes, the families give an insight in the structure of partial cubes and their complexity. The chapter aims to give a bigger picture (or a bigger figure, see Figure 4.1) of how the families are placed in the setting of partial cubes. Partial cube minors are heavily used in this chapter by determining the forbidden minor characterizations of various families: median graphs, almost-median graphs, cellular bipartite graphs, hypercellular graphs, Pasch graphs, Polat graphs, tree-zone graphs, tope graphs of lopsided sets, tope graphs of complexes of oriented matroids, and partial cubes with well-embedded zone graphs. The salient result of this chapter is a characterization of tope graphs of complexes of oriented matroids not only in terms of forbidden minors but also as exactly partial cubes whose antipodal subgraphs are gated. This places the latter family as one of the covering families of partial cubes analyzed here and gives a geometric interpretation of many properties seen in various partial cubes. Another topic from this chapter that we find of great importance the analysis of hypercellular graphs. Results about them present a generalization of many well-known properties of median graphs but also cellular and Polat graphs. We characterize them as gated amalgams obtained from the Cartesian products of even cycles and edges, and derive properties abou them such as the presence of a fixed cell of endomorphisms, median cell properties, and other.

The second last Chapter 5 deals with highly symmetric partial cubes. Since hypercubes are themselves symmetric graphs it is intriguing to consider how this property is translated to their subgraphs. Regular and vertex-transitive partial cubes are discussed while the main results of the chapter are classification of cubic, vertex transitive partial cubes, and a classification of mirror graphs through a connection with hyperplane arrangements and Coxeter groups.

Finally, Chapter 6 considers generating and enumerating partial cubes together with methods and problems occurring with it. The numbers of small partial cubes and their subfamilies are given.

2

Graphs in hypercubes

We start with a chapter providing technical background and a toolbox for understanding the topic of this thesis.

2.1 Hypercubes

The basic structure considered in this thesis is a graph. By a graph $G = (V, E)$ we consider a set V of vertices and a set E of unordered pairs of vertices called edges. With few exceptions considered graphs will be finite and connected, hence if not stated otherwise we assume such a structure. Now we start with a simple definition of hypercube graphs.

Definition 2.1.1. *A* hypercube Q_n *of dimension n is the graph whose vertices are the vectors in $\{1,0\}^n$ and two vertices are adjacent if they differ in exactly one coordinate.*

If the names of the coordinates are needed, we will sometimes denote hypercube Q_n by $Q_{\mathscr{E}}$, where $\mathscr{E}$ is the ground set of coordinates with $|\mathscr{E}| = n$.

This seemingly simple family of graphs present a basic example of non-trivial, highly symmetric, relatively sparse graphs. To unveil their structure we first define the following product structure.

Definition 2.1.2. *The Cartesian product $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and the edge set consisting of all pairs $\{(g_1, h_1), (g_2, h_2)\}$ of vertices with $\{g_1, g_2\} \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$.*

It is straightforward to see that $Q_n \cong K_2^n$, where K_m denotes the complete graph on m vertices and the exponent n in the equation refers to the n-th power in regards to the Cartesian product. In this sense, the structure of hypercubes can be seen as the pure essence of the Cartesian product structure.

Despite the simply describable structure, there are many difficult (open) questions regarding the hypercubes. We point out a few classic ones to convince the reader that the structure of a hypercube is far from understood:

- *Domination number of Q_n.* Problem of domination in hypercubes refers to finding a smallest set D of vertices in Q_n such that each vertex is in D or adjacent to it. Exact sizes of such sets are known up to dimension nine [73] and for dimensions of the form

$n = 2^k - 1$ or $n = 2^k$, for some $k \in \mathbb{N}$ [48]. The domination problem is also known as the problem of covering codes since the vertices of the hypercube can be seen as strings of 0s and 1s, i.e. bits. Thus, the topic is intimately connected to the theory of coding and error correcting codes. It is interesting to notice that the famous Vizing conjecture regarding domination of the product graphs is still wide open, pointing out the difficulty of understanding domination in a product structure.

- *Turán problem in hypercubes.* Let $\text{ex}(Q_n, H)$ denotes the maximum number of edges in a hypercube Q_n, such that they induce a graph without H being its subgraph. Only for certain graphs H the values are known with some open famous conjectures about others [24, 36].

- *Extending a matching in a hypercube to a Hamiltonian cycle (or path).* In [38], Fink proved a long standing question that every perfect matching in a hypercube can be extended to a Hamiltonian cycle. It is not known if the same holds for every matching. Moreover, a similar question of the existence of a Hamiltonian cycle in the induced subgraph of Q_n, n odd, on all the vertices with $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ coordinates equal to 1, called the *middle level graph*, was only recently answered positively in [71].

2.2 Partial cubes

By far the most productive approach in the resent years in understanding the structure of a hypercube is in understanding its subgraphs. To see that the variety of subgraphs in hypercubes is extremely high, consider the following construction. Let G be and arbitrary graph, and, without loss of generality, let its vertices be denoted by integers in $\{1, \dots, n\}$. Let Q_n be a hypercube of dimension n and for each vertex i of G let v_i be the vertex in Q_n with i-th coordinate equal to 1 and all the other coordinates equal to 0. Moreover, for each edge ij of G let v_{ij} be the vertex of Q_n with precisely i-th and j-th coordinate equal to 1, all the other coordinates equal to 0. Then the graph G', induced on all the vertices $\{v_i, v_{ij} \mid i \in V(G), ij \in E(G)\}$ is an induced subgraph of Q_n, isomorphic to the graph obtained from G by subdividing all its edges. Since G was arbitrary, we have proved that the subdivision of every graph can be embedded in a hypercube. Thus subgraphs of hypercubes have surprisingly rich structure. For example, since every group can be realized as the group of symmetries of some graph [39], it follows that every group in fact can be realized as the group of symmetries of a subgraph of a hypercube.

To limit the space of subgraphs of hypercubes to graphs that have more similarities with the hypercubes, the following definition is given. The distance in graphs is defined as the length of the shortest path connecting two vertices.

Definition 2.2.1. *A* partial cube *is a graph G that can be isometrically embedded in a hypercube Q_n, i.e., $d_G(u,v) = d_{Q_n}(u,v)$ for all u, v vertices of G, where d denotes the distance function of the respective graphs.*

Since the metric of a partial cube is inherited from a hypercube, such graphs imitate the properties of hypercubes. Nevertheless, the motivation for studying them by far exceeds the sole fact that they resemble hypercubes. They were introduced by Graham and Pollak [43] in the study of interconnection networks. They form an important graph class in media theory [35], frequently appear in chemical graph theory [34], and quoting [58] present one of the central and most studied classes in metric graph theory. As we will see in Chapter 4, they include many important graph families initially arising in completely different areas of research. Among them are benzenoid graphs, the graphs of regions of hyperplane arrangements in $\mathbb{R}^d$ [13], and, more generally, tope graphs of oriented matroids (OMs) [14], median graphs (alias 1-skeleta of CAT(0) cube complexes) [9, 45], netlike graphs [76, 77, 78, 79, 80], bipartite cellular graphs [7], hypercellular graphs **[22]**, bipartite graphs with S_4 convexity (or Pasch graphs) [21], Peano graphs [75], graphs of lopsided sets [10, 62], 1-skeleta of CAT(0) Coxeter zonotopal complexes [44], and tope graphs of complexes of oriented matroids (COMs) [11].

A key insight in the structure of partial cubes is the following definition. For an edge $e = uv$ of arbitrary graph G, define the sets $W_{uv} = \{x \in V : d(x,u) < d(x,v)\}$ and $W_{vu} = \{x \in V : d(x,v) < d(x,u)\}$. Recall that we call a subgraph (or subset) S *convex* in G if for any two vertices $u, v \in S$ all the shortest u, v-paths in G lie in S. The following is due to Djoković:

Theorem 2.2.2 ([30]). *A graph G is a partial cube if and only if G is bipartite and for any edge $e = uv$ the sets W_{uv} and W_{vu} are convex.*

In the case that G is a partial cube, the sets of the form W_{uv} and $W_{v,u}$ are called *complementary halfspaces* of G. To establish an isometric embedding of G into a hypercube, Djoković [30] introduces the following binary relation Θ – called *Djoković-Winkler relation* – on the edges of G: for two edges $e = uv$ and $e' = u'v'$ we set $e\Theta e'$ iff $u' \in W_{uv}$ and $v' \in W_{vu}$.

Under the conditions of the theorem, it can be shown that $e\Theta e'$ iff $W_{uv} = W_{u'v'}$ and $W_{vu} = W_{v'u'}$, whence Θ is an equivalence relation in partial cubes. Now we point out the following equivalence that follows from the above theorem and was independently proved by Winkler.

Theorem 2.2.3 ([87]). *A graph G is a partial cube if and only if G is bipartite and the relation Θ is transitive.*

Let $\mathscr{E} = \{E_i \mid 1 \leq i \leq n\}$ be the equivalence classes of Θ and let b be an arbitrary fixed vertex taken as the basepoint of partial cube G. For an equivalence class $E_i \in \mathscr{E}$, let $\{H_i^-, H_i^+\}$ be the pair of complementary convex halfspaces of G defined by setting $H_i^- := W_{uv}$ and $H_i^+ := W_{vu}$ for an arbitrary edge $uv \in E_i$ with $b \in W_{uv}$. The embedding of G in a hypercube $Q_{\mathscr{E}}$ is defined by mapping vertex $v \in G$ to vertex of $Q_{\mathscr{E}}$ with coordinate i equal 1 iff $v \in H_i^+$.

Conversely, given G isometrically embedded into a hypercube, the Θ-classes simply are the edges corresponding to the change in a given coordinate. Note that we will denote the elements of $\mathscr{E}$ sometimes by their index f or by the corresponding edge-set E_f. The dimension of the minimal hypercube that G is embedded into or equivalently the number of Θ-classes of G (i.e. $|\mathscr{E}|$) is called the *isometric dimension* of G. One simple corollary of this is a characterization of shortest paths (also called *geodesics*) in a partial cube. Since shortest paths in hypercubes are such that no coordinate is changed more than once, shortest paths in partial cubes correspond to paths having edges in pairwise different Θ-classes.

It is important to notice that the relation Θ is also essential in the recognition of the factors of the Cartesian product of graphs, exposing the interplay between the structure of hypercubes (or partial cubes) and the structure of the Cartesian product of graphs.

From the results above it immediately follows that partial cubes can be recognized in polynomial time. In fact, Eppstein showed that it can be done better that just following the above theorems.

Proposition 2.2.4 ([33])**.** *Partial cubes can be recognized and embedded in quadratic time with respect to the number of their vertices.*

Finally, we notice that by an infinite partial cube we understand a graph isometrically embeddable into an infinite hypercube, i.e. a graph whose vertices are infinite strings of zeros and ones with finitely many ones, and two vertices adjacent if they differ in exactly one coordinate. All results about partial cubes in this section besides the recogizability hold also for such graphs.

2.3 Contractions, restrictions, expansions, and amalgamations

For the rest of this chapter, let G be a partial cube. For $E_f \in \mathscr{E}$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an (f-)*contraction* of G. For a vertex v of G, we will denote by $\pi_f(v)$ the image of v under the f-contraction in G/E_f, i.e., if uv is an edge of E_f, then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subset V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular, we denote the f-*contraction* of G by $\pi_f(G)$, see Figure 2.1.

It is well-known, easy to prove and in particular follows from the proof of the first part of [23, Theorem 3] that $\pi_f(G)$ is an isometric subgraph of $Q_{\mathscr{E}\setminus\{E_f\}}$. Moreover, edge contractions in graphs commute, i.e., the resulting graph does not depend on the order in which a set of edges is contracted. Together we have

Lemma 2.3.1. *Let G be a partial cube and $E_f \in \mathscr{E}$ its Θ-class. Then $\pi_f(G)$ is a partial cube. Moreover if $E_g \in \mathscr{E}$ and $E_f \neq E_g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$.*

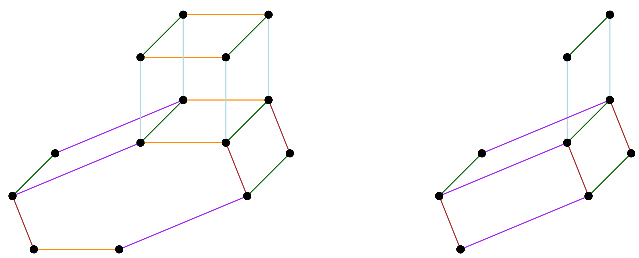


Figure 2.1: Partial cubes with colored Θ-classes. The second is obtained from the first by contracting the orange Θ-class

Given $E_f \in \mathcal{E}$, an *(elementary) restriction* consists in taking one of the subgraphs $G(E_f^-)$ or $G(E_f^+)$ induced by the complementary halfspaces E_f^- and E_f^+, which we will denote by $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$, respectively. These graphs are isometric subgraphs of the hypercube $Q_{\mathcal{E}\setminus\{E_f\}}$. Now applying two elementary restriction with respect to different coordinates f, g, independently of the order of f and g, we will obtain one of the four (possibly empty) subgraphs induced by $E_f^- \cap E_g^-, E_f^- \cap E_g^+, E_f^+ \cap E_g^-$, and $E_f^+ \cap E_g^+$. Since the intersection of convex subsets is convex, each of these four sets is convex in G and consequently induces an isometric subgraph of the hypercube $Q_{\mathcal{E}\setminus\{f,g\}}$. More generally, a *restriction* is a subgraph of G induced by the intersection of a set of (non-complementary) halfspaces of G.

For subset S of the vertices of G and $f \in \mathcal{E}$, we denote $\rho_{f^+}(S) := \rho_{f^+}(G) \cap S$ and $\rho_{f^-}(S) := \rho_{f^-}(G) \cap S$. The smallest convex subgraph of G containing V' is called the *convex hull* of V' and denoted by conv(V'). The following is well-known:

Lemma 2.3.2 ([1, 5, 19]). *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. Indeed, for any subset of vertices V' we have that conv(V') is the intersection of all halfspaces containing V'. In particular, the class of partial cubes is closed under taking restrictions. Moreover, in a bipartite graph G restrictions and convex subgraphs coincide if and only if G is a partial cube.*

It follows easily from Lemma 2.3.2 that halfspaces (elementary restrictions) can be characterized in a partial cube as precisely those convex sets whose complement is also convex. Now we consider the inverse operation of contraction: a partial cube G is an *expansion* of a partial cube G' if $G' = \pi_f(G)$ for some Θ-class E_f of G. In fact, expansions can be described within the smaller graph. Let G' be a partial cube containing two isometric subgraphs G_1' and G_2' such that $G' = G_1' \cup G_2'$, there are no edges from $G_1' \setminus G_2'$ to $G_2' \setminus G_1'$, and denote $G_0' := G_1' \cap G_2'$. A graph G is an expansion of G' with respect to G_0 if G is obtained from G' by replacing each vertex v of G_1' by a vertex v_1 and each vertex v of G_2' by a vertex v_2 such that u_i and v_i, $i = 1, 2$ are adjacent in G if and only if u and v are adjacent vertices of G_i', and $v_1 v_2$ is an edge of G if and only if v is a vertex of G_0'. Another well-known result is the following:

Lemma 2.3.3 ([19, 23]). *A graph G is a partial cube if and only if G can be obtained by a sequence of expansions from a single vertex.*

Let $E_f \in \mathcal{E}$ be one of Θ-classes of G. Assume that a halfspace E_f^+ (or E_f^-) is such that all its vertices are incident with edges from E_f. Then we call E_f^+ (or E_f^-) *peripheral*. In such a case we will also call E_f a peripheral Θ-class, and call G a *peripheral expansion* of $\pi_f(G)$. Note that an expansion along sets G_1, G_2 is peripheral if and only if one of the sets G_1, G_2 is the whole graph and the other one an isometric subgraph.

We have to consider how operations of contraction, restriction, and expansion interact with each other. By definition, expansion and contraction are inverse operations. On the other hand it is not hard to see the following.

Lemma 2.3.4 ([22]). *Contractions and restrictions commute in partial cubes, i.e., if $f, g \in$ $mathcalE$ and $f \neq g$, then $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$.*

Additionally it holds:

Lemma 2.3.5 ([**61**]). *Assume that we have the following commutative diagram of contractions:*

$$\begin{array}{ccc} G & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(G) \\ \downarrow{\scriptstyle \pi_{f_2}} & & \downarrow{\scriptstyle \pi_{f_2}} \\ \pi_{f_2}(G) & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(\pi_{f_2}(G)) \end{array}$$

Assume that G is expanded from $\pi_{f_1}(G)$ along sets $G_1, G_2 \subseteq \pi_{f_1}(G)$. Then $\pi_{f_2}(G)$ is expanded from $\pi_{f_1}(\pi_{f_2}(G))$ along sets $\pi_{f_2}(G_1)$ and $\pi_{f_2}(G_2)$.

While expansions provide a way of building bigger partial cubes, there exist another method to do so that is not limited to partial cubes. Let G_1 and G_2 be graphs with a non empty intersection, i.e. there exists an induced subgraph H of G_1 and G_2 such that $H = G_1 \cap G_2$. Then the union $G_1 \cup G_2$ of the graphs is called the *amalgamation* of G_1 and G_2 along H.

The amalgamation can be seen gluing together two graphs along an induced subgraph. In case of the two graphs being partial cubes the obtained graphs is not necessarily a partial cube. In the Chapter 4, we will deal with different kinds of amalgamations, in particular if H is so called gated amalgamation of partial cubes G_1 and G_2, then the resulting graph is always a partial cube.

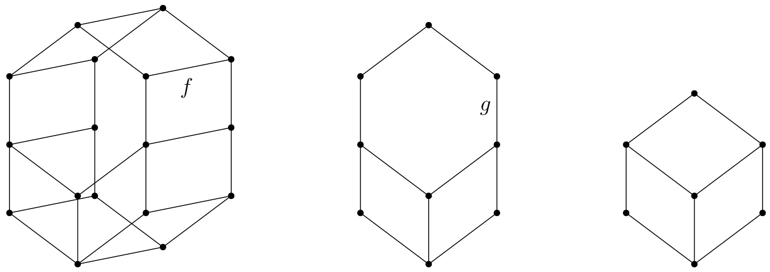
2.4 Minors in partial cubes

A useful tool for understanding the structure of a graph is the concept of a graph minor. A graph H is a minor of G if H can be obtained from G by two operations: contracting arbitrary edges and restricting to an arbitrary subgraph. Using this operations on partial cubes we encounter several problems. A subgraph of a partial cube needs not to be a partial cube and, moreover, a contraction of an edge might not even result in a bipartite graph. This calls for a different definition of the operations.

A geometrical way to avoid contracting even cycles to odd cycles is to define "parallel" edges along which the contraction can be made. Motivated by an example of an even cycle, where we would like to only allow contracting simultaneously both antipodal edges, one can define two edges uv, wz to be parallel if and only if $d(u,w) = d(v,z) = d(u,z) - 1 = d(v,w) - 1$. It is easy to check that this definition in bipartite graphs coincide with the definition of the relation Θ. Moreover, by Theorem 2.2.3, relation Θ is an equivalence relation in a bipartite graph if and only if the graph is a partial cube. Thus the class of partial cubes is a natural class to consider such contraction operations.

Since not all subgraphs of partial cubes are in the class of partial cubes, also the definition of restriction to the arbitrary subgraph must be modified. Allowing only restrictions to convex subgraphs is a natural choice that works well with existing families of partial cubes.

By Lemma 2.3.2, restrictions and contractions defined in this way coincide with restrictions and contractions in partial cubes defined in the previous section. Moreover, the lemmas from the previous section show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and we will call G' a *partial cube-minor* (abbreviated, *pc-minor*) of G (see Figure 2.2).



(a) G (b) $\rho_{f^+}(G)$ (c) $\pi_g(\rho_{f^+}(G))$

Figure 2.2: Minor Q_3^- obtained from a partial cube G.

We will say that a class of partial cubes $\mathscr{C}$ is *pc-minor-closed* if we have that $G \in \mathscr{C}$ and G' is a minor of G imply that $G' \in \mathscr{C}$. A major part of this thesis is devoted to the study of classes of partial cubes that are pc-minor-closed. Clearly, any such class has a (possibly infinite) set X of partial cubes that are not in the class, but every minor of them is. These are the forbidden pc-minors. Vice versa, if $X = \{T_1, T_2, \ldots\}$ is a set of partial cubes, then let $\mathscr{F}(X)$ be the set of all partial cubes G such that *no* $T_i, i = 1, 2, \ldots$, can be obtained as a pc-minor of G. Clearly, such a class is pc-minor-closed. If the list of excluded minors is finite, the following gives the algorithmic recognition of the class.

Proposition 2.4.1 (**[61]**). *Let X be a finite set of partial cubes. It is decidable in polynomial time if a partial cube G is in $\mathscr{F}(X)$.*

In comparison to the "usual" minor relation in graphs, where we have a similar theorem but the constants in the polynomial time complexity are extremely big and algorithms practically not-implementable, the algorithm here is not too complicated. Moreover, for some fixed X corresponding to an established pc-minor closed family (such as median graphs,...) faster algorithms (in comparison to the general one) have been developed. For an infinite X, the hardness of the recognition can be either polynomial or not, as we will see in Chapter 4. Also notice that, by definition of pc-minors, each class of minor closed partial cubes is also pc-minor closed.

We close the section with some simple examples of minor closed families of partial cubes, all of them having only one forbidden minor. Having in mind that every partial cube can be contracted to a vertex (K_1), and every partial cube with an edge can be restricted to an edge (K_2), the first non trivial families are the following (C_n denotes the cycle on n vertices and P_n denotes the path on n vertices):

- $\mathscr{F}(C_4)$ corresponds to the class of trees. While no tree can be contracted and restricted to C_4, if G has a cycle then contracting all the Θ-classes besides two classes crossing the chosen cycle gives a minor C_4.

- $\mathscr{F}(P_3)$ corresponds to precisely all the hypercubes. This follows from many classical results on hypercubes one of them being the characterization that hypercubes are precisely the partial cubes with any two incident edges being on a 4-cycle, see [47].
- $\mathscr{F}(K_2 \square P_3)$ corresponds to all the partial cubes whose blocks are isomorphic to even cycles. This is proved similarly as previous cases.
- The first case generalizes to $\mathscr{F}(C_n)$ corresponding to all the partial cubes having all its convex cycles of length less than n, as it will follow from results in the following chapters.

In Chapter 4 we will deal with less trivial families, which were originally motivated by some other property or structure.

2.5 Subgraphs of partial cubes

Antipodality

For a graph G we say that G is *even* if for every vertex v of G there exists a unique vertex $\bar{v}$ such that $d(v, \bar{v}) = \mathrm{diam}(G)$, where $\mathrm{diam}(G)$ denotes the diameter of G. Such graphs have been studied in [42, 70], for basic examples consider even cycles or hypercubes. Additionally, G is *harmonic-even* if for every adjacent $u, v \in V(G)$ also their diametrical vertices $\bar{u}, \bar{v}$ are adjacent. G is *symmetric-even* if for every $u, v \in V(G)$ holds $d(u, v) + d(v, \bar{u}) = \mathrm{diam}(G)$. In the case G is a partial cube, it is embedded in a hypercube, thus we can talk of vertices at the maximal distance in another way:

Definition 2.5.1. *A partial cube G embedded in a hypercube Q_n is antipodal if for every $v \in G$ embedded as $v = (i_1, i_2, \ldots, i_n)$ there exists $-v \in V(G)$, called the antipode of v, embedded as $-v = (\bar{i_1}, \bar{i_2}, \ldots, \bar{i_n})$, where $\bar{i_j} = 1 - i_j$.*

It follows directly from the definitions that we have implications: G is antipodal $\Rightarrow$ G is symmetric-even $\Rightarrow$ G is harmonic-even $\Rightarrow$ G is even. In [41], Fukuda and Handa proved that if G is a partial cube, then the first two implications are in fact equivalence. The question whether every even partial cube is antipodal is open, see [54].

We call a convex subgraph H of a partial cube G *antipodal* if every vertex x of H has an antipode with respect to H, i.e. H is an antipodal partial cube. In Figure 2.3 is an antipodal partial cube with all its antipodal subgraphs being convex cycles. Their behavior with respect to pc-minors has been described in **[22]** in the following way:

Lemma 2.5.2 (**[22]**). *Let H be an antipodal subgraph of G and $f \in \mathscr{E}$. If E_f is disjoint from H, then $\rho_{f^+}(H)$ is an antipodal subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H, then $\pi_f(H)$ is an antipodal subgraph of $\pi_f(G)$.*

In particular, Lemma 2.5.2 implies that the class of antipodal partial cubes is closed under contractions. Next, we present a characterization of those expansions that generate all antipodal partial cubes from a single vertex, in the same way as Lemma 2.3.3 characterizes

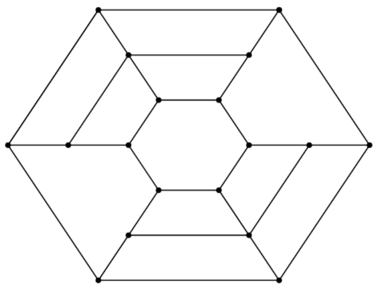


Figure 2.3: An antipodal partial cube.

all partial cubes. Let G be an antipodal partial cube and G_1, G_2 two subgraphs corresponding to an isometric expansion. We say that it is an *antipodal expansion* if and only if $-G_1 = G_2$, where $-G_1$ is defined as the set of antipodes of G_1.

Proposition 2.5.3 ([61]). *Let G be a partial cube and $\pi_e(G)$ antipodal. Then G is an antipodal expansion of $\pi_e(G)$ if and only if G is antipodal. In particular, all antipodal partial cubes arise from a single vertex by a sequence of antipodal expansion.*

Interesting property of antipodal graphs are present is the following.

Lemma 2.5.4 ([22]). *If H is an antipodal subgraph of G, then H contains an isometric cycle C such that $conv(C) = H$.*

Lemma 2.5.5 ([61]). *In an antipodal partial cube G, the* antipodal mapping $v \mapsto -v$ *is a graph automorphism.*

We call a partial cube *affine* if it is a halfspace of an antipodal partial cube. An interesting question is the following: given a partial cube G, is there a simple condition that determines if G is affine, or in other words, does connecting G with $-G$ give a partial cube with both graphs being complementray halfspaces. This question will be particularly interesting in connection with oriented matroids in Section 4.1.

The answer was given [61] by the following intrinsic characterization of affine partial cubes.

Proposition 2.5.6 ([61]). *A partial cube G is affine if and only if for all u, v vertices of G there are $w, -w$ in G such that $conv(u, w)$ and $conv(v, -w)$ are crossed by disjoint sets of Θ-classes.*

By Lemma 4.1.2 a contraction of a halfspace is a halfspace and by Lemma 2.5.2 antipodal partial cubes are closed under contraction, therefore we immediately get:

Lemma 2.5.7. *The class of affine partial cubes is closed under contraction.*

Gated subgraphs

A subgraph H of G, or just a set of vertices of H, is called *gated* (in G) if for every vertex x outside H there exists a vertex x' in H, the *gate* of x, such that each vertex y of H is

Figure 2.4: Gated versus non-gated convex subgraph.

connected with x by a shortest path passing through the gate x' (see Figure 2.4). It is easy to see that if x has a gate in H, then it is unique and that gated subgraphs are convex [45].

In **[22]** we showed that gated subgraphs behave well with respect to pc-minors:

Lemma 2.5.8 ([**22**]). *If H is a gated subgraph of G, then $\rho_{f^+}(H)$ and $\pi_f(H)$ are gated subgraphs of $\rho_{f^+}(G)$ and $\pi_f(G)$, respectively.*

On the other hand, expansions can often cause that gated subgraphs become non-gated.

Lemma 2.5.9 ([**22**]). *Let G be an expansion of $\pi_e(G)$ along sets G_1, G_2. Let H be a subgraph of $\pi_e(G)$, v a vertex of $\pi_e(G)$ and v' the gate of v in H. If $v \in G_1 \cap G_2$, $v' \notin G_1 \cap G_2$ and there exist $v'' \in H$, $v'' \in G_1 \cap G_2$, then the expansion of H in G is not gated.*

The above two lemmas have implications in the following chapters.

3

Cycles in partial cubes

This chapter is mainly based on [65].

3.1 Traverses and cell complexes

Partial cubes inherit metric properties from hypercubes. One of the areas where this is strongly reflected is in the cycle space of partial cubes. Particularly interesting are isometric and convex cycles. Note that in a (general) graph G, every cycle can be obtained as a symmetric difference of isometric cycles. In fact this can be proven by induction on the length of the cycles. Take an arbitrary cycle in G; if it is not an isometric subcycle then there exist a diagonal path connecting two vertices on the cycle of shorter length than the two paths on the cycle. Then the cycle is a symmetric difference of two shorter cycles formed by the diagonal and the two paths on the cycle. By induction, the two cycles can be obtained as a symmetric difference of isometric cycles. We will show that in partial cubes even more holds. We introduced the following definition in **[65]**.

Definition 3.1.1 (**[65]**). *Let $v_1u_1\Theta v_2u_2$ in a partial cube G, with $v_2 \in W_{v_1u_1}$. Let $C^1, \dots, C^n$, $n \geq 1$, be a sequence of isometric cycles such that v_1u_1 lies only on C^1, v_2u_2 lies only on C^n, and each pair C^i and C^{i+1}, for $i \in \{1, \dots, n-1\}$, intersects in exactly one edge and this edge is in $E_{v_1u_1}$, all the other pairs do not intersect. If the shortest path from v_1 to v_2 on the union of $C^1, \dots, C^n$ is a shortest v_1, v_2-path in G, then we call $C^1, \dots, C^n$ a* traverse *from v_1u_1 to v_2u_2.*

Every isometric cycle in a partial cube has its antipodal edges in relation Θ. Using this fact, we see that if $C^1, \dots, C^n$ is a traverse from v_1u_1 to v_2u_2, then also the shortest path from u_1 to u_2 on the union of $C^1, \dots, C^n$ is isometric in G, since it must have all its edges in different Θ-classes. Moreover, notice that the whole traverse is an isometric subgraph. We will call shortest u_1, u_2-path on the traverse the *u_1, u_2-side of the traverse* and, similarly, the shortest v_1, v_2-path the *v_1, v_2-side of the traverse*. The length of these two shortest paths is the *length of the traverse*. If all isometric cycles on a traverse T are convex cycles, we will call T a *convex traverse*.

Lemma 3.1.2 (**[65]**). *Let $v_1u_1\Theta v_2u_2$ in a partial cube G. Then there exists a convex traverse from v_1u_1 to v_2u_2.*

The above lemma is, surprisingly, not too hard to prove, but is a very useful tool. We shall demonstrate its strength in the following. Let $\mathscr{C}(G)$ denote the set of all convex cycles of G and let $\mathbf{C}(G)$ be the 2-dimensional cell complex whose 2-cells are obtained by replacing each convex cycle C of length $2j$ of G by a regular Euclidean polygon $[C]$ with $2j$ sides. Recall that a cell complex $\mathbf{X}$ is *simply connected* if it is connected and if every continuous map of the 1-dimensional sphere S^1 into $\mathbf{X}$ can be extended to a continuous mapping of the disk D^2 with boundary S^1 into $\mathbf{X}$. A direct consequence of Lemma 3.1.2 is the next proposition also proved in a different way in **[22]**.

Proposition 3.1.3. *If G is a partial cube, then $\mathbf{C}(G)$ is simply connected.*

Proof. We will prove that every cycle C is a boundary of an image of continuous map of a disk D^2 in $\mathbf{C}(G)$ by induction on the length of C. Let C be a cycle of G and take an arbitrary edge uv on C. Since uv corresponds to a coordinate change, there has to be another edge $u'v'$ on C that corresponds to a change of the same coordinate. Then uv and $u'v'$ are in relation Θ. By 3.1.2, a convex traverse T between uv and $u'v'$ exists. Let C' be the closed path that consists of the u,u'-side of T and the u,u'-path on C not crossing uv or $u'v'$. Similarly let C'' be the closed path that consists of the v,v'-side of T and the v,v'-path on C not crossing uv or $u'v'$. Then C' and C'' are a union of cycles of smaller length than C since already C' and C'' have smaller length than C. By induction, C', C'' and all the convex cycles on T are boundaries of an image of continuous map of a disk D^2 in $\mathbf{C}(G)$. By construction of C', C'' and T also C is a boundary of an image of continuous map of a disk D^2 in $\mathbf{C}(G)$. □

Notice that a weaker result stating that convex cycles in a partial cube form a basis was proved in [49].

3.2 Enumerative properties of convex cycles and zone graphs

In [58], Klavžar and Shpectorov studied the density of convex cycles in partial cubes by introducing a concept called *convex excess* of a graph. Let G be a partial cube and $\mathscr{C}(G)$ the set of convex cycles in G. The convex excess $ce(G)$ of G is defined as

$$ce(G) = \sum_{C \in \mathscr{C}(G)} \frac{|V(C)| - 4}{2}.$$

Notice that partial cubes cannot have cycles of length less than four, thus the convex excess counts how many (and how much longer) the other convex cycles are.

They proved that the convex excess is closely related with the following definition that they provided.

Definition 3.2.1. *Let G be a partial cube and $E_f \in \mathscr{E}$ one of its Θ-classes. Define the* zone graph *of G with respect to E_f as the graph $\zeta_f(G)$ whose vertices correspond to the edges of E_f and two vertices are connected by an edge if the corresponding edges of E_f lie in a convex cycle of G.*

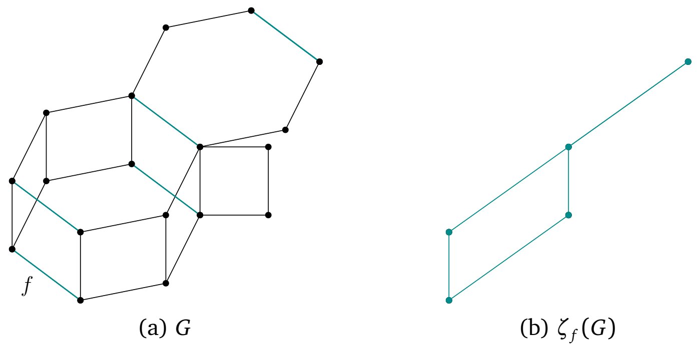


Figure 3.1: A partial cube and its zone graph.

See Figure 3.1 for an example. The zone graphs are defined for partial cubes, but a zone graph itself does not need to be a partial cube. This topic will be thoroughly investigated in Section 4.1, for examples of a partial cube whose not all zone graphs are partial cubes see Figure 3.2. If G is a partial cube with all its zone graphs being trees, then G is called a *tree-zone* graph.

The main result in respect to the convex excess is a Euler type formula connecting the number of vertices, edges, isometric dimension $i(G)$ and convex excess.

Theorem 3.2.2 ([58])**.** *For a partial cube G with n vertices and m edges,*

$$2n - m - i(G) - ce(G) \leq 2.$$

Moreover the equality holds if and only if G is a tree-zone graph.

Observe that ζ_f can be seen as a mapping from edges of G that are not in E_f but lie on a convex cycle crossed by E_f to the edges of $\zeta_f(G)$. If $\zeta_f(G)$ is a partial cube, then we say that $\zeta_f(G)$ is *well-embedded* if for two edges a, b of $\zeta_f(G)$ we have $a\Theta b$ if and only if the sets of Θ-classes crossing $\zeta_f^{-1}(a)$ and $\zeta_f^{-1}(b)$ coincide and if a and b are not in relation Θ then the classes crossing are disjoint.

To understand the definition of well-embeddedness better we state the following lemma.

Lemma 3.2.3 ([61])**.** *Let G be a partial cube and $f \in \mathscr{E}$. Then $\zeta_f(G)$ is a well-embedded partial cube if and only if for any two convex cycles C, C' that are crossed by E_f and some E_g both C and C' are crossed by the same set of Θ-classes.*

Notice an important connection. In partial cubes whose zone graphs are well embedded convex traverses correspond to the shortest paths in zone graphs. It turns out that well-embedded zone graphs work well with contractions and restrictions.

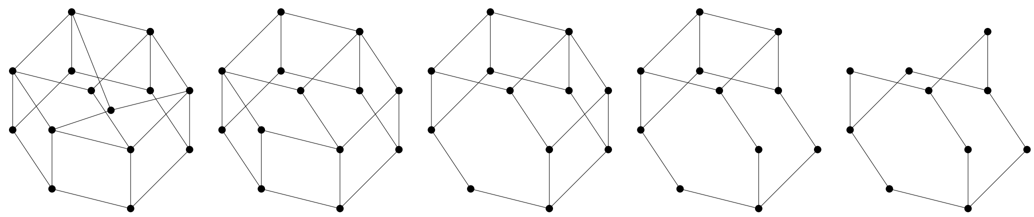
Lemma 3.2.4 ([61])**.** *Let G be a partial cube, $E_f \in \mathscr{E}$ such that $\zeta_f(G)$ is a well-embedded partial cube. Then the zone graph with respect to E_f of a restriction of G is a restriction of $\zeta_f(G)$. Moreover the zone graph with respect to E_f of a contraction of G is a (possibly trivial) contraction of $\zeta_f(G)$.*

A direct consequence is the following.

Lemma 3.2.5. *The family of partial cubes whose all zone graphs are well-embedded partial cubes is a pc-minor closed family.*

Proof. If G has all its zone graphs partial cubes, then all the zone graphs of a restriction or a contraction of G are restrictions or contractions of zone graphs of G, thus partial cubes. □

For a pc-minor closed family we can determine its list of excluded minors. Let $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ be the set of graphs in Figure 3.2 (the construction and a generalization of this graphs will be explained in Section 4.1).



(a) Q_4^{-*} (b) $Q_4^{--}(1)$ (c) $Q_4^{--}(2)$ (d) $Q_4^{--}(3)$ (e) $Q_4^{--}(4)$

Figure 3.2: Graphs $Q_4^{-*}, Q_4^{--}(m)$, for $1 \leq m \leq 4$.

Theorem 3.2.6. *The family of partial cubes whose all zone graphs are well-embedded partial cubes equals* $\mathscr{F}(\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\})$.

Proof. The left inclusion of the families follows from Lemma 3.2.5 and the easy verifiable fact that the graphs in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ have zone graphs that are not partial cubes. The right inclusion was proved in **[61, Lemma 5.4]**. □

3.3 Intertwining

In the previous two sections we have seen that cycles in partial cubes behave nicely. Nevertheless, in certain cases an unexpected behavior emerges, in particular in the following we will present some results when isometric cycles intersect pairwise in more than a vertex or an edge. To analyze such situations we gave the following definition in **[65]**.

Definition 3.3.1. *Let* $C^1 = (v_0 v_1 \dots v_m v_{m+1} \dots v_{2m+2n_1-1})$ *and* $C^2 = (u_0 u_1 \dots u_m u_{m+1} \dots u_{2m+2n_2-1})$ *be isometric cycles with* $u_0 = v_0, \dots, u_m = v_m$ *for* $m \geq 2$*, and all the other vertices pairwise different. Then we say that* C^1 *and* C^2 intertwine.

Notice that in a partial cube, m can be at most half of l_1 or l_2, where l_1 is the length of C^1, l_2 the length of C^2. Let us prove this: If $m > l_1/2$, then the fact that antipodal edges in an isometric cycle are in relation Θ implies that C^1 is determined by the intersection. Moreover, the path in the intersection is not isometric, thus it must cover more than half of C^2, i.e. $m > l_2/2$. Thus also C^2 is determined by the intersection, and consequently we have $C^1 = C^2$.

Even though intertwining is a very particular interaction of two cycles we proved the following.

Lemma 3.3.2 ([**65**]). *Let G be a partial cube and let two isometric cycles intersect in at least two non-adjacent vertices. Then there exist two isometric cycles that intertwine.*

In the case of convex cycles it holds even more.

Lemma 3.3.3 ([**66**]). *If two convex cycles intersect in at least two non-adjacent vertices, then they intertwine.*

The latter two lemmas can be very useful for the study of partial cubes since we can separate partial cubes in two disjoint subfamilies: partial cubes in which any two isometric (or convex) cycles intersect in at most a vertex or an edge, and partial cubes in which we have isometric (or convex) cycles that intertwine.

If two convex cycles intertwine and f is an edge shared by the two cycles, then by Lemma 3.2.3 the zone graph $\zeta_f(G)$ is not well-embedded. We have the following consequence.

Corollary 3.3.4. *If two convex cycles in a partial cube G intertwine, then G has a pc-minor in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$.*

3.4 Partial cubes with girth greater than 4

In this section we present our main results from [**65**], where the intertwining was introduced for the study of partial cubes with high girth, i.e. length of the shortest cycle. We will write $g(G)$ for the girth of G. In the above paper graph $Q_4^{--}(4)$ was defined as the graph from Figure 3.3. Observe how convex cycles in this partial cube intertwine and notice that we have already seen this graph as the forbidden minor for partial cubes whose zone graphs are well-embedded partial cubes.

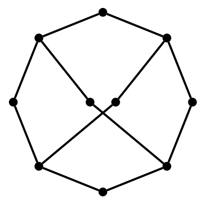


Figure 3.3: Graph $Q_4^{--}(4)$

Proposition 3.4.1 ([**65**]). *If G is a partial cube with $g(G) > 6$, then every pair of isometric cycles in G meets in either exactly one edge, or exactly one vertex, or not at all. Moreover, the same holds if $g(G) = 6$, provided that G contains no isometric subgraph isomorphic to $Q_4^{--}(4)$.*

In fact, if the girth of G is 6 and G contains an isometric subgraph $Q_4^{--}(4)$, the subgraph X is convex. The latter holds since $Q_4^{--}(4)$ can be seen as an isometric subgraph of Q_4, thus if there exist any shortest path in G connecting two verices of $Q_4^{--}(4)$ but completely outside of $Q_4^{--}(4)$, then it lies in the Q_4 containing $Q_4^{--}(4)$. But adding any vertex of Q_4 to $Q_4^{--}(4)$ creates a 4-cycle, by an easy exercise. This proves that the isometric subgraph isomorphic to $Q_4^{--}(4)$ in the last assertion of the proposition can be replaced with a convex subgraph isomorphic to $Q_4^{--}(4)$. Also, recall that by Theorem 3.2.6 all the graphs that have well-embedded zone graphs have no convex $Q_4^{--}(4)$.

The proof of the proposition turns out to be quite technical, while the result implies the opposite: partial cubes with high girth have a convex space that is relatively simple, i.e., with no intertwining. In fact the proposition was used in [11] to show that such graphs are zonotopal complexes of oriented matroids, that we will introduce in Section 4.1. Another way to see the simple structure of partial cubes with high girth is the following corollary of Proposition 3.4.1.

Corollary 3.4.2 ([**65**]). *Every partial cube G with $g(G) > 6$ is a tree-zone graph and hence it holds $2n(G) - m(G) - i(G) - ce(G) = 2$.*

The following results are all motivated and use the same ideas as used in [**65**], but we reprove them since we make small modifications to obtain stronger result that will be used at the end of this section.

Lemma 3.4.3. *Let $P = u_0u_1 \dots u_m$ be a shortest path in a partial cube. If there is some other shortest u_0, u_m-path, then there exists a convex cycle of the form $(u_iu_{i+1} \dots u_jw_{j-1}w_{j-2} \dots w_{j-1})$ for some $0 \le i < j \le m$ and $j - i - 1$ vertices $w_{i+1}, \dots, w_{j-1}$ not on P.*

Proof. Assume this is not the case and let $P = u_0u_1 \dots u_m$ and P' be two different u_0, u_m-geodesics for which the lemma does not apply. Without loss of generality assume the length of P is minimal among all counterexamples of the lemma.

By the minimality assumption, paths P and P' intersect only in u_0 and u_m. Denote the vertices of P' with $u_0z_1z_2 \dots z_{m-1}u_m$ and let C be the cycle formed by P and P'.

Beside u_0z_1 itself, there must be an additional edge on C', that is in relation Θ with u_0z_1. Since P' is a shortest path, this edge is on P. Let $u_{k-1}u_k \in F_{u_0z_1}$, for some $0 < k \le m$. By Lemma 3.1.2, there is a convex traverse from u_0z_1 to $u_{k-1}u_k$. Firstly, assume the path $P'' = u_0u_1 \dots u_{k-1}$ is the u_0, u_{k-1}-side of it. Then the last convex cycle on this traverse is of the form $(u_{k'}u_{k'+1} \dots u_{k-1}u_kw_{k-1} \dots w_{k'+1})$ for some $0 \le k' \le k-2$ and some vertices $w_{k'+1}, \dots, w_{k-1}$ not on P (they do not lie on P since the cycle is isometric). We have found the desired cycle.

On the other hand, assume P'' is not the u_0, u_{k-1}-side of a traverse from u_0z_1 to $u_{k-1}u_k$. The geodesic P'' is shorter than P, and there exists another shortest u_0, u_{k-1}-path, namely the u_0, u_{k-1}-side of the traverse from u_0z_1 to $u_{k-1}u_k$. Then the lemma applies to P'', and since P'' is a subpath of P, the obtained convex cycle is of the form $(u_iu_{i+1} \dots u_jw_{j-1}w_{j-2} \dots w_{i+1})$ for some $0 \le i < j-1 \le m-1$ and some vertices $w_{i+1}, \dots, w_{j-1}$ not on P. □

Lemma 3.4.4. *Let $g(G) > 6$ for a partial cube G. If $u_1v_1\Theta u_2v_2$ with $u_2 \in W_{u_1v_1}$, P_1 being a shortest u_1u_2-path, and P_2 being a shortest v_1v_2-path, then P_1 and P_2 are the sides of the unique (thus convex) traverse from u_1v_1 to u_2v_2. The same holds in G if $g(G) = 6$ and there is no isometric subgraph of G isomorphic to $Q_4^{--}(4)$.*

Proof. Let P_1 be a shortest u_1u_2-path, and let R_1 be the u_1, u_2-sides of some convex traverse T from u_1v_1 to u_2v_2, provided by Lemma 3.1.2. For the sake of contradiction, assume that $R_1 \neq P_1$. By Lemma 3.4.3, there exists a convex cycle $C = (z_k \dots z_{k+l}w_{k+l-1} \dots, w_{k+1})$, where

$z_k, \ldots, z_{k+l}$ are vertices on R_1 and $w_{k+l-1}, \ldots, w_{k+1}$ are some other vertices. If $g(G) \geq 6$, there are two convex cycles, namely C and one of the convex cycles on the traverse from u_1u_2 to v_1v_2, that have at least two edges in common. This is a contradiction with Proposition 3.4.1.

We have proved that R_1 is the only shortest u_1u_2-path, and, similarly, the v_1, v_2-side of T is the only shortest v_1v_2-path. Since it is impossible that two traverses have the same sides, this also proves the uniqueness of the (convex) traverse. □

We are now ready for our main result.

Theorem 3.4.5. *There is no finite partial cube G with $\delta(G) \geq 3$, and $g(G) > 6$ or $g(G) = 6$ and there is no isometric subgraph of G isomorphic to $Q_4^{--}(4)$.*

Proof. Let G be such that $\delta(G) \geq 3$ and $g(G) \geq 6$. We will inductively build an infinite isometric path P in G.

Assume we have built an isometric path P_n of length $n-1$. Notice that this is the same as saying that all the Θ-classes of P_n are in pairwise different Θ-classes. Let u be the last vertex on it. Since $\delta(G) \geq 3$, u has two incident edges, say uu_1, uu_2 that are not on P_n. If one of the E_{uu_1}, E_{uu_2} does not intersect P_n, then we can extend P_n to P_{n+1} with an edge, such that all the edges on P_{n+1} are in pairwise different Θ-classes implying that P_{n+1} is a shortest path.

Hence assume that uu_1 is in relation Θ with an edge a_1b_1 on P_n and uu_2 is in relation Θ with an edge a_2b_2 on P_n. Let T_1, T_2 be convex traverses from uu_1 to a_1b_1 and from u_2 to a_2b_2, respectively. By Lemma 3.4.4, P_n is a side of T_1 as well as T_2. But if $g(G) \geq 6$, then the two first convex cycles on T_1 and T_2, the ones incident with u, share at least two edges. By Proposition 3.4.1, this cannot be. Therefore we can always extend P_n. □

In **[65]**, we proved a variant of Theorem 3.4.5. It is a stronger statement for graphs with $g(G) > 6$, but cannot be generalized to graphs with $g(G) = 6$. Just for this part we assume the graph can be infinite. First a definition is needed: If $d \in \mathbb{N}$, let $B_d(v)$ be the number of vertices at distance at most d from a vertex v of an infinite graph G. If $B_d(v)$ is bounded from below by some exponential function in d, we say that G has *exponential growth*. The definition is independent of the choice of the vertex in G.

Theorem 3.4.6 (**[65]**)**.** *Every partial cube G with $g(G) > 6$ and $\delta(G) \geq 3$ contains an infinite subtree in which vertices have degree 3 or 2. Moreover, any two vertices of degree 2 have distance at least 2. In particular, G is infinite with exponential growth.*

The property from the theorems reminds of the property trees have: every finite tree has vertices of degree 1. In the case of finite partial cubes with $g(G) > 6$ or with $g(G) = 6$ and no isometric subgraph isomorphic to $Q_4^{--}(4)$, they must have vertices of degree 2. The latter has a strong impact on the topic of regular partial cubes, a topic we will consider in more details in Section 5.1.

Corollary 3.4.7. *Let G be a finite regular partial cube with $g(G) > 6$. Then G is K_1, K_2 or an even cycle. The same holds if $g(G) = 6$ and there is no isometric subgraph isomorphic to $Q_4^{--}(4)$.*

To see that the conditions in Theorems 3.4.5, 3.4.6 and Corollary 3.4.7 cannot be weakened, consider the following examples. Recall that the *middle level* graph M_{2n+1}, for $n \geq 1$, is the subgraph of Q_{2n+1} induced on the vertices $(i_1, \ldots, i_{2n+1})$, such that there are exactly n or $n+1$ coordinates equal to 1. In particular, M_3 is the cycle of length 6, while M_5 is known as the Desargues graph (see Figure 5.1 graph $G(10,3)$). Middle level graphs are the only distance-regular partial cubes with girth 6 [86], and they show that the bound $g(G) > 6$ is tight. Notice that in the case $n \geq 2$ these graphs have many isometric subgraphs isomorphic to $Q_4^{--}(4)$.

On the other hand, one could consider the existence of infinite partial cubes with $\delta(G) \geq 3$, $g(G) = 6$, and no convex subgraphs isomorphic to $Q_4^{--}(4)$ but with sub-exponential growth. We know of one such example: an infinite hexagonal net. Simple examination shows that it in fact has polynomial growth.

Another interesting corollary of Theorem 3.4.5 is the following. Consider the famous Erdős–Gyárfás conjecture [37]. The conjecture states that every finite graph G with $\delta(G) \geq 3$ has a cycle of length a power of two. If a partial cube G has $\delta(G) \geq 3$, then it must have girth 4 or 6. In the first case 4 is a power of 2, while in the second case G must include a subgraph isomorphic to $Q_4^{--}(4)$ which has a cycle of length 8. Hence the following holds:

Theorem 3.4.8. *Every partial cube G with $\delta(G) \geq 3$ has a cycle of length 4 or 8. In particular, the Erdős–Gyárfás conjecture holds in partial cubes.*

Another motivation for the study of partial cubes with high minimum degree comes from the theory of oriented matroids that we will introduce in Section 4.1. Every oriented matroid is characterized by its tope graph, formed by its maximal covectors [14]. As we will see, tope graphs are partial cubes and it follows from basic properties of oriented matroids that the minimum degree of a tope graph is at least the rank of the oriented matroid it describes. Since tope graphs of oriented matroids with rank at most 2 are characterized as even cycles, there is a special interest in graphs with high minimum degree.

Theorem 3.4.5 can be seen in another interesting way concerning tope graphs. A famous Sylvester–Gallai theorem states that given a finite number of points in the Euclidean plane, either all the points lie on a single line; or there is a line which contains exactly two of the points. By point-line duality this is equivalent to saying that for any set of lines not all intersecting in a point there exists a point that is crossed by exactly two lines. Now in this dual setting form a graph whose vertices are regions (polygons) between the lines and two regions adjacent if they are separated by exactly one line. Then Sylvester–Gallai theorem states that such a graph has a square. In fact, as we will see in Section 4.1, such a graph is always an affine partial cube without $Q_4^{--}(4)$ isometric subgraphs, half of a 3-connected antipodal partial cube. This is implied also by Theorem 3.4.6, thus it can be seen as a generalization of Sylvester–Gallai theorem to all 3-connected partial cubes (or with $\delta(G) \geq 3$), where the result is not that they must have a square, but rather a square or an isometric $Q_4^{--}(4)$.

4

Families of partial cubes

In this chapter we present various families of partial cubes with the intention to make a map of partial cubes. The families can be seen in two ways. On one hand, the main reason for the popularity of partial cubes is the fact that they present a generalization of many families they contain and that naturally emerge in the study of seemingly non-connected areas of mathematics, chemistry, theory of social choices, genetics and others [6, 34, 35, 58, 60]. Each family is fascinating on its own with particular properties and interests connected to the area of research that was defined in. Nevertheless, many of the properties that were originally proved for a particular family in fact generalize to the whole class of partial cubes.

On the other hand, subfamilies can be seen as hierarchical classes giving an insight in the general structure of partial cubes. In this view subfamilies are just partial cubes with additional properties. The classes we will present are the classes that we find crucial in the understanding of the structure of partial cubes and are also the classes that were of research interest in the recent years. Their hierarchy can be seen in Figure 4.1 and will be in details explained in this chapter. In particular, we will not only settle the inclusions of the families but also determine the positions of intersections of many families by giving examples of such graphs in Figure 4.1 or proving that the intersections are empty. Figure 4.1 gives a simple layout of the subfamilies, but for the price of making them look trivial and completely understood, which is far from the truth.

Even though not all the subfamilies presented here are pc-minor-closed, the pc-minors are the main tools to understand the difference and similarities of the families.

4.1 Tope graphs of (complexes of) oriented matroids

We will start with the most general family of partial cubes studied here called topes of complexes of oriented matroids. This is in contrast with the historical development since this area was in fact only recently introduced [11] and studied **[61]** while the first studied class (besides tress or hypercubes) is the family of median graphs [2, 72, 70, 69]. The reason for choosing this approach is that it gives us a possibility of reviewing old results in a new way and placing long known families in the context of new research.

The topic of complexes of oriented matroids needs an introduction. We will closely follow [61] with additional explanations.

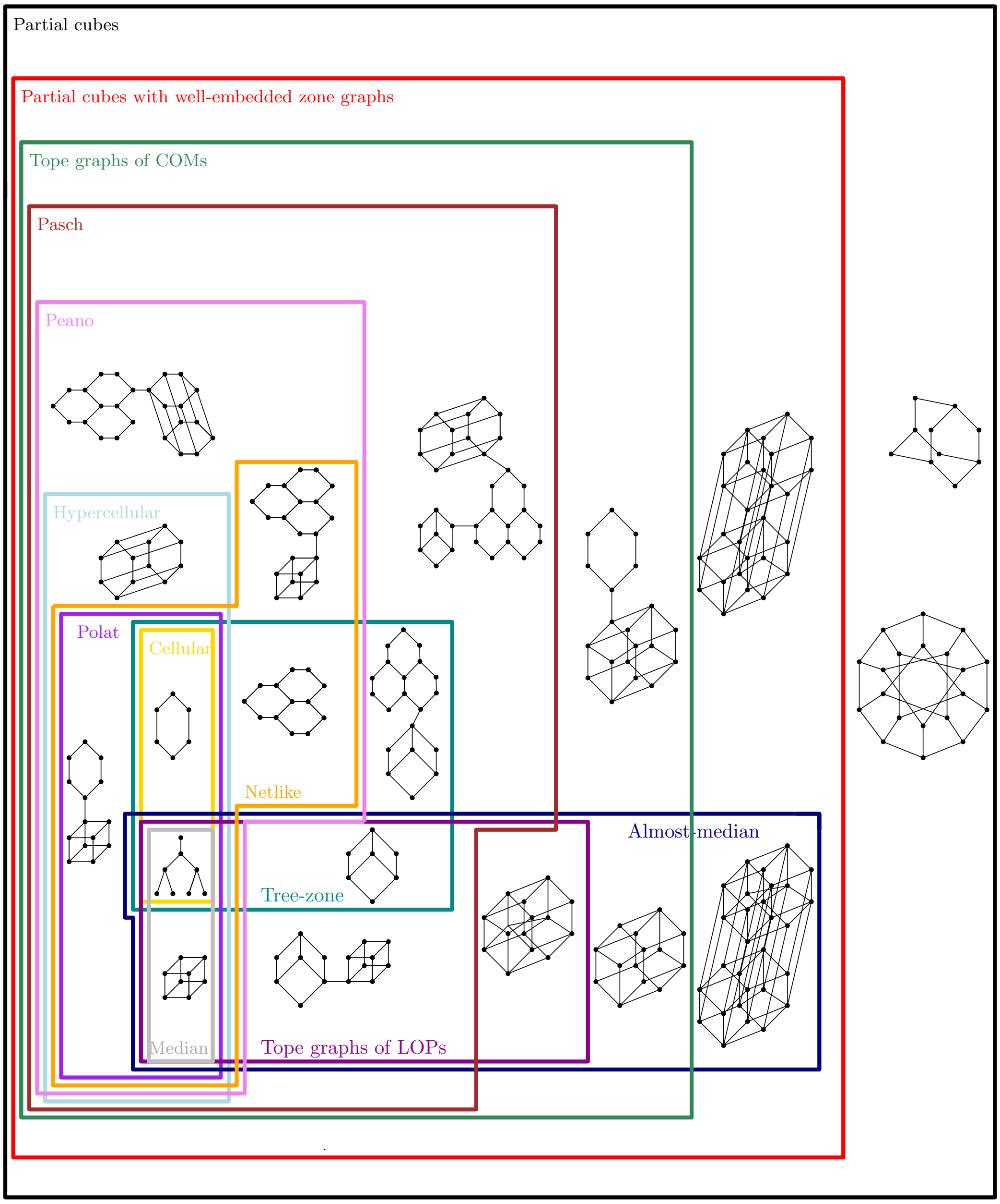


Figure 4.1: A map of families of partial cubes. Each combination of inclusion and exclusion in the families either has an example of such a graph or is empty.

Systems of sign-vectors

We will follow the standard oriented matroid notation from [14] and concerning complexes of oriented matroids we stick to [11]. The topic of systems of sign-vectors is strongly connected with geometry, it can be seen as a combinatorial approach of understanding geometrical objects. Since the geometrical motivation for various concepts regarding systems of sign-vectors is crucial, we will equip the formal definitions of the concepts with some basic translations into the geometrical language.

Let $\mathcal{E}$ be a non-empty finite (ground) set and let $\emptyset \neq \mathcal{L} \subseteq \{\pm, 0\}^{\mathcal{E}}$. The elements of $\mathcal{L}$ are referred to as *covectors*. For the geometrical motivation consider $\mathcal{H} = \{H_1, \ldots, H_n\}$ to be a central arrangement of hyperplanes in an Euclidean space $\mathbb{R}^d$, i.e., hyperplanes in $\mathbb{R}^d$, each including the origin. Chose $v_1, \ldots, v_n$ to be vectors in $\mathbb{R}^d$ such that H_i is precisely the space of vectors orthogonal to v_i, for each $1 \leq i \leq n$. Each vector $v \in \mathbb{R}^d$ gives an element $X_v \in \{\pm, 0\}^n$ whose i-th coordinate is the sign of the scalar product $v \cdot v_i$. Thus the i-th coordinate of X_v tells on which side of the hyperplane H_i vector v is (being 0 if it lies on the hyperplane). The set $\mathcal{L}_{\mathcal{H}}$ of covectors of the hyperplane arrangement is the set of all possible elements of $\{\pm, 0\}^n$ obtained in this way. Note that the elements of $\mathcal{L}_{\mathcal{H}}$ can be regarded as disjoint regions whose union is $\mathbb{R}^d$.

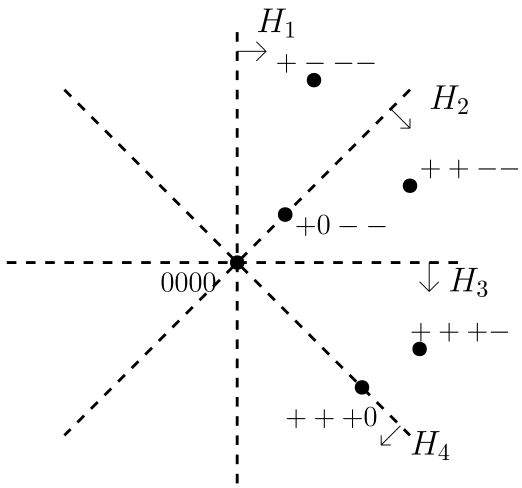


Figure 4.2: Central hyperplane arrangement with some covectors

Back to the general setting, for $X \in \mathcal{L}$, and $e \in \mathcal{E}$, let X_e be the value of X at the coordinate e. The subset $\underline{X} = \{e \in \mathcal{E} : X_e \neq 0\}$ is called the *support* of X and its complement $X^0 = \mathcal{E} \setminus \underline{X} = \{e \in \mathcal{E} : X_e = 0\}$ the *zero set* of X. For $X, Y \in \mathcal{L}$, we call $S(X, Y) = \{f \in \mathcal{E} : X_f Y_f = -\}$ the *separator* of X and Y. For a subset $A \subseteq \mathcal{E}$ and $X \in \mathcal{L}$ the *reorientation* of X with respect to A is the sign-vector defined by

$$({}_AX)_e := \begin{cases} -X_e & \text{if } e \in A \\ X_e & \text{otherwise.} \end{cases}$$

In particular $-X :=_{\mathcal{E}} X$. The *reorientation* of $\mathcal{L}$ with respect to A is defined as ${}_A\mathcal{L} := \{{}_AX \mid X \in \mathcal{L}\}$. In particular, $-\mathcal{L} :=_{\mathcal{E}} \mathcal{L}$. Regarding the geometrical example, these notions are easily explained: let v_x be a vector of $\mathbb{R}^d$ corresponding to the covector $X \in \mathcal{L}_{\mathcal{H}}$. The support of X is the subset of the set of hyperplanes (ground set) for which v_x does not lie on it. The zero set of X is the set of hyperplanes v_x does lie on. If $v_y \in \mathbb{R}^d$ is a vector corresponding to $Y \in \mathcal{L}_{\mathcal{H}}$, then the separator $S(X, Y)$ is the set of hyperplanes separating v_x and v_y. The reorientation of $\mathcal{L}_{\mathcal{H}}$ with respect to $A \subset \mathcal{H}$ corresponds to a different choice of vectors $v_1, \ldots, v_n$, in particular for each $H_i \in A$ we can choose $-v_i$ instead of v_i.

The *composition* of covectors X and Y is the sign-vector $X \circ Y$, where $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ if $X_e = 0$. Notice that this operation is associative, but not commutative. At the first glance a non-intuitive definition can be geometrically explained in the following way. Let $X, Y \in \mathcal{L}_{\mathcal{H}}$ be as in the above example and let v_x, v_y be their corresponding vectors in $\mathbb{R}^d$. The covector $X \circ Y$ corresponds to the vector $v_x + \epsilon v_y$ for some small $\epsilon > 0$. In fact, for each coordinate i with $X_i \neq 0$ vector v_x lies on some side of the hyperplane H_i, thus the same holds for $v_x + \epsilon v_y$ provided ϵ is small enough. On the other hand, if $X_i = 0$, v_x lies on the hyperplane H_i, thus the position relative to H_i of $v_x + \epsilon v_y$ depends only on the position of v_y. This proves that $X \circ Y \in \mathcal{L}_{\mathcal{H}}$ since it corresponds to the vector $v_x + \epsilon v_y$.

We continue with the formal definition of the main axioms. These axioms will be relevant for the definition of various families of sign-vectors. All of them are closed under reorientation.

Composition:

(C) $X \circ Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

Since $\circ$ is associative, arbitrary finite compositions can be written without bracketing $X_1 \circ \ldots \circ X_k$ so that (C) entails that they all belong to $\mathcal{L}$.

Note that contrary to a convention sometimes made in oriented matroids we do not consider compositions over an empty index set, since this would imply that the zero sign-vector belonged to $\mathcal{L}$. The same consideration applies for the following two strengthenings of (C).

Face symmetry:

(FS) $X \circ -Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

By (FS) we first get $X \circ -Y \in \mathcal{L}$ and then $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y) \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$. Thus, (FS) implies (C).

Ideal composition:

(IC) $X \circ Y \in \mathcal{L}$ for all $X \in \mathcal{L}$ and $Y \in \{\pm, 0\}^{\mathcal{E}}$.

Note that (IC) implies (C) and (FS). The following axiom is part of all the systems of sign-vectors discussed in the thesis and the main property that connects systems of sign-vectors with partial cubes. We give the geometrical intuition for it after the definition.

Strong elimination:

(SE) for each pair $X, Y \in \mathcal{L}$ and for each $e \in S(X, Y)$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in \mathcal{E} \setminus S(X, Y)$.

We give a few more axioms.

Symmetry:

(Sym) $-X \in \mathcal{L}$ for all $X \in \mathcal{L}$.

Zero vector:

(Z) the zero sign-vector **0** belongs to $\mathcal{L}$.

Axiom (Z) is a combinatorial equivalent to a geometrical demand that all hyperplanes in an arrangement cross the origin, while (Sym) gives a central symmetry also observed in an Euclidean space.

Finally we add:

$$(X \oplus Y)_e := \begin{cases} 0 & \text{if } e \in S(X,Y) \\ (X \circ Y)_e & \text{otherwise.} \end{cases}$$

Affinity:

(A) $(X \oplus -Y) \circ Z \in \mathcal{L}$ for all $X, Y, Z \in \mathcal{L}$ such that for each $e \in S(X,-Y)$ and $W \in \mathcal{L}$ with $W_e = 0$ there are $f, g \in \mathcal{E} \setminus S(X,-Y)$ such that $W_f \neq (X \circ -Y)_f$ and $W_g \neq (-X \circ Y)_g$.

We are now ready to define the central systems of sign-vectors of the present thesis:

Definition 4.1.1. *A system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is called a:*

- oriented matroid (OM) *if $\mathcal{L}$ satisfies (C), (Sym), and (SE) (or alternatively (SE), (Z) and (FS) [11]),*
- complex of oriented matroids (COM) *if $\mathcal{L}$ satisfies (FS) and (SE),*
- affine matroid (AOM) *if $\mathcal{L}$ satisfies (A), (FS), and (SE),*
- lopsided system (LOP) *if $\mathcal{L}$ satisfies (IC) and (SE).*

Now we explain the axioms with geometrical examples. The system of sign-vectors $(\mathcal{H}, \mathcal{L}_{\mathcal{H}})$ coming from a hyperplane arrangement as defined above is an example of an OM, thus satisfying axioms (C), (Sym), and (SE). The axiom (C) is satisfied for the reasons explained after the definition of composition. Moreover, the axiom (Sym) is satisfied since the arrangement is central: for $X \in \mathcal{L}_{\mathcal{H}}$ with corresponding $v_x \in \mathbb{R}^d$, $-X$ corresponds to $-v_x$, thus $-X \in \mathcal{L}_{\mathcal{H}}$. To see that the axiom (SE) holds in $\mathcal{L}_{\mathcal{H}}$ let again $X, Y \in \mathcal{L}_{\mathcal{H}}$ be covectors and let v_x, v_y be corresponding vectors in $\mathbb{R}^d$. Moreover let $i \in S(X,Y)$, i.e., vectors v_X and v_y are separated by the hyperplane H_i. Then point v_Z on the line connecting v_X and v_Y intersecting H_i gives a vector such that for its covector $Z \in \mathcal{L}_{\mathcal{H}}$ holds $Z_i = 0$ (since it is on the hyperplane H_i) and $Z_f = (X \circ Y)_f$ for all $f \in \mathcal{H} \setminus S(X,Y)$. Oriented matroids obtained from hyperplane arrangements are called *realizable* oriented matroids or sometimes also *zonotopal* oriented matroids.

As noted, axioms for an OM can be replaced by (SE), (Z) and (FS). Thus OMs are a subfamily of COMs and the above example of the hyperplane arrangement is also an example of a COM. Nevertheless, we want to give another example of a COM separating it from an OM. For the latter consider again a central arrangement $\mathcal{H} = \{H_1, \ldots, H_n\}$ of hyperplanes in an Euclidean space $\mathbb{R}^d$. Moreover, let C be an open convex subset of $\mathbb{R}^d$. Now, let $\mathcal{L}_{\mathcal{H},C} \subset \{\pm, 0\}^n$ be the set of covectors defined in the same way as in the case of an OM, but only for those points that lie in the set C. Thus $\mathcal{L}_{\mathcal{H},C} \subset \mathcal{L}_{\mathcal{H}}$ where only those covectors are considered whose corresponding regions intersect C. We claim that the system of sign-vectors $(\mathcal{H}, \mathcal{L}_{\mathcal{H},C})$ is a COM. Let $X, Y \in \mathcal{L}_{\mathcal{H},C}$ be covectors and let v_x, v_y be corresponding vectors in $C \subset \mathbb{R}^d$. Moreover let $i \in S(X,Y)$, i.e., vectors v_X and v_y are separated by the hyperplane H_i. As in the case of OMs, the point v_Z on the line connecting v_X and v_Y intersecting H_i is vector such that for its covector Z it holds that $Z_i = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in \mathcal{H} \setminus S(X,Y)$. Since C is convex, $v_Z \in C$, thus $Z \in \mathcal{L}_{\mathcal{H},C}$. This proves that the axiom (SE) is satisfied. For the axiom (FS), let again $X, Y \in \mathcal{L}_{\mathcal{H},C}$ be covectors. Then let $v_X \in C$ be corresponding vectors of X and $-v_Y$ be the corresponding vector of $-Y$. Covector $-Y$ might not be in $\mathcal{L}_{\mathcal{H},C}$ but as in the case of OMs, there is a vector $-v_Y \in \mathbb{R}^d$ corresponding to it. Then $v_x - \epsilon v_Y \in C$ for sufficiently small ϵ (even though $-v_Y$ might not be in C) since C is an open set and $v_X \in C$. Thus the axiom (FS) is satisfied. As in the case of OMs, call a COM $(\mathcal{H}, \mathcal{L}_{\mathcal{H},C})$ obtained from a central hyperplane arrangement and an open convex subset a *realizable* COM.

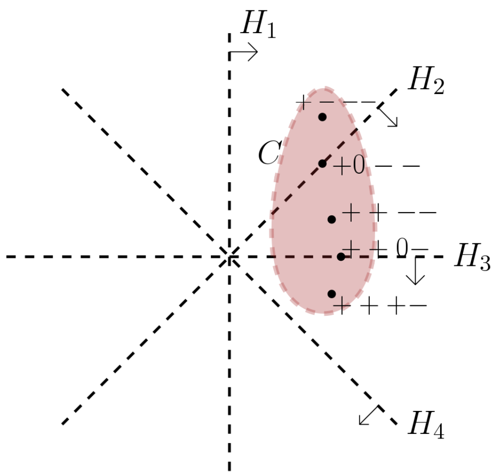


Figure 4.3: Convex subset in central hyperplane arrangement with covectors: a COM

We continue with a few definitions. Let again $\mathcal{L} \subseteq \{0, \pm\}^{\mathcal{E}}$ be a system of sign-vectors and $e \in \mathcal{E}$. For $X \in \mathcal{L}$ let $X \setminus e$ be the element of $\{0, \pm\}^{\mathcal{E} \setminus \{e\}}$ obtained by deleting the coordinate e from X. Define operations $\mathcal{L}/e = \{X \setminus e \mid X \in \mathcal{L}, X_e = 0\}$ as taking the *hyperplane* of e (usually referred to as contraction) and $\mathcal{L} \setminus e = \{X \setminus e \mid X \in \mathcal{L}\}$ as the *deletion* of e. A sign-system that arises by deletion and taking hyperplanes from another one is called a *minor*. Furthermore denote by $\mathcal{L}_e^+ := \{X \setminus e \in \mid X \in \mathcal{L}, X_e = +\}$ and $\mathcal{L}_e^- := \{X \setminus e \mid X \in \mathcal{L}, X_e = -\}$ the positive and negative (open) *halfspaces* with respect to e.

The following is easy to see.

Lemma 4.1.2 ([11]). *For any system of sign-vectors the operations of taking halfspaces, hyperplanes and deletion commute.*

Now we better explain AOMs. Axiom (A) is not easy to be interpreted directly but a theorem due to Karlander [52] characterizes AOMs as exactly the halfspaces of OMs[1].

[1]Note that his proof contains a flaw that has only been observed and fixed recently in [12].

In particular we call an AOM *realizable* if it is obtained from a central arrangement of hyperplanes $\mathscr{H}$ and an open halfspace C in the same way as a realizable COM is obtained, where C is a halfspace defined by one of the hyperplanes in $\mathscr{H}$. Another way to look at realizable AOMs, is to see them as a (not necessarily central) arrangements of hyperplanes. In fact every non-central arrangement of hyperplanes in $\mathbb{R}^d$ can be transformed in a central arrangement by embedding $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ as a hyperplane with the last coordinate fixed and equal to 1, and extending each hyperplane (of $\mathbb{R}^d$) in this subspace to a hyperplane crossing the origin of $\mathbb{R}^{d+1}$ (see Figure 4.4). Adding the hyperplane (of $\mathbb{R}^{d+1}$) with the last coordinate fixed and equal to 0 to the arrangement leads to a realizable AOM as defined above.

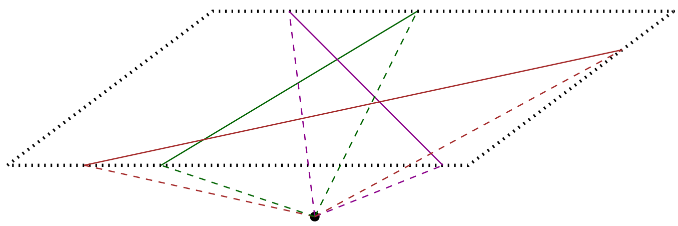


Figure 4.4: Non-central planar arrangement corresponding to open halfspace of a central arrangement in $\mathbb{R}^3$.

Finally, the ideal composition axiom is a bit harder to explain thus a more concrete example is needed. Consider a non-central arrangement of lines in $\mathbb{R}^2$ and let v be a point in it that is in an intersection of k lines, say $\{H_{i_1}, \dots, H_{i_k}\}$. In particular, its corresponding covector X_v as defined in above examples has exactly the coordinates $i_1, \dots, i_k$ equal to 0. Assume that it holds $X_v \circ Y \in \mathscr{L}$ for all $Y \in \{\pm\}^{\mathscr{E}}$. This means that there must be 2^k polygon regions around v. But since v lies in a plane, there are exactly $2k$ polygon regions around v (cf. Figure 4.2), proving that the ideal composition axiom holds in this case only if $k = 2$. In fact, it turns out that the composition axiom holds in a hyperplane arrangement in $\mathbb{R}^d$ iff each point is in the intersection of at most d hyperplanes. More examples and properties of LOPs will be explained in the following subsection.

Our systems of sign-vectors behave well with respect to operations of taking hyperplanes and deletion:

Lemma 4.1.3 (**[61]**). *The classes of COMs, AOMs, OMs, and LOPs are minor closed. Moreover, COMs and LOPs are closed under taking halfspaces.*

The *rank* of a system of sign-vectors $(\mathscr{E}, \mathscr{L})$ is the largest integer r such that there is subset $A \subseteq \mathscr{E}$ of size $|\mathscr{E}| - r$ such that $\mathscr{L} \setminus A = \{\pm, 0\}^r$. In other words, the rank of $(\mathscr{E}, \mathscr{L})$ is just the VC-dimension of $\mathscr{L}$. Note that this definition of rank coincides with the usual rank definition for OMs, see [28].

A system of sign-vectors $(\mathscr{E}, \mathscr{L})$ is *simple* if it satisfies the following two conditions:

(N1*) for each $e \in \mathscr{E}$, $\{\pm, 0\} = \{X_e : X \in \mathscr{L}\}$;

(N2*) for each pair $e \neq f$ in $\mathscr{E}$, there exist $X, Y \in \mathscr{L}$ with $\{X_e X_f, Y_e Y_f\} = \{\pm\}$.

An element $e \in \mathcal{E}$ not satisfying (N1*) is called *redundant*. Two elements $e, f \in \mathcal{E}$ are called *parallel* if they do not satisfy (N2*). Note that parallelism is an equivalence relation on $\mathcal{E}$. We denote by $[e]$ the class of elements parallel to e, for $e \in \mathcal{E}$.

For every COM $(\mathcal{E}, \mathcal{L})$ there exists up to reorientation and relabeling of coordinates a unique simple COM, obtained by successively applying operation $\mathcal{L} \backslash e$ to the redundant coordinates $e \in \mathcal{E}$ and to elements of parallel classes with more than one element. See [11, Proposition 3] for the details. Note that by Lemma 4.1.2 the order in which these operations are taken is irrelevant and by Lemma 4.1.3 all the classes of systems of sign-vectors at consideration here are closed under this operation. We will denote by $\mathcal{S}(\mathcal{E}, \mathcal{L})$ the unique *simplification* of $(\mathcal{E}, \mathcal{L})$.

We finalize this part with the geometrical motivation of the above operations. Let again $\mathcal{H} = \{H_1, \dots, H_n\}$ to be a central arrangement of hyperplanes in $\mathbb{R}^d$ and $\mathcal{L}_{\mathcal{H},C}$ its corresponding realizable COM. Taking a halfspace or a halfplane corresponds to considering only regions on one side of a chosen hyperplane or only regions on the chosen hyperplane, respectively. The deletion corresponds to deleting one of the hyperplanes. If $\mathcal{L}_{\mathcal{H},C}$ does not satisfy the axiom (N1*) then one of the halfplanes in $\mathcal{H}$ does not cross C, thus the halfplane is redundant. If $\mathcal{L}_{\mathcal{H},C}$ satisfies the axiom (N1*), but does not satisfy the axiom (N2*), then two halfspaces coincide, from here the expression that the coordinates are parallel. The simplification $\mathcal{S}(\mathcal{H}, \mathcal{L}_{\mathcal{H},C})$ is obtained by the removal of all redundant hyperplanes and restriction to the equivalence classes of parallel ones.

Systems of sign-vectors and partial cubes

The *topes* of a system of sign-vectors $(\mathcal{E}, \mathcal{L})$ are the elements of $\mathcal{T} := \mathcal{L} \cap \{\pm\}^{\mathcal{E}}$. If $(\mathcal{E}, \mathcal{L})$ is simple, we define the *tope graph* $G(\mathcal{L})$ of $(\mathcal{E}, \mathcal{L})$ as the (unlabeled) subgraph of $Q_{\mathcal{E}}$ induced by $\mathcal{T}$. Note that, for the sake of the convenience, in this case we see the vertices of $Q_{\mathcal{E}}$ as the strings of 1s and -1s rather than 1s and 0s. If $(\mathcal{E}, \mathcal{L})$ is non-simple, we consider $G(\mathcal{L})$ as the tope graph of its simplification $\mathcal{S}(\mathcal{E}, \mathcal{L})$.

In general $G(\mathcal{L})$ is an unlabeled graph and even though it is defined as a subgraph of a hypercube $Q_{\mathcal{E}}$ it could possibly have multiple non-equivalent embeddings in $Q_{\mathcal{E}}$. We call a system $(\mathcal{E}, \mathcal{L})$ a *partial cube system* if its tope graph $G(\mathcal{L})$ is an isometric subgraph of $Q_{\mathcal{E}}$ in which the edges correspond to sign-vectors of $\mathcal{L}$ with a single 0. It is well-known that partial cubes have a unique embedding in $Q_{\mathcal{E}}$ up to automorphisms of $Q_{\mathcal{E}}$, see e.g. [74, Chapter 5]. In other words, the tope graph of a simple partial cube system is invariant under reorientation. For this reason we will, possibly without an explicit note, identify vertices of a partial cube $G(\mathcal{L})$ with subsets of $\{\pm\}^{\mathcal{E}}$. The following was proved in [11, Proposition 2]:

Lemma 4.1.4 ([11]). *Simple COMs are partial cube systems.*

Moreover, in the case of a simple COM that is realizable in $\mathbb{R}^d$ it is crucial to notice that its tope graph corresponds to the regions graph in which the d-dimensional regions of the hyperplane arrangement are its vertices and two regions adjacent iff they are separated by exactly one hyperplane. See Figures 4.2, 4.3 where the tope graphs are an 8-cycle and a path of length 2, respectively.

Before presenting basic results regarding partial cube systems, we discus how the minor operations and taking halfspaces as defined in Section 4.1 affect tope graphs. So let $(\mathscr{E},\mathscr{L})$ be a simple partial cube system.

First note that deletion does not affect the simplicity of $(\mathscr{E},\mathscr{L})$. Furthermore, since $(\mathscr{E},\mathscr{L})$ is a partial cube system, the tope graph $G(\mathscr{L}\backslash e)$ corresponds to $\pi_e(G(\mathscr{L}))$ obtained from $G(\mathscr{L})$ by contracting all the edges in the Θ-class corresponding to coordinate e, as defined in Section 2.4.

Also, the halfspace $\mathscr{L}_e^+$ is easily seen to be simple and its tope graph corresponds to the restriction $\rho_{e^+}(G(\mathscr{L}))$ to the positive halfspace of $E_e \in \mathscr{E}$, as defined in Section 2.4.

The hyperplane $\mathscr{L}/e$ does not need to be a simple system of sign-vectors nor a partial cube system. However, we can establish the following:

Lemma 4.1.5 ([61]). *Let $(\mathscr{E},\mathscr{L})$ be a partial cube system and $e \in \mathscr{E}$. If $\zeta_e(G(\mathscr{L}))$ is a well-embedded partial cube, then $\zeta_e(G(\mathscr{L})) \cong G(\mathscr{L}/e)$.*

The correspondences before the lemma in particular give that deletions and halfspaces of partial cube systems coincide with pc-minors, which together with Lemma 4.1.3 gives:

Proposition 4.1.6. *Let $(\mathscr{E},\mathscr{L})$ and $(\mathscr{E}',\mathscr{L}')$ be simple partial cube systems with tope graphs $G(\mathscr{L})$ and $G(\mathscr{T}')$, respectively. Then $(\mathscr{E}',\mathscr{L}')$ arises from $(\mathscr{E},\mathscr{L})$ by deletion and taking halfspaces iff $G(\mathscr{T}')$ is a pc-minor of $G(\mathscr{L})$. Moreover, the families of tope graphs of COMs and LOPs are pc-minor closed.*

In the following, we will describe further how pc-minors and equivalently deletions and halfspaces of partial cube systems translate metric graph properties as introduced in Subsection 2.4 into properties of sign-vectors.

For $X \in \mathscr{L}$ we set $\mathscr{T}(X) := \{T \in \mathscr{T} \mid X \circ T = T\}$ and denote by $G(X)$ the subgraph of $G(\mathscr{L})$ induced by $\mathscr{T}(X)$. Furthermore, let $\mathscr{H}(\mathscr{L}) = \{G(X) \mid X \in \mathscr{L}\}$ be the set of subgraphs $G(\mathscr{L})$ obtained by considering $G(X)$ for all $X \in \mathscr{L}$. Conversely, given a convex subgraph G' of a partial cube G with Θ-classes $\mathscr{E}$ denote by $X(G')$ the sign-vector with $X(G')_e = +$ or $X(G')_e = -$ if $G' \subseteq E_e^+$ or $G' \subseteq E_e^-$,respectively, and 0 otherwise for all $E_e \in \mathscr{E}$. Note that for each vertex $v \in G(\mathscr{L})$, $X(v) = v$. Furthermore, let $\mathscr{L}(\mathscr{H}) = \{X(G') \mid G' \in \mathscr{H}\}$ for a set $\mathscr{H}$ of convex subgraphs of G.

Proposition 4.1.7 ([61]). *In a simple partial cube system $(\mathscr{E},\mathscr{L})$ for each $X \in \mathscr{L}$ its tope-graph $G(X)$ is a convex subgraph of $G(\mathscr{L})$. Conversely, if $G = (V,E)$ is a partial cube and $\mathscr{H}$ a set of convex subgraphs of G, such that $\mathscr{H}$ includes all the vertices of G, then there is a simple $(\mathscr{E},\mathscr{L})$ such that $G = G(\mathscr{L})$ and $\mathscr{H} = \mathscr{H}(\mathscr{L})$.*

The following establishes a connection between the gates of a convex set and the composition operator.

Lemma 4.1.8 ([61]). *Let G be a partial cube embedded in a hypercube, G' a convex subgraph of G and v a vertex of G. Then w is the gate for v in G' if and only if $X(w) = X(G') \circ X(v)$. Therefore, a subgraph G' is gated if and only if for all $v \in G$ there is a $w \in G$ such that $X(G') \circ X(v) = X(w)$.*

For a partial cube G isometrically embedded in a hypercube $Q_{\mathscr{E}}$ define $\mathscr{L}(G) = \{X \in \{0, \pm\}^{\mathscr{E}} \mid X \circ (-Y) \in \mathscr{T}(X(G)),\ \text{for all } Y \in \mathscr{T}(X(G))\}$.

Lemma 4.1.9 (**[61]**)**.** *Let G be a partial cube. Then $\mathscr{L}(G)$ is a partial cube system that satisfies (FS) (and therefore (C)) and the set $\mathscr{H}(\mathscr{L}(G))$ of corresponding subgraphs coincides with the antipodal gated subgraphs of G.*

Proposition 4.1.7 states that in a simple system of sign-vectors there is a correspondence between its vectors and a subset of the set of convex subgraphs of its tope graph. The following proposition determines which convex subgraphs are in the subset if the system is a COM.

Theorem 4.1.10 (**[61]**)**.** *For a simple COM $(\mathscr{E}, \mathscr{L})$ with tope set $\mathscr{T}$ we have*

$$\begin{aligned} \mathscr{L} &= \{X(G') \mid G' \textit{ antipodal subgraph of } G(\mathscr{L})\} \\ &= \{X(G') \mid G' \textit{ antipodal gated subgraph of } G(\mathscr{L})\} \\ &= \mathscr{L}(G). \end{aligned}$$

In particular, in a tope graph of a COM all antipodal subgraphs are gated.

As a consequence every simple COM is uniquely determined by its tope set, or up to reorientation by its tope graph (for a non-constructive proof see [11, Propositions 1 & 3], for an example of how to obtain a COM out of its tope graph see Figure 4.5. The constructive statement here is in fact a generalization of a theorem known for OMs, usually attributed to Mandel, see [25]. To understand the above theorem geometrically in realizable COMs, recall that in this case $\mathscr{L}$ correspond to regions of $\mathbb{R}^d$ that are defined by some hyperplane arrangement intersected by an open convex set where d-dimensional regions correspond to the vertices of the graph $G(\mathscr{L})$. The theorem states that each region X of dimension less than d defines an antipodal gated subgraph of $G(\mathscr{L})$ of all the d-dimensional regions touching X. In fact the theorem states more, that for each non-trivial antipodal subgraph of $G(\mathscr{L})$ there exist a region touching all the vertices in the subgraph.

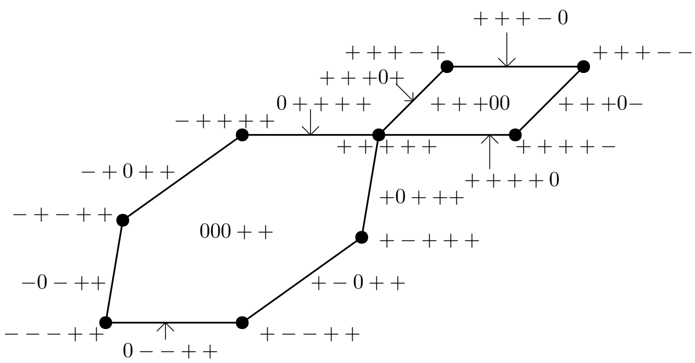


Figure 4.5: A tope graph of a COM in which all antipodal subgraphs are denoted forming a COM.

We will from now on denote the class of tope graphs of COMs by $\mathscr{G}_{\text{COM}}$ and the class of partial cubes in which antipodal subgraphs are gated by AG, i.e. the above theorem implies $\mathscr{G}_{\text{COM}} \subseteq \text{AG}$.

Corollary 4.1.11 ([**61**]). *A partial cube G is in $\mathcal{G}_{COM}$ if and only if $\mathcal{L}(G)$ is a simple COM.*

Let us finally describe how tope graphs of the other systems of sign-vectors from Section 4.1 specialize tope graphs of COMs. We will denote the classes of tope graphs of OMs, AOMs, and LOPs by $\mathcal{G}_{\mathrm{OM}}$, $\mathcal{G}_{\mathrm{AOM}}$, and $\mathcal{G}_{\mathrm{LOP}}$. A consequence of the definition of antipodality in partial cubes is:

Proposition 4.1.12. *A graph is in $\mathcal{G}_{OM}$ if and only if it is antipodal and in $\mathcal{G}_{COM}$.*

A not yet intrinsic description of tope graphs of AOMs follows:

Proposition 4.1.13. *A graph is in $\mathcal{G}_{AOM}$ if and only if it is a halfspace of a graph in $\mathcal{G}_{OM}$.*

Interpreting axiom (IC) in the partial cube model we get: for each $X \in \mathcal{L}$, the antipodal subgraph $G(X)$ must be isomorphic to a hypercube. Thus:

Proposition 4.1.14. *A graph is in $\mathcal{G}_{LOP}$ if and only if all its antipodal subgraphs are hypercubes and it is in $\mathcal{G}_{COM}$.*

Noticing that $\mathcal{L}(Q_r)$, for a hypercube Q_r, equals $\{\pm, 0\}^r$ we immediately get the following lemma from the definition of the rank of a system of sign-vectors.

Lemma 4.1.15. *The rank of a COM $(\mathcal{E}, \mathcal{L})$ is the largest r such that $G(\mathcal{L})$ contracts to Q_r.*

Obstructions for COMs

Let Q_n be the hypercube, $v \in Q_n$ any of its vertices and $-v$ its antipode. Let $Q_n^- := Q_n \setminus -v$ be the hypercube minus one vertex. Consider the set of partial cubes arising from Q_n^- by deleting any subset of $N(v) \cup \{v\}$. It is easy to see that if $n \geq 4$ a graph obtained this way from Q_n^- is a partial cube unless v is not deleted but at least two of its neighbors are deleted. Denote by Q_n^{-*} the partial cube obtained by deleting exactly one neighbor of v, and by $Q_n^{--}(m)$ the graph obtained by deleting v and m neighbors of v, respectively, where for $Q_n^{--}(0)$ we sometimes simply write Q_n^{--}. It is easy to see that Q_n^- and Q_n^{--} are tope graphs of (realizable) COMs. For $n \leq 3$ all the partial cubes arising by the above procedure are isomorphic to Q_n^- or Q_n^{--}, thus the interesting graphs appear for $n \geq 4$. Denote their collection by $\mathcal{Q}^- = \{Q_n^{-*}, Q_n^{--}(m) \mid 4 \leq n; 1 \leq m \leq n\}$. The family $\mathcal{Q}^-$ will turn out to be the list of excluded minors for tope graphs of COMs.

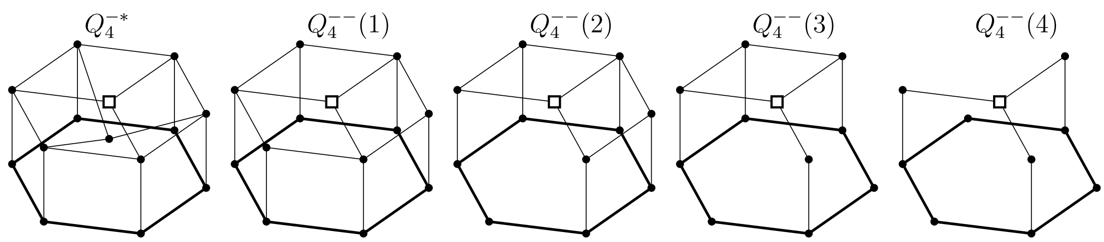


Figure 4.6: Graphs $Q_4^{-*}, Q_4^{--}(m)$, for $1 \leq m \leq 4$. The square vertex has no gate in the bold C_6.

Notice that we have already seen some of the graphs in $\mathcal{Q}^-$. In fact we have encountered the five graphs $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ when characterizing the family of partial cubes whose zone graphs are well-embedded partial cubes. The following lemma says that the class $\mathcal{Q}^-$ is minimal with regards to the minor relation. Moreover, the second assertion of it states that no graph in $\mathcal{Q}^-$ is a tope graph of a COM, since by Theorem 4.1.10 antipodal subgraphs are gated in tope graphs of COMs.

Lemma 4.1.16 (**[61]**). *The class $\mathcal{Q}^-$ is pc-minor minimal, i.e. any pc-minor of a graph in $\mathcal{Q}^-$ is not in $\mathcal{Q}^-$. Furthermore, any graph in $\mathcal{Q}^-$ contains an antipodal subgraph that is not gated, i.e. $\mathcal{Q} \subseteq \overline{\text{AG}}$.*

Characterizations of tope graphs of COMs and corollaries

The main result of this section is a characterization of the tope graphs of COMs that does not depend on axioms of systems of sign vectors but rather graph theoretical properties of this graphs. In fact such a characterization for tope graphs of OMs was an open problem [14, 40] and is implied by our result. We will present two characterizations, one in terms of metric properties of subgraphs and another in terms of excluded pc-minors. A direct corollary is a third characterization in terms of zone graphs. Characterizations of other (geometric) families presented in this section also follow from the main result.

Theorem 4.1.17 (**[61]**). *For a graph G the following conditions are equivalent:*

(i) $G \in \mathcal{G}_{COM}$,

(ii) *G is a partial cube and all its antipodal subgraphs are gated,*

(iii) $G \in \mathcal{F}(\mathcal{Q}^-)$.

We proved the theorem in **[61]** in the following way. Tope graphs have gated antipodal subgraphs by Theorem 4.1.10 thus (i) impies (ii). The second part includes the proof that if G is not a tope graph of a COM, then it must contain a minor in $G \in \mathcal{F}(\mathcal{Q}^-)$ proving (iii) implies (i). The latter is proved in a recursive way, where recursion assumption is used on a smaller graph obtained by well-embedded zone graphs. Recall from Theorem 3.2.6 that if the zone graphs of G are not partial cubes then G has a minor in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\} \subset \mathcal{Q}^-$ and thus is not in $\mathcal{F}(\mathcal{Q}^-)$. Finally, the fact that (ii) implies (iii) seems trivial since all the graphs in $\mathcal{Q}^-$ contain an antipodal subgraph that is not gated, but the catch is hidden in the need to prove that the family AG is a minor closed family. In fact the latter presents the greatest difficulty in the proof of Theorem 4.1.17.

The theorem can be used to obtain a characterization of $\mathcal{G}_{\text{COM}}$ in terms of zone graphs.

Corollary 4.1.18 (**[61]**). *A graph G is in $\mathcal{G}_{\text{COM}}$ if and only if it is a partial cube such that all iterated zone graphs are well-embedded partial cubes.*

It is interesting to compare the families $\mathcal{F}(\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\})$ and $\mathcal{G}_{\text{COM}}$. On one hand, the inclusion of the second in the first is clearly seen through the forbidden subgraphs characterization. On the other hand, Theorem 3.2.6 and Corollary 4.1.18 give

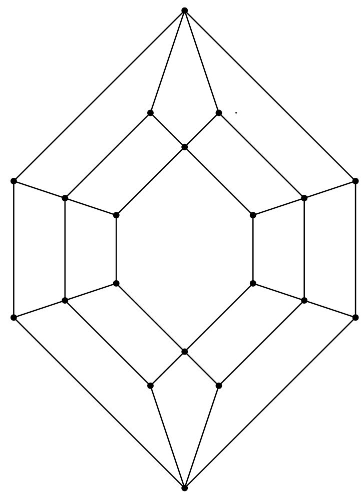


Figure 4.7: An example of a partial cube in $\mathcal{G}_{\mathrm{OM}} \subset \mathcal{G}_{\mathrm{COM}}$, i.e. an antipodal partial cube with all its antipodal subgraphs gated.

the inclusion via the zone graphs, where in the case of the second family all iterated zone graphs are well-embedded partial cubes, while in the first family only the first iteration.

Theorem 4.1.17 and Corollary 4.1.18 specialize to other systems of sign-vectors. Using Proposition 4.1.12 we immediately get:

Corollary 4.1.19 ([**61**]). *For a graph G the following conditions are equivalent:*

(i) $G \in \mathcal{G}_{OM}$,

(ii) G is an antipodal partial cube and all its antipodal subgraphs are gated,

(iii) G is in $\mathcal{F}(\mathcal{Q}^-)$ *and antipodal,*

(iv) G is an antipodal partial cube and all its iterated zone graphs are well-embedded partial cubes.

Note that the equivalence (i)⟺(ii) corresponds to a characterization of tope sets of OMs due to da Silva [28] and (i)⟺(vi) corresponds to a characterization of tope sets of Handa [46]. See Figure 4.7 for an example of the tope graph of an OM, thus also of a COM.

Let us call an affine subgraph G' of an affine partial cube G *conformal* if for all $v \in G'$ we have $-_{G'}v \in G' \Leftrightarrow -_G v \in G$. We give an intrinsic characterization of $\mathcal{G}_{\mathrm{AOM}}$:

Corollary 4.1.20 ([**61**]). *For a graph G the following conditions are equivalent:*

(i) $G \in \mathcal{G}_{AOM}$,

(ii) G is an affine partial cube and all its antipodal and conformal subgraphs are gated,

(iii) G is in $\mathcal{F}(\mathcal{Q}^-)$ *(or* $G \in \mathcal{G}_{COM}$*), affine, and all its conformal subgraphs are gated.*

For the next statement denote $\mathcal{Q}^{--} := \{Q_n^{--} \mid n \geq 3\}$.

Corollary 4.1.21 (**[61]**). *For a graph G the following conditions are equivalent:*

(i) $G \in \mathscr{G}_{LOP}$,

(ii) G is a partial cube and all its antipodal subgraphs are hypercubes,

(iii) G is in $\mathscr{F}(\mathscr{Q}^{--})$.

For lopsided sets the (i)⟺(ii) part of the corollary corresponds to a characterization due to Lawrence [62]. See Figure 4.8 for an example graph.

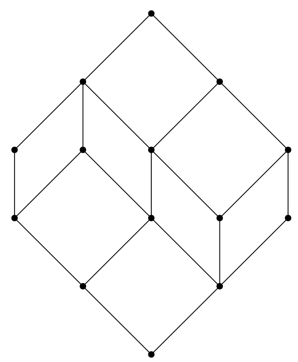


Figure 4.8: The tope graph of a lopsided set.

Recall that by Lemma 4.1.15 the rank of a COM is the dimension of a maximal hypercube to which its tope graph can be contracted. Considering COMs of bounded rank, we can reduce the set of excluded pc-minors to a finite list. For any $r \geq 3$ define the following finite sets

$$\mathscr{Q}_r^- := \{Q_n^{-*}, Q_n^{--}(m), Q_{r+2}^{--}(r+2), Q_{r+1} \mid 4 \leq n \leq r+1; 1 \leq m \leq n\} \subset \mathscr{Q}^- \cup \{Q_{r+1}\},$$

and

$$\mathscr{Q}_r^{--} := \{Q_n^{--}, Q_{r+1} \mid 3 \leq n \leq r+1\} \subset \mathscr{Q}^{--} \cup \{Q_{r+1}\}.$$

Corollary 4.1.22 (**[61]**). *For a graph G and an integer* $r \geq 3$ *we have:*

- $G \in \mathscr{G}_{COM}$ *of rank at most r* $\Leftrightarrow G \in \mathscr{F}(\mathscr{Q}_r^-)$.
- $G \in \mathscr{G}_{OM}$ *of rank at most r* $\Leftrightarrow G \in \mathscr{F}(\mathscr{Q}_r^-)$ *and G is antipodal.*
- $G \in \mathscr{G}_{AOM}$ *of rank at most r* $\Leftrightarrow G \in \mathscr{F}(\mathscr{Q}_r^-)$, *G is affine and all its conformal subgraphs are gated.*
- $G \in \mathscr{G}_{LOP}$ *of rank at most r* $\Leftrightarrow G \in \mathscr{F}(\mathscr{Q}_r^{--})$.

Note that Proposition 2.4.1 directly yields a polynomial time recognition algorithm for the recognition of the bounded rank classes above. However, the recognition of graphs from any of the classes $\mathscr{G}_{\mathrm{COM}}, \mathscr{G}_{\mathrm{AOM}}, \mathscr{G}_{\mathrm{OM}}, \mathscr{G}_{\mathrm{LOP}}$ can be done in polynomial time as in Algorithm 1.

Algorithm 1 Recognition of classes $\mathcal{G}_{\mathrm{COM}}, \mathcal{G}_{\mathrm{AOM}}, \mathcal{G}_{\mathrm{OM}}, \mathcal{G}_{\mathrm{LOP}}$

1. First we isometrically embed the graph in a hypercube. If this cannot be done, then the graph is not in any of the above classes.
2. Next we compute all pairs of vertices u, v of G and store them. We check if each $\mathrm{conv}(u, v)$ is antipodal, and if so, we check if it is gated. If all the antipodal graphs obtained in this way are gated, then $G \in \mathcal{G}_{\mathrm{COM}}$, otherwise we do not proceed. If G is among the antipodal subgraphs, then $\mathcal{G}_{\mathrm{OM}}$.
3. Now we check if G is a LOP. For the latter we just check if all antipodal $\mathrm{conv}(u, v)$ are isomorphic to a hypercube, i.e. $|\mathrm{conv}(u, v)| = 2^{d(u,v)}$. If so, then $G \in \mathcal{G}_{\mathrm{LOP}}$.
4. We continue by checking for each $\mathrm{conv}(u, v)$ if it is an affine subgraphs. For each pair $u', v' \in \mathrm{conv}(u, v)$ such that $|\mathrm{conv}(u', v')| < |\mathrm{conv}(u, v)|$ we store the pair in $NA(u, v)$, and we search for a pair $w, -_{\mathrm{conv}(u,v)}w \in \mathrm{conv}(u, v)$ such that the set of Θ-classes on a shortest (u', w)-path and on a shortest $(v', -_{\mathrm{conv}(u,v)}w)$-path are disjoint. Note that the convex hulls are already computed. If this is the case for all such u', v', we store $\mathrm{conv}(u, v)$ as an affine subgraph.
5. Finally, we check if the whole graph is affine, in this case say $G = \mathrm{conv}(u, v)$. Then for every affine subgraph $\mathrm{conv}(u', v')$ and vertex $w \in \mathrm{conv}(u', v')$, we check if the pair $w, -_{\mathrm{conv}(u',v')}w$ is a pair in $NA(u', v')$ if and only if $w, -_G w$ is a pair in $NA(u, v)$. If this is the case, $\mathrm{conv}(u', v')$ is a conformal subgraph and we check if it is gated. Finally, if all conformal subgraphs are gated, then $G \in \mathcal{G}_{\mathrm{AOM}}$.

Note that by Theroem 2.2.4, partial cubes can be recognized and embedded in a hypercube in quadratic time. For a partial cube embedded in a hypercube checking if it is antipodal can be done in linear time by checking if every vertex has its antipode. The convex hull of any subset can be computed in linear time in the number of edges (for instance by using Lemma 2.3.2) for a graph embedded in a hypercube and checking if a convex subgraph is gated is linear by Lemma 4.1.8. The correctness of the algorithm follows directly from Theorem 4.1.17, Corollaries 4.1.19, 4.1.21, 4.1.20, and Proposition 2.5.6.

Realizability of COMs

As we have seen, examples of tope graph of COMs can be constructed by considering a central hyperplane arrangement together with an open convex subset, and extracting its tope graph. Recall that such COMs are called realizable. In fact, as discussed in the connection with AOMs, one can even take a non-central arrangement since such an arrangement can be extended to a halfspace of a central one (see Figure **??**. Since each convex set in an Euclidean space can be approximated by an intersection of a finite number of halfspaces, notice that the above construction corresponds to constructing a realizable OM and taking a convex subgraph of it, by Lemma 2.3.2. Nevertheless, not all COMs can be constructed in this way, i.e. not all COMs are realizable, see Figure 4.9. One of the reasons for that is that already not all OMs are realizable [14].

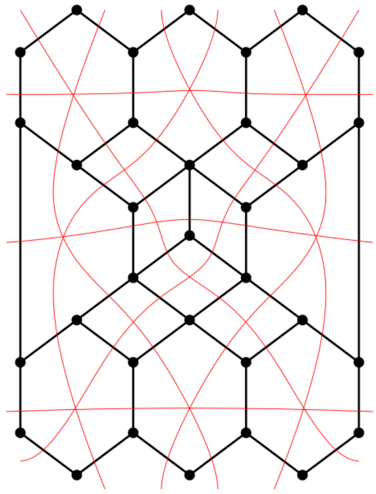


Figure 4.9: Tope graph of an affine COM, halfspace of non-realizable, so called non-Pappus OM.

The Non-Pappus OM form Figure 4.9 (i.e. the full antipodal graph, not just the halfspace) cannot be realized as an OM since the red lines, if stretched, must obey the Pappus theorem, meaning that the three lines in the center of the figure must intersect, and one must obtain a 6-cycle instead of the Q_3^- in the center of the graph (see [14, Section 1.3] for details). Moreover, notice also that the non-Pappus OM cannot be realized as a COM (i.e. realizability of OM and a COM coincides) since if it is a convex subgraph of a realizable COM G, then it must be gated by Theorem 4.1.17. Hence contracting all the Θ-classes of G (i.e. deleting all the hyperplanes in the realization) besides the one intersecting the non-Pappus subgraph gives a realization of the non-Pappus OM. But this cannot exist.

One of the most famous theorems about OMs is their topological representability. First notice that in a realization of a realizable OM all the regions besides the central point intersect the centrally positioned sphere. Hence instead of the whole space $\mathbb{R}^d$ one could consider only its $(d-1)$-dimensional central sphere S^{d-1}, and, instead of the hyperplanes, only $(d-2)$-dimensional subspheres of S^{d-1}. Notice that each subsphere obtained in the above way cuts S^{d-1} into two connected components homeomorphic to a $(d-2)$-dimensional ball. Call subspheres with such a property *pseudospheres*, and connected components obtained by removing a pseudosphere its *sides*.

Definition 4.1.23. *Let S^{d-1} be a sphere. A finite set $A = \{S_e \mid e \in \mathcal{E}\}$ of pseudospheres in S^{d-1} is called an arrangement of pseudospheres if the following conditions hold:*

(1) $S_A = \cap_{e \in A} S_e$ is a sphere, for all $A \subseteq \mathcal{E}$.

(2) If $S_A \nsubseteq S_e$ for some $A \subseteq \mathcal{E}, e \in \mathcal{E}$, and S_e^+, S_e^- are the two sides of S_e, then $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

The arrangement of pseudospheres is *essential* if the intersection $S_{\mathcal{E}}$ (as defined by (1)) is empty. The representability theorem is the following:

Theorem 4.1.24 ([31])**.** *For each essential arrangement of pseudospheres in S^{d-1}, covectors $\mathcal{L}$ obtained from it, together with the zero covector, form an oriented matroid of rank d. Moreover, each simple oriented matroid of rank d can be obtained from an essential arrangement of pseudospheres in S^{d-1}.*

A simplified interpretation of Theorem 4.1.24 is that all simple OMs can be seen as arrangements of pseudospheres, where pseudospheres are (possibly) not straight (as in the case of realizable OMs), but nevertheless have similar properties (as straight ones) on intersections. It is an interesting question of determining which OMs and COMs are realizable. In [82] it was shown that recognizability of realizable OMs is equivalent to the existential theory of the reals, which is a stronger statement than NP-hardness. By Corollary 4.1.19, tope graphs of OMs are just antipodal tope graphs of COMs thus also recognizability of realizable COMs is equivalent to the existential theory of the reals. It is interesting to notice that the tope graphs of realizable COMs are a minor closed family (since contractions correspond to deletions of hyperplanes and restrictions to limiting the convex set to a halfspace of some hyperplane, as it was written before), hence the minimal family of forbidden minors of tope graphs of realizable COMs is infinite and hard to recognize.

On the other hand, the question of topological representability of COMs is an open problem. In [11] it was conjectured that every tope graph of a COM is a convex subgraph of a tope graph of an OM. If this is true, then it implies a topological representation with pseudospheres. Another interesting concept was introduced in [11]: call a tope graph of a COM *locally realizable* if all its antipodal subgraphs are realizable. Graph in Figure 4.9 is locally realizable since all its antipodal subgraphs are even cycles, but not realizable.

Proposition 4.1.25. *If H is a COM that is a convex peripheral expansion of a realizable COM G, then H is realizable.*

Proof. Let $\{H_i \mid 1 \leq i \leq m\}$ be hyperplanes and C an open convex set realizing G in $\mathbb{R}^n$. Let G_1, G_2 be the sets of the peripheral expansion of G, such that $G_1 = G$ and G_2 is a convex subgraph of G. By Lemma 2.3.2, G_2 corresponds to an intersection of open halfspaces. Let C' be a open convex subset such that C' together with $\{H_i \mid 1 \leq i \leq m\}$ realizes G_2 but instead of taking the full intersection of open halfspaces bounding G_2 take C' to be a bit smaller, say ϵ away from the boundary of the intersection of the halfspaces.

Extend $\{H_i \mid 1 \leq i \leq m\}$ in $\mathbb{R}^{n+1}$ so that all the hyperplanes are parallel to the last coordinate axis. Let C be embedded into this space with last coordinate equal to 0 and C' be embedded with the last coordinate equal to 1. Let then C'' be the convex closure of C and C' embedded this way minus the boundary of it. Then C'' is an open convex set. Add a hyperplane H_{m+1} orthogonal to the last axis with the last coordinate fixed to the value $1-\epsilon^2$. Then $\{H_i \mid 1 \leq i \leq m+1\}$ together with C'' is a realization of H assuming ϵ is small enough. □

4.2 Pasch and Peano graphs

We observed in Lemma 2.3.2 that partial cubes are precisely the bipartite graphs in which convex sets correspond to intersection of halfspaces (restrictions). This property is one of the essential features of the metric of Euclidean spaces. In this section we will work with bipartite graphs having additional similarities with Euclidean spaces.

First we define properties we are interested in. We will denote with $I(u,v)$ the *interval* from u to v, i.e., all the vertices that lie on some shortest u,v-path. We defined in partial cubes the sets of the form W_{ab} to be halfspaces, in general graphs halfspaces are defined as those convex sets whose complement is also convex. As noted before, in partial cubes the above two definitions coincide.

Definition 4.2.1. *A graph G is said to have the:*

- Peano property *if for all $u,v,w \in V(G)$, $x \in I(u,w)$ and $y \in I(v,x)$, there exists a point $z \in I(v,w)$ such that $y \in I(u,z)$.*

- Pasch property *if for all $u,v,w \in X$, $v \ \in I(u,w)$ and $w' \in I(u,v)$, the intervals $I(v,v')$ and $I(w,w')$ are non-disjoint.*

- Join-hull commutativity property *if for any convex set $C \subseteq V(G)$ and any $u \in V(G)$, the convex hull of $\{u\} \cup C$ equals the union of the convex hull of $\{u,v\}$ for all $v \in C$.*

- Separation property S_3 *if for every $x \in V(G)$ that does not belong to a some convex set $C \subset V(G)$, there is a halfspace H which separates x form C, that is, $x \in H$ and $C \in V(G) \setminus H$.*

- Separation property S_4 *if for all $C,D \subseteq V(G)$ disjoint convex sets, there is a halfspace H which separates C from D, that is, $C \subset H$ and $D \subset X \setminus H$.*

Now we focus on what the above properties imply in bipartite graphs. Restating Lemma 2.3.2 a bipartite graph has the S_3 property if and only if it is a partial cube.

Lemma 4.2.2 ([21]). *A graph has the S_4 property if and only if it has the Pasch property.*

Since a vertex is a convex set in graphs, the S_4 property implies the S_3 property. For the reasons of the above lemma we define the following family of partial cubes.

Definition 4.2.3. *A bipartite graph G is a Pasch graph if it satisfies the Pasch (or equivalently the S_4) property.*

More equivalences among the properties hold. The first statement is from [85], the second from [20].

Lemma 4.2.4 ([85, 20]). *Let G be a graph in which intervals are convex sets. Then G has the join-hull commutativity property if and only if it has the Peano property. Moreover if G is bipartite and satisfies the latter properties, then it is a Pasch graph.*

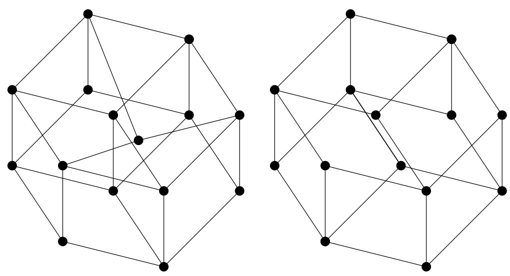
Intervals in hypercubes are always isomorphic to sub-hypercubes, which are convex subgraphs. Now, let u, v be vertices of a partial cube G embedded in a hypercube Q. By the above the interval $I_Q(u, v)$ in Q is a sub-hypercube $Q' \subset Q$. On the other hand, the interval $I_G(u, v)$ of G consists of precisely all the vertices of $G \cap Q'$. Hence also $I_G(u, v)$ is convex, proving that intervals are convex also in partial cubes. Now we can give another definition.

Definition 4.2.5. *A partial cube G is a Peano graph if it satisfies the Peano (or equivalently the join-hull commutativity) property.*

By Lemma 4.2.4, Peano graphs are a subfamily of Pasch graphs. Pasch graphs were explored in detail already in 1994 in [21] while Peano graphs were put on the map only recently in [75].

The main result regarding Pasch graphs explaining their position on the map of Figure 4.1 is the following. The forbidden minors are denoted as in Section 4.1.

Theorem 4.2.6 ([21])**.** *The family of Pasch graphs corresponds to the family* $\mathcal{F}(\{Q_4^-, Q_4^{--}, Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\})$.



(a) Q_4^- (b) Q_4^{--}

Figure 4.10: Graphs Q_4^- and Q_4^{--}.

See Figure 4.10 for the first two forbidden minors and Figure 3.2 for the remaining ones. Theorem 4.2.6 together with Theorem 4.1.17 implies that Pasch graphs are tope graphs of COMs. To see this notice that the graphs in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ are precisely the forbidden minors of COMs of isometric dimension four, while all the other forbidden minors of COMs have a pc-minor in $\{Q_4^-, Q_4^{--}\}$. In fact, we observe that the forbidden minors of isometric dimension five have a convex Q_4^{--}, while all the forbidden minors of higher isometric dimension have a convex Q_4^- subgraph. This implies that any graph that is not a COM is not a Pasch graph since it has a minor among the forbidden minors of COMs, which further have a pc-minor among the forbidden minors of Pasch graphs.

In **[22]** we went further and analyzed isometric cycles in Pasch graphs.

Proposition 4.2.7 (**[22]**)**.** *The convex closure of any isometric cycle of a Pasch graph G is a gated subgraph. If the subgraph is antipodal, then it is isomorphic to a Cartesian product of edges and even cycles.*

The above implies the following.

Proposition 4.2.8. *Every Pasch graph is a tope graph of a locally realizable COM.*

Proof. By Lemma 2.5.4 every antipodal subgraph in a partial cube is a convex closure of some isometric cycle. By Proposition 4.2.7, every antipodal subgraph of a Pasch graph is gated and isomorphic to a Cartesian product of edges and even cycles, i.e., a tope graph of a realizable oriented matroid. □

For a triple of vertices u, v, w of a graph G, a *u-apex* relative to v and w is a vertex $x := (uvw) \in I(u,v) \cap I(u,w)$ such that $I(u,x)$ is maximal with respect to inclusion. A graph G is *apiculate* [8] if and only if for any vertex u the vertex set of G is a meet-semilattice with respect to the base-point order $\preceq_u$ defined by $v \preceq_u v' \iff v \in I(u,v')$, that is, $I(u,v) \cap I(u,w) = I(u,(uvw))$ for any vertices v, w.

Lemma 4.2.9 ([8])**.** *Every Pasch graph G is apiculate.*

Note that many partial cubes are not apiculate. In fact, in [13] it was proved that a tope graph of an OM of rank r is apiculate if and only if the graph is regular with degree r. Also notice that properties from Propositions 4.2.7, 4.2.8 and Lemma 4.2.9, i.e. that the graph has all its antipodal subgraphs gated and isomorphic to the Cartesian product of edges and even cycles, and that the graph is apiculate, are not characterizing properties of Pasch graph. In fact there exist partial cubes with this properties that are not Pasch, but the smallest such graphs have isometric dimension 6 making them too big for a simple presentation. Moreover there exist also apiculate tope graphs of lopsided set (in this case, all their antipodal subgraphs are hypercubes) that are not Pasch. The latter gives a negative answer to the question posed in **[22]**. We found all the above examples by a computer search.

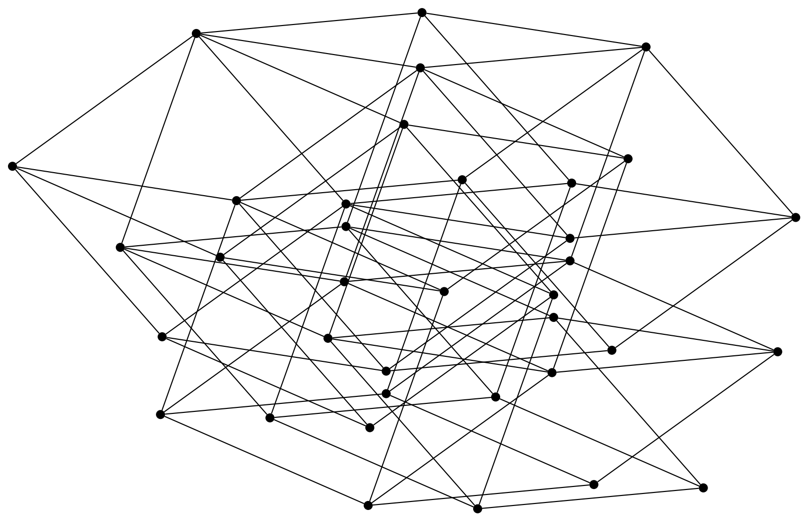
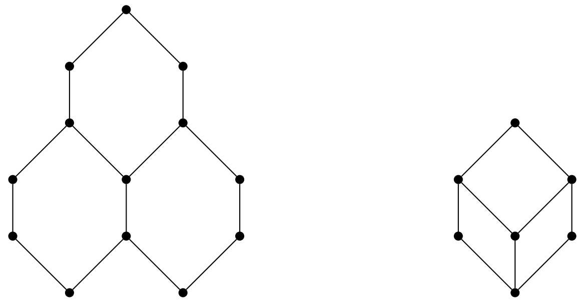


Figure 4.11: A non-Pasch partial cube being apiculate and having all its antipodal subgraphs isomorphic to the Cartesian product of even cycles and edges.

An extensive study of Peano graphs was conducted in [75], here we present the main result of it. Before doing so, we point out that Peano graphs are not a pc-minor closed family proved by the graphs G_1 and $G_2 = Q_3^-$ from Figure 4.12 where G_2 is a pc-minor of G_1 but G_1 is a Peano graph while G_2 is not. Thus different tools are needed to understand Peano graphs.



(a) G_1 (b) G_2

Figure 4.12: A Peano graph G_1 and its non-Peano minor G_2 (which is a Pasch graph).

Recall that a convex closure of a set in a graph can be obtained by iteratively adding all the vertices on the shortest paths between vertices of the set. For a set S in a graph G we will denote with $I(S)$ all the vertices in G lying on shortest path between two vertices of S, i.e., $I(S) = \cup_{u,v \in S} I(u, v)$. A *copoint* at a vertex x is a convex set K which is maximal with respect to the property that $x \notin K$.

Definition 4.2.10. *Let G be a graph. The least non-negative integer n such that* $conv(C \cup \{x\}) = I^n(C \cup \{x\})$ *for each vertex x and each copoint C, is called the pre-hull number of a graph G and is denoted by* $ph(G)$*.*

The pre-hull number gives a measure how complicated the convexity of a graph is. In particular, the bipartite graphs with pre-hull number 0 are characterized as trees [81]. In the case of finite graphs, the characterization of Peano graphs states that they are the second in this measure.

Theorem 4.2.11 ([75]). *A partial cube G is a Peano graph if and only if* $ph(G) \leq 1$.

An important subfamily of Peano graphs is the family of *netlike* partial cubes. The study of them exceeds the scope of this thesis, see [76, 77, 78, 79, 80]. Here we just define them by a characterization presented in [76] (avoiding additional definitions) as the partial cubes with pre-hull number less or equal to 1 and the convex closure of each non-convex isometric cycle isomorphic to a hypercube. In particular, notice that they are locally realizable COMs (since Pasch graphs are) with all its antipodal subgraphs being hypercubes or even cycles. The graphs in Figure 4.12 show that also netlike partial cubes are not a pc-minor closed family since G_1 is a netlike partial cube while G_2 is not.

4.3 Hypercellular graphs

We have seen in the previous chapter that Peano graphs are not a pc-minor closed family. A simplest graph that is not a Peano graph is graph Q_3^-, a 3-dimensional hypercube minus a vertex. This simple motivation lead to the analysis of the family $\mathscr{F}(Q_3^-)$ that turned out to be relevant on the map of partial cubes as the family generalizing many existing families, but with similar properties as them. For the reasons of generalizing bipartite cellular graphs (that we will introduce in Section 4.5) and their cellular structure we called the family *hypercellular graphs* in **[22]**. This section intends to present the most important results from the latter paper.

Many structural properties have been proved for hypercellular graphs, we will start with the property of their isometric cycles:

Theorem 4.3.1 (**[22]**). *The convex closure of any isometric cycle of a graph $G \in \mathscr{F}(Q_3^-)$ is a gated subgraph isomorphic to a Cartesian product of edges and even cycles.*

In the view of Theorem 4.3.1 we will call a subgraph X of a partial cube G a *cell* if X is a convex subgraph of G which is a Cartesian product of edges and even cycles. Note that since a Cartesian product of edges and even cycles is the convex hull of an isometric cycle, by Theorem 4.3.1 the cells of $\mathscr{F}(Q_3^-)$ can be equivalently defined as convex hulls of isometric cycles. An amalgamation defined in Section 2.3 is said to be *gated* if the intersection graph along which we make the amalgamation is gated in both of the graphs. We are prepared for the first structural characterization of hypercellular graphs, showing the importance of cells as the basic building blocks of such graphs (and also motivating their name).

Theorem 4.3.2 (**[22]**). *A partial cube G is hypercellular if G can be obtained by gated amalgams from Cartesian products of edges and even cycles (cells).*

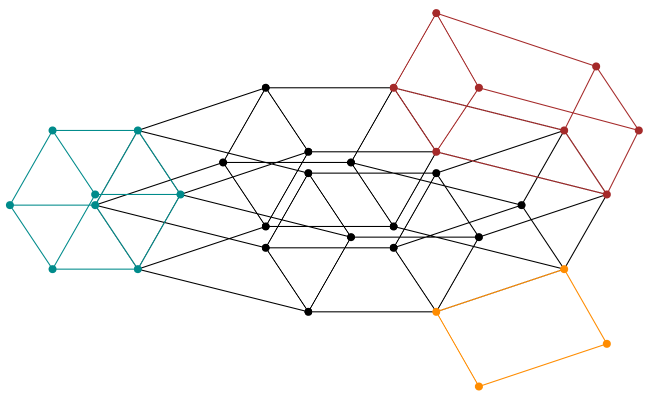


Figure 4.13: A hypercellular graph as a gated amalagamation of three cells.

To emphasize the elegance of this theorem we would like to make a correspondence between hypercellular graphs and chordal graphs. Chordal graphs are considered as one of the most well behaved graphs with nice correspondence to the concepts of tree-width and can be characterized as the graphs obtained as amalgams from cliques where only amalgamations along subgraphs isomorphic to cliques are allowed. In this sense hypercellular graphs can be seen as the bipartite version of chordal graphs.

To better understand convex and gated subgraphs of hypercellular graphs (subgraphs along which we preform amalgamation) we proved the following proposition.

Proposition 4.3.3 (**[22]**). *A connected subgraph H of a hypercellular graph G is convex (respectively, gated) if and only if the intersection of H with each cell of G is convex (respectively, gated).*

The latter gives an interpretation to Theorem 4.3.2 how the amalgamation is done. Say G is the gated amalgamation of G_1 and G_2. Then for each cell of C of G_1 or G_2 the intersection $G_1 \cap G_2 \cap C$ is a subcell of C. Informally we can say that the cells of G_1 and the cells of G_2 meet in the subcells.

Firstly we determine their position in the hierarchy of partial cubes from Figure 4.1.

Corollary 4.3.4 (**[22]**). *Any hypercellular graph G is a Peano graph.*

The latter easily follows from the characterization of hypercellular graphs as gated amalgams of cells, since cells are Peano graphs and the Peano property is preserved by the gated amalgamation. It is also directly seen from the fact that all the forbidden minors of Pasch graphs can be contracted to Q_3^-.

For the further characterizations of hypercellular graphs we need some definitions. We will say that a partial cube G satisfies the *3-convex cycles condition* (abbreviated, *3CC-condition*) if for any three convex cycles C^1, C^2, C^3 that intersect in a vertex and pairwise intersect in three different edges the convex hull of $C^1 \cup C^2 \cup C^3$ is a cell; see Figure 4.14 for an example.

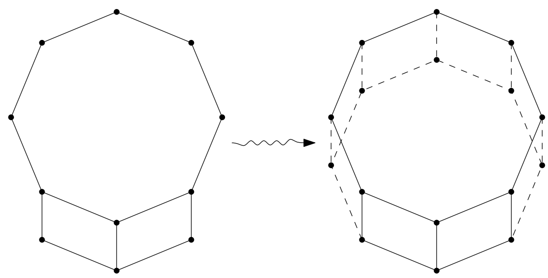


Figure 4.14: The 3-convex cycles condition.

Noticing that the rank of a cell X is the number of edge-factors plus two times the number of cyclic factors one can give a natural generalization of the 3CC-condition. We say that a partial cube G satisfies the *3-cell condition* (abbreviated, *3C-condition*) if for any three cells X_1, X_2, X_3 of rank $k+2$ that intersect in a cell of rank k and pairwise intersect in three different cells of rank $k+1$ the convex hull of $X_1 \cup X_2 \cup X_3$ is a cell.

Theorem 4.3.5 (**[22]**). *For a partial cube $G = (V, E)$, the following conditions are equivalent:*

(i) *$G \in \mathscr{F}(Q_3^-)$, i.e., G is hypercellular;*

(ii) *any cell of G is gated and G satisfies the 3CC-condition;*

(iii) *any cell of G is gated and G satisfies the 3C-condition;*

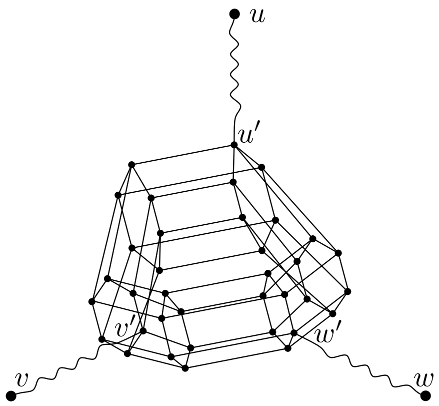


Figure 4.15: A median cell.

The equivalence (i) $\iff$ (ii) of the characterization reminds of the famous characterization of planar graphs as the ones that do not have a minor K_5 or $K_{3,3}$, or equivalently they do not have a subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$. In the case of hypercellular graphs, they do not have a pc-minor Q_3^- or equivalently every three convex cycles forming a subdivision of Q_3^- are a part of a cell. The part (iii) is a generalization of the latter property in a more topological way.

Three (not necessarily distinct) vertices x, y, z of a graph G are said to form a *metric triangle* xyz if the intervals $I(x, y), I(y, z)$, and $I(z, x)$ pairwise intersect only in the common end vertices. A (degenerate) equilateral metric triangle of size 0 is simply a single vertex. We say that a metric triangle xyz is a *quasi-median* of the triplet u, v, w if

$$d(u, v) = d(u, x) + d(x, y) + d(y, v),$$

$$d(v, w) = d(v, y) + d(y, z) + d(z, w),$$

$$d(w, u) = d(w, z) + d(z, x) + d(x, u).$$

Observe that, for every triplet u, v, w, a quasi-median xyz can be constructed in the following way: first select any vertex x from $I(u, v) \cap I(u, w)$ at maximal distance to u, then select a vertex y from $I(v, x) \cap I(v, w)$ at maximal distance to v, and finally select any vertex z from $I(w, x) \cap I(w, y)$ at maximal distance to w. In the case that the quasi-median is degenerate ($x = y = z$), it is a median of the triplet u, v, w. By Lemma 4.2.9, Pasch graphs are apiculate, thus also hypercellular graphs are. For any triplet u, v, w of vertices of an apiculate graph G, the vertices u, v, w admit unique apices $x := (uvw), y := (vuw)$, and $z := (wuv)$ and admit a unique quasi-median defined by the metric triangle xyz.

We say that a triplet u, v, w of vertices in an apiculate graph G admits a *median cell* (respectively, a *median cycle*) if the gated hull $\langle\langle x, y, z \rangle\rangle$ of the unique quasi-median xyz of u, v, w is a Cartesian product of cycles or a single vertex (respectively, a cycle or a single vertex). A graph G is called *cell-median* (respectively, *cycle-median*) if G is apiculate and any triplet u, v, w of G admits a unique median cell (respectively, unique median cycle or vertex). See Figure 4.15 for a visualization. This rather complicated introduction brings us to a simply expressible theorem.

Theorem 4.3.6 ([**22**]). *A partial cube $G = (V, E)$ is cell-median if and only if G is hypercellular.*

The proofs of Theorems 4.3.2, 4.3.5, and 4.3.6 is in contrast to their simple nature rather long and technical. Therefore we shall prefer to concentrate on the corollaries of these four characterizations of the family of hypercellular graphs.

We continue with geometrical properties of hypercellular graphs. The *Helly number $h(G)$* of a graph G is the smallest number $h \geq 2$ such that every finite family of (geodesically) convex sets meeting h by h has a nonempty intersection. The *Caratheodory number $c(G)$* is the smallest number $c \geq 2$ such that for any set $A \subset V$ the convex hull of A is equal to the union of the convex hulls of all subsets of A of size c. The *Radon number $r(G)$* of a graph G is the smallest number $r \geq 2$ such that any set of vertices A of G containing at least $r+1$ vertices can be partitioned into two sets A_1 and A_2 such that $\text{conv}(A_1) \cap \text{conv}(A_2) \neq \emptyset$. More generally, the *mth partition number (Tverberg number)* is the smallest integer $p_m \geq 2$ such that any set of vertices A of G containing at least $p_m + 1$ vertices can be partitioned into m sets $A_1, \ldots, A_m$ such that $\cap_{i=1}^{m} \text{conv}(A_i) \neq \emptyset$. For a detailed treatment of all these fundamental parameters of abstract and graph convexities, see [85].

Recall that the rank of a COM (thus also of a hypercellular graph) G is the maximal dimension of hypercube that G can be contracted to. In the case of hypercellular graphs this coincides with the greatest rank of a cell in G. The following result is straightforward:

Corollary 4.3.7 ([**22**]). *Let G be a hypercellular graph. Then $h(G) \leq 3$, $c(G) \leq 2 \cdot rank(G)$, and $r(G) \leq 10 \cdot rank(G) + 1$. More generally, $p_m \leq (6m-2) \cdot rank(G) + 1$.*

A *star* $\text{St}(v)$ of a vertex v (or a star $\text{St}(X)$ of a cell X) is the union of all cells of G containing v (respectively, X).

Proposition 4.3.8 ([**22**]). *For any cell X of a hypercellular graph G the star $St(X)$ is gated.*

The *thickening* G^{Δ} of a hypercellular graph G is a graph having the same set of vertices as G and two vertices u, v are adjacent in G^{Δ} if and only if u and v belong to a common cell of G. A graph H is called a *Helly graph* if any collection of pairwise intersecting balls of G has a nonempty intersection [9].

Proposition 4.3.9 ([**22**]). *The thickening G^{Δ} of a hypercellular graph G is a Helly graph.*

We conclude with properties of maps of hypercellular graphs to themselves, and with a proposition on regular ones. All of them were proved in [**22**] using the ideas already established for more specific families, hence giving additional confirmation that hypercellular graphs present a meaningful generalization of those families. Interestingly, they hold also for infinite hypercellular graphs.

Proposition 4.3.10 ([**22**]). *If G is a hypercellular graph not containing infinite isometric rays, then there exists a finite cell X in G fixed by every automorphism of G.*

Recall that a *non-expansive map* from a graph G to a graph H is a map $f : V(G) \to V(H)$ such that for any $x, y \in V(G)$ it holds $d_H(f(x), f(y)) \leq d_G(x, y)$.

Proposition 4.3.11 (**[22]**). *Let G be a hypercellular graph and let f be a non-expansive map from G to itself such that $f(S) = S$ for some finite set S of vertices of G. Then there exists a finite cell X of G that is fixed by f. In particular, if G is a finite hypercellular graph, then it has a fixed cell.*

An endomorphism r of G with $r(G) = H$ and $r(v) = v$ for all vertices v in H is called a *retraction* of G and H is called a *retract* of G.

Proposition 4.3.12 (**[22]**). *A retract H of a hypercellular graph G is a hypercellular graph.*

Proposition 4.3.13 (**[22]**). *If G is a finite regular hypercellular graph, then G is a single cell, i.e., G is isomorphic to a Cartesian product of edges and even cycles.*

These results prove that hypercellular graphs are a meaningful generalization of established families since they generalize and unify results about median graph, cellular bipartite graphs and Polat graphs. Further details about these families and their more specific properties will be given in the next chapters.

4.4 Median and almost-median graphs

Median graphs are probably the most studied family of partial cubes, with many applications and generalizations. Their analysis by far exceeds the scope of this thesis, for the summary of the most important results see [45], or for various characterizations see [59]. The aim of this section is only to understand their relations with other families of partial cubes and explore their shared properties. We start with their definition.

Definition 4.4.1. *A graph G is a* median graph *if for every triple $\{u, v, w\}$ of its vertices there is a unique vertex x, called the* median*, such that $d(u, x) + d(x, v) = d(u, v)$, $d(u, x) + d(x, w) = d(u, w)$ and $d(v, x) + d(x, w) = d(v, w)$.*

Se Figure 4.16 for an example of a median graph. We restate the definition in terms of apices defined in Section 4.2. It states that a median graph is such an apiculate graph that for every triple $\{u, v, w\}$ of its vertices apices $(uvw), (vwu), (wuv)$ coincide in one vertex. In particular, by Theorem 4.3.6, median graphs are precisely those hyercellular graphs whose median cells are trivial (isomorphic to a vertex). Furthermore, if a hypercellular graph has no minor isomorphic to a cycle C_6, then all the median cells in it must be trivial, implying that the graph is median. We have the following corollary.

Corollary 4.4.2 (**[22]**). *Median graphs are precisely the graphs in $\mathscr{F}(Q_3^-, C_6)$.*

Since the median graphs are a subfamily of hypercellular graphs, they inherit properties stated in the previous section. Furthermore some properties translate to:

- Median graphs are precisely the graphs obtained from copies of hypercubes by a sequence of gated amalgamations. This was first proved in [85] and [51].
- A partial cube is a median graph if and only if all its cells are hypercubes and 3C-condition (or equivalently 3-CC condition) is satisfied.

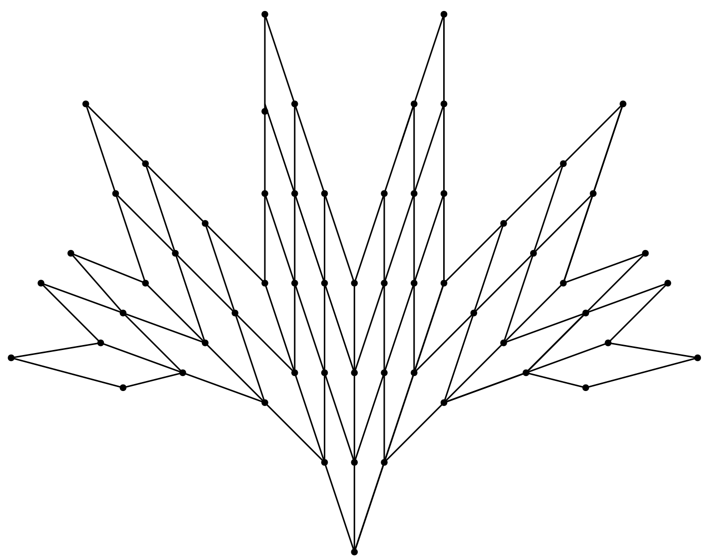


Figure 4.16: A median graph.

- If G is a median graph not containing infinite isometric rays, then there exists a finite hypercube X in G fixed by every automorphism of G. This was first proved in [84].

- Let G be a median graph and let f be a non-expansive map from G to itself such that $f(S) = S$ for some finite set S of vertices of G. Then there exists a finite hypercube X of G that is fixed by f. In particular, if G is a finite median graph, then it has a fixed hypercube. This was also first proved in [84]

- The only regular median graphs are hypercubes. This was first proved in [70].

The above properties of median graphs were known before the introduction of hypercellular graphs and were one of the motivations for the additional studies of the class and its generalizations. We point out one interesting characterization. For an edge ab define U_{ab} and U_{ba} to be the subsets of vertices in W_{ab} and W_{ba}, respectively, that are incident with an edge in relation Θ with ab.

Theorem 4.4.3 ([45])**.** *A partial cube G is a median graph if and only if for each edge ab the sets U_{ab} and U_{ba} induce convex subgraphs.*

This property was a starting point for two generalizations of median graphs. Call partial cubes in which for each edge ab the sets U_{ab} and U_{ba} induce isometric subgraphs *almost-median* graphs. Generalizing even more, call the partial cubes in which for each edge ab the sets U_{ab} and U_{ba} induce connected subgraphs *semi-median* graphs.

Noticing that the graph $Q_4^{--}(1)$ in Figure 3.2 is semi-median but includes a convex 6-cycle which is not semi-median, we conclude that semi-median graphs are not a minor-closed family. Therefore, we focus here on almost-median graphs. The first assertion of the following theorem was first proved in [57] improving the result from [16]. Here we offer an alternative proof and an additional characterization.

Theorem 4.4.4. *A partial cube is an almost-median graph if and only if it has no convex cycle of length at least 6. The later implies that almost-median graphs correspond to the family $\mathscr{F}(C_6)$.*

Proof. First notice that the definition of almost-median graphs is identical as saying that for every two edges in relation Θ there exists a traverse connecting the two edges that consists of only 4-cycles. Let G be an almost-median graph. Every contraction of G contracts traverses made of 4-cycles into traverses of 4-cycles, while every convex subgraph must include also the traverses between its edges. Hence the family of almost-median graphs is a pc-minor closed family. Now assume G includes a convex cycle C of length six or more. Then the antipodal edges of it, say ab and $a'b'$, are in relation Θ and there exists a convex traverse between the two edges made of 4-cycles. In particular the traverse is crossed by precisely the Θ-classes that C is crossed. By Lemma 3.2.3 the zone graph of the Θ-class of ab and $a'b'$ is not a well-embedded partial cube hence G must have a pc-minor in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ by Theorem 3.2.6. But all the graphs in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ are not almost-median graphs, contradicting that it is a pc-minor closed family. Thus G has no convex cycles of length at least 6.

On the other hand, if a partial cube G has no convex cycles of length at least 6, all its convex cycles must be of length 4. In particular, every pair of edges in relation Θ must have a convex traverse connecting them, by 3.1.2, which has to be made of 4-cycles. Then G is an almost-median graph.

Since C_6 is not an almost-median graph and the latter family is minor closed, it must be a subfamily of $\mathscr{F}(C_6)$. On the other hand, if a partial cube is not an almost median graph it includes a convex cycle of length at least six proving it has a pc-minor C_6. This proves the last assertion of the theorem. □

Finally, to fix its position on the map of partial cubes, first notice that all the forbidden minors of partial cubes whose zone graphs are well-embedded have a pc-minor (a convex subgraph) 6-cycle, thus almost-median graphs are a subfamily of such graphs. Another easy corollary of forbidden minor characterization is that tope graphs of LOPs are a subfamily of almost-median graphs, since $C_6 = Q_3^{--}$ is one of the forbidden minors of tope graphs of LOPs. Moreover, almost-median graphs are not a subfamily of tope graphs of COMs as one can observe in Figure 4.17.

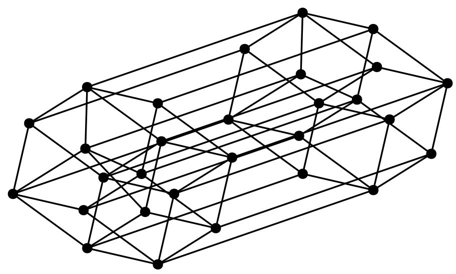


Figure 4.17: An almost-median graph $Q_5^{--}(1)$ that is not in the family of tope graphs of COMs.

We continue with the following two propositions giving more insight in the hierarchicy of families of partial cubes and explaining Figure 4.1.

Proposition 4.4.5. *The intersection of almost-median graphs and Peano graphs are precisely median graphs.*

Proof. Median graphs are a subfamily of almost-median graphs by definition, in [76] it was proved that they are a subfamily of netlike partial cubes, thus a subfamily of peano graphs.

For the sake of contradiction let G be a non-median graph that is netlike and almost-median. By Corollary 4.4.2, G has a minor Q_3^- or C_6. Since it is almost-median the minor must be Q_3^-, by Theorem 6.3.22. This is, G contains a convex subgraph that can be contracted by a sequence of contractions $G_1, \dots, G_k$ to $G_k = Q_3^-$.

Now we consider all possible expansions of Q_3^- along isometric sets G_1 and G_2. The graph can be seen as a union of three 4-cycles. If one of the 4 cycle is such that is not entirely in G_1 or in G_2, then this 4-cycle is expanded to a convex 6-cycle. If all of the 4-cycles are either entirely in G_1 or in G_2, then without loss of generality we can assume that two of the 4-cycles are in G_1. If the third cycle is also in G_1, then the expanded graph will include a convex Q_3^-. If on the other hand, the third cycle is entirely in G_2 but not entirely in G_1, then it has precisely three vertices in G_1. This implies that the third 4-cycles expands into a Q_3^-.

We have proved that every expansion of Q_3^- either contains a convex C_6 or a convex Q_3^-. Now notice that expansion of a convex subgraph is a convex subgraph in the extended graph. Since the graphs $G_1, \dots, G_k$ do not contain a convex C_6, this inductively means that they all contain a convex Q_3^-. In particular, G contains a convex Q_3^-. Let v_1, v_2, v_3 be the vertices of Q_3^- that have degree 2 inside the Q_3^- and v be the central vertex of the Q_3^-. The path P from v_1 to v_2 is a convex subgraph since the Q_3^- is convex but the convex hull of $\{P, v\}$ does not equal to the union of intervals from v to P since it also includes v_3. Hence the join-hull commutativity property does not hold in G, a contradiction with the graph being Peano. □

Proposition 4.4.6. *Intersection of Pasch graphs and almost-median graphs is included in the family of tope graphs of LOPs.*

Proof. Considering the union of the sets of the forbidden minors of Pasch graphs and almost median graphs we see that some of the graphs are not minimal with respect to the minors inclusion. In particularly all the graphs $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ have a minor C_6, thus the minimal set of the forbidden minors is $\{C_6, Q_4^-, Q_4^{--}\}$. Now each of the forbidden minors of tope graphs of LOPs $\{Q_n^{--} \mid n \geq 3\}$ has a minor in $\{C_6, Q_4^-, Q_4^{--}\}$ proving the proposition. □

We conclude this chapter with a new result about median graphs.

Proposition 4.4.7. *Median graphs are realizable COMs.*

Proof. We prove the assertion with the induction on the size of a median graph G. As proved in [45] and easily seen using the fact that for each edge ab the sets U_{ab} and U_{ba} are convex in a median graph, it holds that there exists an edge uv such that W_{ab} is peripheral. Then G is a peripheral expansion of a smaller median graph, thus by induction and Proposition 4.1.25, G is realizable. □

4.5 Tree-zone and cellular graphs

In Section 3.2 tree-zone graphs were introduced as those graphs whose all zone graphs are trees. Such graphs naturally emerged in the study the cycle space of partial cubes. Here we further analyze this graphs.

Lemma 4.5.1. *Let G be a tree-zone graph. Then no two convex cycles are both crossed by some Θ-classes E_f and E_g, $E_f \neq E_g$.*

Proof. Assume that there exist C^1, C^2 both crossed by E_f and E_g. Let a_1b_1, a_2b_2 be edges on C^1 in E_f and a_3b_3, a_4b_4 be edges on C^2 in E_f. Without loss of generality $a_1b_1, a_3b_3 \in E_g^+$ and $a_2b_2, a_4b_4 \in E_g^-$. By Lemma 3.1.2 there exists a convex traverse T_1 from a_1b_1 to a_3b_3 and a convex traverse T_2 from a_2b_2 to a_4b_4. By definition of traverse, $T_1 \in E_g^-$ and $T_2 \in E_g^+$. Then C^1, T_1, C^2, T_2 together form a cycle in the zone graph $\zeta_f(G)$. A contradiction. □

Corollary 4.5.2. *Tree-zone graphs are a pc-minor closed family.*

Proof. By Lemma 4.5.1, the assumptions of Lemma 3.2.3 are trivially satisfied in a tree-zone graph G, thus all the zone graphs of G are well-embedded partial cubes. By Lemma 3.2.4, zone graphs of every pc-minor of G are pc-minors of zone graphs of G. The only minors of trees are trees. □

Theorem 4.5.3. *Tree-zone graphs correspond to $\mathscr{F}(Q_3, Q_4^{--}(4))$.*

Proof. Both graphs $Q_3, Q_4^{--}(4)$ have a zone graph which is a cycle, thus tree-zone graphs are a subfamily of $\mathscr{F}(Q_3, Q_4^{--}(4))$. Let G be a graph that is not a tree-zone graph. If G has a zone graph that is not a well-embedded partial cube, then G has a minor $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$. All the graphs in the latter set have a minor Q_3 besides $Q_4^{--}(4)$. Thus in this case G has a minor in $\{Q_3, Q_4^{--}(4)\}$.

Now assume that all the zone graphs of G are well-embedded. Let $\zeta_f(G)$ be such that it contains a cycle. Contract all Θ-classes of G besides E_f and two classes crossing two incident convex cycles corresponding to two incident edges on the cycle of $\zeta_f(G)$. The obtained partial cube G' must have isometric dimension three, moreover by Lemma 3.2.4 its zone graph $\zeta_f(G')$ must have a 4-cycle. Then G must be isomorphic to Q_3. This finishes the proof. □

An interesting example of tree-zone graphs are so called *benzenoid* structures. These graphs, motivated by chemistry, are defined as the unions of 6-cycles lying inside a hexagonal grid without forming a hole. It is easy to see that all the zone graphs of benzenoids are isomorphic to paths (see Figure 4.18).

A similar structure called *cellular* bipartite graphs was introduced in [7]. As proved it can be defined in one of the following equivalent ways.

Theorem 4.5.4 ([7])**.** *A bipartite graph G is cellular if it satisfies one of the following equivalent conditions:*

(a) G is obtained as a gated amalgamation from even cycles and edges.

(b) *Every isometric cycle in G is gated and there are no tree isometric cycles sharing exactly a vertex and pairwise sharing exactly an edge in G.*

(c) *conv*$(S) = \cup_{v,u \in S} I(u,v)$

(d) *conv*$(\{u,v,w\}) = I(u,v) \cup I(v,w) \cup I(w,u)$

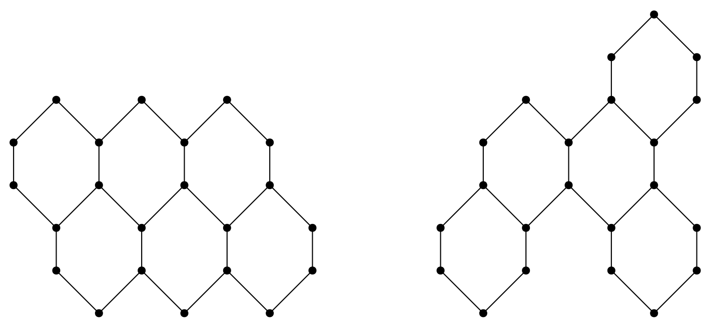


Figure 4.18: Tree-zone graph with only the second being cellular bipartite graph.

As it is clearly seen by the characterization (a) cellular graphs are precisely those hypercellular graphs whose cells are even cycles, edges, or vertices. Thus the characterization (b) could also be strengthen by Theorem 4.3.5 to:

(b') *There are no three convex cycles sharing exactly a vertex and pairwise sharing exactly an edge in G.*

Also notice that (c) is a generalization of the join-hull commutativity property, a direct way to see that cellular graphs are Peano graphs in contrast to seeing them as a subfamily of hypercellular graphs. The position of cellular graphs as hypercellular with no high-dimensional cubes immediately gives the following characterization:

Theorem 4.5.5 (**[22]**). *The cellular bipartite graphs correspond exactly to the family* $\mathscr{F}(Q_3^-, Q_3)$.

Similar as in the case of median graphs, properties of cellular graphs inherited from hypercellular graphs translate to:

- Cellular bipartite graphs have the median cycle property.
- If G is a cellular bipartite graph not containing infinite isometric rays, then there exists a finite cycle, an edge or a vertex in G fixed by every automorphism of G.
- Let G be a cellular bipartite graph and let f be a non-expansive map from G to itself such that $f(S) = S$ for some finite set S of vertices of G. Then there exists a finite cycle, edge or a vertex of G that is fixed by f. In particular, if G is a finite graph, then it has a fixed cycle, edge or a vertex.
- The only regular median graphs are even cycles, an edge or a vertex.

To our knowledge all the above results have not been observed before **[22]**. Interestingly, Polat in [78] studied netlike partial cubes that have a median cycle property and showed that these are precisely graphs obtained by gated amalgamation from hypercubes and even cycles. We will call them *Polat* graphs. Similarly as above we have:

Theorem 4.5.6 (**[22]**). *Polat graphs correspond to the family* $\mathscr{F}(Q_3^-, K_2 \square C_6)$.

Notice that it is easily seen from the above, that all cellular bipartite graphs and median graphs are Polat graphs, thus netlike partial cubes. Also, Polat graphs inherit properties of hypercellular graphs that were not observed before: a fixed hypercube of an automorphism or an endomorphism with $f(S) = S$ for some finite set S, classification of the regular cellular graphs and a characterization as partial cubes with with the median cycle property.

Finally we settle the last position of the intersection of families of partial cubes in the next proposition.

Proposition 4.5.7. *The intersection of tree-zone graphs and Peano graphs is a subfamily of netlike partial cubes.*

Proof. Let G be a tree-zone graph with the Peano property. By Theorem 4.2.11, $\mathrm{ph}(G) \leq 1$. Thus by the definition of netlike partial cubes, we only need to prove that the convex hull of any non-convex isometric cycle is a hypercube. But since hypercubes of dimension three or more cannot be convex subgraphs of tree-zone graphs by Theorem 4.5.3, we in fact need to prove that all isometric cycles of G are convex.

Let C be an isometric cycle of G and $u_1v_1\Theta u_2v_2$ two edges on it. Let T be a convex traverse from u_1v_1 to u_2v_2. If the shortest u_1,u_2- and v_1,v_2-paths in C are the sides of the traverse T, then T must consist from exactly one convex cycle, namely C. Hence assume that, say the u_1,u_2-path in C is not the side of T. It follows from Lemma 3.4.3 that then there exists a convex cycle on vertices $w_0,\dots,w_{2k}$ such that vertices $w_0,\dots,w_n$ induce a path that lies on a side of T. Since Peano graphs and also three-zone graphs are a subfamly of well-embedded partial cubes, no two convex cycles intertwine. In particular, this implies that C shares an edge with two convex cycles C^1, C^2 and all three share a vertex. Moreover, at least one of them must be a 4-cycle, since C has half of its edges on the side of T and there is no intertwining in G. Without loss of generality assume C is a 4-cycle, otherwise rename the cycles.

Let v be the vertex in the intersection $C \cap C^1 \cap C^2$ and vu_1, vu_2, vu_3 edges in the intersection $C \cap C^1, C \cap C^2, C^1 \cap C^2$, respectively. Then W_{u_3v} is a maximal convex set that does not include v, by Lemma 2.3.2. Since G is a Peano graph, $\mathrm{ph}(G) \leq 1$, by Theorem 4.2.11. In particular, $\mathrm{conv}(W_{u_3v} \cup \{v\}) = I(W_{u_3v} \cup \{v\})$. Clearly C^1 and C^2 are inside $\mathrm{conv}(W_{u_3v} \cup \{v\})$, hence u_1 and u_2 are included. But since C is a 4-cycle, also $C \subseteq \mathrm{conv}(W_{u_3v} \cup \{v\})$. Let u_4 be the vertex of C adjacent to u_1, u_2 and different from v. Then there must be a shortest path P from W_{u_3v} to v passing through u_4. The latter implies that there exist an edge in E_{u_3v} on P that is in $W_{u_1v} \cap W_{u_2v}$. Moreover the edges on $C^1 \setminus C^2, C^1 \cap C^2, C^2 \setminus C^1$ in E_{vu_3} are in $W_{u_1v} \cap W_{vu_2}, W_{vu_1} \cap W_{vu_2}, W_{vu_1} \cap W_{u_2v}$, respectively. Hence contracting all the Θ-classes in G besides $E_{vu_1}, E_{vu_2}, E_{vu_3}$ gives a hypercube, contradictory to the assumption that G is a tree-zone graph. □

5

Highly symmetric partial cubes

Hypercubs are considered to be one of the classic examples of graphs that possess many symmetries. It is a fundamental question to ask how those symmetries are preserved on their subgraphs. To our knowledge the first ones who addressed this question were Brouwer, Dejter and Thomassen in 1992 in [18]. They provided many surprising and diverse examples of vertex-transitive subgraphs of hypercubes, but did not make a classification. Based on their results, examples are very diverse hence a classification seems too ambitious. They suggested that one of the reasons for the latter is that the group of symmetries of a subgraph of a hypercube need not be induced by the group of symmetries of the hypercube.

On the other hand, a complete opposite occurs if one considers only median graphs. As seen in Section 4.4, but first proved in [70], the hypercubes are the only finite regular median graphs, thus the only finite vertex-transitive. As observed in Section 4.3, the complete classification can be obtained also in the case of hypercellular graphs, where cells are the only examples. The structure of median (and possibly also hypercellular) graphs is so limiting that the latter results can even be generalized to infinite graphs of bounded growth, see [63, 64]. Symmetric partial cubes seem a perfect balance between the (too large) variety of symmetric subgraphs of hypercubes and (too well-structured) regular median graphs.

In this chapter we first focus on regular partial cubes describing the history of their research and then present results concerning vertex-transitive partial cubes.

5.1 Regular partial cubes

The study of regular partial cubes began in 1992 by Weichsel in [86], independent of studies of median graphs. He considered distance-regular subgraphs of hypercubes, i.e., he did not in advance limit himself only to isometric subgraphs, but he imposed a striker criterion on the regularity. He derived certain properties of them, and noticed that all his examples are not just subgraphs, but isometric subgraphs of hypercubes. It was thus a natural decision to focus on the symmetries of partial cubes. He classified all distance-regular partial cubes based on their girth: hypercubes are the only ones with girth four, the six cycle and the middle level graphs are the only ones with girth six, and even cycles of length at least eight are the only ones with higher girths. Notice that all these graphs are vertex-transitive, therefore they are a subfamily of vertex-transitive subgraphs of hypercubes.

A hunt for regular partial cubes began in more recent years, particularly focusing on the cubic case. The topic became interesting after a computer search was made in [15], showing that besides prisms only three other cubic partial cubes on at most 30 vertices exist. Moreover they managed to construct more cubic partial cubes by modifying known ones. On the other hand, Brešar et al. took a different approach in [17], and used so called cubic inflation – a method for constructing cubic graphs on surfaces – to find a few new regular (even vertex-transitive) partial cubes. Klavžar et al. analyzed the family of generalized Petersen graphs in [55] to prove that there is only one cubic partial cube among them, namely $G(10,3)$.

A connection between partial cubes and geometric structures given by the subfamily of COMs (Section 4.1) was used in [29, 32] where other regular partial cubes were found. In particular, Eppstein analyzed arrangements of (pseudo) lines in a plane leading to cubic partial cubes. Finally, a systematic approach was taken in [56] to group together various examples and to form so called tribes of cubic partial cubes. Furthermore, they used their method to find some new examples and explained how to produce even more of them.

With [56] it became clear that the variety of regular partial cubes is probably too big to allow an explicit classification of its graphs. Nevertheless, the examples show certain unexpected properties. In Section 3.4, we showed that partial cubes with minimum degree at least 3 have girth at most 6 hence Corollary 3.4.7 states that there are no regular partial cubes with girth 8 or more besides even cycles, an edge, or a vertex. Other properties remain unproven for several years:

Are all cubic partial cubes besides $G(10,3)$ planar graphs?

Barnette's conjecture states that all cubic, 3-connected, planar bipartite graphs are Hamiltonian. Since all the examples of cubic partial cubes are 3-connected and also $G(10,3)$ is Hamiltonian, the following is also in question:

Are all cubic partial cubes Hamiltonian?

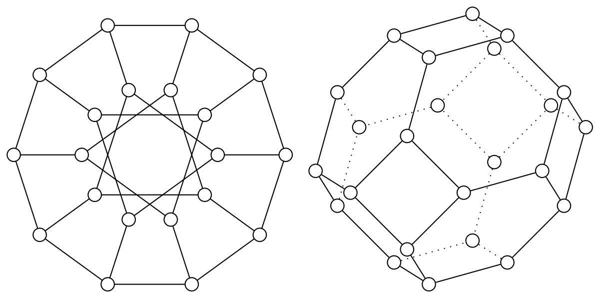
5.2 Vertex-transitive partial cubes

The main result presented in this section is the classification of cubic, vertex-transitive partial cubes. Furthermore, we summarize the ideas that we used to prove the theorem in [66].

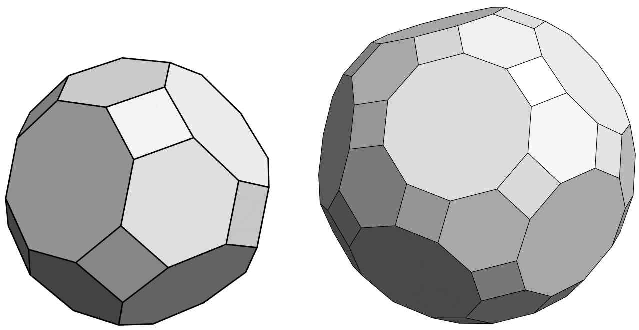
Let K_2 denote the complete graph of order 2, C_k the cycle of length k, and $G(n,k)$ the generalized Petersen graph with parameters $3 \leq n, 1 \leq k < n/2$. The main result of the mentioned paper is the following:

Theorem 5.2.1 ([**66**]). *If G is a finite, cubic, vertex-transitive partial cube, then G is isomorphic to one of the following: $K_2 \square C_{2n}$, for $n \geq 2$, $G(10,3)$, the cubic permutahedron, the truncated cuboctahedron, or the truncated icosidodecahedron.*

To our surprise, the variety of the graphs from Theorem 5.2.1 (cf. Figure 5.1) is small, and all the graphs are classical graphs that were studied from many (especially geometric) views. We point out that the cubic permutahedron, the truncated cuboctahedron, and the truncated icosidodecahedron are cubic inflations of graphs of platonic surfaces [17], $K_2 \square C_{2n}$ are the only cubic Cartesian products of (vertex-transitive) partial cubes (this



(a) $G(10,3)$ (b) Cubic permutahedron



(c) Truncated cuboctahedron (d) Truncated icosidodecahedron

Figure 5.1: The four sporadic examples of cubic, vertex-transitive partial cubes

includes also the hypercube $Q_3 \cong K_2 \square C_4$), while $G(10,3)$ is, as already stated, the only known non-planar cubic partial cube and is isomorphic to the middle level graph of valence three [55]. All the graphs besides $G(10,3)$ are tope graphs of OMs of rank 3.

The main idea how to prove Theorem 5.2.1 is to consider convex cycles incident with some vertex (since the graphs are vertex-transitive this local property is independent of the choice of the vertex). In particular we proceeded in the following way:

- First we analyzed the case if every vertex is incident with two (convex) 4-cycles. This is the easy case and it immediately implies that the graph in this case must be isomorphic to $K_2 \square C_{2n}$, for some $n \geq 2$.
- On the other hand, if the girth of the graph is 6, then Proposition 3.4.1 implies that each vertex must be incident with at least two convex 6-cycles and a simple case analysis leads to the proof that in this case the graph must be isomorphic to $G(10,3)$.
- By Corollary 3.4.7, the girth of a regular cubic partial cube can only be 4 or 6. Thus in the remaining cubic, vertex-transitive partial cubes every vertex must be incident with precisely one 4-cycle. It turns out that the only remaining cases are that every vertex is incident with additionally two convex 6-cycles, or a convex 6-cycle and some convex cycle of greater length.
- The case when every vertex is incident with a 4-cycle and two convex 6-cycles leads to the graph being the cubic permutahedron, by embedding it on surface and counting its vertices.

- The final case when every vertex is adjacent with a 4-cycle, a convex 6-cycle and some convex cycle of greater length is the hardest. To complete the proof, an analysis of intertwining (Section 3.3) is needed to prove that the third cycle must be an 8- or 10-cycle and then to establish a connection with Cayley graphs. In this case the graph must be isomorphic to the truncated cuboctahedron, or the truncated icosidodecahedron.

Since the variety of such graphs is rather small, it suggests that a similar classification can be done for graphs with higher valencies. The latter problem is wide open, the classifications are not known even for Cayley graphs that are partial cubes. The regular graphs in the subcubic cases can be seen as the beginnings of greater families of vertex-transitive partial cubes:

- Middle level graphs were in Section 2.1 defined as the induced subgraphs of Q_n, $n > 1$ odd, on all the vertices with $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ coordinates equal to 1. It is clear that such graphs are vertex transitive, it is not hard to see that they are partial cubes. $G(10,3)$ is the second simplest middle level graph for $n = 5$ right after a 6-cycle for $n = 3$.

- The cubic permutahedron, the truncated cuboctahedron, the truncated icosidodecahedron, even cycles, and K_2 are Cayley graphs of finite Coxeter groups as we will see in Section 5.3 in more details.

- The Cartesian product of vertex-transitive graphs is clearly vertex-transitive. Even prisms are just the Cartesian products of cycles and edges, similarly we can construct many others.

To our knowledge these are the only known examples of vertex-transitive partial cubes.

5.3 Mirror graphs

Brešar et al. gave the following definition in [17]. Let G be a simple, connected graph. Call a partition $P = \{E_1, E_2, \dots, E_k\}$ of edges in G a *mirror partition* if for every $i \in \{1, \dots, k\}$, there exists an automorphism α_i of G such that:

(i) for every edge $uv \in E_i$: $\alpha_i(u) = v$ and $\alpha_i(v) = u$

(ii) $G - E_i$ consists of two connected components G_i^1 and G_i^2, and α_i maps G_i^1 to G_i^2

A graph that has a mirror partition is called a *mirror graph*. By definition they are highly symmetrical graphs. In [17] it was shown that all mirror graphs are vertex-transitive, and certain connections with regular maps and polytope structures were established, indicating strong geometric properties of these graphs. An even more surprising result that they provided is that every mirror graph is a partial cube and that the mirror partition of the edges corresponds to partition in the Θ-classes. Therefore the study of mirror graphs is in fact the study of partial cubes with symmetrical halfspaces.

In **[67]** we characterized mirror graphs in a way that led to the complete classification of such graphs. We present the result here, but first a few definitions. Recall from Section

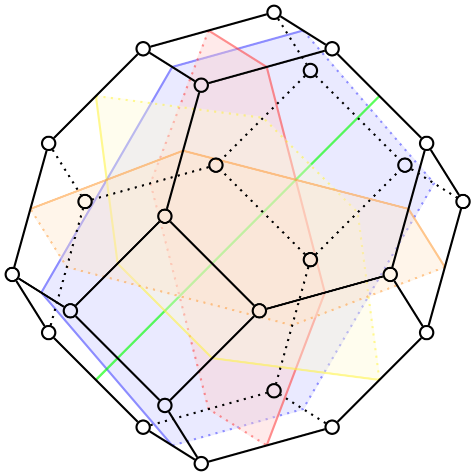


Figure 5.2: Example of a mirror graph: the cubic permutahedron with the corresponding hyperplane arrangement

4.1 that a partial cube is a tope graphs of realizable oriented matroid if it can be obtained from a central arrangement of hyperplanes in a Euclidean space. For a hyperplane H_i with an orthogonal vector v_i its *reflection* is the map $\sigma_{H_i}(x) = x - 2\frac{x \cdot v_i}{v_i \cdot v_i} v_i$. A central arrangement of hyperplanes $\{H_1, \dots, H_m\}$ in $\mathbb{R}^n$ such that for every $i \in \{1, \dots, m\}$ the reflection of H_i permutes the hyperplanes $\{H_1, \dots, H_m\}$ are called *reflection arrangements*. A simple example is a collection of m central lines in a plane such that the angle between lines and a chosen axis is $\frac{i}{\pi}$, while for a more complicated example see Figure 5.2.

A *Coxeter group* is a group which can be presented by generators and relations as $\langle \alpha_1, \dots, \alpha_m \mid (\alpha_i \alpha_j)^{k_{ij}} = 1$ for all $1 \leq i, j \leq m\rangle$, where $k_{ii} = 1$ and $k_{ij} \geq 2$ for all $0 \leq i < j \leq m$. By a classical result [14, Theorem 2.3.7], reflection arrangements are in one to one correspondence with finite Coxeter groups since the tope graphs of reflection arrangements are the Cayley graphs of finite Coxeter groups and vice versa. Moreover finite Coxeter groups were classified by Coxeter [26]. They give rise to four infinite families and six exceptional cases of irreducible reflection arrangements. Here irreducible means that there is no non-trivial partition of the hyperplanes in two mutually orthogonal classes; equivalently, their tope graphs are not the Cartesian product of smaller tope graphs.

The characterization of mirror graphs we provided is the following.

Theorem 5.3.1 (**[67]**). *For a graph G the following statements are equivalent:*

(i) G is a mirror graph.

(ii) G is the Cayley graph of a finite Coxeter group with canonical generators.

(iii) G is the tope graph of a reflection arrangement.

As stated above, the finite Coxeter groups are classified. The classification is usually denoted by $A_n, B_n, D_n, I_2(p), E_6, E_7, E_8, F_4, H_3, H_4$, where the first four are infinite families and the rest are sporadic cases. This translates to the following partial cubes. The family

$I_2(p)$ corresponds to even cycles. Families A_n, B_n, D_n give rise to three infinite families of partial cubes, for each $n \geq 3$ the corresponding graphs are n-valent, with rank n and of order $(n+1)!, 2^n n!, 2^{n-1} n!$, respectively. Particularly, in the case $n = 3$ the Cayley graph of A_3 and D_3 are both isomorphic to the cubic permutahedron and the Cayley graph of B_3 is isomorphic to the truncated cuboctahedron (Figure 5.1). For H_3, its Cayley graph is the truncated icosidodecahedron while for E_6, E_7, E_8, F_4, H_4 their Cayley graphs are: a 6-valent graph on 51840 vertices, a 7-valent graph on 2903040 vertices, an 8-valent graph on 696729600 vertices, a 4-valent graph on 1152 vertices, and a 4-valent graph on 14400 vertices, respectively.

The idea how to prove Theorem 5.3.1 is to prove the following results.

Lemma 5.3.2 (**[67]**)**.** *A mirror graph G is an antipodal partial cube.*

Lemma 5.3.3 (**[67]**)**.** *Let G be an antipodal partial cube, and E_{ab}, E_{cd} two of its Θ-classes. Then there exists a convex cycle in G that includes edges from E_{ab} and E_{cd}.*

Lemma 5.3.4 (**[67]**)**.** *For an antipodal partial cube G and a Θ-class E_{ab} in G, there exists at most one automorphism α_{ab} of G such that for each $uv \in E_{ab}$ it holds $v^{\alpha_{ab}} = u$ and $u^{\alpha_{ab}} = v$. Moreover, $\alpha_{ab}^2 = 1$.*

Recalling that the mirror partition in fact corresponds to the partition into Θ-classes, the above states that the mirror automorphisms are very limited.

Observation 1 (**[67]**)**.** An automorphism α of a partial cube G is completely determined by the permutation of the Θ-classes of G and the image v^α of some vertex v of G.

Additionally, the space of convex cycles needs to be analyzed to prove the correspondence with the Cayley graphs of Coxeter groups. The Lemmas 5.3.2 and 5.3.4 and Observation 1 led to a polynomial algorithm presented in Algorithm 2, that for a graph G with n vertices and m edges decides if G is a mirror graph, and in the positive case outputs its mirror partition and mirror automorphisms **[67]**.

An interesting problem regarding mirror graphs is to generalize Theorem 5.3.1 to infinite graphs. In fact, it is not known if the Cayley graphs of infinite Coxeter groups with canonical generators are partial cubes (and in this case if they are mirror graphs), or if infinite mirror graphs are a subfamily of the Cayley graphs of infinite Coxeter groups.

Algorithm 2 Recognition of mirror graphs **[67]**

1. Check if G is a partial cube by calculating the Θ-classes and obtaining its embedding in a hypercube. This can be done in $O(n^2)$ by Proposition 2.2.4. The Θ-classes are candidates for the mirror partition of G. If G is not a partial cube, it is not a mirror graph.

2. For each Θ-class F_{ab}, its corresponding mirror automorphism α_{ab}, if existent, must map all the convex cycles crossed by F_{ab} to themselves. By Lemma 5.3.3 this determines the image of each Θ-class, and thus by Observation 1 gives a candidate for the mirror automorphism. Convex cycles of G can be found in $O(mn^2)$ by [49], obtaining at most $O(nm)$ of them by [3]. Iterating through convex cycles we can determine for each Θ-class how its corresponding mirror automorphism permutes the other Θ-classes.

3. Considering G embedded in a hypercube, each permutation of Θ-classes can be seen as a permutation of coordinates of the hypercube that G is embedded into, and thus as an automorphism of the hypercube. Hence it can be checked if the candidates for the mirror automorphisms in fact define automorphisms of G by checking if they map the vertices of G to vertices of G. If so, we output the Θ-classes and the corresponding mirror automorphisms.

6

Computational and computed properties

6.1 Generating partial cubes

In this section we present an approach to generating and enumerating partial cubes up to reasonable sizes. Even though the structure of partial cubes is very limiting their variety is large. In particular, notice that the number of non-isomorphic trees on 20 vertices is 823065 and on 30 vertices already 40330829030. Hence generating partial cubes bounded in the number of vertices in this way leads to time and space complexity difficulties already before generating, say, a simple 5-cube on 32 vertices. For this reason rather than bounding the number of vertices in partial cubes we want to generate, we bound the isometric dimension of the cubes. This way we can obtain partial cubes with already complicated structure, but not too many of them.

The idea that works good is to generate isometric subgraphs of a fixed hypercube by a mixed integer-linear program searching for possible solutions. For a hypercube Q_n and vertices $v, u \in Q_n$ we denote with $N_v(u)$ the set of all the neighbors of u that are closer to v that u is. We write conditions of a mixed integer-linear program without a maximization function in the following way:

$$\begin{aligned} x_v &\in \{0,1\} && \text{for all } v \in Q_n \\ \Big(\sum_{w\in N_v(u)} x_w\Big) - x_v - x_u &\geq -1 && \text{for all non-identical } v,u \in Q_n \end{aligned}$$

The variables x_v for $v \in Q_n$ indicate which vertices of the hypercube we want to include in the partial cube. On the other hand, the second argument states that if for some pair $u, v \in Q_n$ both the vertices are included, then there must be a neighbor of u closer to v than u also included. Demanding the latter for all the pairs of vertices, for each two included vertices also a shortest path connecting them must be included.

We use Cplex [27] for obtaining the pool of all possible solution to the above condition. After obtaining the solutions we reduce them to the isomorphism classes using Sagemath [83]. For hypercube Q_5, the program gives 2345 non-isomorphic partial cubes embeddable into it. To find all partial cubes embeddable in Q_6 an additional trick is needed: for each of 2345 partial cubes embeddable in Q_5, say G, we generate the above mixed integer-linear program with vertices in Q_6 and fix the variables corresponding to vertices with the first coordinate equal to 1 to induce precisely G. This way only 2^5 variables are undetermined. This procedure leads to 13491182 non-isomorphic partial cube embeddable in Q_6, taking

Isometric dimension	0	1	2	3	4	5	6
All partial cubes	1	1	2	7	48	2286	13488837
Trees	1	1	1	2	3	6	11
Median graphs	1	1	2	5	18	90	736
Cellular graphs	1	1	2	5	17	77	501
Polat graphs	1	1	2	6	21	112	925
Netlike partial cubs	1	1	2	6	21	112	926
Hypercellular graphs	1	1	2	6	22	119	1025
Peano graphs	1	1	2	6	22	119	1026
Tree-zone graphs	1	1	2	6	33	935	265136
Pasch graphs	1	1	2	7	41	1257	845609
Tope graphs of LOP	1	1	2	6	36	1249	2933377
Tope graphs of COMs	1	1	2	7	43	1476	3204305
Almost-median graphs	1	1	2	6	38	1395	3153906
Well-embedded partial cubes	1	1	2	7	43	1482	3269284
Tope graphs of OMs	1	1	1	2	4	9	35
Antipodal partial cubes	1	1	1	2	4	13	115

Table 6.1: Enumerative properies of families of partial cubes up to isometric dimension 6.

approximately 1.4GB of memory to save. Generating all partial cubes embeddable in Q_7 seems implausible with respect to time and memory suitable for contemporary computers.

6.2 Computed properties

A list of partial cubes from the previous section allows us to analyze small partial cubes. Particularly we are interested in enumerating partial cubes being part of the families presented in Chapter 4. Since most of the families presented are minor closed families with finite list of excluded minors, we can by Proposition 2.4.1 use a polynomial algorithm for their recognition. Notice that for certain classes, more efficient algorithms have been developed, but were not needed here. The results are presented in Table 6.1.

6.3 Computational properties of isomorphisms of partial cubes

For many computational aspects of partial cubes it is crucial to compute if two partial cubes are isomorphic or not. A practical solution is offered by implementation called *nauty* [68], while in this chapter we settle the theoretical properties of this problem. Recall that the problem of deciding if two graphs are isomorphic is up to date best solved in quasipolynomial time $e^{(\log n)^{O(1)}}$ where n is the number of vertices [4]. A problem equivalent to the latter problem is said to be *GI-complete*. We find the following result surprising since median graph are often seen as a generalization of hypercubes and trees, isomorphism problem of which can be solved in linear time [53].

Theorem 6.3.1. *Deciding if two median graphs are isomorphic is GI-complete.*

Proof. Say G_1 and G_2 are two connected graphs. First form new graphs G_1' and G_2' from G_1 and G_2, respectively, by subdividing each edge. Subdivision vertices can easily be recognized as a half of a bipartition whose all vertices have degree two. Such recognition is unique if and only if the graphs are not cycles in which case the graphs are isomorphic if they have the same number of vertices. This implies that G_1' and G_2' are isomorphic iff G_1 and G_2 are.

Now form new graphs G_1'' and G_2'' from G_1' and G_2' in the following way: vertices of each new graph are vertices and edges of the old graph plus an additional special vertex. Two vertices in the new graph are adjacent if one is a vertex and the other is an incident edge in the old graph or if one is the special vertex and the other is an arbitrary vertex in the old graph. In [50], where such a construction was introduced, it was proved that that the obtained graphs are median if and only if the starting graphs are triangle-free. Moreover the special vertex is uniquely recognizable as the maximum degree vertex (and thus also vertices and edges in the starting graph are uniquely recognizable) if the starting graph is not a star. Since graphs G_1' and G_2' were subdivision graphs, they are triangle-free and are stars only if G_1, G_2 are isomorphic to K_2. It follows that G_1'' and G_2'' are isomorphic if and only if G_1 and G_2 are isomorphic. Since the final graphs have $n+3m+1$ vertices, this is a polynomial reduction proving that the problem is GI-complete. □

maybe add some conjectures about partial cubes

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Povzetek

Grafi v hiperkockah

Teza se ukvarja s podgrafi, ki jih lahko najdemo v grafu hiperkocke:

Definicija 2.1.1 Hiperkocka Q_n *dimenzije* n *je graf z vozlišči* $\{1,0\}^n$ *in povezavami med tistimi pari vozlišč, ki se razlikujejo v natanko eni koordinati.*

Ta navidezno preprosta struktura v sebi skriva veliko neintuitivnih lastnosti povezanih z višje dimenzijskimi prostori in veliko nerešenih problemov. V zadnjem času najbolj razvijajoče področje teorije hiperkock je raziskovanje njenih podgrafov, ki imajo podobne metrične lastnost kot sama hiperkocka. Definirajmo take podgafe:

Definicija 2.2.1 Delna kocka *je graf* G*, ki ga lahko izometrično vložimo v graf hiperkocke* Q_n*, tj. vložitev je taka, da velja* $d_G(u,v) = d_{Q_n}(u,v)$ *za poljubni vozlišči* u *in* v *grafa* G*, kjer* d *označuje funkcijo razdalje v grafu.*

Najpomembnejša definicija v študiju delnih kock je definicija relacije Θ. Za povezavo $e = uv$ poljubnega grafa G, definirajmo $W_{uv} = \{x \in V : d(x,u) < d(x,v)\}$ in $W_{vu} = \{x \in V : d(x,v) < d(x,u)\}$ in ju imenujmo *polprostora*. Povezavi uv in $u'v'$ sta v relaciji Θ, $uv\Theta u'v'$, če velja $u' \in W_{uv}$ and $v' \in W_{vu}$.

Imenujmo podgraf H grafa G *konveksen*, če so za poljubni vozlišči u, v iz H vse najkrajše poti med u in v v G tudi vsebovane v H. Djoković in Winkler sta dokazala, da je G delna kocka natanko tedaj, ko je graf dvodelen in je relacija tranzitivna. Še več, v dvodelnem grafu G je lastnost biti delna kocka ekvivalentna temu, da sta za vsako povezavo uv polprostora W_{uv} in W_{vu} konveksna. Če imamo delno kocko izometrično vloženo v hiperkocko, potem so Θ-razredi preprosto povezave, ki ustrezajo spremembi neke koordinate hiperkocke in polprostora W_{uv}, W_{vu} sta razdelitev vozlišč v množici, ki imajo ustrezno koordinato 1 oziroma 0. Število Θ-razredov grafa G, oziroma ekvivalentno dimenzijo najmanjše hiperkocke, v katero je G izometrično vložljiv, imenujemo izometrična dimenzija G-ja.

Ker je presek konveksnih množic konveksen, da je vsak presek polprostorov konveksen. Imenujmo take preseke polprostorov *restrikcije*. Velja še več:

Lema 2.3.2 *Množica vseh restrikcij v delni kocki se ujema z množico vseh konveksnih podgrafov.*

Po drugi strani graf G/E_f dobljen s s skrčitvijo vseh povezav v delni kocki G, ki so v

Θ-razredu E_f, imenujemo *skrčitev* G-ja. Ni težko videti, da je vsaka skrčitev delne kocke prav tako delna kocka vložljiva v hiperkocko z izometrično dimenzijo ena manj kot jo ima G.

Zgornje definicije omogočajo osnovno orodje za delo z delnimi kockami. Naj bo graf G' dobljen iz grafa G s pomočjo zaporedja skrčitev in restrikcij. Tedaj rečemo, da je graf G' *minor* grafa G. Za poddružino delnih kock $\mathscr{C}$, v kateri velja, da če $G \in \mathscr{C}$ in G' je minor G-ja, potem je tudi $G' \in \mathscr{C}$, pravimo, da je *zaprta za minorje*. Vsaka taka družina ima (mogoče neskončno) množico delnih kock X, za katere velja, da niso v družini, ampak vsak njihov minor pa je. Obratno, za množico delnih kock $X = \{T_1, T_2, \ldots\}$ označimo s $\mathscr{F}(X)$ množico vseh delnih kock, ki nimajo minorja v X. Taka množica je seveda zaprta za minorje, če je število prepovedanih minorjev končno, jo lahko tudi algoritmično prepoznamo:

Lema 2.4.1 *Naj bo X končna množica delnih kock. Za vsako delno kocko G lahko v polinomskem času odločimo, če je v $\mathscr{F}(X)$.*

Predstavimo še par definicij potrebnih za razumevanje glavnih rezultatov te teze. Delna kocka G vložena v hiperkocko Q_n je *antipodalna*, če za vsako vozlišče $v = (i_1, i_2, \ldots, i_n)$ obstaja vozlišče $-v$ v G vloženo kot $-v = (\bar{i_1}, \bar{i_2}, \ldots, \bar{i_n})$, kjer $\bar{i_j} = 1 - i_j$. Tudi konveksen podgraf H imenujemo antipodalen, če ima vsako vozlišče v H svoje antipodalno vozlišče v H. Antipodalne delne kocke imajo zanimive lastnosti, kot na primer, da je vsaka njihova skrčitev tudi antipodalna ter:

Lema 2.5.5 *V antipodalne delni kocki G je* antipodalna *preslikava $v \mapsto -v$ avtomorfizem grafa.*

Podgraf H grafa G imenujemo *zastražen*, če za vsako vozlišče x v G izven H obstaja neko vozlišče v_x v H, tako da lahko iz v pridemo po najkrajši pot do poljubnega vozlišča v H preko x_v. Tudi zastraženi grafi v delnih kockah imajo lastnosti, ki se dobro povezujejo z minorji, saj je vsaka slika zastraženega podgrafa restrikcije ali skrčitve prav tako zastražen podgraf.

Cikli v delnih kockah

Delne kocke si delijo mnogo podobnosti s hiperkockami. Urejenost ciklov je ena izmed takih lastnosti. Definirajmo eno izmed struktur, ki jo tvorijo cikli v delnih kockah. Naj bosta povezavi $v_1u_1 \Theta v_2u_2$ v delni kocki G, kjer $v_2 \in W_{v_1u_1}$. Naj bo $C^1, \ldots, C^n$, $n \geq 1$ zaporedje izometričnih ciklov, kjer v_1u_1 leži samo na C^1, v_2u_2 leži samo na C^n in je presek vsakega para C^i ter C^{i+1} za $i \in \{1, \ldots, n-1\}$ natanko povezava, ki leži v $E_{v_1u_1}$, ostale pari se ne sečejo. Če je najkrajša pod iz v_1 v v_2 na uniji ciklov $C^1, \ldots, C^n$ tudi najkrajša pot iz v_1 do v_2 v G, potem imenujemo $C^1, \ldots, C^n$ *traverza* iz v_1u_1 do v_2u_2. Če so cikli na traverzi vsi konveksni, potem jo imenujemo *konveksna traverza*. Presenetljiv in izjemno uporaben rezultat je sledeči:

Lema 3.1.2 *Naj velja $v_1u_1 \Theta v_2u_2$ v delni kocki G. Potem obstaja konveksna traverza iz v_1u_1 v v_2u_2.*

Naj bo $\mathscr{C}(G)$ celični kompleks dobljen tako, da nadomestimo vsak konveksen cikel C dolžine $2j$ v G s pravilnim dvodimenzionalnim $2j$-kotnikom $[C]$. S pomočjo konveksnih traverz se da pokazati sledeče:

Trditev 3.1.3 *Za delno kocko G je celični kompleks $\mathbf{C}(G)$ enostavno povezan.*

S pomočjo konveksnih ciklov lahko tvorimo tudi nove grafe iz delne kocke G. Naj bo E_f eden izmed Θ-razredov v G. *Graf con* $\zeta_f(G)$ grafa G in Θ-razreda E_f je graf, katerega vozlišča so povezave v E_f in dve vozlišči povezani, če ustrezni povezavi ležita na konveksnem ciklu v G. Izkaže se, da so grafi con vedno povezani, v najpreprostejšem primeru so to drevesa. Grafom, v katerih so vsi grafi con drevesa pravimo *drevesne* delne kocke.

Operacijo ζ_f lahko vidimo kot preslikavo iz povezav v E_f v vozlišča novega grafa ter iz povezav grafa G, ki niso v E_f, ampak ležijo na konveksem ciklu s povezavama iz E_f, v povezave novega grafa. Vendar tako dobljeni graf ni nujno delna kocka. Če pa velja, da je, in da za vsaki povezavi $a\Theta b$ iz grafa con $\zeta_f(G)$ vsi Θ-razredi, ki prečijo $\zeta_f^{-1}(a)$, ustrezajo Θ-razredom, ki prečijo $\zeta_f^{-1}(b)$, potem rečemo, da je graf con *dobro vložen*. Naj bodo grafi $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$ definirani kot na Sliki 3.2.

Izrek 3.2.6 *Družina delnih kock, katerih vsi grafi con so dobro vloženi je zaprta za minorje in ustreza družini $\mathscr{F}(\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\})$.*

Cikli v delnih kocka pa lahko tvorijo tudi bolj prepletene strukture. Bodita $C^1 = (v_0v_1 \dots v_m v_{m+1} \dots v_{2m+2n_1-1})$ in $C^2 = (u_0u_1 \dots\ u_m u_{m+1} \dots\ u_{2m+2n_2-1})$ izometrična cikla z $u_0 = v_0, \dots, u_m = v_m$ za $m \geq 2$, vsa druga vozlišča so paroma različna. Potem pravimo, da se C^1 in C^2 *prepletata*.

Lema 3.3.2 *Če obstajata v G izometrična cikla, ki se sečeta v vsaj dveh nezaporednih vozliščih, potem obstajata tudi izometrična cikla, ki se prepletata.*

Dodatno še velja, da če se dva konveksna cikla sečeta v vsaj dveh nezaporednih vozliščih, potem se taka dva cikla prepletata. S počjo analize prepletanja je moč analizirati grafe, ki vsebujejo samo daljše cikle. Označimo z $g(G)$ *ožino* grafa G, tj. dolžino najkrajšega cikla v G.

Posledica 3.4.2 *Vsaka delna kocka G z $g(G) > 6$ je drevesna delna kocka.*

Pokazati se tudi da, da delna kocka ne more imeti hkrati samo dolgih ciklov in visokih stopenj vozlišč.

Izrek 3.4.5 *Ne obstaja nobena delna kocka G z $\delta(G) \geq 3$ in $g(G) > 6$. Prav tako ne obstaja nobena z $g(G) = 6$, $\delta(G) \geq 3$ in brez izometričnega podgrafa $Q_4^{--}(4)$.*

Slednje ima zanimive posledice. Regularna delna kocka G z $g(G) > 6$ je lahko izomorfna le K_1, K_2 ali sodemu ciklu. Obstajajo regularne delne kocke z $g(G) = 6$, primer take je posplošeni Petersonov graf $G(10, 3)$. Še lažje najdemo grafe z $g(G) = 4$, primer take je recimo sama hiperkocka. Še ena zanimiva posledica Izreka 3.4.5 je, da v vsaki delni kocki G z $\delta(G) \geq 3$ obstaja cikle dolžine 4 ali pa cikel dolžine 8, saj v grafu $Q_4^{--}(4)$ obstaja cikel dolžine 8. To implicira, da slaven odprti problem Erdős–Gyárfás o obstoju cikla dolžine 2^n, za nek n, v vsakem grafu H z $\delta(H) \geq 3$ drži v delnih kockah.

Poddružine delnih kock

Delne kocke vsebujejo veliko poddružin, ki so se neodvisno pojavile pri študiju navidez nepovezanih področij, kot so različna področja matematike, kemije, teorije družbenih odločitev in drugih. Te družine lahko vidimo kot ena izmed glavnih motivacij za študij delnih kock. Poleg tega pa poddružine omogočajo boljši vpogled v lastnosti delnih kock. Namen tega poglavja je predstaviti izbrane poddružine, ki se pojavljajo v raziskavah v zadnjih letih, ter razumeti relacije med družinami. V nadaljevanju bomo predstavili različne karakterizacije teh družin, ki razlagajo zemljevid delnih kock predstavljen na Sliki 4.1.

Tope grafi (kompleksov) orientiranih matroidov

Orientirani matroidi so kombinatorični objekti, ki omogočajo razumevanje raznolikosti postavitev vektorjev v evklidski prostor in objektov povezanih s tem. Naj bodo $v_1, \ldots, v_n$ vektorji v $\mathbb{R}^d$ in za vsak $1 \leq i \leq n$ naj bo H_i centralna hiperravnina v $\mathbb{R}^d$ ortogonalna na v_i. Vsakemu $v \in \mathbb{R}^d$ lahko priredimo element $X_v \in \{\pm, 0\}^n$ katerega i-ta koordinata ustreza predznaku skalarnega produkta $v \cdot v_i$. Torej i-ta koordinata X_v pove na kateri strani hiperravnine H_i se vektor v nahaja (kjer vrednost 0 pomeni, da leži na hiperravnini). Množica $\mathcal{L}_{\mathcal{H}}$ imenovana *kovektorji* razporeditve hiperravnin je podmnožica $\{\pm, 0\}^n$ vseh elemtov, ki jih lahko pridobimo na zgoraj opisan način. Take možice kovektorjev imajo lastnosti, s katerimi lahko aksiomatiziramo kombinatorične objekte $\mathcal{L}$. Za kovektor X označimo z X_i vrednost njegove i-te koordinate.

Kompozicija:

(C) $X \circ Y \in \mathcal{L}$ za vse $X, Y \in \mathcal{L}$, kjer je $(X \circ Y)_e = X_e$, če $X_e \neq 0$, in $(X \circ Y)_e = Y_e$, če $X_e = 0$.

Obrazna simetrija:

(FS) $X \circ -Y \in \mathcal{L}$ za vse $X, Y \in \mathcal{L}$, kjer je $-Y$ dobljen iz Y z množenjem z -1 vseh svojih koordinat.

Z uporabo (FS) dobimo $X \circ -Y \in \mathcal{L}$, torej $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y) \in \mathcal{L}$ za vse $X, Y \in \mathcal{L}$. Torej (FS) implicira (C).

Za par $X, Y \in \mathcal{L}$ definiramo $S(X, Y)$ kot množico vseh koordinat e, za katere velja $X_e Y_e = -1$.

Stroga eliminacija:

(SE) za vsak par $X, Y \in \mathcal{L}$ in za vsak $e \in S(X, Y)$ obstaja $Z \in \mathcal{L}$ da velja $Z_e = 0$ in $Z_f = (X \circ Y)_f$ za vse kordinate f, ki niso v $S(X, Y)$.

Simetrija:

(Sym) $-X \in \mathscr{L}$ za vse $X \in \mathscr{L}$.

Ničelni kovektor:

(Z) Ničelni kovektor $\mathbf{0}$ je vsebovan v $\mathscr{L}$.

Kompozicija idealov:

(IC) $X \circ Y \in \mathscr{L}$ za vse $X \in \mathscr{L}$ in $Y \in \{\pm, 0\}^n$.

Slednje zadošča za definicijo:

Definicija 4.1.1 *Množici kovektorjev $\mathscr{L} \subset \{\pm, 0\}^n$ pravimo:*

- orientirani matroid (OM)*, če $\mathscr{L}$ zadošča (C), (Sym) in (SE) (ali ekvivalentno (SE), (Z) in (FS)),*
- kompleks orientiranih matroidov (COM)*, če $\mathscr{L}$ zadošča (FS) in (SE),*
- neuravnovešen sistem (LOP)*, če $\mathscr{L}$ zadošča (IC) in (SE).*

Kot omenjeno primere (kompleksov) orientiranih matroidov lahko konstruiramo s pomočjo razporeditve hiperravnin v evklidski prostor. Natančneje, s postopkom opisanim na začetku te sekcije lahko vsaki razporeditvi hiperravnin priredimo množico kovektorjev. Če izhajamo iz centralne razporeditve, potem dobljeni kovektorji tvorijo orientirani matroid. Če pa se omejimo samo na vektorje v neki odprti konveksni podmnožici evklidskega prostora in le tem priredimo kovektorje, je dobljena struktura vedno kompleks orientiranih matroidov. Kompleksom orientiranih matroidov dobljenih na ta način pravimo *realizabilni*.

Izkaže se, da za opis (kompleksov) orientiranih matroidov ne potrebujem celotne strukture, zadošča poznavanje le nekaterih kovektorjev. Naj bo $\mathscr{L}$ množica kovektorjev. Tiste kovektorje, ki nimajo koordinat enakih 0 lahko vidimo kot vozlišča v hiperkocki enake dimenzije kot so dimezionalni kovektorji. Torej tvorijo induciran podgraf hiperkocke. Imenujmo ta graf *tope* graf in ga označimo z $G(\mathscr{L})$. Eden izmed osnovnih rezultatov teorije (kompleksov) orientranih matroidov pravi, da je tope graf $G(\mathscr{L})$ delna kocka v primeru, ko je $\mathscr{L}$ COM, OM ali LOP. Še več, tak sistem je enolično določen z njegovim tope grafom. Naš glavni rezultat je karakterizacija tope grafov v jeziku teorije grafov. Označimo z $\mathscr{G}_{\mathrm{COM}}$ množico vseh delnih kock, ki so tope grafi kakega kompleksa orientiranega matroida, z $\mathscr{G}_{\mathrm{OM}}$ množico vseh delnih kock, ki so tope grafi kakega orientiranega matroida in z $\mathscr{G}_{\mathrm{LOP}}$ množico vseh delnih kock, ki so tope grafi kakega neuravnovešenega sistema.

Uvedimo množico posebnih delnih kock. Naj bo Q_n hiperkocka, $v \in Q_n$ njeno vozlišče in $-v$ antipodalno vozlišče izbranega vozlišča. Najprej definirajmo $Q_n^- := Q_n \setminus -v$ kot hiperkocko brez enega vozlišča. Če dodatno obravnavamo množico grafov dobljenih z odstranitvijo neke podmnožice vozlišč $N(v) \cup \{v\}$ v Q_n^-, $n \geq 4$, opazimo, da je dobljeni graf delna kocka natanko tedaj, ko odstranimo v in poljubno podmnožico $N(v)$ ali pa v ohranimo in odstranimo samo en element iz $N(v)$. Označimo z Q_n^{-*} graf dobljen na slednji način in z $Q_n^{--}(m)$ graf dobljen z odstranitvijo v in m njegovih sosedov. Namesto $Q_n^{--}(0)$ bomo pisali kar Q_n^{--}. Izkaže se, da so tako dobljeni grafi ključni za razumevanje tope grafov kompleksov orientiranih matroidov. Definirajmo $\mathcal{Q}^- = \{Q_n^{-*}, Q_n^{--}(m) \mid 4 \leq n; 1 \leq m \leq n\}$.

Ključna lastnost delnih kock v $\mathcal{Q}^-$ je, da vse vsebujejo antipodalni graf, ki ni zastražen (kot je na primer vidno na Sliki 4.6). Prav tako velja, da noben graf iz te družine ni minor kakega drugega, saj je vsaka skrčitev ali restrikcija grafa iz družine enaka manjši hiperkocki ali hiperkocki brez enega vozlišča. Torej je družina minimalna glede na relacijo minorjev. Izkaže se, da so ti grafi prepovedani minorji družine $\mathcal{G}_{\mathrm{COM}}$.

Izrek 4.1.17 *Za graf G so naslednji pogoji ekvivalentni:*

(i) $G \in \mathcal{G}_{COM}$,

(ii) *G je delna kocka in vsi njeni antipodalni grafi so zastraženi,*

(iii) $G \in \mathcal{F}(\mathcal{Q}^-)$.

Iz gornjih karakterizacij je moč pokazati še eno. Za delno kocko G imenujmo *iterirani grafi con* množico vseh grafov con dobljenih iz G, množico vseh grafov con dobljenih iz teh grafov in tako dalje.

Posledica 4.1.18 *Graf G je v $\mathcal{G}_{COM}$ natanko tedaj, ko je delna kocka in vsi njegovi iterirani grafi con so dobro vložene delne kocke.*

Iz zgornjega takoj sledi tudi karakterizacija tope grafov dobljenih iz orientiranih matroidov ter neuravnovešenih sistemov.

Posledica 4.1.19 *Za graf G so naslednji pogoji ekvivalentni:*

(i) $G \in \mathcal{G}_{OM}$,

(ii) *G je antipodalna delna kocka in vsi njeni antipodalni grafi so zastraženi,*

(iii) *G je v* $\mathcal{F}(\mathcal{Q}^-)$ *in antipodalen,*

(iv) *G je antipodalna delna kocka in vsi njeni iterirani grafi con so dobro vložene delne kocke.*

Označimo $\mathcal{Q}^{--} := \{Q_n^{--} \mid n \geq 3\}$.

Posledica 4.1.21 *Za graf G so naslednji pogoji ekvivalentni:*

(i) $G \in \mathcal{G}_{LOP}$,

(ii) *G je delna kocka in vsi njegovi antipodalni grafi so izomorfni hiperkockam,*

(iii) *G je v* $\mathcal{F}(\mathcal{Q}^{--})$.

Iz teh rezultatov sledi, da lahko družino grafov v $\mathscr{G}_{\text{COM}}$, v $\mathscr{G}_{\text{OM}}$ ali v $\mathscr{G}_{\text{LOP}}$ prepoznamo v polinomskem času, saj zadošča, da najdemo antipodalne grafe in preverimo njihovo zastraženost.

Pasch in Peano grafi

V evklidski geometriji naletimo na več lastnosti, ki bi jih želeli tudi v grafu G. Imenujmo konveksno množico, katere komplement je prav tako konveksen, *polprostor* (kar ustreza definiciji polprostora v delni kocki).

- *Peanova lastnost:* za vse $u, v, w \in V(G)$, $x \in I(u, w)$ in $y \in I(v, x)$, obstaja točka $z \in I(v, w)$, da $y \in I(u, z)$.
- *Pascheva lastnost:* za vse $u, v, w \in X$, $v \in I(u, w)$ in $w' \in I(u, v)$, se intervala $I(v, v')$ in $I(w, w')$ sekata.
- *Komutativnost ogrinjače in unije:* za poljubno konvekso množico $C \subseteq V(G)$ in vsak $u \in V(G)$ je konveksna ogrinjača $\{u\} \cup C$ enaka uniji konveksnih ogrinjač $\{u, v\}$ za vsak $v \in C$.
- *Separacijska lastnost S_3:* za poljubno točko $x \in V(G)$, ki ni v konveksi množici $C \subset V(G)$, obstaja polprostor H, ki ločuje x od C, tj. $x \in H$ in $C \in V(G) - H$.
- *Separacijska lastnost S_4:* za poljubni disjunktni konveksi množici $C, D \subseteq V(G)$ obstaja polprostor H, ki loči C od D, tj. $C \subset H$ in $D \subset X - H$.

V grafih se izkaže, da sta separacijska lastnost S_4 in Pascheva lastnost ekvivalentni. V dvodelnih grafih pa je lastnost S_3 ekvivalentna temu, da je graf delna kocka. Še več, v delnih kockah sta Peanova lastnost in komutativnost ogrinjače in unije ekvivalentni. Torej zgornje lastnosti definirajo dva zanimiva razreda delnih kock: Paschevi grafi, če zadoščajo Paschevi lastnosti (ekvivalentno lastnosti S_4), in Peanovi grafi, če zadoščajo Peanovi lastnosti (ekvivalentno lastnosti komutiranja ogrinjače in unije). Ni težko videti, da so Penovi grafi podmnožica Paschevih.

Karkaterizacija, ki pojasni pozicijo Paschevi grafov na zemljevidu delnih kock je sledeča:

Izrek 4.2.6 *Družina Paschevih grafov ustreza družini* $\mathscr{F}(\{Q_4^-, Q_4^{--}, Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\})$.

Iz izreka sledi, da so Paschevi grafi (in zato tudi Peonovi grafi) tope grafi kompleksov orientiranih matroidov, kajti vsi prepovedani minorji slednje družine imajo minorja v $\{Q_4^-, Q_4^{--}, Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$. Poleg tega imajo Paschevi grafi več zanimivih lastnosti, kot recimo, da je vsak antipodalen podgraf v Paschevem grafu kartezični produkt ciklov in povezav. Najmanjši primer Paschevega grafa, ki ni Peanov je Q_3^-.

Hipercelični grafi

Hipercelični grafi so naravna posplošitev več družin delnih kock, v katerih ne najdemo konveksnega graf Q_3^- in imajo veliko lepih lastnosti. Definirali smo jih kot družina $\mathscr{F}(Q_3^-)$, izkaže pa se, da so lahko definirani na veliko različnih načinov. *Amalgam* nedisjunktnih grafov G_1, G_2 je graf na njuni uniji vozlišč in povezav. Amalgam je *zastražen*, če je $G_1 \cap G_2$ zastražen podgraf tako v G_1 kot v G_2.

Izrek 4.3.2 *Delna kocka G je hipercelična natanko tedaj, ko je dobljena iz zaporednih zastraženih amalgamov, kjer začnemo z grafi izomorfnimi kartezičnemu produktu sodih ciklov in povezav.*

Zgornja karakterizacija razloži ime hiperceličnih grafov, saj so slednji sestavljeni iz tako imenovanih hipercelic, tj. kartezičnih produktov ciklov in povezav. Prav tako iz zgornjega izreka sledi, da so hipercelični grafi Peanovi grafi, saj se Peanova lastnost ohranja pri zastraženih amalgamacijah.

Graf G zadošča pogoju *treh konveksnih ciklov*, če za vsake tri konveksne cikle C^1, C^2, C^3, ki se sečejo v enem vozlišču in paroma v povezavah, velja, da je konveksna ogrinjača množice $C^1 \cup C^2 \cup C^3$ izomorfna hipercelici (glej Sliko 4.14). To lastnost lahko tudi posplošimo v višje dimezije. Če definiramo *rank* hipercelice kot vsoto števila faktorjev hipercelice, ki so izomorfni povezavi, in dva krat števila faktorjev izomorfnih ciklom, lahko posplošimo: Graf G zadošča pogoju *treh konveksnih celic*, če za vsake tri hipercelice X_1, X_2, X_3 dimenzije $k+2$, ki se sečejo v celici dimenzije k, paroma pa v celicah dimenzije $k+1$, velja, da je konveksna ogrinjača $X_1 \cup X_2 \cup X_3$ izomorfna hipercelici.

Izrek 4.3.5 *Za delno kocko G so naslednje lastnosti ekvivalentne:*

(i) $G \in \mathscr{F}(Q_3^-)$*, tj. G je hipercelična;*

(ii) vsaka celica v G je zastražena in G zadošča pogoju treh konveksnih ciklov;

(iii) vsaka celica v G je zastražena in G zadošča pogoju treh konveksnih celic.

Hipercelični grafi imajo več zanimivih lastnosti. Za vsaka tri vozlišča hiperceličnega grafa obstaja enolično določena hipercelica, t.i. medianska celica, tako da lahko najdemo najkrajšo pot med poljubnima dvema izmed izbranih treh tako, da preči mediansko celico. Nadalje vsak endomorfizem končnega hiperceličnega grafa ima neko celico, ki se preslika sama vase. Prav tako velja, da so vsi regularni hipercelični grafi izomorfen eni sami hipercelicam.

Medianski in skoraj-medianski grafi

Medianski grafi so verjetno najbolj raziskovana družina delnih kock. Njihova definicija je sledeča: Graf G je *medianski*, če za vsako trojico vozlišč $\{u, v, w\}$ v G obstaja enolično določeno vozlišče x, imenovano *mediana*, da velja $d(u,x) + d(x,v) = d(u,v)$, $d(u,x) + d(x,w) = d(u,w)$ in $d(v,x) + d(x,w) = d(v,w)$. S pomočjo študije hiperceličnih grafov smo pokazali, da lahko medianske grafe klasificiramo tudi v jeziku minorjev.

Posledica 4.4.2 *Medianski grafi so natanko družina $\mathscr{F}(Q_3^-, C_6)$.*

Torej medianski grafi podedujejo vse lastnost hiperceličnih grafov, kjer dodatno velja, da so edine hipercelice hiperkocke. Ena izmed zanimivih karakterizacij medianskih grafov je sledeča. Označimo z U_{uv} in U_{vu}, za vsako povezavo uv v delni kocki G, množici vozlišč v W_{uv}, ki imajo soseda v W_{vu}, in obratno. Izkaže se, da je G delna kocka natanko tedaj, ko so množice U_{uv} in U_{vu} konveksne za vsak uv. To motivira sledečo posplošitev. Imenujmo delno kocko, v kateri so množice U_{uv} in U_{vu} izometrične za vsak uv, *skoraj-medianski* graf. Dokazali smo:

Izrek 6.3.22 *Delna kocka je skoraj medianska natanko tedaj, ko nima konveksih ciklov dolžine šest ali več, kar je natanko tedaj, ko je v $\mathscr{F}(C_6)$.*

Še več, izkaže se, da so medianski grafi natanko presek skoraj-medianskih grafov in Peanovih grafov. Dokazali smo tudi, da so medianski grafi realizabilni kompleksi orientiranih matroidov.

Drevesne delne kocke in celični grafi

Drevesne delne kocke smo definirali kot delne kocke, ki imajo vse grafe con izomorfne drevesom. Izkaže se, da je taka družina zaprta za minorje in preprosto opisljiva:

Izrek 4.5.3 *Drevesne delne kocke ustrezajo družini $\mathscr{F}(Q_3, Q_4^{--}(4))$.*

Eden izmed zanimivih primerov drevesnih delnih kock, ki se pogosto pojavljajo v kemiji, so tako imenovani benzeoidi, tj. unije 6-ciklov v heksagonalni mreži brez pravih lukenj. Ni težko videti, da so grafi poti edini grafi con v benzenoidih. Podobna struktura imenovana *celični* grafi so bili definirani na več ekvivalentnih načinov. Omenimo dva in sicer, da so grafi dobljeni z zastraženo amalgamacijo povezav in sodih ciklov, oziroma ekvivalentno, da so dvodelni grafi, v katerih je konveksna ogrinjača vsake množice S enaka uniji intervalov med vozlišči v S, tj. $\text{conv}(S) = \cup_{v,u\in S} I(u, v)$. Iz prve definicije neposredno sledi, da so celični grafi poddružina hiperceličnih.

S pomočjo analize hiperceličnih grafov lahko dodamo še eno klasifikacijo celičnih grafov:

Izrek 4.5.5 *Celični grafi ustrezajo družini $\mathscr{F}(Q_3^-, Q_3)$.*

Kot v primeru medianskih grafov tudi celični grafi podedujejo veliko lepih lastnosti od hiperceličnih, na primer lastnost fiksne celice pri vsakem endomorfizmu, klasifikacijo regularnih, itd.

Simetrične delne kock

Hiperkocke so grafi, ki vsebujejo ogromno simetrij. Vprašanje, kako se te simetrije prenesejo na podgrafe, se izkaže za precej težko. Prvi rezultati o obstoju vozliščno tranzitivnih podgrafov hiperkock segajo v leto 1992, ko so Brower, Dejter in Thomassen predstavili veliko raznolikih primerov in izpostavili, da se klasifikacija le teh zdi nedosegljiva. Po drugi strani, če se omejimo samo na, recimo, medianske grafe, je nabor le teh zelo omejen, saj so le hiperkocke regularni medianski grafi.

Študij regularnih delnih kock je požel precej zanimanja. Poleg rezultatov o medianski grafih, so v devetdesetih klasificirali razdaljno-regularne grafe. Šele v naslednjem desetletju je bilo opaženo, da to niso edine regularne delne kocke. S pomočjo računalnika so klasificirali vse kubične delne kocke na največ 30 vozliščih ter našli nove in večje primere. Z metodami kot je kubična inflacija ter razporeditvami premic v ravnino (torej v povezavi s kompleksi orientiranih matroidov) so raziskovalci dobili nove zanimive kubične delne kocke. S slednjim se je nakazalo, da je tudi klasifikacija regularnih delnih kock težko dosegljiva.

Vozliščno tranzitivne delne kocke

V naših raziskavah smo se omejili na vozliščno tranzitivne, kubične delne kocke. Naš glavni rezultat je popolna klasifikacija le teh.

Izrek 5.2.1 *Če je G končna, kubična, vozliščno tranzitivna delna kocka, potem je G izomorfna enemu izmed naslednjih grafov: $K_2 \square C_{2n}$, za $n \geq 2$, $G(10,3)$, kubičnemu permutaedru, prisekanemu kubooktaedru ali prisekanemu ikozidodekaedru.*

Grafi $K_2 \square C_{2n}$, za $n \geq 2$, so prizme, graf $G(10,3)$ je posplošeni Petersonov graf z danimi parametri, medtem ko so kubični permutaeder, prisekani kubooktaeder in prisekani ikozidodekaeder klasični grafi geometrijskih teles. Vse grafe lahko najdemo na Sliki 5.1. Kot zanimivost navedimo, da je $G(10,3)$ edini znan primer kubične, neravninske delne kocke. Ker je število vozliščno tranzitivnih, kubičnih delnih kock precej omejeno, slednje daje upanje za podobno klasifikacijo v primeru višjih stopenj. Zgornje grafe lahko vidimo kot začetnike večjih družin vozliščno tranzitivnih grafov.

- Kartezični produkt vozliščno tranzitivnih delnih kock je vedno vozliščno tranzitivna delna kocka. Prizme nad sodimi cikli so kartezični produkt povezav in sodih ciklov, edinih vozliščno tranzitivnih grafov s stopnjami manj kot dva. Podobno lahko konstruiramo nove vozliščno tranzitivne grafe.

- *Middle level* grafi so definirani kot inducirani podgrafi hiperkock Q_n, za lih $n > 1$, na vseh vozliščih ki imaj natanko $\lfloor n/2 \rfloor$ ali $\lceil n/2 \rceil$ koordinat enakih 1. Taki grafi so vedno vozliščno tranzitivne delne kocke. Graf $G(10,3)$ je najpreprostejši primer za $n = 5$, takoj za 6-ciklom v primeru $n = 3$.

- Kubični permutaeder, prisekani kubooktaeder in prisekani ikozidodekaeder so Cayleyjevi grafi končnih Coxeterjevih grup, kot bomo podrobneje razložili v nadaljevanju.

Zgoraj opisane družine so edine znane vozliščno tranzitivne delne kocke.

Zrcalni grafi

Definicija zrcalnih grafov je sledeča. Naj bo G povezan graf. Imenujmo particijo $P = \{E_1, E_2, \ldots, E_k\}$ povezav v G *zrcalna particija*, če za vsak $i \in \{1, \ldots, k\}$ obstaja avtomorfizem α_i grafa G, da velja:

(i) za vsako povezavo $uv \in E_i$ velja: $\alpha_i(u) = v$ in $\alpha_i(v) = u$.

(ii) $G - E_i$ je sestavljen iz natanko dveh povezanih komponent G_i^1 in G_i^2 ter α_i preslika G_i^1 v G_i^2.

Graf, ki poseduje zrcalno particijo je imenovan *zrcalni graf*. Ni težko pokazati, da so zrcalni grafi vozliščno tranzitivni, manj pričakovan rezultat je, da so delne kocke, v katerih zrcalna particija ustreza particiji povezav v Θ-razrede.

V nadeljevanju bomo predstavili popolno klasifikacijo zrcalnih grafov. Začnimo z definicijami. Spomnimo se, da tope grafi realizabilnih orientiranih matroidov izhajajo iz centralnih postavitev hiperravnin v evklidski prostor. Za hiperravnino H_i z vektorjem v_i ortogonalnim na hiperravnino definirajmo *zrcaljenje* kot preslikavo $\sigma_{H_i}(x) = x - 2\frac{x \cdot v_i}{v_i \cdot v_i} v_i$ v evklidskem prostoru. Centralna razporeditev hiperravnin $\{H_1, \ldots, H_m\}$ v $\mathbb{R}^n$ se imenuje *zrcalna razporeditev*, če za vsak $i \in \{1, \ldots, m\}$ zrcaljenje čez hiperravnino H_i permutira hiperravnine (primer na Sliki 5.2).

Nadalje imenujmo grupo *Coxeterjeva*, če jo lahko predstavimo z generatorji in relacijami v sledeči oblike: $\langle \alpha_1, \ldots, \alpha_m \mid (\alpha_i \alpha_j)^{k_{ij}} = 1$ za vse $1 \leq i, j \leq m \rangle$, kjer je $k_{ii} = 1$ in $k_{ij} \geq 2$ za vse $0 \leq i < j \leq m$.

Znano je, da obstaja bijektivna korespondenca med zrcalnimi razporeditvami in končnimi Coxeterjevimi grupami. S sledečim smo povezali ti dve strukturi z zrcalnimi grafi.

Izrek 5.3.1 *Za končen graf G so naslednje izjave ekvivalentne:*

(i) G je zrcalni graf.

(ii) G je Cayleyjev graf končne Coxeterjeve grupe s kanoničnimi generatorji.

(iii) G je tope graf zrcalne razporeditve hiperravnin.

Končne Coxeterjeve grupe so povsem klasificirane, običajno jih označimo z $A_n, B_n, D_n, I_2(p), E_6, E_7, E_8, F_4, H_3, H_4$, kjer prve štiri označujejo neskončne družine, ostalih šest pa je sporadičnih primerov. V jeziku delnih kock se prevedejo na naslednje. Družina $I_2(p)$ ustreza sodim ciklom. Družine A_n, B_n, D_n definirajo neskončne družine delnih kock, za vsak n dobimo n-regularen graf na $(n+1)!, 2^n n!$ oziroma $2^{n-1} n!$ vozliščih. Natančneje, Cayleyeva grafa A_3 in D_3 ustrezata kubičnemu permutaedru, B_3 pa ustreza prisekanemu kubooktaedru. Cayleyev graf H_3 ustreza prisekanemu ikozidodekaedru, medtem ko E_6, E_7, E_8, F_4, H_4 ustrezajo: 6-regularnem grafu na 51840 vozliščih, 7-regularnemu grafu na 2903040 vozliščih, 8-regularnemu grafu na 696729600 vozliščih, 4-regularnemu grafu na 1152 vozliščih in 4-regularnemu grafu na 14400 vozliščih.

Iz samega dokaza zgornjega izreka tudi sledi, da se da zrcalne grafe prepoznati ter najti njihovo zrcalno particijo in zrcalne avtomorfizme v polinomskem času.

Računske in izračunane lastnosti

S pomočjo celoštevilskega linearnega programa smo generirali delne kocke vložljive v hiperkocko dimenzije največ šest. Takih je kar 13491182 neizomorfnih delnih kock. Za vsako smo preverili, če je vsebovana v kaki izmed poddružin opisanih v prejšnjih odstavkih in nekaj drugih lastnosti. Rezultati so podani v Tabeli 6.1.

Eden izmed osnovnih vprašanj, na katerega naletimo pri računski analizi delnih kock, je vprašanje, kako preveriti, če sta dve delni kocki izomorfni. Ali sta dva grafa izomorfna, se po dosedanjih vedenjih da preveriti najhitreje v $e^{(\log n)^{O(1)}}$ operacijah, kjer je n število vozlišč v obeh grafih. Problemom, ki so temu ekvivalentni, rečemo, da so GI polni. Za konec smo pokazali zanimiv izrek:

Izrek 6.3.1 *Problem odločanja, ali sta dva medianska grafa izomorfna, je GI poln.*