

Colouring bottomless rectangles and arborescences

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Abstract

We study problems related to colouring bottomless rectangles. One of our main results shows that for any positive integers m, k , there is no semi-online algorithm that can k -colour bottomless rectangles with disjoint boundaries in increasing order of their top sides, so that any m -fold covered point is covered by at least two colours. This is, surprisingly, a corollary of a stronger result for arborescence colourings. Any semi-online colouring algorithm that colours an arborescence in leaf-to-root order with a bounded number of colours produces arbitrarily long monochromatic paths. This is complemented by optimal upper bounds given by simple online colouring algorithms from other directions.

Our other main results study configurations of bottomless rectangles in an attempt to improve the *polychromatic k -colouring number*, m_k^* . We show that for many families of bottomless rectangles, such as unit-width bottomless rectangles, m_k^* is linear in k . We also present an improved lower bound for general families: $m_k^* \geq 2k - 1$.

1 Introduction

The systematic study of polychromatic colourings and cover-decomposition of geometric ranges was initiated by Pach over 30 years ago [17, 18]. The field has gained popularity in the new millennium, with several breakthrough results; for a (slightly outdated) survey, see [19], or see the up-to-date interactive webpage <http://coge.elte.hu/cogezoo.html> (maintained by Keszegh and Pálvölgyi).

A family of geometric regions \mathcal{F} and a point set P in some \mathbb{R}^d naturally define a primal hypergraph, $H(P, \mathcal{F})$. The vertex set of H is the points of P , where $\mathcal{F}' \subset \mathcal{F}$ is an edge if for some $p \in P$, p is covered by exactly the regions in \mathcal{F}' . We are interested in the dual hypergraph, $H(\mathcal{F}, P)$, on the vertex set \mathcal{F} , where $p \in P$ is an edge if for some $\mathcal{F}' \subset \mathcal{F}$, p is covered by exactly the regions in \mathcal{F}' .

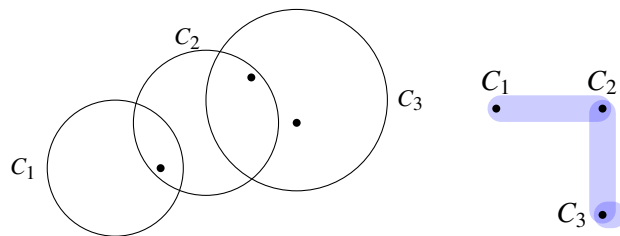


Figure 1: A family of circles, a finite point set, and the corresponding dual hypergraph.

We can then define the chromatic number $\chi_{\mathcal{F}}$ of any family \mathcal{F} of geometric regions. This is the minimum number of colours needed to colour any *finite* point set P so that any region containing at least two points contains at least two colours. The dual chromatic number, $\chi_{\mathcal{F}}^*$, is defined analogously as the number of

28 colours needed to properly colour any *finite* subfamily of \mathcal{F} . In this paper, we will study the *polychromatic*
 29 *colouring numbers*.

30 **Definition 1.1.** The k -th (primal) *polychromatic colouring number* $m_k(\mathcal{F})$ is the smallest number needed
 31 to k -colour any finite point set P so that any region containing at least $m_k(\mathcal{F})$ points contains all k colours.

32 Its dual $m_k^*(\mathcal{F})$ is the smallest number needed to k -colour any finite subfamily of \mathcal{F} so that any point
 33 covered by m_k^* regions is covered by all k colours.

34 The polychromatic colouring problem is partly motivated by the sensor cover problem; given a set of
 35 sensors covering an area, can we decompose them into k sets so that any area covered by m_k^* sensors is
 36 covered in each of these sets? When $k = 2$, this is called the cover-decomposability problem. In particular,
 37 we say a set $P \subset \mathbb{R}^2$ is *cover-decomposable* if $m_2^*(\mathcal{F}_P) < \infty$, where \mathcal{F}_P is the family of all *translates* of P .
 38 In [18] it was shown that every centrally-symmetric open convex polygon is cover-decomposable, and this
 39 was extended to all open convex polygons in [21]. The bound $m_k^*(\mathcal{F}_P) = O(k)$ was proved for any convex
 40 polygon P in [7].

41 The problem becomes more complicated if we consider *homothets* of a convex polygon. For example,
 42 if \mathcal{F}_{\square} denotes the family of axis-parallel squares in the plane, $m_2(\mathcal{F}_{\square}) \leq 215$ [1]. On the other hand,
 43 $m_2^*(\mathcal{F}_{\square}) = \infty$, that is, for any number m there is a family \mathcal{F}_m of axis-parallel squares such that (1) each point
 44 in the plane is covered by at least m squares, but (2) any 2-colouring of \mathcal{F}_m produces a point covered by
 45 squares of exactly one colour [15]. Furthermore, if \mathcal{F}_{\square} denotes the family of axis-parallel rectangles in the
 46 plane, $m_2(\mathcal{F}_{\square}) = \infty$ [5]. Consequently, $m_k(\mathcal{F}_{\square}) = \infty$ for any k . The dual $m_k^*(\mathcal{F}_{\square})$ is infinite as well; there
 47 is a constant $C > 0$ such that for any numbers $n \geq r \geq 2$, there is a family of n axis-parallel rectangles for
 48 which any colouring with at most $C \log n (r \log r)^{-1}$ colours produces a point covered by r monochromatic
 49 axis-parallel rectangles [20].

50 This paper focuses on one particular family: *bottomless rectangles*.

51 **Definition 1.2.** A subset of \mathbb{R}^2 is called a (closed) bottomless rectangle if it is of the form $\{(x, y) : l \leq x \leq$
 52 $r, y \leq t\}$. We simply refer to a bottomless rectangle by these parameters (l, r, t) .

53 These range spaces were first defined by Keszegh [11], who showed $m_2 = 4$ and $m_2^* = 3$. Later Asinowski
 54 et al. [2] showed that for any positive integer k , any finite set of points in \mathbb{R}^2 can be k -coloured such that
 55 any bottomless rectangle with at least $3k - 2$ points contains all k colors. They also showed that the optimal
 56 number that can be written in place of $3k - 2$ in the above statement is at least $1.67k$. In our language, if \mathcal{F}_{\square}
 57 denotes the family of all bottomless rectangles in the plane, $1.67k \leq m_k(\mathcal{F}_{\square}) \leq 3k - 2$.

58 Our paper studies the dual problem: we would like to determine the optimal $m_k^*(\mathcal{F}_{\square})$. About this question
 59 much less is known; $m_2^* = 3$ [11], while the best general upper bound is $m_k^* = O(k^{5.09})$, a corollary of a more
 60 general result [4] about *octants* (combined with an improvement of the base case [13] that slightly lowered
 61 the exponent). The general conjecture, however, is that $m_k^* = O(k)$ for any family for which m_2^* is finite
 62 [19]. It was also proved in [4] that there is no semi-online algorithm “from above” for colouring bottomless
 63 rectangles. One of our main results is a generalisation of this negative statement.

64 **Theorem 1.3.** *For any k and m , and any semi-online algorithm that k -colours bottomless rectangles from*
 65 *below (resp. from above, from the right, or from the left), there is a family of bottomless rectangles such that*
 66 *the algorithm will produce an m -fold covered point that is covered by at most one colour.*

67 Our proof is much more complicated than the one in [4]; while there an Erdős-Szekeres [6] type
 68 incremental argument is used, we need a certain diagonalisation method. In particular, we reduce the semi-
 69 online bottomless rectangle colouring problem to a question about semi-online colourings of arborescences,
 70 which is interesting in its own right.

71 **Theorem 1.4.** *For any k and m , and any semi-online colouring algorithm that k -colours the vertices of*
 72 *an arborescence in a leaf-to-root order, there is an arborescence that has a leaf-to-root order such that the*
 73 *algorithm will produce a directed path of length m that contains at most one colour.*

74 We apply this theorem to four natural configurations of bottomless rectangles to show that for each
 75 configuration, there is a direction from which a semi-online algorithm fails. Furthermore, we show in
 76 Theorem 4.1 that bottomless rectangles do not admit *shallow hitting sets*, which are another standard tool to
 77 bound polychromatic colouring numbers.

78 These negative results are complemented by optimal upper bounds given by online algorithms from the
 79 other directions. We obtain linear bounds for m_k^* for the following families of bottomless rectangles:

- 80 • $m_k^*(\mathcal{F}_{\text{unit}}) \leq 2k - 1$ for *unit-width* families, Proposition 2.5,
- 81 • $m_k^*(\mathcal{F}_{\text{hanging}}) \leq 2k - 1$ for *hanging* families, Theorem 2.8, and
- 82 • $m_k^*(\mathcal{F}_{\text{int}}) \leq 3k$ for *intersecting* families. Theorem 2.9,

83 We also improve the current lower bound for m_k^* , by showing $m_k^* \geq 2k - 1$ (Theorem 4.2).

84 In section 2, we introduce certain configurations of bottomless rectangles, and define the corresponding
 85 colouring problem. We also prove our other main results: that for many families of bottomless rectangles,
 86 such as *unit-width*¹ and *intersecting*² families, m_k^* is linear. In section 3, we prove Theorem 1.4 and deduce
 87 Theorem 1.3 as a corollary of it. In section 4, we look at other methods to improve the upper bound on m_k^* ,
 88 and improve the lower bound for general families to $m_k^* \geq 2k - 1$.

89 2 Configurations of bottomless rectangles

90 2.1 Erdős-Szekeres configurations

91 We would like to improve the upper bound $m_k^*(\mathcal{F}_{\square}) = O(k^{5.09})$ for general families by classifying some
 92 configurations of bottomless rectangles, finding a colouring for each configuration, and combining these
 93 to obtain a good colouring for general families. To this end, we will use the classical result of Erdős and
 94 Szekeres [6] that any sequence of length $(k - 1)^2 + 1$ contains a monotone subsequence of length k .

95 Recall that we associated to each rectangle its parameters (l, r, t) . We refer to l as its left-coordinate, r
 96 its right-coordinate, and t its height. Let p be a point covered by $(m - 1)^4 + 1$ rectangles. Ordering these
 97 rectangles by left endpoint, we find a subsequence of length $(m - 1)^2 + 1$ whose right endpoints are monotone.
 98 Applying the result of Erdős and Szekeres again, we find a (sub)subsequence of length m whose heights are
 99 monotone. This proves that any point that is contained in $(m - 1)^4 + 1$ bottomless rectangles is contained
 100 in m bottomless rectangles such that each of the three parameters of these m bottomless rectangles are
 101 in increasing or decreasing order. We name these configurations, respectively, *increasing/decreasing steps*,
 102 *towers* and *nested rectangles* (see Figure 2).

103 We are interested in colouring families with respect to a *fixed* configuration. For example, can we k -colour
 104 a finite family \mathcal{F} so that any point covered by an m -tower is covered by all k colours? We refer to least such
 105 m as m_k^* for *towers*, and analogously for the other configurations.

106 **Theorem 2.1.** $m_k^* = k$ for each fixed configuration.

¹See subsection 2.2

²See subsection 2.3

107 A result of Berge [3] shows that for any family \mathcal{F} of geometric regions, $m_2^*(\mathcal{F}) = 2$ if and only if
 108 $m_k^*(\mathcal{F}) = k$ for all k . It is not hard to show that $m_2^* = 2$ for each configuration, and then apply the result of
 109 Berge. Nevertheless, it is valuable to see that there is a simple online algorithm for each configuration.

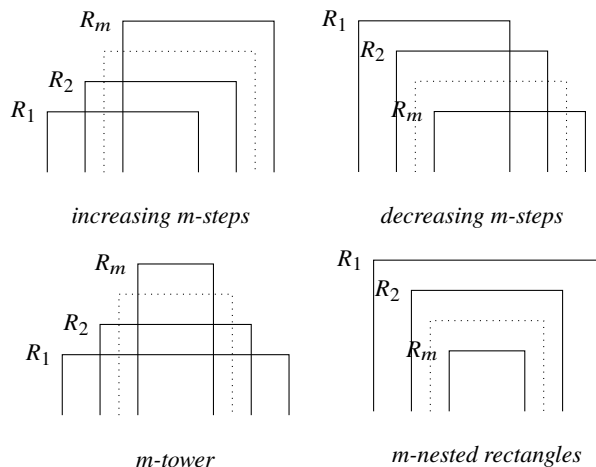


Figure 2: Erdős-Szekeres configurations

110 **Definition 2.2.** In this paper, we will consider the following two types of colouring algorithms for hyper-
 111 graphs, which receive the vertices in some order. Both types must colour the vertices *irrevocably* – they are
 112 not allowed to recolour vertices.

- 113 (1) An *online algorithm* must colour each vertex immediately so that at each step, there is no conflict in
 114 the partial colouring. ³
- 115 (2) A *semi-online algorithm* need not colour each vertex immediately, but must ensure that at each step
 116 there is no conflict in the partial colouring.

117 This condition on the partial colouring means that, for example, at every step the algorithm must maintain
 118 that a point covered by m rectangles is covered by all k colours, but not all points have to be colored.

119 *Proof of Theorem 2.1.* We first present a colouring algorithm for towers. We colour the rectangles in increas-
 120 ing order of height, i.e. from below, so that at every step the following property holds.

121 (*) If a point p is covered by a j -tower for $j \leq k$, then p is covered by at least j different colours.

122 At step 1, colour the rectangle of least height arbitrarily. Suppose the first $t - 1$ rectangles have been
 123 coloured so that (*) holds. We colour the rectangle R_t as follows. For each $1 \leq i \leq k$, let y_i be the largest
 124 number so that if $p \in R_t$ has y -coordinate less than y_i , then p is covered by colour i . (This corresponds to
 125 a tallest rectangle S of colour i such that (S, R_t) is a tower.) If y_i does not exist for some colour i , colour R_t
 126 with colour i . Otherwise, suppose $y_1 > \dots > y_k$, and colour R_t with colour k .

127 Property (*) holds: if p is covered by a j -tower, then either p was already covered by j colours, or we
 128 added a new colour to the set of colours covering p . We use the same algorithm to colour k -nested sets, only
 129 we colour the rectangles from above. It is easy to check that with this ordering, the same property holds.

³There is a large literature on online algorithms [8, 9, 16].

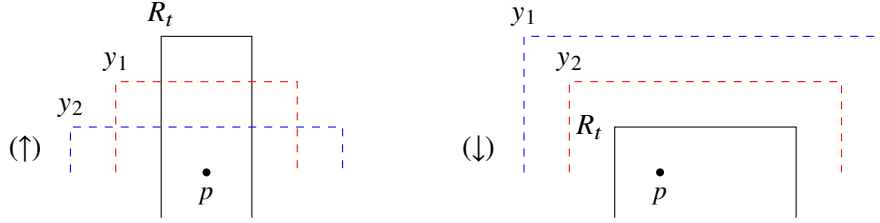


Figure 3: Colouring algorithms for k -towers and k -nested sets respectively

130 The algorithm for increasing k -steps is only slightly different. We colour the rectangles in decreasing
 131 order of right endpoint (from the right). At step t , for $1 \leq i \leq k$, let x_i be the least number so that if $p \in R_t$
 132 has x -coordinate greater than x_i , then p is covered by a rectangle of colour i (corresponding to the leftmost
 133 rectangle S of colour i such that (R_t, S) form increasing steps). As earlier, if some x_i does not exist, give R_t
 134 colour i . Otherwise, if $x_1 < \dots < x_k$, give R_t colour k .

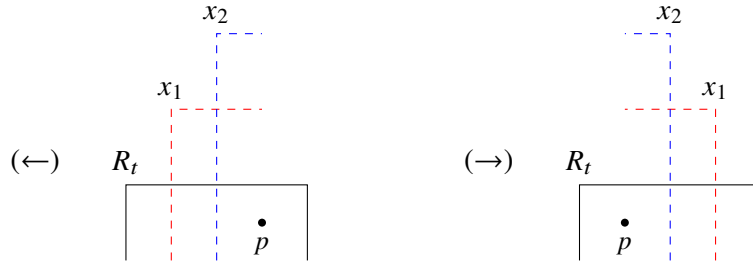


Figure 4: Colouring algorithms for increasing and decreasing k -steps respectively

135 □

136 Note that ordering a tower in increasing order of height (from below) is the same as ordering it in
 137 decreasing order of right endpoint (from the right), or increasing order of left endpoint (from the left). We
 138 may repeat the same algorithm for towers, colouring the rectangles from the left (or right) and we will still
 139 obtain a good colouring. Similarly, ordering a nested set from above is the same as ordering it from the left or
 140 right. Ordering increasing steps from the right (resp. decreasing steps from the left) is the same as ordering
 141 them from above. This is all to say that the same algorithm can be used from these “good” directions for
 142 each fixed configuration.

143

	left (\rightarrow)	right (\leftarrow)	below (\uparrow)	above (\downarrow)
inc. steps	∞	$= k$	∞	$= k$
dec. steps	$= k$	∞	∞	$= k$
towers	$= k$	$= k$	$= k$	∞
nested	$= k$	$= k$	∞	$= k$

144 Table 1: m_k^* values for each configuration given by semi-online algorithms from different directions.

145 The value ∞ indicates the non-existence of semi-online colouring algorithms, which we prove in the next
146 section.

147 The next natural question to ask is: how can we combine these colourings? Nested rectangles seem to
148 have the “simplest” structure of the four configurations. Indeed, ordering nested rectangles from above is
149 the same as ordering them from the left or right. Further, if (R_1, R_2) are nested (R_1 contains R_2), then any
150 point in R_2 is necessarily contained in R_1 . This shows that we can modify the algorithms for the other
151 configurations to also colour nested rectangles.

152 **Proposition 2.3.** $m_k^* = k$ if we k -colour any finite family \mathcal{F} with respect to

- 153 (a) towers and nested sets
- 154 (b) increasing steps and nested sets
- 155 (c) decreasing steps and nested sets

156 *Proof.* We first present the algorithm for towers and nested sets. The precise statement is that any finite
157 family \mathcal{F} can be k -coloured so that any point contained in a k -tower or a k -nested set is covered by all k
158 colours. We colour the rectangles from the right (this can also be done from the left). We maintain the same
159 property as earlier.

160 (*) If a point p is covered by a j -tower or a j -nested set for $j \leq k$, then p is covered by at least j different
161 colours.

162 At step t , for $1 \leq i \leq k$, let y_i be the greatest number so that if $p \in R_t$ has y -coordinate less than y_i , then p
163 is covered by a rectangle of colour i . As earlier, if some y_i does not exist, give R_t colour i . Otherwise, suppose
164 $y_1 > \dots > y_k$, and give R_t colour k . To prove that (*) holds is not as straightforward as in Theorem 2.1. Let
165 y denote the height of R_t . If $y_k > y$, or $y > y_1$, (*) holds by the same argument as in Theorem 2.1. If not, we
166 have $y_1 > \dots > y_{l-1} > y > y_l > \dots > y_k$. Suppose $p \in R_t$ is covered by a j -nested set R_1, \dots, R_{j-1}, R_t .
167 Since each y_i is maximal, (*) holds by the same argument as earlier. The only essentially different case is
168 when $p \in R_t$ is covered by a j -tower. If we did not add a new colour to the set of rectangles containing p ,
169 this means that p was already covered by a rectangle of colour k . However, as y_k was chosen to be maximal,
170 the y -coordinate of p must be less than y_k , so p is already covered by all k colours.

171 The algorithms for increasing and decreasing steps are modified in the exact same way. □

172 2.2 Unit-width rectangles

173 The next natural pair of Erdős-Szekeres configurations to attempt to combine is the increasing and
174 decreasing steps. For this, we consider a different setup; refer to *steps* as the case when we assume that \mathcal{F}
175 does not contain any towers or nested sets. What is $m_{k, \text{steps}}^*$?

176 Let *unit bottomless* be the case when all the rectangles in \mathcal{F} have the same width, or unit width. It is clear
177 that “*unit bottomless* \subset *steps*”, as any family of unit bottomless rectangles cannot contain towers or nested
178 sets.

179 **Proposition 2.4.** “*steps* = *unit bottomless*”, i.e. any family of tower- and nested set-free rectangles can be
180 realised as a family of unit width bottomless rectangles so that the corresponding dual hypergraphs are
181 isomorphic.

182 *Proof.* We prove the inclusion steps \subseteq unit bottomless by our favourite method, induction on $|\mathcal{F}|$. Suppose
 183 any family \mathcal{F} of $n - 1$ rectangles that do not contain towers or nested sets can be realised as a family \mathcal{F}_{unit} of
 184 unit bottomless rectangles (with an isomorphic hypergraph), and that this realisation preserves heights and
 185 the ordering of left endpoints. That is, the height of a rectangle R in \mathcal{F} is the same as its realisation in \mathcal{F}_{unit} .

186 Let $|\mathcal{F}| = n$, and let R be the leftmost rectangle in \mathcal{F} . Take any realisation of $\mathcal{F} \setminus R$ as a family \mathcal{G} , and
 187 let R_1, \dots, R_m be the rectangles that intersect R , and R'_1, \dots, R'_m their realisations. Assume without loss of
 188 generality that $l(R_1) < \dots < l(R_m)$.

189 In particular, $l(R_m) < r(R) < r(R_1)$ (as they intersect), so R_1, \dots, R_m also intersect each other. This
 190 implies that the interval $[l(R'_1), \dots, l(R'_m)]$ has length strictly less than 1. Thus for ϵ small enough, if we
 191 realise R as a unit width rectangle R' with $r(R') = l(R'_m) + \epsilon$ with the same height, then R' will intersect
 192 exactly the rectangles R'_1, \dots, R'_m (with the same hypergraph structure).

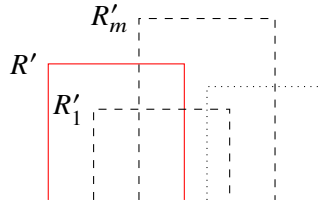


Figure 5: We ensure that the realisation of R preserves the hypergraph structure.

193 □

194 So instead of considering colouring points with respect to bottomless rectangles, we may consider
 195 colouring steps with respect to points.

196 **Proposition 2.5.** *For steps, $m_k^* \leq 2k - 1$.*

197 The proof of the proposition will use ABA-free hypergraphs [14]. We say a hypergraph \mathcal{H} with an
 198 ordering $<$ of its vertex set is ABA-free if there are no hyperedges A and B and vertices $x < y < z$ with
 199 $x, z \in A \setminus B$ and $y \in B \setminus A$. For example, interval hypergraphs - where the vertices are points in \mathbb{R} and
 200 the hyperedges are the subsets induced by some intervals - are ABA-free. A result of [14] tells us that for
 201 ABA-free hypergraphs, $m_k \leq 2k - 1$.

202 *Proof of Proposition 2.5.* Let \mathcal{F} be a family containing no nested sets or towers and P a finite point set. We
 203 claim that by ordering the rectangles by left endpoint, the resulting hypergraph on the vertex set \mathcal{F} with edges
 204 induced by P is ABA-free. Suppose for contradiction we have three rectangles with $l(R_1) < l(R_2) < l(R_3)$,
 205 and points p and q so that $p \in (R_1 \cap R_3) \setminus R_2$, and $q \in R_2 \setminus (R_1 \cap R_2)$.

206 Recall that a point (x, y) is in a rectangle R if and only if $x \in [l(R), r(R)]$ and $y < y(R)$. Let $p = (x_p, y_p)$
 207 and $q = (x_q, y_q)$. Then, $p \in R_1, R_3$ but $p \notin R_2$ implies,

$$l(R_1) < l(R_2) < l(R_3) < x_p < r(R_1) < r(R_2) < r(R_3), \text{ and}$$

$$y(R_1), y(R_3) > y_p > y(R_2)$$

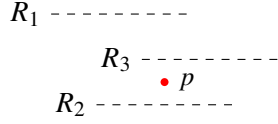


Figure 6: An arrangement of p and the three rectangles implied by the above conditions.

209 As a result, $[l(R_2), r(R_2)]$ is covered by the intervals $[l(R_1), r(R_1)]$ and $[l(R_3), r(R_3)]$, and R_2 is below
 210 R_1 and R_3 ; the “top side” of R_2 is covered by the top sides of R_1 and R_3 as in Figure 6. Thus any point in R_2
 211 is contained in at least one of R_1 and R_3 , contradicting that $q \in R_2$ but $q \notin R_1$ or R_3 .
 212 □

213 We end this subsection by extending this to families that do not contain towers.

214 **Theorem 2.6.** *For families \mathcal{F} that do not contain towers, $m_k^* \leq 2k - 1$.*

215 *Proof.* As earlier, we want to show that the corresponding hypergraph is ABA-free. Suppose again that we
 216 have three rectangles with $l(R_1) < l(R_2) < l(R_3)$, and points p and q so that $p \in R_1, R_3$, $q \notin R_1, R_3$, and
 217 $q \in R_2$, $p \notin R_2$.

218 The previous proposition shows R_1, R_2, R_3 must contain at least one nested set. It is also easy to see that
 219 not all three of them can form a nested set, so exactly two of them do. Further, the condition $p \in R_1, R_3$ but
 220 $q \notin R_1, R_3$ implies that (R_1, R_3) must form a nested set (where R_1 contains R_3). In this case, it is easy to
 221 check that it is not possible to have a rectangle R_2 that forms steps with both R_1 and R_3 , and contains q but
 222 not p . □

223 2.3 Other families

224 For families that are tower-free, m_k^* is linear. What of families that do contain towers?

225 We say a family of bottomless rectangles is *hanging* if their left endpoints lie on the line $y = x$. It is clear
 226 that we can choose any line with positive slope, as rotating the line $y = x$ and moving the left endpoints along
 227 with it preserves the hypergraph structure. So more generally, a *hanging family* is one whose left endpoints
 228 all lie on a fixed line with positive slope, which will be $y = x$ for convenience. A tower, for example, can be
 229 realised as a hanging arrangement by “stretching” the left endpoints.

230 **Proposition 2.7.** *For hanging families, $m_k^* \leq 3k - 2$.*

231 This proof relies on a reduction to the primal problem for general families of bottomless rectangles,
 232 when we colour points with respect to bottomless rectangles. We want to show that any hanging family \mathcal{F}
 233 and point set P can be realised as a family $\mathcal{F}(P)$ and point set $P(\mathcal{F})$ so that a rectangle $R \in \mathcal{F}$ covers a point
 234 $q \in P$ if and only if the corresponding point $r \in P(\mathcal{F})$ is contained in the rectangle $Q \in \mathcal{F}(P)$.

235 To each rectangle $R \in \mathcal{F}$, we associate its right endpoint $r(R)$, and to each point $q = (x, y) \in P$, we
 236 associate an infinite hanging rectangle from the point (x, x) . Of course, when we say infinite, we simply
 237 mean that the right endpoint of the corresponding rectangle in $\mathcal{F}(P)$ is sufficiently large.

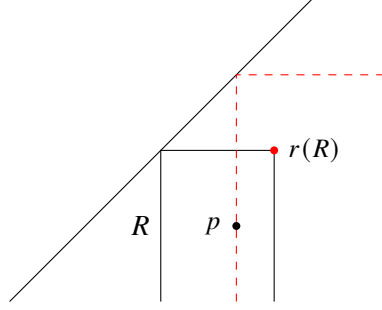


Figure 7: The infinite hanging rectangle from p contains $r(R)$ if and only if R contains p .

238 Since the best upper bound for the primal problem is $3k - 2$, we have the desired bound for hanging
 239 families. However, by modifying the proof of this upper bound from [2], we can improve it to $2k - 1$. By the
 240 duality we observed, it suffices to consider colouring points with respect to infinite hanging rectangles. Order
 241 the points in increasing order of y -coordinate (from below). Note that if p is to the left of some hanging
 242 rectangle R , then R does not contain p , so once we start colouring points of R we may “disregard” p . We
 243 present the points in increasing order of y -coordinate as follows. At step t , suppose q is to be presented, and
 244 p is the leftmost point at this step. If q is covered by a rectangle to the *right* of p , first we “delete” p , then
 245 we present q . If not, then we present q without deleting any points. This enables us to only care about the
 246 leftmost $2k - 1$ points, as we discard the nonessential ones.

247 More precisely, we frame this as a *dynamic colouring problem* on the line. We wish to colour a dynamically
 248 appearing point set P where one of the following kinds of events can occur.

- 249 (1) A new point appears.
- 250 (2) The leftmost point disappears.

251 **Theorem 2.8.** *For hanging arrangements, $m_k^* \leq 2k - 1$.*

252 *Proof.* Given a dynamically appearing point set P as discussed, we want to k -colour it so that at every step,
 253 the leftmost $2k - 1$ points contain at least one point of each colour. For $i = 1, \dots, k$, define an i -gap to be an
 254 inclusion-maximal set of points between two points of colour i , and an i -prefix to be the set of points before
 255 the first point of colour i . It suffices to maintain the following invariants at each step.

- 256 (a) All i -gaps have size at least $k - 1$, and
- 257 (b) all i -prefixes have size at most $2k - 2$.

258 Suppose that these invariants are satisfied at some step, and then an event of type (1) occurs. This can only
 259 harm (b) by creating an i -prefix of size $2k - 1$ for some i . At most $k - 1$ colours occur in the leftmost k
 260 points, and by (a), no colour occurs more than once. So there is some uncoloured point which we can colour
 261 with colour i ; by construction, this is separated from the next point of colour i by at least $k - 1$ points, so (a)
 262 is preserved.

263 Now suppose an event of type (2) occurs: the first point disappears. Again, this can only harm (b) by
 264 creating an i -prefix of size $2k - 1$. This means that the leftmost point had colour i , so we may once again
 265 find an uncoloured point in the leftmost k points of the i -prefix, and colour it with colour i . \square

266 Another type of family we consider is the *intersecting family*. These are families where any two rectangles
 267 are pairwise intersecting. In particular, there is a point v contained in the intersection of all the rectangles.
 268 We call the vertical line through v the *spine* of the family.

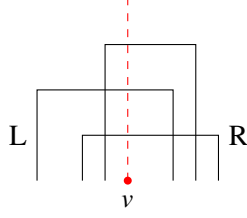


Figure 8: The *spine* through v defines the sides L and R.

269 **Theorem 2.9.** For intersecting families, $m_k^* \leq 3k$.

270 Our proof will rely on *shallow hitting sets*.

271 **Definition 2.10.** A subset $S \subset \mathcal{F}$ is a *c-shallow hitting set for depth d* if every point that is covered by exactly
272 d rectangles is covered by at least 1 and at most c rectangles of S .

273 Suppose for fixed c and a family \mathcal{F} of bottomless rectangles, we have a c -shallow hitting set for any
274 depth d . First we construct a c -shallow hitting set S_1 for depth ck , then remove S_1 from \mathcal{F} and construct a
275 c -shallow hitting set S_2 for depth $ck - c$, then remove S_2 from \mathcal{F} and so on, yielding some disjoint subfamilies
276 S_1, \dots, S_k . It is now easy to check that any point covered by ck rectangles is covered by at least 1 rectangle
277 from each of S_1, \dots, S_k : $m_k^* \leq ck$ for \mathcal{F} .

278 *Proof of Theorem 2.9.* Let d be arbitrary. We will construct hitting sets S_L and S_R , and show that $S = S_L \cup S_R$
279 is a 3-shallow hitting set for depth d . Order the points at depth exactly d from below, and we will add rectangles
280 to S_L and S_R in this order. If p is a point at depth d on side s that is not yet covered by any rectangle of
281 S_s , add to S_s the rectangle covering p whose extension to the other side is shortest. Once we are finished
282 constructing S_L and S_R in this order, we reduce $S = S_L \cup S_R$ so that it is a minimal hitting set for depth d .

283 To show that it is 3-shallow, we show that every point p at depth d on side, say L , is covered by at most 2
284 rectangles from S_L and at most 1 from S_R . This is by minimality; if p is covered by $T_1, T_2, T_3 \in S_L$, removing
285 the one of the lower two whose left endpoint is closer to the spine preserves that S is a hitting set. (This is
286 not true without the fact that \mathcal{F} is intersecting.) If p is contained by two rectangles $R_1, R_2 \in S_R$, suppose
287 that $y(R_1) > y(R_2)$. Since we chose R_1 for S_R from below, there must be a point q on the right side that is
288 covered by R_1 but is above R_2 . As a result, there are $d - 1$ other rectangles covering q whose heights are
289 between R_1 and R_2 . By the choice of R_2 for q by minimality of its left endpoint, the left endpoints of these
290 $d - 1$ rectangles extend beyond the left endpoint of R_2 , so they cover p . This gives $d + 1$ rectangles that cover
291 p , a contradiction. \square

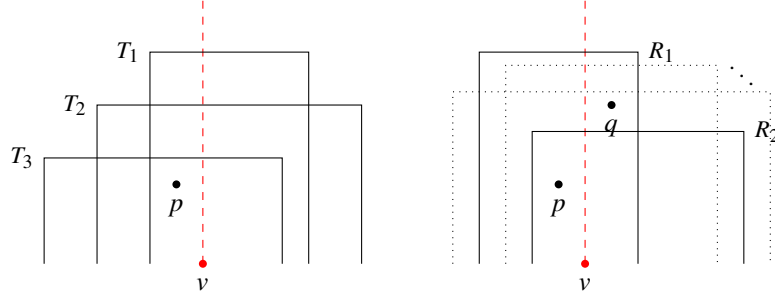


Figure 9: (left) T_2 is not needed to hit points at depth d on the left, and (right) there are $d - 1$ rectangles between R_1 and R_2 that cover p .

3 Arborescences

The goal for this section is to prove Theorem 1.4,

Theorem 1.4. *For any k and m , and any semi-online colouring algorithm that k -colours the vertices of an arborescence in a leaf-to-root order, there is an arborescence that has a leaf-to-root order such that the algorithm will produce a directed path of length m that contains at most one colour.*

and to show that Theorem 1.3 can be deduced from it.

Theorem 1.3. *For any k and m , and any semi-online algorithm that k -colours bottomless rectangles from below (resp. from above, from the right, or from the left), there is a family of bottomless rectangles such that the algorithm will produce an m -fold covered point that is covered by at most one colour.*

We would like to associate to each family \mathcal{F} of rectangles a simple graph, and derive a polychromatic colouring of \mathcal{F} from a suitable colouring of this graph. First we will define the family of graphs that we consider (arborescences), prove Theorem 1.4, then show how these graphs are obtained from bottomless rectangles.

3.1 The setup

An arborescence is a directed tree with a distinguished *root* vertex such that all edges are directed away from the root, i.e. there is a unique directed path from the root to any vertex. We denote the length of the shortest directed path from u to v , if it exists, by $dist(u, v)$. Recall that the length of a path is the number of edges, or one less than the number of vertices. A disjoint union of arborescences is called a *branching*. We say that an ordering of the vertices of a branching is *root-to-leaf* if every vertex is preceded by its in-neighbors and succeeded by its out-neighbors; in particular, from every component first the root is presented and last a leaf.

Claim 1. *The vertices of any branching can be k -coloured by an online algorithm in a root-to-leaf order such that any directed path on k vertices contains all k colours.*

Proof. If a root is presented, colour it with colour 1. Every time a new vertex v is presented in the component with root r , colour v according to the parity of $dist(r, v) \pmod{k}$ (which can be determined from a root-to-leaf ordering). \square

318 We call the reversal of a root-to-leaf ordering a *leaf-to-root* ordering; from each component, first a leaf
 319 is presented and last the root. Our main result, Theorem 1.4, shows that the converse of the above claim
 320 fails: any semi-online algorithm will in fact leave arbitrarily long monochromatic paths. In order to apply
 321 this result to bottomless rectangles, however, we will need a stronger condition on the leaf-to-root ordering.

322 For two vertices u and v of a branching, say $u < v$ if they are in the same component and there is a
 323 directed path from u to v . This forms a partial order where the roots are the minimal elements and the leaves
 324 the maximal. A leaf-to-root ordering is a linear extension of this partial order that presents the $<$ -maximal
 325 element first.

326 If $u < v$ and there are no other vertices between them, i.e. uv is a directed edge, write $u < v$ and say that
 327 v is a *parent* of u . (Thus, somewhat contradicting the laws of nature, every vertex can have only one child,
 328 but several parents.) When presenting the vertices of a branching in a leaf-to-root order, the newly presented
 329 vertex u will always form a root, while its parents were all roots of the branching before u was presented.

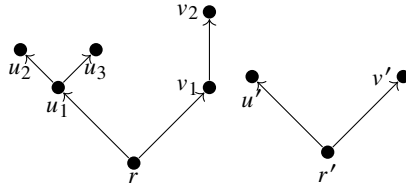


Figure 10: A branching with roots r and r' . In this example, $u_1 \succ r$, i.e. u_1 is a parent of r , but u_2 is not a parent of r even though $u_2 \succ r$ (u_2 is a “grandparent” of r), and $v' \not\succeq r$. A linear extension of this (or a leaf-to-root ordering) might present the vertices u', v' and r' before u_3 , so it is not necessary that the roots of the branching are the last vertices presented.

330 Denote the roots of the branching before a new vertex u is presented by v_1, v_2, \dots indexed in the order
 331 in which they were presented. We say that a *leaf-to-root* ordering is *geometric* if the parents of u form an
 332 interval in this order, i.e., for every u , $\{v_i \mid u < v_i\} = \{v_i \mid l < i < r\}$ for some l and r . Intuitively, we do not
 333 allow an ordering of the following type.

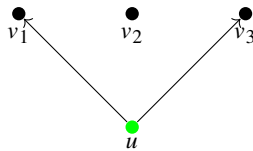


Figure 11: The order v_1, v_2, v_3, u is a leaf-to-root order but it is not *geometric*, because u is adjacent to v_1 and v_3 but not to v_2 .

334 Now we state a stronger form of Theorem 1.4.

335 **Theorem 3.1.** *For any numbers m, k , there is no semi-online k -colouring algorithm that receives the vertices
 336 of an arborescence in a geometric leaf-to-root order and maintains that at every stage, all directed paths of
 337 length m contain all 2 colours.*

338 Call a semi-online k -colouring algorithm *m-proper* if any path on m vertices contains at least two colours.
 339 The theorem states there is no m -proper semi-online k -colouring algorithm for arborescences presented in
 340 geometric leaf-to-root order. The idea of the proof is that for any vertex u , there are only finitely many
 341 possibilities for all directed paths of length m from u . However, we can always force the algorithm to produce
 342 a new “type” of path, leading to a contradiction.

343 Fix k colours, C_1, \dots, C_k , a branching \mathcal{F} with a geometric leaf-to-root order, a point $p \in V(\mathcal{F})$, and the
 344 time t at which p appears. To ease future notation, let us get some (many) definitions out of the way.

- 345 • p_u is a u -parent of p if there is a directed path (p, p_1, \dots, p_u) , i.e. $\text{dist}(p, p_u) = u$ in the graph. We
 346 refer to the subpath (p_1, \dots, p_u) as the *chain* corresponding to p_u .
- 347 • A u -parent p_u of p is *in* C_i if p_u is a u -parent of p and every point in the chain (p_1, \dots, p_u) is coloured
 348 with C_i at time t . (Note that p itself need not have colour C_i .)
- 349 • A u -parent p_u of p in C_i is *maximal* if there is no $p_{u+1} \succ p_u$ that is also coloured with C_i at time t
 350 (note that this depends *only* on t , even if some such p_{u+1} is coloured later).
- 351 • Similarly, p_u is an *uncoloured* u -parent of p if every point of (p_1, \dots, p_u) is uncoloured, and it is a
 352 *maximal* uncoloured u -parent if there is no $p_{u+1} \succ p_u$ that is also uncoloured.
- 353 • The *type* of p , $tp(p)$ is defined as the vector $(t_1, \dots, t_k) \in \mathbb{N}^k$, where

$$t_i = \max\{u : p \text{ has a maximal } u\text{-parent in } C_i\}$$

- 354 • If two partially coloured trees, t_1 and t_2 , are isomorphic, we write $t_1 \cong t_2$. Note that for the
 355 isomorphism we require that vertices coloured, say red, must be mapped to red vertices - we do not
 356 allow the isomorphism to permute the colours.

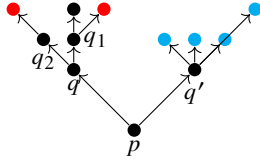


Figure 12: Suppose this is the partially 2-coloured arborescence when p appears. q is an uncoloured 1-parent of p , while q_2 is a *maximal* uncoloured 2-parent. The type of p is $(0, 0)$. We cannot say anything about the type of q' , because this depends on the time at which q' appears. If the blue points were coloured *when* q' appeared, then $tp(q') = (0, 2)$. If they were only coloured when p appeared, then $tp(q') = (0, 0)$, even though it now has blue points above it.

357 Let S_t be the set of points that have appeared by time t in the same connected component of \mathcal{F} as p (or
 358 in the subtree rooted at p at time t). We now associate to p a tree $t(p)$ by “trimming” the induced subgraph
 359 $\mathcal{F}[S_t]$ in the following steps. (See Figure 13.)

- 360 1. If q is uncoloured and $\text{dist}(p, q) > m$, delete q .
- 361 2. If q_1 and q_2 are both maximal t_i -parents in C_i for some remaining q , delete q_2 and all points that are
 362 $> q_2$.
- 363 3. For $i = 1, \dots, m$, if q is a $(m - i)$ -parent of p , and $q_1 \succ q$ and $q_2 \succ q$ are such that the subtrees rooted
 364 at q_1 and q_2 are isomorphic, delete q_2 .

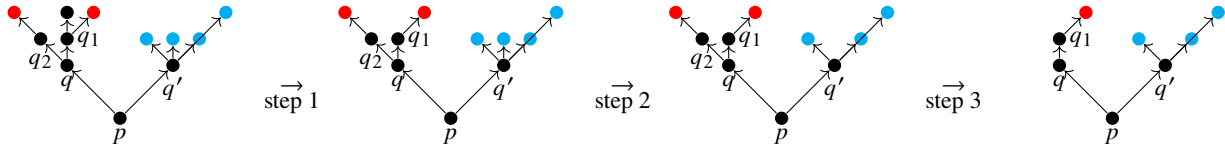


Figure 13: Example for trimming with $m = k = 2$. In step 1, we delete the uncoloured 3-parent of p , but preserve the red parent of q_1 . In step 2, we “trim” the blue parents of q' . In step 3, the subtrees rooted at q_1 and q_2 are isomorphic, so we delete q_2 .

365 The idea of this trimming process is to retain only the “essential” information about the colouring when
 366 p appears and reduce the number of possible $t(p)$ to a bounded number of options. If we assume that the
 367 algorithm has produced a m -proper colouring until the time that p appears, then we can disregard vertices at
 368 distance $> m$ from p . If a vertex was not deleted during the trimming, we say that it was *preserved*.

369 We could modify step 1 to delete *all* points at distance $> m$ from p . However, in the proof we will use
 370 the fact that the type of any point at distance $\leq m$ from p is preserved. Of course, if the algorithm is good,
 371 then any directed path of length m contains at least 2 colours, so deleting only the uncoloured points is just a
 372 technical condition that simplifies notation. Finally, in step 3, we ensure that we do not have any “repetitions”.
 373 For example, if all the branches rooted at p are isomorphic, by considering only one of them we do not lose
 374 any important information.

375 We emphasise that $t(p)$ depends only on the time at which p appears. For instance, in the above figure,
 376 even if q' is coloured blue at a later time, $t(p)$ does not change.

377 3.2 Proof of the main theorem

378 The crucial result of the trimming process is that the following lemma holds.

379 **Lemma 3.2.** *Suppose that a semi-online colouring algorithm as in the statement of the theorem exists. Then*
 380 *the following hold.*

- 381 1. *The set $\{t(p) : \mathcal{F} \text{ is a branching, } p \in V(\mathcal{F})\}$ is finite.*
- 382 2. *If $q \in S_t$ is preserved after the trimming, and q had an t_i -parent in C_i in \mathcal{F} , then q has an t_i -parent in*
 383 *C_i in $t(p)$. In particular, the type of q is preserved.*
- 384 3. *Suppose $p' \prec p$ is presented, and q was an uncoloured u -parent of p in $t(p)$ for $u < d$. If none of the*
 385 *points on the chain from p' to q are coloured when p' is presented, then q is preserved in $t(p')$.*

386 *Proof.* We show that there are only finitely many possibilities for $t(p)$. In step 1 of the trimming we delete
 387 uncoloured points at distance $> m$ from p . In step 2 we preserve only maximal parents in C_i of p for
 388 each colour C_i . Since the algorithm is m -proper, $t(p)$ will have depth at most m . In step 2, we also delete
 389 “repetitions” so there are only finitely many possibilities for each of the branches above p . And in step 3,
 390 we delete isomorphic subtrees, so no two of the branches above p are isomorphic. Thus $t(p)$ can take only
 391 finitely many values.

392 The second claim follows from our earlier argument. For the third claim, we only need to consider the
 393 case when the algorithm produces an uncoloured u -parent q' such that one of q and q' must be trimmed (i.e.,
 394 the subtrees rooted at q and q' are isomorphic). In this case, we can assume without loss of generality that
 395 q' is trimmed so the second property holds. \square

396 **Lemma 3.3.** *At any stage of the algorithm, suppose that we have a collection of trees with roots p_1, \dots, p_s*
 397 *presented in this order such that no two $t(p_i)$ and $t(p_j)$ are isomorphic. Then presenting a vertex p with*
 398 *parents p_1, \dots, p_s , will give a tree $t(p)$ that is non-isomorphic to any $t(p_i)$.*

399 *Proof.* Suppose for contradiction that for some p_i , $t(p) \cong t(p_i)$. Let $\varphi : t(p) \rightarrow t(p_i)$ be an isomorphism
 400 (preserving colourings). We prove by induction for all $u < m$ that there is a chain $p = r_0 \triangleleft r_1 \triangleleft \dots \triangleleft r_u$ in
 401 $t(p)$ such that for all $i \leq u$ we have $\varphi(r_{i-1}) = r_i$, and r_i is uncoloured.

402 First suppose p is coloured, say with C_1 , in $t(p)$, and let t_1 be maximal such that p has a t_1 -parent in C_1 .
 403 $r_1 = p_i = \varphi(p)$ was coloured with C_1 in $t(r_1)$, and by the isomorphism r_1 has a t_1 -parent in C_1 . Since we
 404 did not recolour any points, this produces a $(t_1 + 1)$ -parent in C_1 of p in $t(p)$, contradicting the maximality
 405 of t_1 .

406 So p must be uncoloured in $t(p)$, which implies that r_1 was uncoloured in $t(r_1)$. To complete the base
 407 case of the induction hypothesis, we need to show that r_1 remains uncoloured in $t(p)$, i.e., when p appears.
 408 Let t_1 be as earlier, and suppose again that r_1 is coloured with C_1 in $t(p)$. Then p has an $(t_1 + 1)$ -parent in
 409 C_1 in $t(p)$, again a contradiction.

410 Suppose we have produced a chain $p = r_0 \triangleleft r_1 \triangleleft \dots \triangleleft r_{u-1}$ from the induction hypothesis. If $u - 1 =$
 411 m , then we are done. Otherwise, let $r_u = \varphi(r_{u-1})$. Then r_u is uncoloured in $t(r_1)$. Since $r_{u-1} \triangleright r_{u-2}$,
 412 $r_u \triangleright \varphi(r_{u-2}) = r_{u-1}$, so $p = r_0 \triangleleft r_1 \triangleleft \dots \triangleleft r_u$ is a chain, and it remains to show that r_u is uncoloured in
 413 $t(p)$. Suppose r_u is coloured in $t(p)$ with C_1 . If s_1 is maximal so that r_{u-1} has an s_1 -parent in C_1 in $t(p)$,
 414 then r_u has an s_1 -parent in C_1 in $t(p_1)$, producing an $(s_1 + 1)$ -parent in C_1 for r_{u-1} in $t(p)$. This contradicts
 415 the maximality of s_1 .

416 This eventually produces a chain of m uncoloured points, which contradicts the correctness of the
 417 semi-online algorithm. \square

418 *Proof.* From here we can finish the proof of Theorem 3.1 with an infinite descent argument as follows. Order
 419 the finite sequences of naturals, $\mathbb{N}^{<\omega}$, such that $(s_1, s_2, \dots, s_l) > (s'_1, s'_2, \dots, s'_l)$ if there is some i such that
 420 for all $j < i$ we have $s_j = s'_j$ but $s_i > s'_i$, or $l > l'$ and for all $j \leq l'$ we have $s_j = s'_j$. For a branching \mathcal{F} ,
 421 we define its *associated sequence* as follows. For each root p_i of \mathcal{F} , consider the sequence of trees $t(p_i)$
 422 in the order their roots were presented. Let i_1 be the smallest index such that for every $t(p_i)$ there is an
 423 $i' \leq i_1$ such that $t(p_i) \cong t(p_{i'})$. The number of different trees $t(p_i)$ (same as the number of different trees
 424 up to i_1) is denoted by s_1 . In general, after i_{j-1} has been defined, let i_j be the smallest index such that for
 425 every $t(p_i)$ with $i > i_{j-1}$ there is an $i_{j-1} < i' \leq i_j$ such that $t(p_i) \cong t(p_{i'})$. The number of different trees
 426 $t(p_i)$ for $i_{j-1} < i \leq i_j$ is denoted by s_j . We repeat this for N steps, where N denotes the number of possible
 427 different (i.e., non-isomorphic) trees t , or until there are no more roots in \mathcal{F} . The numbers (s_1, \dots, s_l) are
 428 the associated sequence of \mathcal{F} .

429 Note that there are finitely many associated sequences, as each $N \geq s_1 \geq s_2 \geq \dots \geq s_l$, and also $l \leq N$.
 430 Applying Lemma 3.3 to the largest associated sequence that can be attained during the run of the semi-online
 431 algorithm, we get a contradiction as follows. Let \mathcal{F} be a branching whose associated sequence, (s_1, \dots, s_l) ,
 432 is the largest.

433 Case 1: If $s_1 = N$, then we present a new point p whose parents are the roots of \mathcal{F} , and by Lemma 3.3
 434 we produce a new tree, which is not possible.

435 Case 2: If $N > s_1 > \dots > s_l$, then $l < N$. Introduce a new vertex disjoint from all vertices of \mathcal{F} . This
 436 will either increase an earlier s_i , or give a new $s_{l+1} = 1$, but both of these contradict the maximality of
 437 (s_1, \dots, s_l) .

438 Case 3: There is some j for which $s_j = s_{j+1}$. This is only possible if all the trees $t(p_i)$ for $i_{j-1} < i \leq i_j$
 439 have an isomorphic copy $t(p_{i'})$ for some $i_j < i' \leq i_{j+1}$. Introduce a new vertex p under all the roots p_i of \mathcal{F}
 440 with index $i > i_j$ to obtain a new branching \mathcal{F}' . By Lemma 3.3, the tree $t(p)$ is non-isomorphic to any $t(p_i)$

441 with $i_{j-1} < i \leq i_j$. Therefore, the associated sequence of \mathcal{F}' will be larger than (s_1, \dots, s_l) , contradicting its
 442 maximality.

443 In summary, it is not possible for a semi-online k -colouring algorithm to produce finitely many associated
 444 sequences, so it cannot be m -proper. □

445 3.3 Application to bottomless rectangles

446 In this section, we apply Theorem 3.1 to semi-online colouring algorithms for Erdős-Szekeres configu-
 447 rations. We start with towers.

448 **Corollary 3.4.** *There is no semi-online colouring algorithm for towers from above, i.e., for any numbers k
 449 and m , for any semi-online algorithm that k -colours bottomless rectangles from above, there is a family of
 450 bottomless rectangles such that any two intersecting rectangles form a tower, and the algorithm produces an
 451 m -fold covered point that is covered by at most one colour.*

452 *Proof.* In order to apply Theorem 3.1, we need to show that any branching can be realised as a family of
 453 towers so that

- 454 1. ordering the rectangles from above corresponds to a geometric leaf-to-root order of the branching, and
- 455 2. a semi-online colouring algorithm for towers from above corresponds to an m -proper semi-online
 456 k -colouring algorithm for branchings in this order.

457 For any arborescence \mathcal{F} in geometric leaf-to-root order, we show by induction on $|\mathcal{F}|$ that it can be
 458 realised as a family of towers with this order. For $|\mathcal{F}| = 1$ this is clear. For the inductive step, we will need
 459 to use the fact that the ordering is geometric. For example, suppose we have a non-geometric order and
 460 three roots p, q, r that are realised as disjoint rectangles, with q between p and r . Then if the next root s is
 461 presented with $s < p$ and $s < r$, but $s \not\prec q$, s cannot be realised as a rectangle.

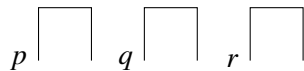


Figure 14: There is no way to present a new rectangle s that intersects p and r but not q .

462 Now we prove the induction step. Let $|\mathcal{F}| = n$, and r be the last element in the ordering of $V(\mathcal{F})$. Take
 463 any realisation of $\mathcal{F} \setminus \{r\}$ as a family of towers. If r is an isolated vertex in \mathcal{F} , present r as a disjoint rectangle
 464 to the right of the realisation $\mathcal{F} \setminus \{r\}$. Otherwise, since the order is geometric, r will only intersect some
 465 geometrically adjacent rectangles of \mathcal{F} (by construction). Hence r can be realised as a minimal rectangle.

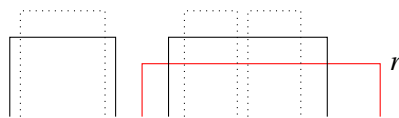


Figure 15: By the geometric ordering, we can realise r as a minimal element.

466 The proof for nested rectangles from below is analogous. □

467 **Corollary 3.5.** *There is no semi-online k -colouring algorithm from the left or from below for increasing
 468 steps. More precisely, for any integers k and m , there is no semi-online algorithm to k -colour rectangles*

469 from the left (or from below) so that at every step, any point covered by m -increasing steps is covered by at
 470 least 2 colours. Similarly, there is no semi-online colouring algorithm for decreasing steps from the right or
 471 from below.

472 Note that this statement is slightly weaker than Theorem 1.3 or Corollary 3.4 because we do not exclude
 473 the other kind of configurations from the family.

474 *Proof.* We first prove the statement for increasing steps from the left. Again, we will prove the corollary by
 475 induction on $|\mathcal{F}|$. However, we also weaken our requirements for the colouring of the steps. That is, we need
 476 not assume that every directed path of length m in the branching corresponds to a point covered by exactly
 477 m increasing steps. It is easy to see that a semi-online colouring algorithm of \mathcal{F} is m -proper if and only if
 478 when any point $p \in \mathcal{F}$ is presented, any directed path of length m from p contains at least 2 colours. So it
 479 suffices to prove the following by induction.

480 Any branching \mathcal{F} with a geometric leaf-to-root order can be realised as a family of bottomless rectangles
 481 so that

- 482 1. when $p \in \mathcal{F}$ is presented, we realise p as a rectangle so that any directed path of length m from p
 483 corresponds to a point covered by exactly m increasing steps, and
- 484 2. any two rectangles intersect either as increasing or as decreasing steps.

485 The second assumption is a technical condition to ensure that q covers the top-right corner of r if and
 486 only if (r, q) form increasing steps, so we can choose the top-right corner of an appropriate rectangle as the
 487 point satisfying the first induction hypothesis.

488 The case $|\mathcal{F}| = 1$ is trivial. Let $|\mathcal{F}| = n$, and r be the last element in the ordering of \mathcal{F} . Take any realisation
 489 of $\mathcal{F} \setminus \{r\}$ satisfying the induction hypotheses. If r is an isolated vertex, let q be the last element presented
 490 (thus a root), and realise r as a rectangle so that (q, r) form decreasing 2-steps (see Figure 16). There are no
 491 directed paths of length m from r so both induction hypotheses are satisfied.

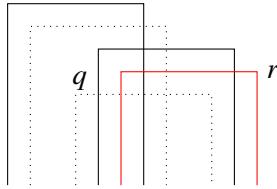


Figure 16: The rectangles in decreasing steps correspond to roots of the branching.

492 Otherwise, since the ordering is geometric, r will only intersect the rightmost rectangles (by construction),
 493 thus can be realised as a rectangle that forms increasing steps with these rightmost roots, and decreasing
 494 steps with the other roots (see Figure 17).

495 To see that the first hypothesis is satisfied, consider the rectangles corresponding to any directed path of
 496 length m from r , say (r_1, \dots, r_{m-1}, r) . Then the top-right corner of r_1 will not be covered by any rectangle
 497 other than the ones in this chain - this follows from the induction hypothesis and the fact that \mathcal{F} is a branching,
 498 so r_2 is the unique child of r_1 .

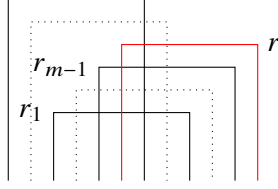


Figure 17: The new rectangle r is presented in increasing steps with r_{m-1} , and decreasing steps with the other minimal elements.

499 The proof for increasing steps from below follows analogously, except we change the second induction
500 hypothesis to assume that any two rectangles intersect either as increasing steps or as a tower. In this
501 construction, the roots of the branching at any time will correspond to a tower, and the geometric ordering
502 ensures that a new root can be placed in increasing steps with the top-most rectangles of the tower.

503 The proof for decreasing steps is exactly the same, only interchanging left and right. \square

504 4 Further results for bottomless rectangles

505 4.1 No shallow hitting sets

506 To prove an upper bound for intersecting families in Theorem 2.9, we used the fact that if a family admits
507 a c -shallow hitting set for arbitrary depths d , then $m_k^* \leq ck$. Unfortunately, general families of bottomless
508 rectangles do not admit such hitting sets.

509 **Theorem 4.1.** *For every integer $c \geq 0$, there exists a real number $r < 1$ and an integer $D \geq 1$ such that for*
510 *every integer $d \geq D$, there is a family $\mathcal{F} = \mathcal{F}(h, d)$ such that the following holds. For any hitting set H of*
511 *the d -cells of \mathcal{F} , there is a vertical line ℓ such that $|\ell \cap \mathcal{F}| \leq \lceil rd \rceil$ and $|\ell \cap H| \geq h$.*

512 In particular, $\mathcal{F}(h, d)$ witnesses that there are no $(h - 1)$ -shallow hitting sets for depth d .

513 *Proof.* We construct $\mathcal{F}(h, d)$ by induction on h . For $h = 0$, letting $r = 0$, $D = 1$, and $\mathcal{F}(h, d)$ be the empty
514 family, the conditions are satisfied. Let r_{h-1} and D_{h-1} be the values given by the induction hypothesis for
515 $h - 1$. Choose s to be a real number such that $0 < s < 1 - r_{h-1}$. Set

$$D > \max\left(\frac{D_{h-1}}{s}, \frac{1}{s(1 - r_{h-1})}\right); \quad r = \max\left(r_{h-1} + s, 1 - s(1 - r_{h-1}) - \frac{1}{D}\right).$$

516 For $d \geq D$, we define $\mathcal{F} = \mathcal{F}(h, d)$ as follows. Take d rectangles (R_1, \dots, R_d) in increasing d -steps. We will
517 insert families of the form $\mathcal{F}(h - 1, i)$ for some $D_{h-1} \leq i \leq d$. First, set $i_0 = \lfloor sd \rfloor$, so that $i_0 \geq \lfloor sD \rfloor D_{h-1} \geq 1$.
518 Insert a copy of $\mathcal{F}(h - 1, d)$ so that it intersects only the rectangles R_1, \dots, R_{i_0} , and the top sides of all the
519 rectangles are above that of R_{i_0} . Next, for every $i_0 < i \leq d$, insert a copy of $\mathcal{F}(h - 1, i)$ so that it intersects
520 exactly the rectangles R_i, \dots, R_d , and all the top sides lie between those of R_i and R_{i+1} (or just above that of
521 R_d when $i = d$).

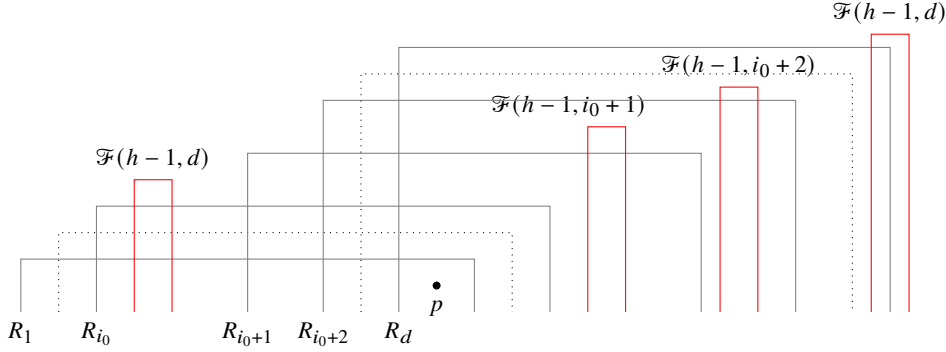


Figure 18: The construction of $\mathcal{F} = \mathcal{F}(h, d)$.

522 Now, let H be a hitting set for the d -cells of \mathcal{F} . Since the point p is in a d -cell, H must contain at least
 523 one of R_1, \dots, R_d . First, suppose that $R_i \in H$ for some $i \leq i_0$. The intersection of H with the leftmost
 524 copy of $\mathcal{F}(h-1, d)$ is a hitting set for the d -cells in $\mathcal{F}(h-1, d)$. By induction, there is a line ℓ such that
 525 $|\ell \cap \mathcal{F}(h-1, d)| \leq r_{h-1}d$ and $|\ell \cap H \cap \mathcal{F}(h-1, d)| \geq h-1$. Then,

$$|\ell \cap \mathcal{F}| \leq r_{h-1}d + i_0 \leq (r_{h-1} + s)d < rd$$

526 and

$$|\ell \cap H| \geq (h-1) + 1 \geq h.$$

527 On the other hand, suppose H does not contain any rectangle R_i for $i \leq i_0$. Let i be maximal so that $R_i \in H$.
 528 Then, $H \cap \mathcal{F}(h-1, i)$ is a hitting set for the i -cells in $\mathcal{F}(h-1, i)$. Again, we have a vertical line ℓ such that
 529 $|\ell \cap \mathcal{F}(h-1, i)| \leq r_{h-1}i$ and $|\ell \cap H \cap \mathcal{F}(h-1, i)| \geq h-1$. Similarly,

$$|\ell \cap H| \geq h$$

and

$$\begin{aligned} |\ell \cap \mathcal{F}| &\leq r_{h-1}i + (d-i) + 1 \\ &= d + 1 - i(1 - r_{h-1}) \\ &\leq d + 1 - sd(1 - r_{h-1}) \\ &= d(1 - s(1 - r_{h-1}) + 1/d) \\ &\leq rd. \end{aligned}$$

530 □

531 4.2 An improved lower bound

532 Finally, we present an improved lower bound for general bottomless rectangle families, and a weaker
 533 lower bound that can be applied to the steps problem.

534 **Theorem 4.2.** $m_k^*(\mathcal{F}_\square) \geq 2k - 1$ for general families of bottomless rectangles.

535 *Proof.* Our lower bound construction proceeds in two steps.

536 1. If $m_k^* < m_{k-1}^* + 2$, then every family has a polychromatic k -colouring that is proper.

537 2. There is a family so that no polychromatic k -colouring is proper.

538 This contradiction shows that $m_k^* \geq m_{k-1}^* + 2$, so by induction $m_k^* \geq 2k - 1$.

539 1. Suppose for some family \mathcal{F} , no polychromatic k -colouring of \mathcal{F} is proper. Let \mathcal{G} be a witness to the
 540 sharpness of m_{k-1}^* , i.e. any $(k-1)$ -colouring of \mathcal{G} produces a point covered by $m_{k-1}^* - 1$ rectangles but not all
 541 k colours. In a small interval around every 2-covered point in \mathcal{F} , we place a thin copy of \mathcal{G} (see Figure 19).

542 Any polychromatic colouring of this new family \mathcal{F}' must induce a polychromatic colouring of \mathcal{F} , so
 543 some copy of \mathcal{G} is covered by 2 rectangles of the same colour, say red. By hypothesis, any point in this
 544 copy of \mathcal{G} covered by at least m_k^* rectangles is covered by all k colours. Since every such point is covered
 545 by exactly two red rectangles from \mathcal{F} , recolouring every red rectangle in \mathcal{G} blue cannot ruin this property.
 546 However, this induces a $(k-1)$ -colouring of \mathcal{G} so that any point in $m_{k-1}^* - 1$ rectangles is covered by all
 547 $k-1$ colours, a contradiction.

548 So every family must have a polychromatic colouring that is proper.

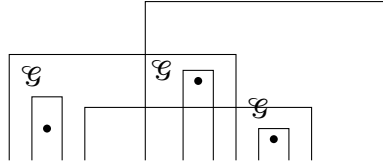


Figure 19: Place disjoint thin copies of \mathcal{G} around every 2-covered point in \mathcal{F} .

549 2. Consider the family in Figure 20.

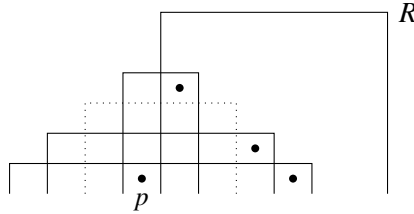


Figure 20: No polychromatic colouring of this family will be proper.

550 We have an m -tower (where m may be arbitrarily large), so that each rectangle from the tower meets
 551 R in a 2-covered point. Suppose without loss of generality that R is coloured red in some polychromatic
 552 k -colouring. For this colouring to be proper, no rectangle of the tower can be red - however the point p will
 553 then be covered by m rectangles, none of which are red, so the colouring cannot be polychromatic. This
 554 completes our proof. \square

555 This lower bound cannot be applied to *unit bottomless*, as this construction relies heavily on towers. For
 556 these, we prove the following weaker lower bound.

557 **Proposition 4.3.** $m_k^*(\mathcal{F}_{\text{unit}}) \geq 2\lfloor \frac{2k-1}{3} \rfloor + 1$ for unit bottomless.

558 *Proof.* This is a generalisation of the construction that shows that $m_k^* = 3$. Let \mathcal{F} be a family of $2k-1$
 559 rectangles partitioned into 3 almost equal subfamilies, $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 as follows.

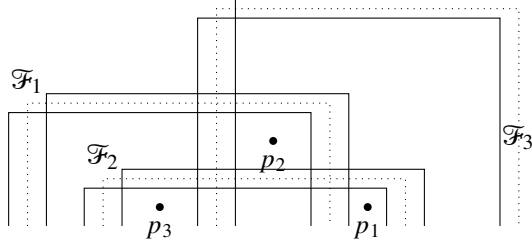


Figure 21: This construction shows that $m_k \geq 2\lfloor \frac{2k-1}{3} \rfloor + 1$.

560 Consider any k -colouring of \mathcal{F} . Some colour, say red, is used at most once, so it appears in at most one
 561 of $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 , say \mathcal{F}_i . Then the point p_i is covered by the other two subfamilies, and no red rectangle.
 562 Since $\lfloor \frac{2k-1}{3} \rfloor \leq |\mathcal{F}_i| \leq \lceil \frac{2k-1}{3} \rceil$, this proves the lower bound. \square

563 Note that the family in the figure does not contain any towers or nested sets. This gives a lower bound to
 564 complement Proposition 2.5, namely that for steps, $m_k^* \geq 2\lfloor \frac{2k-1}{3} \rfloor + 1$.

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