BOUNDEDNESS FOR PROPER CONFLICT-FREE AND ODD COLORINGS

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ABSTRACT. The *proper conflict-free chromatic number*, $\chi_{\mathrm{pcf}}(G)$, of a graph G is the least k such that G has a proper k-coloring in which for each non-isolated vertex there is a color appearing exactly once among its neighbors. The *proper odd chromatic number*, $\chi_{\mathrm{o}}(G)$, of G is the least k such that G has a proper coloring in which for every non-isolated vertex there is a color appearing an odd number of times among its neighbors. We clearly have $\chi(G) \leq \chi_{\mathrm{o}}(G) \leq \chi_{\mathrm{pcf}}(G)$. We say that a graph class $\mathcal{G}$ is χ_{pcf}*-bounded* (χ_{o}*-bounded*) if there is a function f such that $\chi_{\mathrm{pcf}}(G) \leq f(\chi(G))$ ($\chi_{\mathrm{o}}(G) \leq f(\chi(G))$) for every $G \in \mathcal{G}$. Caro, Petruševski, and Škrekovski (2022) asked for classes that are linearly χ_{pcf}-bounded (χ_{pcf}-bounded), and as a starting point, they showed that every claw-free graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq 2\Delta(G) + 1$, which implies $\chi_{\mathrm{pcf}}(G) \leq 4\chi(G) + 1$. In addition, they conjectured that any graph G with $\Delta(G) \geq 3$ satisfies $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 1$. In this paper, we improve the bound for claw-free graphs to a nearly tight bound by showing that such a graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 6$, and even $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 4$ if it is a quasi-line graph. Moreover, we show that convex-round graphs and permutation graphs are linearly χ_{pcf}-bounded. For these last two results, we prove a lemma that reduces the problem of deciding if a hereditary class is linearly χ_{pcf}-bounded to deciding if the bipartite graphs in the class are χ_{pcf}-bounded by an absolute constant. This lemma complements a theorem of Liu (2022) and motivates us to further study boundedness in bipartite graphs. So among other results, we show that convex bipartite graphs are not χ_{o}-bounded, and a class of bipartite circle graphs that is linearly χ_{o}-bounded but not χ_{pcf}-bounded.

§1. INTRODUCTION

The frequency assignment problem has motivated much research about 2-distance colorings, in which vertices get a different color if their distance is at most 2. In such a coloring, each color appears at most once in each neighborhood. A weakening of this, introduced for similar applications [14], is that of *conflict-free coloring*, where for each non-isolated vertex there is a color appearing exactly once among its neighbors. Conflict-free colorings are not necessarily proper: indeed three colors suffice to conflict-free color any planar graph [1], and, as proved by Pyber [26], four colors suffice to conflict-free color any line-graph.

Siting between the notion of proper coloring and 2-distance coloring are *proper conflict-free colorings*, introduced by Fabrici, Lužar, Rindošová, and Soták [15]. The *proper conflict-free chromatic number* of a graph G is the smallest k such that the graph has a conflict-free proper k-coloring; it is denoted by $\chi_{\mathrm{pcf}}(G)$. Fabrici *et al.* showed that there are planar graphs that cannot be proper conflict-free colored with 5 colors, and conjectured this result to be tight. They also proved that

$\chi_{\mathrm{pcf}}(G) \leq 8$ if G is planar, a bound that was improved by Caro, Petruševski, and Škrekovski [5] if G has high girth or small maximum average degree (see also [8]).

A variant of proper conflict-free colorings is that of *proper odd colorings*[1], which are proper colorings in which for every non-isolated vertex there is a color appearing an odd number of times among its neighbors. The least k such that G has a proper odd k-coloring is called the *proper odd chromatic number* of G and is denoted by $\chi_{\mathrm{o}}(G)$.

We clearly have $\chi(G) \leq \chi_{\mathrm{o}}(G) \leq \chi_{\mathrm{pcf}}(G)$, where $\chi(G)$ is the proper chromatic number of G. We say that a graph class $\mathcal{G}$ is *linearly χ_{pcf}-bounded* (*linearly χ_{o}-bounded*) if there is a linear function f such that $\chi_{\mathrm{pcf}}(G) \leq f(\chi(G))$ ($\chi_{\mathrm{o}}(G) \leq f(\chi(G))$) for every $G \in \mathcal{G}$.

Problem 1.1 (Caro *et al.* [5])**.** *Find "generic" graph classes that are linearly χ_{pcf}-bounded.*

As a starting point for their problem, Caro *et al.* [5] showed that the class of claw-free graphs (graphs excluding $K_{1,3}$ as an induced subgraph) is linearly χ_{pcf}-bounded. Namely, they showed that every claw-free graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq 2\Delta(G) + 1$. Since every claw-free graph satisfies $\Delta(G)/2 \leq \chi(G)$, this gives $\chi_{\mathrm{pcf}}(G) \leq 4\chi(G) + 1$. This paper improves this bound for claw-free (and more so for quasi-line graphs) to a near-tight bound. Moreover, we show that convex-round graphs and permutations graphs are linearly χ_{pcf}-bounded, and also give a class of circle graphs that are χ_{o}-bounded but not χ_{pcf}-bounded. We present these last results in Section 1.1, where we also present a lemma that reduces the problem of deciding if a hereditary class is linearly χ_{pcf}-bounded to deciding if the bipartite graphs in the class are χ_{pcf}-bounded by an absolute constant. Our results on claw-free graphs and quasi-line graphs are independent of this lemma and also give evidence to a conjecture of Caro *et al.* [5]; these are presented in Section 1.2.

1.1. **Boundedness and bipartite graphs.** Towards negative results concerning χ_{o}-boundedness, it has been known [25] that the *full subdivision* $S(G)$ of a graph G, i.e., the graph obtained from G by subdividing each edge once, is a bipartite graph with $\chi_{\mathrm{pcf}}(S(G)) \geq \chi_{\mathrm{o}}(S(G)) \geq \chi(G)$, hence providing classes that are not χ_{o}-bounded. First, we study this phenomenon in depth.

Proposition 1.2. *If G is a graph and $S(G)$ is its full subdivision, we have that*

(i) $\chi_{\mathrm{pcf}}(S(G)) \leq \max(5, \chi(G))$,
(ii) *if* $\chi(G) \leq 3$, *then* $\chi_{\mathrm{pcf}}(S(G)) \leq 4$,
(iii) *if* $\chi(G) \leq 4$, *then* $\chi_{\mathrm{o}}(S(G)) \leq 4$.

There exist bipartite graphs G with $\chi_{\mathrm{o}}(S(G)) = 4$.

Note that the above results are all tight, except the result that shows that $\chi_{\mathrm{pcf}}(S(G)) \leq 5$ when $\chi(G) = 4$. In this case, it is natural to ask whether this bound is tight, see Question 3.2. It turns out, as we will prove, that bipartite graphs are *the* obstacle for linear χ_{o}- and χ_{pcf}-boundedness.

[1] This notion was introduced as *odd colorings* by Petruševski and Škrekovski [25], but there is a previous notion of "odd colorings" in the literature, the difference being that the present one is proper and the previous one is not necessarily so.

Giving a step in this direction, and as a way of answering Problem 1.1, Liu [22] recently proved the following (indeed for the choice version of χ_{pcf}).

Theorem 1.3 (Liu [22])**.** *For every positive integer ℓ, there exists an integer c_ℓ such that for every graph G with no odd K_ℓ-minor, if $\chi_{\text{pcf}}(H) \leq t$ for every induced bipartite subgraph H of G, then $\chi_{\text{pcf}}(G) \leq t + c_\ell$.*

Graphs excluding an odd minor include those excluding a minor, but can also be arbitrarily dense. However, graphs with excluded odd minors have bounded chromatic number (see, for example, [17]). Theorem 1.3 essentially tells us that if $\chi_{\text{pcf}}(G)$ is large, then either G contains a large clique as an odd minor or it has an induced bipartite subgraph H with $\chi_{\text{pcf}}(H)$ being large. The following lemma guarantees that if $\chi_{\text{pcf}}(G)$ is large, then either G has large chromatic number or it contains an induced bipartite subgraph H with $\chi_{\text{pcf}}(H)$ being large. Hence, our result (qualitatively) generalizes Theorem 1.3, while in the latter the dependence on t is better.

Lemma 1.4. *If every induced (A, B)-bipartite subgraph of a graph G has a proper conflict-free (proper odd) coloring c with $|c(A)|, |c(B)| \leq t$, then $\chi_{\text{pcf}}(G) \leq t^2\chi(G)$ ($\chi_{\text{o}}(G) \leq t^2\chi(G)$).*

Lemma 1.4 provides the following way to attack Problem 1.1: if we want to know whether a hereditary class $\mathcal{G}$ of graphs is linearly χ_{pcf}-bounded, then it is enough to check that there is an absolute constant k such that every bipartite graph $G \in \mathcal{G}$ satisfies $\chi_{\text{pcf}}(G) \leq k$. With this approach, we obtain linearly χ_{pcf}-boundedness for two classes. These classes have arbitrarily large cliques and are therefore outside of the range of Theorem 1.3.

A graph is a *permutation graph* if it is the intersection graph of segments having endpoints on two parallel lines. Hickingbotham [18] proved that every graph satisfies $\chi_{\text{pcf}}(G) \leq 2\text{col}_2(G) - 1$, where $\text{col}_2(G)$ denotes the 2-coloring number of G (see [20] for a definition), while [13, Lemma 1(a)] implies the bound $\text{col}_2(G) \leq 25\chi(G)$ if G is a permutation graph. Thus $\chi_{\text{pcf}}(G) \leq 50\chi(G) - 1$ for every permutation graph G. However, we are able to improve this as follows.

Theorem 1.5. *For every permutation graph G, we have $\chi_{\text{pcf}}(G) \leq 3\chi(G)$.*

A graph is called *convex-round* if its vertices have a circular ordering such that every neighborhood is an interval [3]. Note that the result of Hickingbotham cannot be applied to convex-round graphs, since balanced complete bipartite graphs are convex-round and have unbounded 2-coloring number. However, Lemma 1.4 allows us to present a new linearly χ_{pcf}-boundedness result for this class.

Theorem 1.6. *For every convex-round graph G, we have $\chi_{\text{pcf}}(G) \leq 9\chi(G)$.*

In Proposition 3.6 we show that for every n there is a convex-round and permutation graph G_n with $\chi(G_n) = n$ requiring at least $n + 2$ colors in any proper odd coloring. We then look at the limits of χ_{o}- and χ_{pcf}-boundedness. Since permutation graphs are the comparability graphs of 2-dimensional posets, it is natural to wonder whether Theorem 1.5 extends. However, for every natural r there is a graph G with $\chi(G) \geq r$ and the dimension of the poset whose comparability graph is $S(G)$ is at most 4 [23, 31]. Hence, with Proposition 1.2, we have bipartite comparability graphs of 4-dimensional posets of unbounded proper odd chromatic number. However, if the dimension of the poset whose comparability graph is $S(G)$ is at most 3, then G is planar [28] and, in particular, has bounded chromatic number and, by Proposition 1.2, also $S(G)$ has bounded proper odd chromatic number. We show below that comparability graphs of 3-dimensional posets are not χ_o-bounded.

Another class containing permutation graphs is the class of *circle graphs*, i.e. intersection graphs of chords of a circle. For their part, *segment intersection graphs*, i.e. intersection graphs of horizontal and vertical segments in the plane, contain bipartite permutation graphs. On the other hand, any (A, B)-bipartite graph G that is a permutation graph [30] or convex-round [3] is *biconvex*. This means, for $V \in \{A, B\}$ and $W \in \{A, B\} \setminus \{V\}$, there is a linear order L of V such that for every $w \in W$ the vertices in $N(w)$ appear consecutively, as an interval $I_L(w)$, in L. A "one-sided" generalization of biconvex is the notion of *convex bipartite*, i.e., for one of A, B, say A, there is a linear order L such that for every $w \in B$ the vertices in $N(w)$ appear consecutively, as an interval $I_L(w)$, in L [21]. The following establishes some limits for χ_{pcf}- and χ_o-boundedness.

Theorem 1.7. *(a) There is a class $\mathcal{G}$ of convex bipartite grid-intersection graphs that are comparability graphs of* 3*-dimensional posets with unbounded proper odd chromatic number.*

(b) There is a class $\mathcal{H}$ of bipartite circle graphs of bounded proper odd chromatic number and unbounded proper conflict-free chromatic number.

Our construction in Theorem 1.7 (b) is identical with one from [19] (given in a different representation), where it is shown that even the (improper) conflict-free chromatic number of circle graphs is unbounded. However, here we emphasize that this class separates proper odd and proper conflict-free chromatic number. Whether circle graphs are χ_o-bounded remains open, see Question 3.10.

1.2. **Claw-free and quasi-line graphs.** The study of (improper) conflict-free edge coloring dates back to the work of Pyber [26], who showed that for every simple graph G, its line graph can be conflict-free colored with at most 4 colors. This, together with Vizing's Theorem, tells us that the line graph of every simple graph G can be properly conflict-free colored with $4\Delta(G) + 4$ colors.

Going beyond line graphs, (improper) conflict-free coloring has been considered for the class of claw-free graphs. For instance, Bhyravarapu, Kalyanasundaram, and Mathew [4] proved that every claw-free graph G can be conflict-free colored with $O(\log(\Delta(G)))$ colors, thus extending a result of Dębski and Przybyło [12]. For proper conflict-free colorings, Caro *et al.* [5] proved that, for every claw-free graph G, we have $\chi_{pcf}(G) \leq 2\Delta(G) + 1$. For line graphs, this improves (only) the additive constant of the bound obtained through Pyber's result and Vizing's Theorem (since if H is the line graph of a simple graph G, then we could have $\Delta(H) = 2\Delta(G) - 2$).

Note that for every claw-free graph G we have $\chi_{pcf}(G) = \chi_o(G)$, because each neighborhood has independence number at most two, and thus a color appearing an odd number of times in a neighborhood must appear exactly once. Therefore, two very recent results of Dai, Ouyang, and Pirot [11] provide different improvements on the above bound of Caro *et al*. They proved that any graph G with maximum degree Δ satisfies both $\chi_o(G) \leq \Delta + \lceil 4(\ln \Delta + \ln \ln \Delta + 3) \rceil$, and $\chi_o(G) \leq \lfloor \frac{3\Delta}{2} \rfloor + 2$.

We improve all these bounds for claw-free graphs as follows.

Theorem 1.8. *Every claw-free graph G satisfies $\chi_{pcf}(G) \leq \Delta(G) + 6$.*

Our proof is constructive, and a polynomial-time algorithm for finding the coloring can be derived from it.

Besides being an improvement on the upper bounds cited above, Theorem 1.8 also gives support to the existence of a constant C such that $\chi_{pcf}(G) \leq \Delta(G) + C$ for each graph G, a belief expressed by Caro *et al.* [5], who conjectured that such a constant should be 1. We mention that a proof of this conjecture for general graphs seems still far from reach: towards this, Caro *et al.* proved that every graph G satisfies $\chi_{pcf}(G) \leq 5\Delta(G)/2$, while Cranston and Liu [10] proved that if $\Delta(G) \geq 10^9$, then $\chi_{pcf}(G) \leq \lceil 1.6550826\,\Delta(G) + \sqrt{\Delta(G)} \rceil$, with slightly weaker bounds when $\Delta(G) \geq 750$.

A graph G is said to be a *quasi-line graph* if, for every $v \in V(G)$, the set $N(v)$ can be expressed as the union of two cliques. The class of quasi-line graphs is a proper superset of the class of line graphs (of multigraphs) and a proper subset of the class of claw-free graphs. It is also a superset of the class of *concave-round graphs* which are the complements of convex-round graphs considered in Theorem 1.6, see [27]. Chudnovsky and Ovetsky [9] proved that if G is a quasi-line graph, then $\chi(G) \leq \frac{3}{2}\omega(G)$, generalizing a classic result by Shannon [29]. We further improve Theorem 1.8 for quasi-line graphs as follows.

Theorem 1.9. *Every quasi-line graph G satisfies $\chi_{pcf}(G) \leq \Delta(G) + 4$.*

Since quasi-line graphs satisfy $\Delta(G) \leq 2\omega(G) - 2$, any such graph G satisfies $\chi_{pcf}(G) \leq 2\chi(G) + 2$.

1.3. **Organization of the paper and notation.** The rest of the paper is organized as follows. In Section 2, we prove Theorems 1.8 and 1.9. In Section 3.1, we prove Proposition 1.2. In Section 3.2, we prove Lemma 1.4 and apply it to prove Theorems 1.5 and 1.6. Sections 3.3 and 3.4 are devoted to prove Theorem 1.7 (a) and Theorem 1.7 (b).

§2. Claw-free graphs and Quasi-line graphs

In this section, we prove that any claw-free graph G has a conflict-free (equivalently, an odd) $(\Delta(G) + 6)$-coloring. We also improve to $\Delta(G) + 4$ the previous upper bound for the class of quasi-line graphs. Subsection 2.1 contains general terminology and statements regarding odd colorings in graphs that are used in the proofs and in the rest of the paper.

2.1. **Preliminaries.** If c is a coloring of a graph G and t is a color that appears exactly once (or an odd number of times, depending on the context) in the neighborhood of a vertex v, then we say that t is a *witness* for v. We denote the set of witnesses of v by $W(c, v)$. If $W(c, v) = \emptyset$, we say *v is unhappy in c*; otherwise, we say *v is happy in c*. Note that $|W(c, v)|$ and $d(v)$ have the same parity. We also denote by $U(c)$ the set of all unhappy vertices of c and call it the *core* of c.

We write $N[u]$ for the *closed neighborhood* of u, defined as $N(u) \cup \{u\}$. For each vertex v and each color $i \notin c(N[v])$, we denote by c_v^i the coloring obtained from c by recoloring v with color i. Notice that, as $i \notin c(N[v])$, the coloring c_v^i is also a proper coloring of G.

Note that, due to parity, if c is any proper vertex coloring of a graph, then any vertex of odd degree has a witness, but a vertex of even degree may not have a witness.

In the proof of our main result of this section, for a given graph G, we construct a sequence of proper colorings such that their associated cores decrease (with respect to inclusion) along the sequence. The sequence starts with an arbitrary proper $\Delta(G) + 6$ coloring of G and continues unless the core of the current coloring is empty. To obtain the next coloring c', we take any vertex v in the core of the current coloring c and recolor one of its neighbors u with a color $i \notin c(N[u])$. By definition, this coloring is c_u^i. We shall prove that v is not in the core of the coloring c_u^i. In fact, we get a bit more.

Lemma 2.1. *For each vertex u and each $i \notin c(N[u])$ we have that*

$$U(c) \cap U(c_u^i) \cap N(u) = \emptyset.$$

Proof. Let $v \in U(c) \cap N(u)$. Then we know that each color j is used by the coloring c an even number of times in $N(v)$. Particularly, $c(u)$ and i are used by coloring c an even number of times in $N(v)$. The colorings c and c_u^i differ only at the color of vertex u. Then, colors $c(u)$ and i are used in $N(v)$ by coloring c_u^i an odd number of times. Hence $W(c_u^i, v) = \{c(u), i\}$ and then $v \notin U(c_u^i)$. □

If the set $U(c_u^i) \setminus U(c)$ is empty, then the new coloring is c_u^i. The main technical obstacle that we have to overcome is dealing with vertices in $U(c_u^i) \setminus U(c)$. The reason why a vertex w belongs to $U(c_u^i)$ but not to $U(c)$ is the following: w is a neighbor of u such that colors $c(u)$ and i are the only colors appearing an odd number of times among its neighbors; namely, $W(c, w) = \{c(u), i\}$. By the choice of color i, no neighbor of u has color i. Hence, some neighbor z of w has color i and z and u are not adjacent. This motivates the following definition.

A vertex w is *c-critical* if it has two neighbors, w_1 and w_2, with $w_1 \notin N(w_2)$, $c(w_1) \neq c(w_2)$, and $W(c, w) = \{c(w_1), c(w_2)\}$. A c-critical vertex creates a problem because it has only two colors in its neighborhood appearing an odd number of times. However, the same fact limits the number of colors that appears in its neighborhood, as proved in the next result.

Lemma 2.2. *Let c be a proper vertex coloring of a graph G. If w is a c-critical vertex, then $|c(N[w])| \leq d(w)/2 + 2$.*

Proof. Direct from the definition of c-critical: $|W(c,w)|$ and $d(w)$ have the same parity, and colors in $c(N[w]) \setminus W(c,w)$ appear in at least two neighbors of w. $\square$

A color i is *c-safe* for a vertex u if $i \notin c(N[u])$ and $W(c,w) \neq \{c(u), i\}$ for each $w \in N(u)$.

Lemma 2.3. *Let c be a proper vertex coloring of a graph G. Let u be a vertex and $i \notin c(N[u])$. The color i is c-safe for u if and only if $U(c_u^i) \setminus U(c)$ is empty. Moreover, $w \in U(c_u^i) \setminus U(c)$ if and only if $W(c,w) = \{c(u), i\}$ and $w \in N(u)$.*

Proof. Let us assume that i is not c-safe for u. Then, there is a neighbor w of u such that $W(c,w) = \{c(u), i\}$. This implies that $c(u)$ and i are the only colors in $c(N(w))$ that appear an odd number of times. In coloring c_u^i, the color $c(u)$ is replaced by color i, and then both colors appear an even number of times in $c_v^i(N(w))$. Hence, $W(c_u^i, w)$ is empty. Whence, $w \in U(c_u^i) \setminus U(c)$.

Conversely, let $w \in U(c_u^i) \setminus U(c)$. Then $W(c_u^i, w)$ is empty, and $W(c,w)$ is not empty. Since $W(c_u^i, z) = W(c,z)$ for each $z \notin N(u)$, we get that $w \in N(u)$. Let us assume that there is a color $j \in W(c,w) \setminus \{c(u), i\}$. But then we would have j in $W(c_u^i, w)$, since c_u^i and c differ only in the colors $c(u)$ and i. Hence, $W(c,w) \subseteq \{c(u), i\}$. If some color $j \in \{c(u), i\}$ does not belong to $W(c,w)$, then it appears an even number of times in $N(w)$. But as $u \in N(w)$ and it changes from color $c(u)$ to color i in c_u^i, the color j would appear an odd number of times in $c_u^i(N(w))$, which is not possible, as $W(c_u^i, w)$ is empty. We conclude that $W(c,w) = \{c(u), i\}$. $\square$

From Lemma 2.3, we have that for each vertex u and each color $i \notin c(N[u])$, the vertices in the set $U(c_u^i) \setminus U(c)$ are c-critical. In fact, if $w \in U(c_u^i) \setminus U(c)$, then $W(c,w) = \{c(u), i\}$. Hence, there is a neighbor z of w with $c(z) = i$. Since $i \notin c(N[u])$, we get that z is not adjacent to u.

Lemma 2.4. *Let c be a proper vertex coloring of a graph G and let $u \in V(G)$. For each $i, j \notin c(N[u])$, $i \neq j$, the sets $U(c_u^i) \setminus U(c)$ and $U(c_u^j) \setminus U(c)$ are disjoint. Moreover, for each $j \neq i$, each vertex in $U(c_u^j) \setminus U(c)$ is c_u^i-critical.*

Proof. Let $w_i, w_j \in N(u)$ such that $w_i \in U(c_u^i) \setminus U(c)$ and $w_j \in U(c_u^j) \setminus U(c)$. From Lemma 2.3 we know that $W(c, w_i) = \{c(u), i\}$ and $W(c, w_j) = \{c(u), j\}$. Since $i, j \notin c(N[u])$ and $i \neq j$, we get that $w_i \neq w_j$.

Now, with respect to the coloring c_u^i, we have that $W(c_u^i, w) = \{i, j\}$ for each neighbor w of u with $W(c,w) = \{c(u), j\}$. Thus, w is c_u^i-critical. $\square$

From the previous lemmas, we have that if i is not c-safe for u, then $U(c_u^i) \setminus U(c)$ is non-empty and $(U(c_u^i) \cap U(c_u^j)) \setminus U(c)$ is empty, if $i \neq j$. Moreover, the set

$$\bigcup_{i \notin c(N[u])} (U(c_u^i) \setminus U(c))$$

is a subset of $N(u)$ and it contains only c-critical vertices.

2.2. **Claw-free graphs.** A graph G is *claw-free* if every vertex $v \in V(G)$ does not have an independent set of size 3 in $N(v)$. Then, a graph G is claw-free if and only if for every two adjacent vertices v and u, $N(u) \setminus N[v]$ is a clique.

Notice that in any proper vertex coloring c of a claw-free graph, any color i appears at most twice in the neighborhood of each vertex. Then, for each vertex v, a color $i \in W(c, v)$ if and only if there is exactly one neighbor of v with color i. Therefore, a coloring of a claw-free graph is odd if and only if it is conflict-free.

Lemma 2.5. *Let $c\colon V(G) \to [\Delta(G) + 6]$ be a proper vertex coloring of a claw-free graph G. If a vertex v has t neighbors which are c-critical, then it has at least $\min\{2, t\}$ neighbors which are c-critical and have a c-safe color.*

Proof. We first prove that if a c-critical neighbor of v does not have a c-safe color, then it has at least four neighbors which are c-critical and also neighbors of v. This, in particular, proves the statement when $t = 1$.

Let z be a c-critical neighbor of v and let $\Delta = \Delta(G)$. By Lemma 2.3, we know that for each color i not in $c(N[z])$ there is a vertex $z_i \in U(c_z^i) \setminus U(c) \subseteq N(z)$ with $W(c, z_i) = \{c(z), i\}$. As c is a $(\Delta + 6)$-coloring and z is c-critical, Lemma 2.2 implies that there are at least $(\Delta + 6) - d(z)/2 - 2$ colors not in $c(N[z])$, which says that z has at least $\Delta + 4 - d(z)/2$ neighbors that are c-critical.

We now prove that at least four of these c-critical neighbors of z also belong to $N(v)$. To this purpose, it is enough to prove that the set K given by

$$K := N(z) \setminus N[v]\,,$$

contains at most $d(z)/2$ vertices, since then, at least $\Delta + 4 - d(z) \geq 4$ of the c-critical neighbors of z also belong to $N(v)$.

We use the above-mentioned fact that, in a claw-free graph, the set K forms a clique in G; and let $W(c, z) = \{c(z_1), c(z_2)\}$, with $z_1, z_2 \in N(z)$ and $z_1 \notin N(z_2)$. This implies, on the one hand, that at most one of z_1 or z_2 belongs to K, as they are non-adjacent. On the other hand, it implies that each vertex of K must have a different color in the coloring c. Then, K can contain at most one vertex of those whose color appears twice in $N(z)$ and one of the vertices z_1 or z_2. Therefore, $|K| \leq |c(N[z])| - 2 = d(z)/2$.

We now prove that when $t \geq 2$, vertex v has at least two c-critical neighbors, both having a c-safe color.

Let z be any c-critical neighbor of v and let C be the set of all neighbors of v which are c-critical, except for z. Then, as $t \geq 2$, we get that C is not empty. We claim that C has a vertex that has a c-safe color. For a contradiction, let us assume that no vertex in C has a c-safe color.

Let us define the digraph D with $V(D) = C$ and such that $(x, y) \in E(D)$ if and only if $c(x) \in W(c, y)$. For each $x \in C$ we know that its in-degree in D is at most two since $|W(c, x)| = 2$. Hence, $|A| \leq 2|C|$.

We have proved that if a c-critical neighbor of v does not have a c-safe color, then it has at least four neighbors which are c-critical and belong to $N(v)$. Hence, under our assumption that no vertex in C has a c-safe color, we get that each vertex in C has out-degree at least three in D. This implies that $|A| \geq 3|C|$, which is a contradiction with $|A| \leq 2|C|$ since, by hypothesis, C is not empty.

To finish the proof, we repeat the previous argument but now taking as z a c-critical neighbor of v having a c-safe color whose existence we already proved. □

Lemma 2.6. *Let $c\colon V(G) \to [\Delta(G)+6]$ be a proper vertex coloring of a claw-free graph G. Let v be a vertex such that no vertex in $N[v]$ is c-critical. Then, each $w \in N(v)$ with $|W(c,w)| \leq 4$, has a c-safe color.*

Proof. For any vertex $w \in V(G)$, we have that

$$|c(N[w])| = \frac{d(w) - |W(c,w)|}{2} + |W(c,w)| + 1 = \frac{d(w) + |W(c,w)|}{2} + 1\,.$$

Moreover, any clique contained in $N(w)$ has size at most $|c(N[w])| - 1$. As we are assuming that G is claw-free, the set $N(w) \setminus N[v]$ forms a clique, which implies that it has at most $|c(N[w])| - 1$ vertices. Furthermore, for each $i \notin c(N[w])$, the set $U(c^i_w) \setminus U(c)$ is contained in $N(w) \setminus N[v]$, because each vertex in $U(c^i_w) \setminus U(c)$ is c-critical and no vertex in $N[v]$ is c-critical, by hypothesis.

Now, if no color is c-safe for w, then we get that the set $N(w) \setminus N[v]$ has at least $\Delta(G) + 6 - |c(N[w])|$ vertices, one for each color not in $c(N[w])$. Therefore, if no color is c-safe for w, then we get the contradiction

$$\Delta(G) + 6 \leq |N(w) \setminus N[v]| + |c(N[w])| \leq 2|c(N[w])| - 1 = d(w) + W(c,w) + 1 < \Delta(G) + 6\,.$$

The result follows. □

At last, we can state our main result for this section.

Theorem 1.8. *Every claw-free graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 6$.*

Proof. Let $c\colon V(G) \to [\Delta(G)+6]$ be a proper vertex coloring of G. We shall prove that if $|U(c)| > 0$, then there is another coloring c' with $|U(c')| < |U(c)|$.

Let $v \in U(c)$. Suppose that there exists a neighbor u of v having a c-safe color i. By Lemma 2.3, $U(c^i_u) \setminus U(c) = \emptyset$. As $v \in U(c) \cap N(u)$, then by Lemma 2.1, $v \notin U(c^i_u)$. Thus, $|U(c^i_u)| \leq |U(c)| - 1$. Hence, we can continue under the assumption that no neighbor of v has a c-safe color.

Let $u \in N(v)$. Note that every $i \notin c(N[u])$ is not c-safe for u, which means, by Lemma 2.3, that $U(c^i_u) \setminus U(c)$ is not empty. Also note that the number of colors not appearing in $c(N[u])$ is $\Delta(G) + 6 - |c(N[u])| \geq \Delta(G) + 6 - (\Delta(G) + 1) = 5$. Fix some $i \notin c(N[u])$. For $j \notin c(N[u])$, with $j \neq i$, we know, by Lemma 2.4, that all vertices in $U(c^j_u) \setminus U(c)$ are c^i_u-critical and that $U(c^j_u) \setminus U(c)$ is disjoint from $U(c^i_u) \setminus U(c)$. Since $\bigcup_{j \notin c(N[u])} (U(c^j_u) \setminus U(c)) \subseteq N(u)$, we conclude that u has at least 5 neighbors which are c^i_u-critical.

Thus, from Lemma 2.5, we obtain that u has at least two c_u^i-critical neighbors both having a c_u^i-safe color. Then, u has a c_u^i-critical neighbor $w \neq v$ that has a c_u^i-safe color ℓ.

We show now that $w \notin N(v)$. Indeed, notice that the sets $W(c_u^i, w)$ and $W(c, w)$ differ exactly in colors $c(u)$ and i, when they are not the same set. Then,

$$|W(c,w)| \leq |W(c_u^i, w)| + 2 = 4\,.$$

Since no neighbor of v has a c-safe color, from Lemma 2.5 we know that no neighbor of v is c-critical. Hence, w can not be a neighbor of v as otherwise, as we have that $|W(c,w)| \leq 4$, it would have a c-safe color, by Lemma 2.6.

We prove now that the coloring $c' = (c_u^i)_w^\ell$ is such that $U(c')$ is a proper subset of $U(c)$. For this purpose, we first show that $U(c_u^i) \setminus U(c)$ is contained in $N(w)$.

Since no vertex in the set $N[v]$ has a c-safe color, due to Lemma 2.3 we know that $U(c_u^i) \setminus U(c) \subseteq (N(u) \setminus N[v])$. We already proved that $w \in N(u) \setminus N[v]$.

We use again the fact that when G is a claw-free graph, the set $N(u) \setminus N[v]$ forms a clique to conclude that the set $U(c_u^i) \setminus U(c)$ is a subset of $N(w)$. This implies that

$$U(c_u^i) = (U(c_u^i) \cap U(c)) \cup (U(c_u^i) \setminus U(c)) \subseteq (U(c_u^i) \cap U(c)) \cup (U(c_u^i) \cap N(w)).$$

We also have that

$$U(c') = (U(c') \cap U(c_u^i)) \cup (U(c') \setminus U(c_u^i)) = U(c') \cap U(c_u^i)\,,$$

since, by Lemma 2.3, we have that $U(c') \setminus U(c_u^i)$ is empty, as the color ℓ is c_u^i-safe for w.

Thus,

$$U(c') = U(c') \cap U(c_u^i) \subseteq U(c') \cap U(c_u^i) \cap U(c)\,,$$

since, by Lemma 2.1, we have that $U(c') \cap U(c_u^i) \cap N(w)$ is empty. Again by Lemma 2.1, we know that $v \notin U(c_u^i)$, since $i \notin c(N[u])$ and $v \in U(c) \cap N(u)$. Therefore, $U(c')$ is a subset of $U(c_u^i) \cap U(c)$ which, as $v \in U(c) \setminus U(c_u^i)$, is a proper subset of $U(c)$. □

2.3. **Quasi-line graphs.** A graph G is a *quasi-line* graph if, for each vertex $v \in V(G)$, there is a partition $\{K_v^1, K_v^2\}$ of $N(v)$ into cliques. Then, any quasi-line graph is a claw-free graph, and any line graph is a quasi-line graph.

Lemma 2.7. *Let G be a quasi-line graph and let $c\colon V(G) \to [\Delta(G) + 4]$ be a proper vertex coloring of G. Let u be a c-critical vertex. Then $d(u)$ is even and $|c(N[u])| = d(u)/2 + 2$. Moreover, for each c-critical vertex $w \in N(u)$, $d(w) = d(u)$.*

Proof. To prove the first statement remember that by definition of u being c-critical we have that $W(c,u) = \{c(u_1), c(u_2)\}$, where u_1 and u_2 are neighbors of u which are not adjacent. As G is a quasi-line graph, there is a partition $\{K_u^1, K_u^2\}$ of $N(u)$ into cliques and, since u_1, u_2 are not adjacent, we can assume that $u_1 \in K_u^1$ and $u_2 \in K_u^2$.

As K_u^1 and K_u^2 are cliques, we know that $|c(K_u^1)| = |K_u^1|$ and $|c(K_u^2)| = |K_u^2|$. Due to $W(c,u) = \{c(u_1), c(u_2)\}$, we actually have that $c(K_u^1) \setminus \{c(u_1)\} = c(K_u^2) \setminus \{c(u_2)\}$ and, thus, $|K_u^1| = |K_u^2|$. Therefore, $d(u) = 2|K_u^1|$ and $|c(N(u))| = |K_u^1| + 1 = d(u)/2 + 1$. Hence, $|c(N[u])| = d(u)/2 + 2$.

Let $w \in N(u)$ be a c-critical vertex. We just proved that the degree of any c-critical vertex is even and that $|c(N[w])| = d(w)/2 + 2$. Let w_1 and w_2 be non-adjacent neighbors of w such that $W(c,w) = \{c(w_1), c(w_2)\}$.

We can assume that the vertex w belongs to K_u^1. Then, $K := (K_u^1 \cup \{u\}) \setminus \{w\}$ is contained in $N(w)$. Since K is a clique, we have that $|c(K)| = |K|$ and one of the vertices w_1 or w_2 does not belong to K. Hence, $|c(K)| \leq |c(N(w))| - 1 = d(w)/2$. Since $d(u)/2 = |K_u^1| = |K|$, we get that $d(u)/2 \leq d(w)/2$.

By interchanging the role of u and w in the previous argument, we get that $d(u) \geq d(w)$, then proving the second statement. □

Lemma 2.8. *Let G be a quasi-line graph and let $c\colon V(G) \to [\Delta(G)+4]$ be a proper vertex coloring of G. Let u be a c-critical vertex. If u does not have a c-safe color, then for each partition $\{K^1, K^2\}$ of $N(u)$ into cliques and each $j = 1, 2$, the vertex u has a neighbor in K^j which is c-critical and has a c-safe color.*

Proof. Let $i \in \{1,2\}$ and let C be the subset of c-critical vertices in $K^i \cup \{u\}$. We consider the digraph D with $V(D) = C$ and such that $(x,y) \in E(D)$ whenever $c(x) \in W(c,y)$.

If the arcs (x,y) and (z,y) are in $E(D)$, then $c(x), c(z) \in W(c,y)$. As C is a clique in G, we have that $c(x) \neq c(z)$ and thus we get that $W(c,y) = \{c(x), c(z)\}$. But then, y would not be c-critical, as x and z are adjacent. We conclude that the in-degree of each vertex in D is at most one and thus $|E(D)| \leq |C|$.

Now, we show that every vertex in C not having a c-safe color has out-degree at least two in D. Let $z \in C$ be a vertex with no c-safe color. By Lemma 2.3, we know that for each color i not in $c(N[z])$ there is $z_i \in C(c_z^i) \setminus C(c) \subseteq N(z)$. As c is a $(\Delta(G)+4)$-coloring and z is c-critical, Lemma 2.7 implies that there are $(\Delta(G)+4) - d(z)/2 - 2$ colors not in $c(N[z])$, which says that z has $\Delta(G) + 2 - d(z)/2$ neighbors which are c-critical.

From Lemma 2.7 we get that $d(z) = d(u)$ and hence, as z has $|K^j|$ neighbors in $K^j \cup \{u\}$ and $|K^j| = d(u)/2 = d(z)/2$, we know that z has $d(z)/2$ neighbors not in $K^j \cup \{u\}$. Hence, at most $d(z)/2$ of the $\Delta(G) + 2 - d(z)/2$ neighbors of z which are c-critical do not belong to C. Whence, at least $\Delta(G) + 4 - d(z) - 2 \geq 2$ of these neighbors belong to C. Therefore, the out-degree of a vertex z in C not having a c-safe color is at least two. As $|E(D)| \leq |C|$, we get that at least one vertex in U has a c-safe color. □

Theorem 1.9. *Every quasi-line graph G satisfies $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + 4$.*

Proof. Let $c\colon V(G) \to [\Delta(G)+4]$ be a proper coloring of G. We shall prove that if $|U(c)| > 0$, then there is another coloring c' with $|U(c')| < |U(c)|$.

Let $v \in U(c)$. Suppose that there exists a neighbor u of v having a c-safe color i. By Lemma 2.3, $U(c_u^i) \setminus U(c) = \emptyset$. As $v \in U(c) \cap N(u)$, then by Lemma 2.1, $v \notin U(c_u^i)$. Thus, $|U(c_u^i)| \leq$

$|U(c)| - 1$. Hence, we can continue under the assumption that no neighbor of v has a c-safe color. In particular, from Lemma 2.8, we also can assume that no neighbor of v is c-critical.

Let $u \in N(v)$ and $i \notin c(N[u])$. We prove that u has at least 3 neighbors that are c_u^i-critical. In fact, v is one of them. Moreover, we know that $\Delta(G) + 4 - |c(N[u])| \geq 3$ and that for each $j \notin c(N[u])$, $j \neq i$, the set $U(c_u^j) \setminus U(c)$ is not empty and by Lemma 2.4, that its vertices are c_u^i-critical.

Let $\{K^1, K^2\}$ be a partition of $N(u)$ into cliques. We can assume that $v \in K^2$. Hence, by Lemma 2.8 we obtain that u has a c_u^i-critical neighbor w in K^1 having a c_u^i-safe color ℓ.

Since v does not have c-critical neighbors, and each vertex in $U(c_u^i) \setminus U(c)$ is c-critical, we have that $(U(c_u^i) \setminus U(c)) \cap N(v)$ is empty, and thus $(U(c_u^i) \setminus U(c)) \subseteq K^1$.

We prove now that the coloring $c' = (c_u^i)_w^\ell$ is such that $U(c')$ is a proper subset of $U(c)$. It is clear that $U(c_u^i) \setminus U(c)$ is contained in $N(w)$, since both $(U(c_u^i) \setminus U(c))$ and w are contained in the clique K^1. This implies that

$$U(c_u^i) = (U(c_u^i) \cap U(c)) \cup (U(c_u^i) \setminus U(c)) \subseteq (U(c_u^i) \cap U(c)) \cup (U(c_u^i) \cap N(w)) .$$

We also have that

$$U(c') = (U(c') \cap U(c_u^i)) \cup (U(c') \setminus U(c_u^i)) = U(c') \cap U(c_u^i) ,$$

since by Lemma 2.3, we have that $U(c') \setminus U(c_u^i)$ is empty, as the color ℓ is c_u^i-safe for w. Thus,

$$U(c') = U(c') \cap U(c_u^i) \subseteq U(c') \cap U(c_u^i) \cap U(c) ,$$

since by Lemma 2.1 we have that $U(c') \cap U(c_u^i) \cap N(w)$ is empty. Therefore, $U(c')$ is a subset of $U(c_u^i) \cap U(c)$ which, as $v \in U(c) \setminus U(c_u^i)$, is a proper subset of $U(c)$. □

§3. BOUNDEDNESS AND BIPARTITE GRAPHS

As mentioned earlier, one has $\chi(G) \leq \chi_{\mathrm{o}}(G) \leq \chi_{\mathrm{pcf}}(G)$ for every graph G. In this section, we investigate whether, for a family of graphs $\mathcal{G}$, there exists a function f such that $\chi_{\mathrm{o}}(G) \leq f(\chi(G))$ or even $\chi_{\mathrm{pcf}}(G) \leq f(\chi(G))$ for every graph $G \in \mathcal{G}$. Indeed, concerning proper conflict-free coloring [5, Question 6.2], it has been asked for generic χ_{pcf}-bounded families.

3.1. **Full subdivisions and chromatic number.** First, we will present a result that, in particular, shows the known result that no such function f exists for large classes of bipartite graphs. For this, given a graph G, let $S(G)$ be the (bipartite) graph obtained from G by subdividing every edge of G exactly once. Clearly, the graphs $S(G)$ are not χ_o-bounded. In what follows, we prove some more specific relations between $\chi(G)$ and $\chi_{\mathrm{pcf}}(S(G))$ (or $\chi_{\mathrm{o}}(S(G))$). These results, show, in particular, that $\chi_{\mathrm{pcf}}(S(G))$ is at most $\chi(G) + 2$. Note that Proposition 1.2(i) generalizes [5, Observation 2.7].

Proposition 1.2. *If G is a graph and $S(G)$ is its full subdivision, we have that*

(i) $\chi_{\mathrm{pcf}}(S(G)) \leq \max(5, \chi(G))$,

(ii) if $\chi(G) \leq 3$, *then* $\chi_{\mathrm{pcf}}(S(G)) \leq 4$,

(iii) if $\chi(G) \leq 4$, then $\chi_{\mathrm{o}}(S(G)) \leq 4$.

There exist bipartite graphs G with $\chi_{\mathrm{o}}(S(G)) = 4$.

Proof. Given G and $S(G)$ as stated, let $V_S = \{s_e \colon e \in E(G)\}$, that is, S_V contains the vertices yielded by the subdivision of the edges of G. Note that $S(G)$ is a $(V(G), S_V)$-bipartite graph.

Now let $k = \max\{5, \chi(G)\}$. We prove (i) by showing a proper conflict-free k-coloring of $S(G)$. Let $c\colon V(G) \to [\chi(G)]$ be a coloring of G. Now we give an algorithm to build a proper conflict-free coloring $c'\colon V(G) \cup S_V \to [k]$ for $S(G)$. For each $u \in V(G) \cup S_V$, we denote by $w(c', u)$ a color which appears only once in the neighborhood of u (and thus makes u happy). Initially, we put $w(c', u) = 0$ for all $u \in V(G) \cup S_V$.

First, let $c'(u) = c(u)$ for all $u \in V(G)$. Note that this makes every vertex s_{uv} happy, since s_{uv} has degree 2 and $c'(u) = c(u) \neq c(v) = c'(v)$, so $W(c', s_{uv}) = \{c(u), c(v)\}$. We can thus put any of $c(u)$ or $c(v)$ in $w(c', s_{uv})$. It remains to color the vertices in S_V so the resulting coloring is proper and all the vertices in $V(G)$ are happy.

We proceed through S_V in any order taking into account that for any vertex $v \in V(G)$ possibly already $w(c', v) \neq 0$. Let $s_{uv} \in S_V$ be an uncolored vertex. We cannot assign to s_{uv} the colors $c(u)$ and $c(v)$ nor the colors $w(c', u)$ and $w(c', v)$. Therefore, there are at most four forbidden colors to s_{uv}. Since $k \geq 5$, there is a color a available, and so we put $c'(s_{uv}) = a$. If $w(c', u) = 0$, then we put $w(c', u) = a$; if $w(c', v) = 0$, then we put $w(c', v) = a$.

Note that the witness $w(c', u)$ defined for a vertex $u \in V(G)$ does not color any other vertex in the neighborhood of u, since it is forbidden to be used by any other s_{uv}. Thus c' is indeed a proper conflict-free k-coloring of G.

We prove (ii) by showing a proper conflict-free coloring $c'\colon V(G) \cup S_V \to [4]$ of $S(G)$ from a $c\colon V(G) \to [\chi(G)]$ coloring of G. Recall that $\chi(G) \leq 3$.

Let M be a maximum matching in G and denote by $V(M)$ the set of vertices covered by M in G. We start by putting $c'(u) = c(u)$ for all $u \in V(M)$. Now we put $c'(s_{uv}) = 4$ for all $uv \in M$. Then, we put in $c'(s_{uv})$ a color from $[3] \setminus \{c(u), c(v)\}$ for all $uv \in E(G) \setminus M$ such that $u \in V(M)$ or $v \in V(M)$. If M is a perfect matching, then the coloring defined so far is a proper conflict-free coloring of $S(G)$, with color 4 being the witness of every vertex of $V(G)$.

If M is not a perfect matching, then let $X = V(G) \setminus V(M)$ be the set of vertices not covered by M. Notice that X is an independent set. Put $c'(v) = 4$ for every $v \in X$. Then, for each $v \in X$, choose one $u \in N(v)$ and let $c'(s_{vu}) = c(v)$. For all other $x \in N(v) \setminus \{u\}$, we put in $c'(s_{vx})$ a color from $[3] \setminus \{c(v), c(x)\}$. This guarantees that the color $c(v)$ appears only once in the neighborhood of v in the coloring c'. Hence c' is a conflict-free coloring. This concludes the proof that $\chi_{\mathrm{pcf}}(S(G)) \leq 4$.

We now prove (iii). We can assume that G is a connected graph with $\chi(G) = 4$. We start by showing the following result.

Claim 3.1. *Let T be a spanning tree of G. Let $c\colon V(G) \to [4]$ be a proper coloring of G and let c' be a coloring of $\{s_e \colon e \notin E(T)\}$ such that $c'(s_{uv}) \in [4] \setminus \{c(u), c(v)\}$. Then, for each vertex $r \in V(G)$, there* *is an extension of c and c' to a proper 4-coloring $\hat{c}$ of $S(G)$ such that for each $w \in V(S(G)) \setminus \{r\}$ there is a color a_w that is a witness for w in $\hat{c}$.*

Proof. Start by making $\hat{c}(u) = c(u)$ for all $u \in V(G)$ and $\hat{c}(s_e) = c'(s_e)$ for all $e \notin E(T)$. At this moment, note that all s_e with $e \notin E(T)$ are happy in $\hat{c}$. Now let $r \in V(G)$. We assign to each vertex s_{uv} with $uv \in E(T)$ a color in $[4] \setminus \{c(u), c(v)\}$ according to the following iterative process.

Pick $uv \in E(T)$ with $v \neq r$ being a leaf of T. Let $\{a, b\} = [4] \setminus \{c(u), c(v)\}$. Recall that $W(\hat{c}, v)$ denotes the set of witnesses for v in $\hat{c}$. If $a \notin W(\hat{c}, v)$, then let $\hat{c}(s_{uv}) = a$; otherwise, let $\hat{c}(s_{uv}) = b$. Update T with $T - uv$ and repeat the process.

Because each vertex s_{uv} has received a color in $[4] \setminus \{c(u), c(v)\}$, the resulting coloring $\hat{c}$ is a proper coloring of $S(G)$ which extends the colorings c and c'.

Furthermore, note that after attributing a color to s_{uv} in the process, all neighbors if v in $S(G)$ are colored, and color a is a witness for v in $\hat{c}$. Also, since the degree of each vertex s_{uv} is two and $c(u) \neq c(v)$, both colors are witnesses for s_{uv}. Thus, all $w \in V(S(G)) \setminus \{r\}$ have $W(\hat{c}, w) \neq \emptyset$. $\Diamond$

Let $c\colon V(G) \to [4]$ be a proper 4-coloring of G such that there is a vertex r whose neighborhood is not monochromatic. Note that, because $\chi(G) = 4$, such 4-coloring always exists. Now the result follows by induction on the number of blocks of G.

First, assume that G is 2-connected and take any spanning tree T of G such that r is a leaf of T (just take any spanning tree of $G - r$ and add r as a leaf). We define a coloring c' of the set $\{s_e \colon e \notin E(T)\}$ as follows. Let p be the vertex adjacent to r in T. As we are assuming that the neighborhood of r is not monochromatic in c, there is a neighbor q of r such that $c(p) \neq c(q)$. We first color s_{rq} with color $c(p)$ and each vertex s_{rx}, with $x \neq p, q$, with some color in $[4] \setminus \{c(r), c(p), c(x)\}$. Notice that with this coloring, the color $c(p) = c'(s_{rq})$ appears exactly once among the neighbors of r in $S(G)$ which are not s_{rp}. We then color each uncolored vertex s_{uv}, with $uv \notin E(T)$ with an arbitrary color in $[4] \setminus \{c(u), c(v)\}$.

From our previous claim, there is a proper coloring $\hat{c}$ of $S(G)$ which extends c and c' and such that each vertex $w \in V(S(G)) \setminus \{r\}$ has a witness. Moreover, $\hat{c}(s_{rp}) \in [4] \setminus \{c(r), c(p)\}$. Since the color $c(p)$ appears exactly once among the neighbors of r in $S(G)$ which are not s_{rp}, we conclude that $c(p) \in W(\hat{c}, r)$. Therefore, $\hat{c}$ is an odd 4-coloring of $S(G)$.

Now, consider that G is not 2-connected. Note that if r is not a cut-vertex, then we can also find a spanning tree T of G where r is a leaf of T and use the same arguments as above to find an odd 4-coloring of $S(G)$. Thus the only vertices that have non-monochromatic neighborhoods are cut-vertices.

Let $B \subseteq G$ be a block of G containing exactly one cut-vertex r of G. By the previous argument, all vertices in $N(r) \cap V(B)$ have monochromatic neighborhood (particularly, they only have color $c(r)$ in their neighborhood). Let $H := G - (B - r)$. Clearly, $\chi(H) \leq 4$ so, by hypothesis, we know that there is an odd 4-coloring $\hat{c}_1$ of $S(H)$. We now show how to extend $\hat{c}_1$ to a 4-coloring $\hat{c}$ of $S(G)$. For that, assume w.l.o.g that $\hat{c}_1(r) = c(r)$.

Let $b \in W(\hat{c}_1, r)$ and let us first define a coloring $c_2 \colon V(B) \to [4]$. Put $c_2(u) = b$ for every $u \in N(r) \cap V(B)$ and $c_2(u) = c(u)$ for every $u \in V(B) \setminus N(r)$. Note that c_2 is a proper 4-coloring of B, since $b \neq \hat{c}_1(r)$ and each $u \in N(r) \cap V(B)$ only has color $\hat{c}_1(r)$ in its neighborhood. By taking a spanning tree T of B, any 4-coloring c_2' of $\{s_e \colon e \in E(B) \setminus E(T)\}$ with $c_2'(s_{uv}) \in [4] \setminus \{c_2(u), c_2(v)\}$ and applying our previous claim, we get that there is a proper 4-coloring $\hat{c}_2$ of $S(B)$ such that each vertex $w \in V(S(B)) \setminus \{r\}$ has a witness in $\hat{c}_2$.

At last, let $\hat{c}$ be the proper 4-coloring of $S(G)$ defined by $\hat{c}_1$ and $\hat{c}_2$. Since all vertices s_{rx} with $rx \in E(B)$ have color $c_2'(s_{rx}) \neq c_2(x) = b$, the color b is still a witness for r in $\hat{c}$. Therefore, $\hat{c}$ is an odd proper 4-coloring of $S(G)$.

Finally, let us show that the upper bounds given in (ii) and (iii) are tight. For that, consider the complete bipartite graph $G = K_{2k,2k}$, and let $V(G) = X \cup Y$, where X and Y are independent. We show that $\chi_{\mathrm{o}}(S(G)) \geq 4$ for all $k \geq 1$. Let $c' \colon V(G) \cup S_V \to \{1, 2, \ldots, p\}$ be an odd coloring of $S(G)$. Since every vertex s_{xy} in S_V has exactly two neighbors, it follows that the sets X and Y use together at least two different colors. Suppose only one color, say 1, is used in X, and a vertex y in Y gets color 2. Since y has even degree, all subdivision vertices which are neighbors of y cannot receive the same color. Thus two other colors different from 1 and 2 are needed to color these neighbors of y, and therefore, $p \geq 4$. Now suppose, without loss of generality, that X uses at least 2 different colors. Since G is a complete bipartite graph, the colors used (by c') in X have to be different from the colors used in Y. Thus, $\chi_{\mathrm{o}}(S(G)) \geq 4$, and therefore, $\chi_{\mathrm{pcf}}(S(G)) \geq 4$. □

The following remains open.

Question 3.2. *Is there a graph G with $\chi(G) = 4$ and $\chi_{\mathrm{pcf}}(S(G)) = 5$?*

3.2. **Reducing linear boundedness to bipartite graphs.** The following result justifies that for questions on χ_{o}-boundedness (as well as for χ_{pcf}-boundedness), we can restrict our attention to bipartite graphs.

Lemma 1.4. *If every induced (A, B)-bipartite subgraph of a graph G has a proper conflict-free (proper odd) coloring c with $|c(A)|, |c(B)| \leq t$, then $\chi_{\mathrm{pcf}}(G) \leq t^2\chi(G)$ ($\chi_{\mathrm{o}}(G) \leq t^2\chi(G)$).*

Proof. We give the proof for odd colorings since the proof for proper conflict-free colorings is identical. Let G be as in the statement $c' \colon V(G) \to [\chi(G)]$ be a proper $\chi(G)$-coloring of G and assume without loss of generality that G has no isolated vertices. Order the vertices of G by increasing color and compute a greedy $\chi(G)$-coloring c of G along this ordering (it is not hard to see that c will indeed be a $\chi(G)$-coloring). Note that coloring c has the property that every vertex of color i is incident to some vertex of color $i-1$ for all $i > 1$. Let $V_i = \{v \in V(H) \colon c(v) = i\}$.

Now for every $i \in [\chi(G)] \setminus \{1\}$ consider the induced bipartite subgraph $H := G[V_{i-1} \cup V_i]$. By hypothesis, there is a proper odd coloring h_1 of V_{i-1}, which we can assume uses colors in $[t]$, that guarantees that every vertex in V_i has an odd number of neighbors of some color from this set. (Note that the graph induced in G by $V_{i-1} \cup V_i$ contains no isolated vertices in V_i by construction.) Thus, every vertex outside of V_1 is happy.

Hence, we still need to ensure that all vertices in V_1 have some odd color in their neighborhood. Since G has no isolated vertices, every $v \in V_1$ has some neighbor in some other set V_i, associate v to the smallest such i and denote it by $f(v)$. Now for every $i \in [\chi(H)] \setminus \{1\}$ consider the induced bipartite subgraph $H := G[\bigcup_{v \in V_1 : f(v)=i}(\{v\} \cup (N(v) \cap V_i))]$. Again, by hypothesis, there is a proper odd coloring $h_2 \colon \bigcup_{v \in V_1 : f(v)=i}(N(v) \cap V_i) \to [t]$ such that every vertex v in V_1 has an odd number of neighbors of some color in each V_i which intersects $N(v)$. The coloring h_2 extends to a partial coloring on V and we can define artificially as 1 on all remaining vertices of v. Taking the product coloring $c''(v) = (h_1(v), h_2(v))$ yields an odd coloring of G. Since h_1 and h_2 both use at most t colors, c'' uses at most t^2 colors. □

Lemma 3.3. *Every biconvex graph G with bipartition (A, B) has a proper conflict-free 6-coloring c such that $|c(A)|, |c(B)| \leq 3$.*

Proof. Let L_1 and L_2 be orderings of A and B, respectively, that exist because G is biconvex. It suffices to show that A can be 3-colored in a way that every vertex in B has a witness that colors only one of its neighbors.

For every vertex $w \in B$ let $I_w = I_{L_1}(w)$, and let $\mathcal{I} = \{I_w \colon w \in B\}$. Also, for any $I \in \mathcal{I}$, let $\mathcal{I}_{|I} = \{I \cap I_w \colon w \in B, I \cap I_w \neq \emptyset\}$. We need the following claim.

Claim 3.4. *For each interval $I \in \mathcal{I}$, the set $\mathcal{I}_{|I}$ has at most two inclusion-minimal intervals.*

Proof. Let $I = I_w$, and suppose that there are three inclusion-minimal intervals $I \cap I_{w_1}, I \cap I_{w_2}, I \cap I_{w_3}$ in $\mathcal{I}_{|I}$. Clearly, none of them corresponds to I. For $i \in \{1, 2, 3\}$, let v_i be the left endpoint of I_{w_i}. Since the intervals $I \cap I_{w_i}$ are inclusion-minimal, the v_i are mutually distinct, but each v_i also appears in I which imply that for each i, $N(v_i) \cap \{w_1, w_2, w_3, w\} = \{w_i, w\}$. By the interval property in L_2, it follows that w needs to be in the same interval with each of the three w_1, w_2, and w_3 in L_2, which is not possible. ◊

We are ready to describe the coloring of A. Consider any inclusion-maximal set $\mathcal{I}_M$ of mutually disjoint intervals in $\mathcal{I}$. (To obtain this, we can simply go from left to right through L_1, and every time we see a vertex uncovered by our set, we add an interval having this vertex as its left endpoint.) For each $I \in \mathcal{I}_M$, take the leftmost and rightmost vertices contained in a minimal interval of $\mathcal{I}_{|I}$. Collect these vertices in a set $S \subseteq A$ and color them alternatingly from left to right with colors 1 and 2. All other points in A are colored 3.

To conclude, let us show that every $J \in \mathcal{I}$ contains one color exactly once. By the choice of $\mathcal{I}_M$, we know that J intersects at least one element of $\mathcal{I}_M$. If $I \in \mathcal{I}_M$ and $I \cap J \neq \emptyset$, then $I \cap J$ is considered in $\mathcal{I}_{|I}$, and thus J contains a vertex of color 1 or 2. In fact, $I \cap J$ contains both a vertex colored 1 and a vertex colored 2 only if it contains the two inclusion-minimal intervals in $\mathcal{I}_{|I}$. If J moreover had another vertex colored 2, then it would also contain a vertex colored 1 and, in particular, for some $I' \in \mathcal{I}_M \setminus \{I\}$, we would have $I' \cup J \neq \emptyset$. But this would imply that J contains three inclusion-minimal intervals, contradicting Claim 3.4. Therefore, the result follows. □

Theorem 1.6. *For every convex-round graph G, we have $\chi_{\text{pcf}}(G) \leq 9\chi(G)$.*

Proof. The class of convex round graphs is closed under taking induced subgraphs and the bipartite convex-round graphs are exactly the biconvex graphs [3]. The result follows from Lemmas 1.4 and 3.3. □

A *comparability graph* is a graph that connects pairs of elements that are comparable to each other in a partial order. A partial order P has dimension two if and only if there exists a partial order Q on the same set of elements, such that every pair of distinct elements is comparable in exactly one of these two partial orders. Therefore, the permutation graphs are equivalent to the comparability graphs of 2-dimensional partially ordered sets (posets). As another consequence of Lemma 3.3, we show that permutation graphs are also χ_{pcf}-bounded.

Theorem 1.5. *For every permutation graph G, we have $\chi_{\text{pcf}}(G) \leq 3\chi(G)$.*

Proof. Bipartite permutation graphs are biconvex [30]. This allows us to use Lemma 3.3, but in order to obtain an improved bound on the χ_{pcf}-binding function, we use an argument different from Lemma 1.4. Suppose that G is the comparability graph of a poset P. Take a greedy antichain partition of P by iteratively picking all the maxima. It is well-known (sometimes called dual Dilworth's Theorem) that the number of antichains corresponds to the height $h(P)$, i.e., the length of the longest chain of P. Let $(A_1, \ldots, A_{h(P)})$ be the antichains in our partition of P, where $A_{h(P)}$ contains all the maxima of P. By Lemma 3.3 we can color each A_i, with $i \in [h(P)] \setminus \{1\}$, using colors from $\{3i-2, 3i-1, 3i\}$ in a way that every vertex of A_{i-1} has one of these colors appearing once in its neighborhood. (Note that the bipartite permutation graph induced by $A_i \cup A_{i-1}$ has no isolated vertices in A_{i-1} by construction.) Finally, consider $G[A_{h(P)} \cup A_1]$ and note that all isolated vertices of this graph are isolated vertices of G. Hence, using Lemma 3.3 again, we can color A_1 with colors from $\{1, 2, 3\}$ in a way that every non-isolated vertex of $A_{h(P)}$ has one of these colors appearing once in its neighborhood. Since we have $h(P) \leq \chi(G)$, the result follows. □

Remark 3.5. Note that in both Theorems 1.6 and 1.5 we also get an upper bound in terms of the clique number ω. Namely, convex-round graphs are circular-perfect [2] and any circular-perfect graph G satisfies $\chi(G) \leq \omega(G) + 1$ [24]. Furthermore, comparability graphs are perfect, hence any permutation graph satisfies $\chi(G) \leq \omega(G)$.

We do not know how good our upper bounds from this section are. Let us merely establish, that on the considered classes we cannot have equality of $\chi(G)$ and $\chi_{\text{o}}(G)$ for all G.

For each natural number $n \geq 2$ we define *the cocktail party graph* CP_n as the graph obtained from the complete graph K_{2n} on $\{0, \ldots, 2n-1\}$ with a perfect matching $M = \{(2i, 2i+1) \mid i \in \{0, \ldots n-1\}$ removed.

Proposition 3.6. *For every $n \geq 2$ the cocktail party graph CP_n on $2n$ vertices is a convex-round permutation graph with $\chi(CP_n) + 2 \leq \chi_o(CP_n)$.*

Proof. Ordering the vertices of CP_n by their indices circularly shows that this graph is convex-round. The partial order P given by the rule $i <_P j$ whenever $i < j - 1$ and $i <_P i + 1$ if i is odd yields a poset of width 2 and hence dimension 2, whose comparability graph is CP_n.

Clearly, $\chi(CP_n) = n$ and let us prove by induction on n that $\chi_o(CP_n) \geq n + 2$. For $n = 2$ we have that $CP_n = C_4$ for which it is known that 4 colors are needed in an odd coloring. Now let $n \geq 3$ and consider CP_n with an odd coloring c. Remove the two independent vertices $2n-2, 2n-1$ from CP_n obtaining CP_{n-1}. If the restriction of c to G_{n-1} uses at least $n+1$ colors, then since both $2n-2, 2n-1$ are connected to all vertices of CP_{n-1}, the coloring c needs at least one other color, i.e., it uses at least $n+2$ colors.

If the restriction of c to CP_{n-1} uses at most n colors, then by induction hypothesis there is a vertex CP_{n-1} such that all the colors appearing in its neighborhood appear an even number of times. By symmetry, we can assume that 0 is such a vertex. Since $N(0)$ is the union of two cliques of the same size, a set of equally colored vertices in $N(0)$ must be exactly a pair $2j, 2j+1$ for $1 \leq j \leq n-2$. And on $N(0)$ at least $n-2$ colors are used and 0 uses yet another color. Then without loss of generality, the vertex $2n-2$ must make 0 happy in c and $2n-1$ must be colored differently from $2n-2$. But none of them can use any of the already used $n-1$ colors, hence $n+1$ colors are used. Moreover, since $2n-2, 2n-1$ are happy in c, the vertex 1 must use yet another color. Hence at least $n+2$ are used. ☐

3.3. **Unboundedness of χ_o for convex bipartite graphs.** The graphs that we construct now will be convex bipartite, hence showing that Lemma 3.3 does not extend to this class. For every positive integer k, let G_k be the (A, B)-bipartite graph where $A = [2^k]$, $\mathcal{I}$ is the set of all non-empty intervals in A, and every $I \in \mathcal{I}$ is the neighborhood of a vertex in B. The following corresponds to Theorem 1.7 a).

Proposition 3.7. *For every positive integer k, the graph G_k:*

(i) *is convex bipartite,*
(ii) *has $2^{k+1} + \binom{2^k}{2}$ vertices,*
(iii) *is a grid-intersection graph,*
(iv) *is the comparability graph of a 3-dimensional poset,*
(v) *has $\chi_{\mathrm{pcf}}(G_k) \leq k + 2$,*
(vi) *has $\chi_o(G_k) \geq k + 2$.*

Proof. It is easy to see that (i) follows from the definition of G_k.

To see (ii), note that $|A| = 2^k$ and there are $2^k + \binom{2^k}{2}$ intervals in $\mathcal{I}$.

To see (iii), simply draw every vertex of $v \in A = [2^k]$ as a vertical ray starting at the coordinate $(v, 0)$ every vertex of B as the horizontal interval $I_w \in \mathcal{I}$ corresponding to its neighborhood at a height such that no two horizontal intervals intersect.

Now to prove (iv), observe that the horizontal intervals representing B can furthermore be drawn such that if an interval is contained in another one, then it is drawn lower. This is possible since these intervals do not intersect. Such graphs were studied *SegRay* graphs (segmentation-ray intersection graphs), with a special representation for which it is known that they are comparability graphs of 3-dimensional posets, see [6, Lemma 1] and also [7].

For (v), we prove by induction on k that $\chi_{\text{pcf}}(G_k) \leq k+2$ with the additional property that only $k+1$ colors are used on A. This is clearly true for G_1, which is a path on 5 vertices, so consider $k > 1$. Let $A_1 = \{1, \ldots, 2^{k-1}\}$ and $A_2 = \{2^{k-1}+1, \ldots, 2^k\}$, and for $i = 1, 2$ let $B_i = \{v \in B \colon N(v) \subseteq A_i\}$ and $H_i := G_k[A_i \cup B_i]$. Note that each H_i is isomorphic to G_{k-1} so, by inductive hypothesis, let $c_i \colon V(H_i) \to [k+1]$ be a vertex coloring for H_i such that A_i only uses colors in $[k]$. Exchange the labels of colors $k+1$ and $c_2(2^{k-1}+1)$ in c_2 and observe that color $k+1$ appears only once in A_2. Now we build a coloring c for G_k. Put $c(v) = c_i(v)$ if $v \in V(H_i)$, and put $c(v) = k+2$ if $v \notin V(H_1) \cup V(H_2)$. Note that all vertices in A, as well as all vertices in B whose neighborhood is completely in some A_i, are happy by induction. All other vertices in B are neighbors of vertex $2^{k-1}+1$ of A, and thus they are happy because see the color $k+1$ exactly once.

For (vi), we will show that at least $k+1$ colors are needed by A. Since B contains a vertex that is adjacent to all vertices of A, this will imply the claim. So suppose that we have an odd coloring where A is colored with only k colors. Consider all intervals of the form $[2i-1, 2i] \subseteq A$, for $i \in [2^{k-1}]$. In order to have an odd coloring, any such interval must use at least two different colors $r, s \in [k]$. We code this by associating to the interval $[2i-1, 2i]$ the vector $x^i \in \mathbb{Z}_2^k$ which is 0 everywhere except in the entries r, s, where it is 1.

Note that, since we have an odd coloring, for each pair $j < \ell$, $\sum_{i=j}^{\ell} x^i \neq 0$, because otherwise the interval $[2j-1, 2\ell]$ would only have colors appearing at an even number of times. Moreover, this implies that for every pair $j < \ell$, we have that $\sum_{i=j}^{2^{k-1}-1} x^i \neq \sum_{i=\ell+1}^{2^{k-1}-1} x^i$. Thus, the set $X = \{\sum_{i=j}^{2^{k-1}-1} x^i : j \in [2^{k-1}-1]\}$ has $2^{k-1}-1$ different elements. On the other hand, we know $X \subseteq \{x \in \mathbb{Z}_2^k \colon \sum_{r=1}^{k} x_r = 0\}$, which is a subspace of $\mathbb{Z}_2^k$ of codimension 1, hence has 2^{k-1} elements. Consider now the vector $x^{2^{k-1}}$. Since the interval $[2j-1, 2^k]$ must have a color a odd number of times, then for each $j < 2^{k-1}$ we have $\sum_{i=1}^{2^{k-1}-1} x^i \neq x^{2^{k-1}}$, which implies that vector $x^{2^{k-1}}$ does not belong to X. Then is must be equal to 0, which is a contradiction to the existence of the odd coloring. □

3.4. **Separating χ_{o} and χ_{pcf} in bipartite circle graphs.** In this subsection, we will construct a class of bipartite circle graphs on which the proper conflict-free chromatic number is unbounded, while the odd chromatic number is bounded. To the best of our knowledge, no graphs with such properties have been shown before. Biconvex graphs are also a proper subclass of bipartite circle graphs [32] but in light of our previous results, here we will work on the difference between these classes. Before getting to the construction, let us give a definition and a lemma. A family $\mathcal{F}$ of subsets of a set V is *nested* if $\emptyset \notin \mathcal{F}$ and $I \cap J \neq \emptyset \implies I \subseteq J$ or $I \supseteq J$ for all $I, J \in \mathcal{F}$.

Lemma 3.8. *If $\mathcal{F} \subseteq 2^V$ is a nested family, then V can be 3-colored such that every member $I \in \mathcal{F}$ contains one color an odd number of times and one color an even number of times.*

Proof. We order $\mathcal{F}$ by inclusion and construct a coloring bottom to top in this poset by induction. If $I \in \mathcal{F}$ is minimal, then we can clearly 3-color its elements such that one color appears an odd number of times and another one an even number of times.

Now suppose $I \in \mathcal{F}$ and the elements of V covered by the (mutually disjoint) inclusion maximal intervals $I_1, \dots, I_k$ contained in I are colored such that each interval contained in some I_i, $1 \leq i \leq k$, contains some color an odd number of times.

Permute the coloring of the elements contained in each I_i, $1 \leq i \leq k$, such that for an odd number of I_i, $1 \leq i \leq k$, their odd color is 1 and their even color is 2 and for an even number of the I_i, $1 \leq i \leq k$, their odd color is 2 and their even color is 1. Finally, color all elements covered by I but by none of the intervals contained in I by 3. In the resulting coloring, I has an odd number of elements of color 1 and an even number of elements of color 2. □

For every positive integer k, define H_k as an (A, B)-bipartite graph where $A = [2^k]$, $\mathcal{I}$ is the maximum nested set of intervals of length 2^i for all $1 \leq i \leq k$ in A, i.e., greedily take intervals of the specified length starting from the beginning, and every $I \in \mathcal{I}$ is the neighborhood of a vertex in B. See Figure 1 for an illustration. The following corresponds to Theorem 1.7 b).

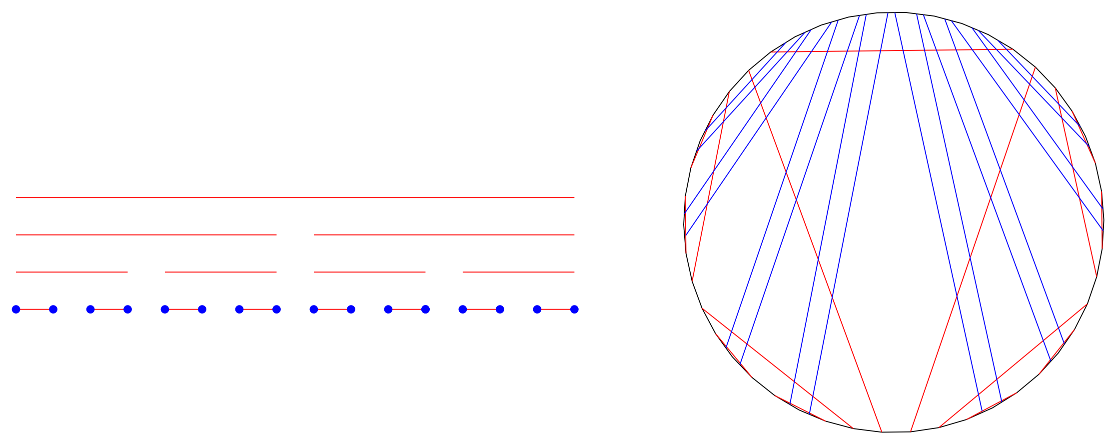

FIGURE 1. The interval representation for H_4 and its representations as chords in a circle.

Proposition 3.9. *For every positive integer k, the graph H_k satisfies the following properties.*

(i) *is a bipartite circle graph,*
(ii) *has $2^{k+1} - 1$ vertices,*
(iii) *has $\chi_{\mathrm{pcf}}(H_k) \geq k + 2$,*
(iv) *has $\chi_{\mathrm{o}}(H_k) \leq 4$.*

Proof. Note that (i) follows from the definition of H_k. Figure 1 illustrates how to represent H_k as a circle graph.

To see (ii), note that A has 2^k vertices and the vertices of B corresponding to the intervals in $\mathcal{I}$ ordered by inclusion form (the closure of) a full binary tree with 2^{k-1} leaves. Hence, H_k has $1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1$ vertices.

To show (iii), we prove by induction on k that at least $k + 1$ colors are needed to color A in any conflict-free coloring. Then, since B contains a vertex incident to all vertices of A, this implies that, in total, at least $k + 2$ colors are needed.

The claim clearly holds for H_1, which is a path on 3 vertices and needs 2 colors on its endpoints. Now let $k > 1$ and suppose we have a conflict-free coloring of H_k such that A uses only colors in $[k]$. Because of the vertex in B whose neighborhood is A, we can also assume, w.l.o.g, that color k appears exactly once in A. Then, in one of the sets $A_1 = \{1, \ldots, 2^{k-1}\}$, $A_2 = \{2^{k-1} + 1, \ldots, 2^k\}$, the color k does not appear, w.l.o.g., suppose A_1. Let $B_1 = \{v \in B \colon N(v) \subseteq A_1\}$ and $G := H_k[A_1 \cup B_1]$. Note that G is isomorphic to H_{k-1} and that the odd coloring for H_k restricted to G is an odd coloring for G where A_1 uses only colors in $[k-1]$, which is a contradiction to the induction hypothesis.

To see (iv), note that, by Lemma 3.8, we can color A with 3 colors. Now for every interval $I_w \in \mathcal{I}$ of size two, color the corresponding vertex $w \in B$ with a color different from the two colors used in I_w. Color all remaining vertices of B with color 4. This is an odd coloring for H_k. □

While we have separated odd and proper conflict-free chromatic numbers on the class of circle graphs, the following remains open.

Question 3.10. *Are circle graphs χ_o-bounded?*

In order to answer this question, it is sufficient to study whether bipartite circle graphs have bounded odd chromatic number. A possibly helpful representation of such graphs through spanning trees of planar graphs is due to de Fraysseix [16, Proposition 6].

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