

# Sandpiles prediction and crossover on $\mathbb{Z}^2$ within Moore neighborhood

Pablo Concha-Vega<sup>1,2\*</sup>, Eric Goles<sup>1</sup>, Pedro Montealegre<sup>1</sup>,  
Kévin Perrot<sup>2,3</sup>

<sup>1\*</sup>Facultad de Ciencias y Tecnología, Universidad Adolfo Ibáñez,  
Santiago, 7941169, Chile.

<sup>2</sup>LIS, CNRS, Aix Marseille Université Toulon, Marseille, 13009, France.

<sup>3</sup>Université publique, France.

\*Corresponding author(s). E-mail(s): [pabloconcha@alumnos.uai.cl](mailto:pabloconcha@alumnos.uai.cl);  
Contributing authors: [eric.chacc@uai.cl](mailto:eric.chacc@uai.cl); [p.montealegre@uai.cl](mailto:p.montealegre@uai.cl);  
[kevin.perrot@lis-lab.fr](mailto:kevin.perrot@lis-lab.fr);

## Abstract

The computational complexity of predicting sandpiles on  $\mathbb{Z}^2$  is not settled yet, neither for von Neumann nor for Moore neighborhood (is it in **NC**? is it **P**-complete?). In this work we study the sandpile model considering all the **256** possible sub-neighborhoods within the Moore neighborhood. Surprisingly, we found that **12** of them have a **P**-complete prediction problem, while for the remaining **244** neighborhoods, we prove that they do not admit a crossover gate, *i.e.*, for them, it is impossible to cross information, if the bit of information is the presence (or absence) of an avalanche.

**Keywords:** Sandpile models, discrete dynamical system, computational complexity.

## 1 Introduction

In the field of computer science, the study of discrete dynamical systems has emerged as a fascinating tool that offers a unique lens through which we can understand complex phenomena. One of the most iconic and influential branches of these systems, characterized by their discrete space and time, is cellular automata. Cellular automata are simple yet powerful mathematical constructs that have captured the imagination of

computer scientists, mathematicians, and physicists for decades. A cellular automaton consists of a grid of cells, each of which has a state (from a finite set) that evolves following a local rule (homogeneous in space and time). The study of discrete dynamical systems, and specially cellular automata, has not only enriched our understanding of fundamental computational principles but has also paved the way for the development of innovative models and applications across various domains [1–6].

In this article we center our attention on sandpile models, a particular class of cellular automata introduced by Bak, Tang and Wiesenfeld as the first example of a dynamical system exhibiting self-organized criticality [7]. As its name suggests, sandpile cellular automata simulate the behavior of sand grains on a 2-dimensional grid. Each grid cell has a capacity threshold. When a cell accumulates more sand grains than its capacity, it becomes *unstable* and redistributes some of its grains to its neighboring cells. This redistribution process, often referred to as an *avalanche*, occurs iteratively, with unstable cells toppling and causing their neighbors to become unstable in a cascading fashion. Although it was originally formulated over a 2-dimensional lattice with the von Neumann neighborhood (the cells surrounding a cell in the four cardinal directions), the sandpile model can be naturally extended to arbitrary graph topologies, hence to  $d$ -dimensional grids. The remarkable feature of the sandpile model over arbitrary graphs is its Turing universality [8]. This means that it has the capacity to simulate any given Turing machine, a foundational concept in computer science representing a general-purpose computing device. Shortly thereafter, a pivotal question in the field of computer science arose, as articulated by Moore and Nilsson [9]: *given an initial distribution of grains in a sandpile, what is the eventual state it converges to after all possible topplings have occurred?* This problem, commonly referred to as the *prediction problem* for sandpiles, serves as a cornerstone in the study of sandpile dynamics. Depending on the computational complexity of solving this problem, it offers valuable insights into the most efficient algorithms that can be employed for computing the sandpile dynamics. The initial results of Moore and Nilsson unveil fundamental differences in the computational capabilities of the sandpile model depending on dimensionality. They demonstrated that in three or more dimensions, the sandpile dynamics is intrinsically sequential (P-complete), implying that solving the prediction problem inherently requires a naive simulation approach (unless  $P = NC$ ). On the other hand, in the one-dimensional case, they presented a fast parallel algorithm ( $NC^3$ ). Determining the computational complexity of the 2-dimensional case is still an open problem today.

When establishing the P-completeness of the prediction problem associated with a cellular automaton, the prevailing framework in the literature is the well-known *Banks approach*. This approach hinges on reducing from the *monotone circuit value problem* (MCVP), via constructing a series of circuitry gadgets. These gadgets typically consist of *wires* and *logic gates* (AND and OR). Nevertheless, when the support of the dynamics is symmetric and planar, it may be necessary to design more devices, such as *diodes* and *crossover gates* [10–12]. Crossover gates, in particular, hold a significant role as they often present the greatest challenge in terms of construction in a 2-dimensional context. In essence, crossover gates consist of a rectangular finite configuration that facilitates the unimpeded transmission of information (commonly

referred to as a *signal*) from one edge to the opposite (e.g., from north to south), without interfering with information traveling in the perpendicular direction (e.g., from west to east). The underlying challenge in the context of 2-dimensional sandpile models is the absence of an evident construction of a crossover gate. Furthermore, the work of Goles and Gajardo [13] has sustained this difficulty, proving that a crossover gate cannot exist for the 2-dimensional sandpile, whether one considers the von Neumann neighborhood or the Moore neighborhood (the eight adjacent cells surrounding a cell). It is crucial to emphasize that the impossibility to construct a crossover gate solely signifies the limitation of the Banks approach, therefore the classification of the prediction problem remains unsolved, as we mentioned before.

This leads us to the focus of our work, where we extend notions in relation to the complexity of 2-dimensional sandpiles by meticulously studying all the sub-neighborhoods within the Moore neighborhood, *i.e.*, all the 256 possible combinations of the eight cells included in the Moore neighborhood. Surprisingly, we found that there are 12 neighborhoods that admit a crossover gate. For them, we construct all the necessary circuitry for proving the P-completeness of their prediction problem. In contrast, for the remaining sub-neighborhoods, we extend the findings of Goles and Gajardo by establishing the impossibility of a crossover gate. While some of these proofs present fairly straightforward conclusions, others require deeper analysis and geometric arguments.

### Outline

This article is organized as follows. In Section 2, we present an overview of the essential concepts including definitions and previously established results relevant to our work. Subsequently, our results are presented Section 3, which conducts an exhaustive analysis of the prediction problem associated to every possible sub-neighborhood of the Moore neighborhood. Concluding our work, Section 4 presents a comprehensive discussion concerning prediction problems within the context of sandpiles. It considers the analytical tools employed in their study and explores potential model variations, providing perspectives on the subject.

## 2 Definitions and state of the art

Let  $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ . Let  $\subset_{\text{fin}}$  denote finite subsets. For  $d \in \mathbb{N}_+$  let  $0^d$  denote the  $d$ -tuple  $(0, \dots, 0)$ . Let  $\mathbf{H}(\alpha) = 1$  if  $\alpha \geq 0$  and  $\mathbf{H}(\alpha) = 0$  if  $\alpha < 0$  denote the Heaviside step function. We denote  $[n] = \{1, \dots, n\}$  and  $\llbracket n \rrbracket = \{0, \dots, n-1\}$ .

### Sandpile models

In this work we study sandpile dynamics exclusively on square lattices  $\mathbb{Z}^d$  of dimension  $d = 2$ . A (uniform) *sandpile model* is defined by a *neighborhood*  $\mathcal{N} \subset_{\text{fin}} \mathbb{Z}^2$ . We denote its size by  $\theta_{\mathcal{N}} = |\mathcal{N}|$ . A *configuration*  $c : \mathbb{Z}^2 \rightarrow \mathbb{N}$  assigns  $c(p)$  grains to cell  $p \in \mathbb{Z}^2$ . Its dynamics  $F_{\mathcal{N}} : \mathbb{N}^{\mathbb{Z}^2} \rightarrow \mathbb{N}^{\mathbb{Z}^2}$  is defined as:

$$\forall p \in \mathbb{Z}^d, F_{\mathcal{N}}(c)(p) = c(p) - \theta_{\mathcal{N}} \cdot \mathbf{H}(c(p) - \theta_{\mathcal{N}}) + \sum_{p' \in \mathcal{N}} \mathbf{H}(c(p - p') - \theta_{\mathcal{N}}).$$

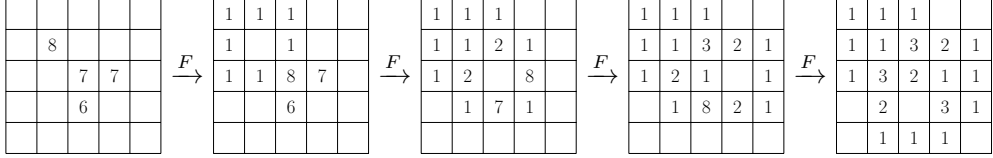


**Fig. 1:** Example of sandpile dynamics for Moore neighborhood  $\mathcal{N}_m$ . The numbers indicate the amount of grains in every cell. A blank site represents an empty cell.

In words, each cell reaching the threshold  $\theta_{\mathcal{N}}$  *topples* or *fires*, sending one grain to each out-neighbor which relative positions are given by  $\mathcal{N}$ . This local rule is applied in parallel at every cell of the lattice. The most famous neighborhoods, for their simplicity and symmetries, are *von Neumann* and *Moore*, respectively defined as:

$$\mathcal{N}_{\text{vn}} = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| = 1\} \quad \text{and} \quad \mathcal{N}_m = \{(x, y) \in \mathbb{Z}^2 \mid 1 \leq x^2 + y^2 \leq 2\}.$$

See an example on Figure 1.

When it is clear from the context, we drop the subscript notations and simply denote  $\theta$  and  $F$  the threshold and the evolution rule. A configuration  $c$  is *finite* when it contains a finite number of sand grains, denoted  $\mathcal{G}(c) = \sum_{p \in \mathbb{Z}^2} c(p)$ . This quantity is invariant by  $F$  for any sandpile model:  $\mathcal{G}(c) = \mathcal{G}(F(c))$ . A configuration  $c$  is *stable* when no toppling occurs ( $\forall p \in \mathbb{Z}^2, c(p) < \theta$ ). From any finite configuration  $c$ , the dynamics converges to a stable configuration denoted  $c^\circ = \lim_{t \rightarrow \infty} F^t(c)$ , see for example [14]. Note that it is custom to exclude the cell itself (at relative position  $0^d$ ) from its neighborhood, in order to avoid “trapping” one sand grain at each cell.

We restrict our study to neighborhoods  $\mathcal{N}_{\subset \text{fin}} \mathbb{Z}^2$  that span  $\mathbb{Z}^2$  *i.e.*, such that  $\mathcal{N} = \{p_1, p_2, \dots, p_k\}$  and:

$$\{a_1 \cdot p_1 + a_2 \cdot p_2 \cdots + a_k \cdot p_k \mid a_1, a_2, \dots, a_k \in \mathbb{Z}\} = \mathbb{Z}^2.$$

Indeed, otherwise we have independent subdynamics, meaning that  $\mathbb{Z}^2$  can be decomposed into sublattices for which the sand contents never interact. Consequently it is equivalent to consider the product of sandpile dynamics for smaller neighborhoods. This would be problematic for our considerations, because intuitively crossovers would be possible for signals evolving within separate sublattices.

### **Prediction problems and crossovers**

The problems are expressed for each sandpile model  $\mathcal{N}_{\subset \text{fin}} \mathbb{Z}^2$ . The asymptotic prediction of sandpile dynamics is the problem of computing the stable configuration  $c^\circ$  reached from a given configuration  $c$ . It is classically restricted to an input configuration  $c$  which is finite and stable, and a position  $p \in \mathbb{Z}^2$  on which *one* grain is added, and formalized as the following decision problems [14]. For any  $p \in \mathbb{Z}^2$ , let  $1p$  be the configuration containing a unique grain at position  $p$ , and let  $+$  denote the cell-wise addition of two configurations ( $\forall p \in \mathbb{Z}^2, (c + c')(p) = c(p) + c'(p)$ ). It asks whether cell  $q \in \mathbb{Z}^2$  topples during the evolution from  $c + 1p$  to  $(c + 1p)^\circ$ .

$\mathcal{N}$ -sandpile prediction problem ( $\mathcal{N}$ -PRED)

Input: a finite stable configuration  $c \in \mathbb{N}^{\mathbb{Z}^2}$ , and two positions  $p, q \in \mathbb{Z}^d$ .

Question: does  $\exists t \in \mathbb{N}, F^t(c + 1p)(q) \geq \theta$  ?

The general goal is to dichotomize the complexity of prediction problems according to the neighborhood  $\mathcal{N}$ : either in NC or P-complete for logspace reductions (see [15, 16] for the definitions of NC, P, and their meaning within the theory of computational complexity). Indeed, it is proven in [14] that all such problems are in P (more precisely, the quantification over  $t$  can be upper bounded to a value polynomial in the size of  $c$ ). On the one hand, proofs that the problem is in NC either consists in providing an efficient parallel algorithm, or reducing the problem to another one for which such an algorithm is known [9, 14, 17, 18]. On the other hand, proofs that the problem is P-complete rely on so-called *Banks approach* [19], which consists in reducing from *monotone circuit value problem* (MCVP) by implementing the circuit computation via chains of reactions in the sandpile dynamics. In this context, it is required to build *wires, turns, and gates, or gates, diodes and crossover gates* [10]. It turns out that all gates are usually straightforward to design, except crossover gates. The inexistence of crossover gates is therefore a strong indication that such a reduction may not be possible, but it is relative to a precise definition of crossover gate (to prove that none exist) which enforces a precise (hence restrictive) consideration of what a *wire*, or more generally a *signal*, is [20].

**Definition 1** (Crossover gate). A *crossover gate* is a rectangular subset of the lattice on which a stable configuration is set, such that it transports signals from two pairs of opposite sides, independently from each other: two wires cross each other. Given a sandpile model  $\mathcal{N}$ , a crossover gate of size  $M \times N$  is formally defined as a configuration  $g : \llbracket M \rrbracket \times \llbracket N \rrbracket \rightarrow \{0, \dots, \theta - 1\}$  together with four cells  $n, e, s, w \in (\llbracket M \rrbracket \times \llbracket N \rrbracket) \setminus \{(0, 0), (M - 1, 0), (0, N - 1), (M - 1, N - 1)\}$  not in the corners, verifying the following constraints 1–5. For convenience, denote the four sides of the gate as  $S_0 = \{(0, y) \mid y \in \llbracket N \rrbracket\}$ ,  $S_1 = \{(x, 0) \mid x \in \llbracket M \rrbracket\}$ ,  $S_2 = \{(M - 1, y) \mid y \in \llbracket N \rrbracket\}$ ,  $S_3 = \{(x, N - 1) \mid x \in \llbracket M \rrbracket\}$ .

1.  $n, s$  are on opposite sides, *i.e.*  $\exists i_n, n \in S_{i_n}, \exists i_s, s \in S_{i_s}$  and  $|i_n - i_s| = 2$ ;
2.  $w, e$  are on opposite sides, *i.e.*  $\exists i_w, w \in S_{i_w}, \exists i_e, e \in S_{i_e}$  and  $|i_w - i_e| = 2$ ;
3.  $n, e, s, w$  are on different sides, *i.e.*  $\{i_n, i_e, i_s, i_w\} = \{0, 1, 2, 3\}$ ;
4. adding a grain at  $n$  eventually topples  $s$ , without toppling any cell on the other sides, *i.e.*  $\exists t \in \mathbb{N}, F^t(g + 1_n)(s) \geq \theta$ , but  $\forall p \in S_{i_w} \cup S_{i_e}, \forall t \in \mathbb{N}, F^t(g + 1_n)(p) < \theta$ ;
5. adding a grain at  $w$  eventually topples  $e$ , without toppling any cell on the other sides, *i.e.*  $\exists t \in \mathbb{N}, F^t(g + 1_w)(e) \geq \theta$ , but  $\forall p \in S_{i_n} \cup S_{i_s}, \forall t \in \mathbb{N}, F^t(g + 1_w)(p) < \theta$ .

An example of crossover gate is depicted on Figure 2, with  $n = (5, 10), s = (5, 0), w = (0, 5), e = (10, 5)$ : adding a grain at  $n$  eventually topples  $s$  without toppling any cell from the east nor west sides, and adding a grain at  $w$  eventually topples  $e$  without toppling any cell from the north nor south sides. It is necessary to provide this formal definition (because we will state some impossibility results), although it is difficult to use in practice, and in the following no precise reference to its constituents will be made. The main tool we will employ is the *pair of firing graphs*, presented at the beginning of Section 3.

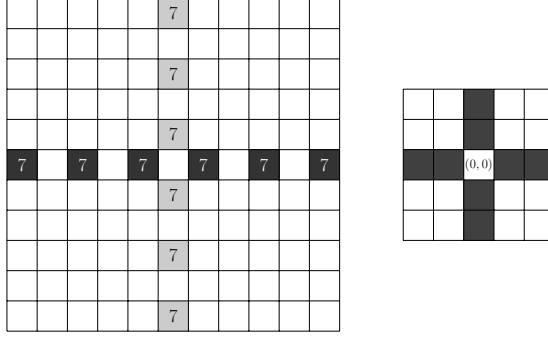


**Fig. 2:** Example (from [13]) of crossover gate of size  $11 \times 11$ , for the von Neumann neighborhood of radius two whose eight cells are depicted on the right:

$$\mathcal{N}_{\text{vn}2} = \bigcup_{i \in \{-2, -1, 1, 2\}} \{(i, 0), (0, i)\}.$$

Adding a grain at the grey cell on the north side (resp. black cell on the west side) triggers a chain reaction of topplings eventually toppling the grey cell on the south side (resp. black cell on the east side).

### Known results

The study of the computational complexity of sandpile models on lattices started with the work of Moore and Nilssen in 1999 [9], where it is proven that the prediction problem with von Neumann neighborhood is in **NC** in dimension one (the bound has been improved to **AC**<sup>1</sup> in [21]), and is **P**-complete in dimension three and above. The two dimensional case is left open in this preliminary work, and has been the subject of subsequent works in many directions. A survey [14] proves that all prediction problems on lattices  $(\mathbb{Z}^d)$  are solvable in polynomial time, generalizing a result of Tardos [22] on finite undirected graphs. The survey [14] also generalizes the dimension sensitivity, by proving that the prediction problem is in **NC** for all one dimensional sandpile model, and is **P**-complete for all three or more dimensional sandpile model.

Towards the understanding of the computational complexity of the two dimensional sandpile model with von Neumann neighborhood, Goles and Gajardo [13] proved that it is impossible to construct a crossover gate (with elementary signals following the approach initiated by Banks on cellular automata [19]). In the present work, we develop on this standard approach and on obstacles to the simulation of monotone circuit value problem within sandpiles (crossover impossibility). Out of this context, the design of unconventional signals and circuit simulations is largely open [20]. The authors of [13] also introduced the firing graph, which is a central concept of Section 3 (Definition 3). Despite the fact that crossover gates are known to be impossible in two dimensional sandpile model with von Neumann neighborhood of radius one  $\mathcal{N}_{\text{vn}}$ , no **NC** algorithm is known. The situation is identical for the Moore neighborhood  $\mathcal{N}_{\text{m}}$  [13, 23].

Other two dimensional sandpile models have been proven to have **P**-complete prediction problems, by reduction from **MCVP**. It turns out that slight modifications of von Neumann neighborhood, such as that of radius two or more, are **P**-complete [13]. It is also the case for Kadanoff sandpile models of radius two or more [24]. More generally, the shape of the neighborhood is not an obstacle to the existence of a crossover

gate. Indeed, it is proven in [23] that increasing the radius of any shape leads to a neighborhood that is eventually able to perform a crossover. This is in particular the case of Moore neighborhood of big enough radius, and even of a discrete circular neighborhood of big enough radius.

Fungal sandpiles received a recent interest, it consists in splitting the rule application into an horizontal and a vertical phase. Crossover gates have been proven to exist even on very restricted version of the model, which has required subtle considerations on delays in the composition of semi-crossover gates [11, 12].

A general framework for the design of P-completeness reductions in discrete dynamical systems is presented in [10], in terms of cellular automata and threshold networks. A focus is put on intrinsic universality, and the power of reusable wires (which is presumably impossible to implement in sandpiles, because of the abelian property that the order of topplings commute) is emphasized through the discovery of a planar crossover gate in this context.

On overall, many neighborhoods in dimension two are known to have a P-complete sandpile prediction problem, and are therefore intrinsically sequential. On the other hand, few others are known to have a sandpile prediction problem in NC, and are therefore efficiently parallelizable. More precisely, it is known that there are infinitely many neighborhoods  $\mathcal{N}_{\subset_{\text{fin}} \mathbb{Z}^2}$  having a P-complete  $\mathcal{N}$ -PRED problem. To the best of our knowledge, it is currently not known whether there is an infinity of neighborhoods  $\mathcal{N}_{\subset_{\text{fin}} \mathbb{Z}^2}$  having a  $\mathcal{N}$ -PRED problem in NC.

The freezing world also received a dedicated attention. In the context of sandpiles, its complexity is still open for the von Neumann neighborhood of radius one, although important restrictions are proven to be either equivalent, or in NC [25]. This track of research builds on early results on majority like cellular automata, which are P-complete to predict on graphs of degree at least five, but in NC on graphs of degree at most four (which includes the  $\mathbb{Z}^2$  case) [17, 18]. Contrary to direct NC algorithms employed to predict sandpiles in dimension one [26, 27], the contributions on majority like cellular automata make use of clever parallel algorithms from [16], for example regarding the computation of the connected components of an undirected graph.

### 3 Crossover among subsets of Moore

In this section we present an exhaustive study of whether each subset of the Moore neighborhood  $\mathcal{N}_{\text{m}}$  admits a crossover gate or not, in the (uniform) sandpile model. Observe that some of the  $2^8$  subsets of  $\mathcal{N}_{\text{m}}$  (which has 8 elements) do not span  $\mathbb{Z}^2$ , and the existence of a crossover gate is invariant by rotation and axial symmetries of the neighborhood. It reduces the number of neighborhoods to study to 43 equivalence classes, presented in Table 1. The numbering of the neighborhoods is presented below.

**Definition 2** (Subset of Moore neighborhood). *Let  $\mathbf{n}, \mathbf{ne}, \mathbf{e}, \mathbf{se}, \mathbf{s}, \mathbf{sw}, \mathbf{w}, \mathbf{nw}$  denote the eight cardinal coordinates, starting with north and clockwise. We will denote a subset of Moore neighborhood as an 8-bits string, where the  $i$ -th bit encodes whether the  $i$ -th coordinate in the list  $\mathbf{n}, \mathbf{e}, \mathbf{s}, \mathbf{w}, \mathbf{nw}, \mathbf{ne}, \mathbf{se}, \mathbf{sw}$ , is part of the neighborhood (bit 1), or not (bit 0). We may also interpret 8-bits strings as numbers (converted to decimal) with the least significant bit on the right, see Figure 3.*

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| <div><div>152</div><div>10011000</div><div></div></div> <div><div>196</div><div>11000100</div><div></div></div> <div><div>44</div><div>00101100</div><div></div></div> <div><div>73</div><div>01001001</div><div></div></div> <div><div>70</div><div>01000110</div><div></div></div> <div><div>140</div><div>10001100</div><div></div></div> <div><div>184</div><div>10111000</div><div></div></div> <div><div>212</div><div>11010100</div><div></div></div> <div><div>216</div><div>11011000</div><div></div></div> <div><div>228</div><div>11100100</div><div></div></div> <div><div>173</div><div>10101011</div><div></div></div>   |
| <div><div>210</div><div>11010010</div><div></div></div> <div><div>116</div><div>01110100</div><div></div></div> <div><div>120</div><div>01111000</div><div></div></div> <div><div>178</div><div>10110010</div><div></div></div> <div><div>202</div><div>11001010</div><div></div></div> <div><div>58</div><div>00111010</div><div></div></div> <div><div>106</div><div>01101010</div><div></div></div> <div><div>53</div><div>00110101</div><div></div></div> <div><div>90</div><div>01011010</div><div></div></div> <div><div>85</div><div>01010101</div><div></div></div> <div><div>151</div><div>10010111</div><div></div></div>    |
| <div><div>186</div><div>10110100</div><div></div></div> <div><div>209</div><div>11010001</div><div></div></div> <div><div>225</div><div>11100001</div><div></div></div> <div><div>232</div><div>11101000</div><div></div></div> <div><div>103</div><div>01100101</div><div></div></div> <div><div>149</div><div>10010101</div><div></div></div> <div><div>154</div><div>10011010</div><div></div></div> <div><div>197</div><div>11000101</div><div></div></div> <div><div>165</div><div>10100101</div><div></div></div> <div><div>170</div><div>10101010</div><div></div></div> <div><div>92</div><div>10011101</div><div></div></div> |
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| <div><div>89</div><div>01011001</div><div></div></div> <div><div>172</div><div>10101100</div><div></div></div> <div><div>108</div><div>01101100</div><div></div></div> <div><div>547</div><div>10010011</div><div></div></div> <div><div>156</div><div>10010110</div><div></div></div> <div><div>201</div><div>11001001</div><div></div></div> <div><div>153</div><div>10011001</div><div></div></div> <div><div>156</div><div>10011100</div><div></div></div> <div><div>198</div><div>11000110</div><div></div></div> <div><div>204</div><div>11001100</div><div></div></div> <div><div>200</div><div>11001011</div><div></div></div> |

**Table 1:** Equivalence classes of neighborhoods among subsets of  $\mathcal{N}_m$ . The representative of each class is highlighted. (Part 1/2)

In this numbering scheme, the von Neumann neighborhood ( $\mathcal{N}_{vn}$ ) corresponds to  $11110000 = 240$ , while the Moore neighborhood ( $\mathcal{N}_m$ ) corresponds to  $11111111 = 255$ . It is worth noting that neighborhoods with numbers  $\leq 15$  consist exclusively of diagonal neighbors. Consequently, these neighborhoods do not span  $\mathbb{Z}^2$ .

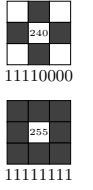
We illustrate with a small figure next to each Theorem statement which neighborhoods we are considering. They are named in the same order they are shown.

Let us start with the early results that motivate the present study.

**Theorem 1** ([13, 23]). *Neighborhoods 240 (von Neumann  $\mathcal{N}_{vn}$ ) and 255 (Moore  $\mathcal{N}_m$ ) do not admit a crossover gate.*

This reduces the equivalence classes from 43 to 41.

The causal structure of topplings is captured by the concept of firing graph, introduced in [13]. It is at the basis of our reasonings in this work. Given a neighborhood



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**Table 1:** Equivalence classes of neighborhoods among subsets of  $\mathcal{N}_m$ . The representative of each class is highlighted. (Part 2/2)

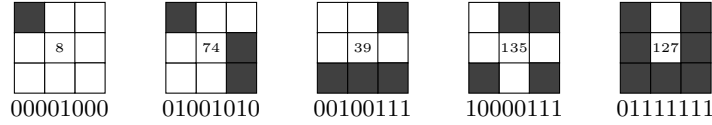


**Fig. 3:** From left to right, subsets of Moore neighborhoods corresponding to  $00001000 = 8$ ,  $01001010 = 74$ ,  $00100111 = 39$ ,  $10000111 = 135$ ,  $01111111 = 127$ .

$\mathcal{N}$ , we say that cell  $v_1$  is an *in-neighbor* of cell  $v_2$  when  $(v_2 - v_1) \in \mathcal{N}$ . In the symmetric situation  $(v_1 - v_2) \in \mathcal{N}$  it is called an *out-neighbor*. Observe that  $v_1$  may be both an in-neighbor and an out-neighbor of  $v_2$ . In particular, this is always the case for symmetric neighborhoods, such as von Neumann  $\mathcal{N}_{vn}$  and Moore  $\mathcal{N}_m$ .

**Definition 3** (Firing graphs). Given a crossover gate  $g$  on cells  $n, s, w, e \in \llbracket m \rrbracket^2$ , we define its two *firing graphs*  $G_{ns}^g = (V_{ns}, A_{ns})$  and  $G_{we}^g = (V_{we}, A_{we})$  as:

- $V_{ns}$  (resp.  $V_{we}$ ) is the set of fired cells after adding a grain at cell  $n$  (resp.  $w$ );
- there is an arc  $(v_1, v_2) \in A_{ns}$  (resp.  $A_{we}$ ) when  $v_1, v_2 \in V_{ns}$  (resp.  $V_{we}$ ), and both:
  - $v_1$  is an in-neighbor of  $v_2$  for  $\mathcal{N}$ , and
  - $v_1$  is fired strictly before  $v_2$  after adding a grain at cell  $n$  (resp.  $w$ ).

The endpoints of a firing graph are called *starting cells* ( $n$  and  $w$ ) and *ending cells* ( $s$  and  $e$ ) respectively. For every arc  $(v_1, v_2)$  in a firing graph, we call  $v_1$  a *predecessor* of  $v_2$ . Naturally, for every arc  $(v_1, v_2)$ , we call  $v_2$  a *successor* of  $v_1$ . The transitive and reflexive closure of the predecessor (resp. successor) relationship is the *ancestor* (resp. *descendant*) relationship. By definition, a predecessor is necessarily an in-neighbor, and a successor is necessarily an out-neighbor. The firing graph captures the *effective* causal relationships among cells within a given crossover gate, whereas the in-neighbor and out-neighbor relationships can be seen as *potential*.

### Section outline

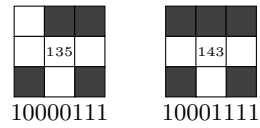
We present our findings across distinct subsections to facilitate a structured exploration of our results. Initially, our intention was to group numerous neighborhoods into collective proofs. However, we encountered challenges in grouping them together because there is no evident factorization of the arguments, *i.e.*, almost all the proofs are unique for each neighborhood. As a result, we opted to categorize them into separate subsections based on their distinctive characteristics. In Subsection 3.1, we show a striking discovery, two sub-neighbors of  $\mathcal{N}_m$ , identified by the numbers 135 and 143, unexpectedly exhibit the capability for a crossover gate. We prove that both problems 135-PRED and 143-PRED are P-complete by crafting all the necessary circuitry for reducing from MCVP. In Subsection 3.2, we introduce the framework used for proving the impossibility of a crossover gate for all the other neighborhoods. This framework hinges on a pivotal concept, the “crossing constraint”, which serves as a primary component in the subsequent proofs presented in this section. In Subsection 3.3, we prove that all the neighborhoods generating planar graphs on the grid do not admit a crossover gate. Moving to Subsection 3.4, our attention turns to neighborhoods 247 and 127. These neighborhoods stand out for their size of 7, which means they are only missing one cell out of the eight found in the Moore neighborhood. Subsection 3.5 is dedicated to the study of neighborhoods featuring only two cells on the diagonals, while Subsection 3.6 focuses on neighborhoods with three and four diagonal cells. At this stage, we are left with just two remaining equivalence classes, represented by neighborhoods 95 and 39. We have assigned individual subsections to these classes due to their distinct proofs involving the new concept of “temporal minimality of escape cells”. In Subsection 3.7, we exploit it to prove that a crossover is impossible for neighborhood 95. Finally, Subsection 3.8 is dedicated in its entirety to the comprehensive study of neighborhood 39, undoubtedly presenting the most intricate proof among all the Moore sub-neighbors. This subsection entails not only a synthesis of prior concepts but also the introduction of new ones, all within a proof structure that relies on nested cases and inductions under narrow hypotheses. The intricacy of the proof can be attributed to the intuitive notion that neighborhood 39 almost permits a crossover gate, which is discussed at the beginning of the Subsection. Table 2 summarizes the results of this section.

### 3.1 Crossover possibility

Neighborhoods studied in this subsection:

| Section               | 3.1                                                                                                                                                                                            | 3.3                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   | 3.4                                                                                                                                                                                            | 3.5                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | 3.6                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  | 3.7                                                                                             | 3.8                                                                                             |
|-----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| Type                  | P-complete                                                                                                                                                                                     | Planar                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | Too many neighbors                                                                                                                                                                             | 2 diagonals                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | 3 and 4 diagonals                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    | Escape cell                                                                                     | Almost crossing                                                                                 |
| Neighborhoods         | <br>10000111<br><br>10001111 | <br>01000010<br><br>10000010<br><br>10010010<br><br>01010010<br><br>01001010<br><br>11010010<br><br>01101010<br><br>11110010<br><br>11111010<br><br>11000000<br><br>11010000<br><br>11001000<br><br>01100010<br><br>11100010<br><br>11001010<br><br>01011010<br><br>11101010 | <br>11110111<br><br>01111111 | <br>00100011<br><br>01000011<br><br>10000011<br><br>10100011<br><br>01100011<br><br>11000011<br><br>01010011<br><br>01110011<br><br>11100011<br><br>11010011<br><br>11110011 | <br>01100111<br><br>10010111<br><br>01010111<br><br>11000111<br><br>11010111<br><br>01110111<br><br>01101111 | <br>01011111 | <br>00100111 |
| Have a crossover gate | ✓                                                                                                                                                                                              | ✗                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                                                                                                                                                |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |                                                                                                 |                                                                                                 |

Table 2: Summary of results exposed in Section 3.



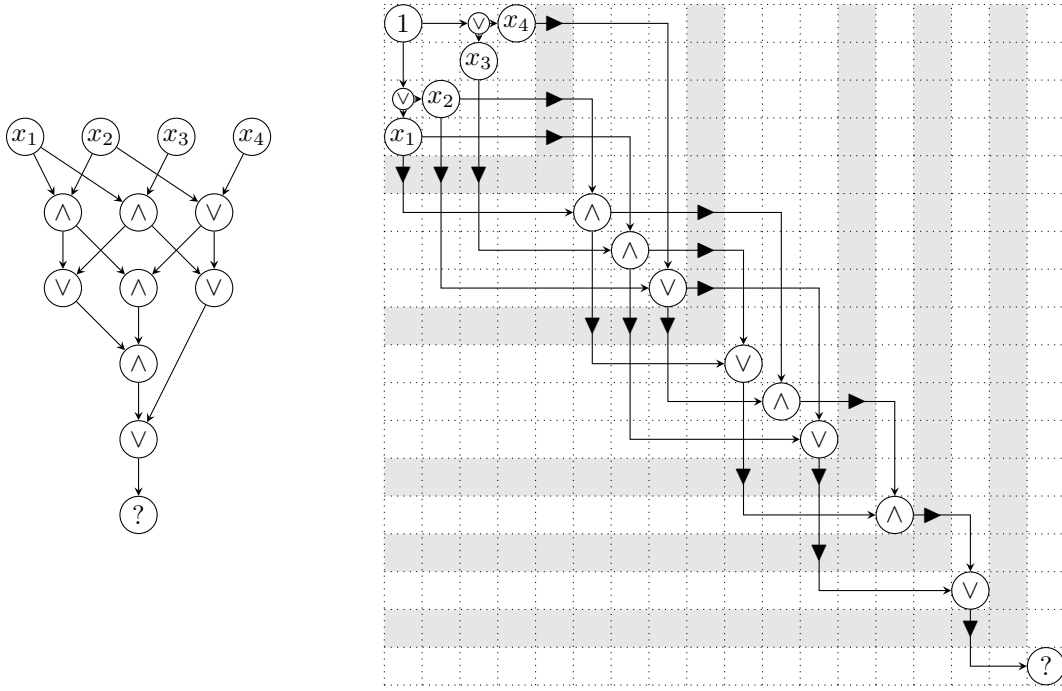
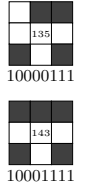


**Fig. 4:** Embedding of a **MCVP** instance of fan-in and fan-out two (left) on the grid (right). Diodes (represented as  $\blacktriangleright$  and  $\blacktriangledown$ ) are used systematically between layers in order to enforce the flow of information (and prevent it to go backward).

Two non-equivalent subsets of Moore neighborhood admit a crossover gate, and are consequently P-complete to predict, as formalized by  $\mathcal{N}$ -**PRED** problem. They are neighborhoods 135 and 143, for which we present gadgets for the purpose of implementing arbitrary Boolean circuits. Following one grain addition (at  $p$ ), the dynamics of the sandpile then simulates the computation of the circuit, until the output bit causes (bit 1) or not (bit 0) the toppling of a given cell (at  $q$ ).

**Theorem 2.**  $\mathcal{N}$ -**PRED** is P-complete for neighborhoods 135 and 143.

*Proof.* We prove P-completeness by reduction from monotone circuit value problem (**MCVP**) with gates of fan-in and fan-out two (problem **AM2CVP** in [15]). We consider an embedding of the monotone circuit on the two dimensional grid (see Figure 4), and replace all the elements by the corresponding sandpile gadgets of fixed size. For neighborhoods 135 and 143 we explain how to build: *background*, *constants* 0 and 1, *west-east* and *north-south wires*, *west-south* and *north-east turns*, *signal-duplications*, *and gates*, *or gates*, *crossover gates* and *diodes*. All these elements are of size  $10 \times 10$ , and are presented for both sandpile models on Figure 5: signals 1 are chains of topplings, signals 0 are stable, and gates plug neatly along their sides. A single constant 1 gate is used, whose central cell is reset to 3 and is the vertex  $p$  on which a sand grain will be added. It triggers the avalanche process (chain reaction of topplings), whose signal is duplicated (using *or gates*) so as to reach the variable gadgets  $x_i$ . These



variable gadgets are either *or* gates (for inputs set to 1 in the **MCVP** instance) or constant zero gates (for inputs set to 0). The questioned cell is a wire, with one vertex containing 3 sand grains being the vertex  $q$  on which  $\mathcal{N}$ -**PRED** asks whether it topples. The answer is equivalent to having the circuit outputting 1. The reduction can be performed in logarithmic space, because it amounts to computing positions in  $\mathbb{Z}^2$  for each of the elements of the circuit.  $\square$

In the following subsections we prove that none of the remaining 39 equivalence classes, *i.e.*, no other sub-neighborhood within the Moore neighborhood, admit a crossover gate.

### 3.2 Crossover impossibility: crossing constraint

In order to prove impossibility of crossover gates, our proofs will proceed by contradiction, assuming a crossover gate exists. In this subsection we give a general setup that will be used in most proofs. We take a result from [23], proving that if a crossover gate exists, then there also exists a crossover gate where the two firing graphs have disjoint vertex sets (they have no cell in common). In [23] it is stated for Eulerian graphs, a property verified by the sandpile models as introduced in the present work.

**Lemma 1** ([23]). *If a sandpile model  $\mathcal{N}$  admits a crossover gate, then it also admits a crossover gate  $g$  with firing graphs  $G_{ns}^g = (V_{ns}, A_{ns})$  and  $G_{we}^g = (V_{we}, A_{we})$ , such that  $V_{ns} \cap V_{we} = \emptyset$ .*

Because of Lemma 1, from now on, everytime time we refer to firing graphs of a crossover gate, we assume that they are disjoint. The following Lemma is a very basic consideration used intensively in the coming proofs, describing the initial setup for further reasonings.

**Lemma 2** (Crossing constraint). *If  $\mathcal{N}_{\subset_{fin} \mathcal{N}_m}$  admits a crossover gate  $g$ , then there exists  $(i, j) \in \mathbb{Z}^2$  such that the two firing graphs  $G_{ns}^g, G_{we}^g$  verify, up to rotation:*

- $(i, j), (i+1, j+1) \in V(G_{ns}^g)$  and  $((i+1, j+1), (i, j)) \in E(G_{ns}^g)$ ,
- $(i+1, j), (i, j+1) \in V(G_{we}^g)$  and  $((i, j+1), (i+1, j)) \in E(G_{we}^g)$ .

*Proof.* Since at some point at least one edge of  $G_{ns}^g$  has to go over at least one edge of  $G_{we}^g$ , and considering the fact that  $\mathcal{N} \subseteq \mathcal{N}_m$ , the only way of crossing is in a diagonal fashion. Otherwise,  $\mathcal{N}$  is not a sub-neighborhood of  $\mathcal{N}_m$ .  $\square$

In the proofs, we will denote the four cells given by Lemma 2 as  $v_0 = (i, j)$ ,  $u_0 = (i+1, j)$ ,  $u_1 = (i, j+1)$ ,  $v_1 = (i+1, j+1)$ , as illustrated on Figure 6. Besides, depending of the orientation of the crossing constraint, we say that a crossover gate  $g$  with firing graphs  $G_{ns}^g$  and  $G_{we}^g$ , can cross:

- downwards, if  $(u_1, u_0) \in E(G_{ns}^g)$  and  $(v_1, v_0) \in E(G_{we}^g)$ ;
- rightwards, if  $(u_1, u_0) \in E(G_{ns}^g)$  and  $(v_0, v_1) \in E(G_{we}^g)$ ;
- upwards, if  $(u_0, u_1) \in E(G_{ns}^g)$  and  $(v_0, v_1) \in E(G_{we}^g)$ ; or
- leftwards if  $(u_0, u_1) \in E(G_{ns}^g)$  and  $(v_1, v_0) \in E(G_{we}^g)$ .

Most of our reasonings will be based on the fact that, if some vertex  $v$  from one firing graph is an in-neighbor of vertex  $u$  from the other firing graph, then vertex  $u$  must have at least two predecessors in its firing graph (otherwise it is also fired in the other firing graph, contradicting our assumption that the two firing graphs are

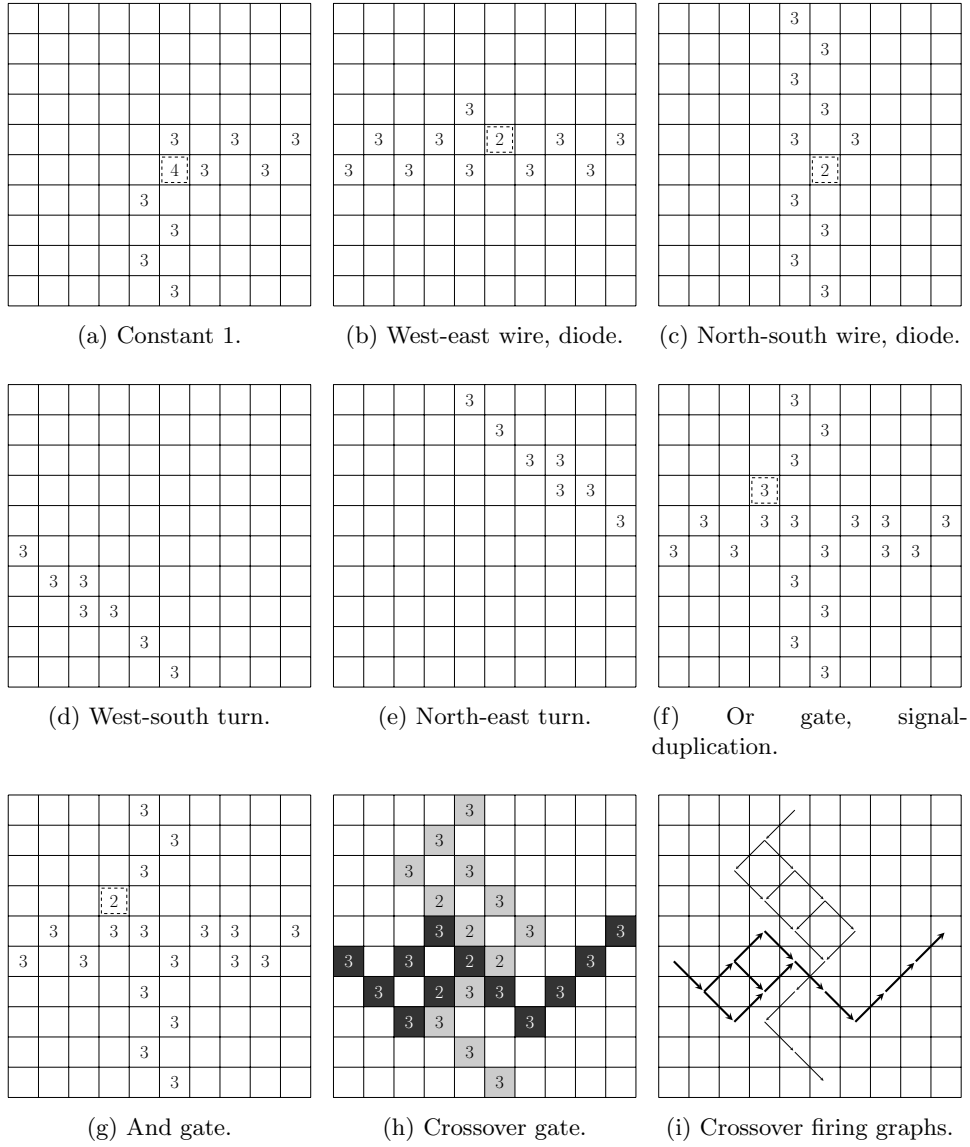


**Fig. 5:** Wiring toolkit and gates for 135. For 143, all non-empty cells must have one additional grain. No number means 0 grains (empty). Important cells are highlighted. The background and constant 0 gates are simply full of 0s. Diodes and wires coincide. Note that the constant 1 gate sends its signal to both east and south. Gates take their inputs from west and north then send their output to both east and south. The crossover gate is bicolored relative to the *west to east* and *north to south* signals, whose firing graphs are also presented (last figure).

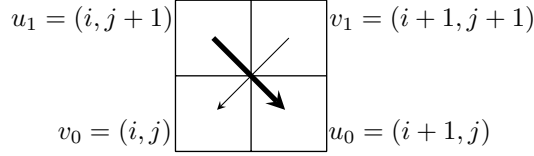
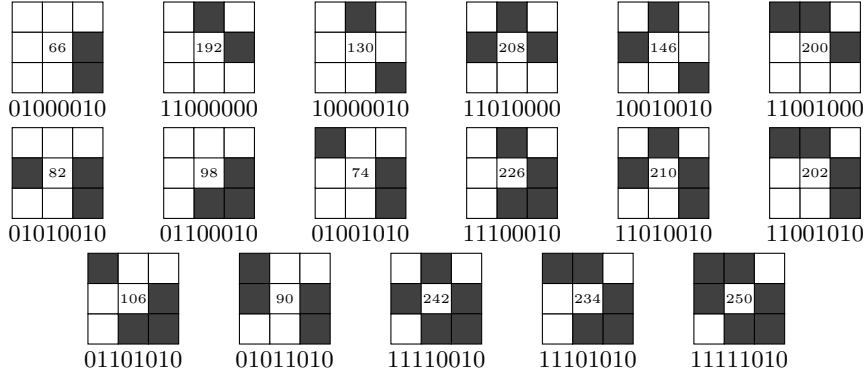


**Fig. 6:** Illustration of the crossing constraint (Lemma 2), up to rotation. One firing graph with plain arcs and vertices  $v_i$ , the other one with bold arcs and vertices  $u_i$  (effective causalities). All subsequent figures will use the same convention, with dashed arrows usually representing in-neighbors (potential causality).

disjoint). The arguments applies to any number of in-neighbors from the other firing graph: a vertex must have strictly more predecessors from its own firing graph than in-neighbors from the other firing graph.

### 3.3 Crossover impossibility: planar neighborhoods

Neighborhoods studied in this subsection:



Among the 39 remaining non-equivalent neighborhoods, a first pruning can be done looking at those neighborhoods that describe a planar graph. Indeed, the crossover impossibility is a corollary of Lemma 1 in this case. We first introduce formally the sandpile graph corresponding to a given neighborhood, which is an infinite graph representing the potential chain of reactions that can occur in the sandpile dynamics of a given neighborhood.

**Definition 4** (Sandpile graph). Given a neighborhood  $\mathcal{N} \subset_{\text{fin}} \mathbb{Z}^2$ , its *sandpile graph* has vertex set  $\mathbb{Z}^2$ , and arcs  $(v_1, v_2)$  when  $(v_2 - v_1) \in \mathcal{N}$ .

**Corollary 1.** *If neighborhood  $\mathcal{N}$  has a planar sandpile graph, then it does not admit a crossover gate.*

*Proof.* It follows directly from Lemma 1, which states that the two firing graphs (which are finite subgraphs of the sandpile graph) of a crossover gate can have disjoint sets of vertices (cells). If the sandpile graph is planar, then it is impossible to have a path from  $n$  to  $s$  and a path from  $w$  to  $e$  without any cell in common.  $\square$

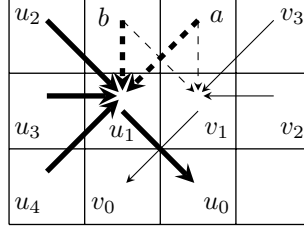


**Fig. 7:** Proof of Theorem 4 for neighborhood 247, downward crossing. Only one of the cells  $a$  and  $b$  can be a predecessor of  $v_1$ .

There are 17 non equivalent planar neighborhoods, for which Corollary 1 can be applied to deduce the impossibility of crossover.

**Theorem 3.** *Neighborhoods 66, 192, 130, 208, 146, 200, 82, 98, 74, 226, 210, 202, 106, 90, 242, 234, and 250 are planar, and therefore do not admit a crossover gate.*

### 3.4 Crossover impossibility: too many neighbors

Neighborhoods studied in this subsection:



In this subsection we study the two neighborhoods which are missing only one cell among the 8 cells of the Moore neighborhood. Neighborhood 247 misses one cell on the diagonal, and neighborhood 127 misses one cell on the side. Starting from the crossing constraint of Lemma 2, the argument in this case is that the predecessors of  $u_1$  and  $v_1$  cannot be chosen without interacting too much with each other (via the in-neighbor relationship). For the former we quickly reach an impossibility of crossover, whereas the latter requires substantial case analysis.

**Theorem 4.** *Neighborhood 247 does not admit a crossover gate.*

*Proof.* The four cells given by Lemma 2 can be oriented as a downward or rightward crossing.

**Downward crossing.** This case is depicted on Figure 7. Both cells  $u_0$  and  $u_1$  are in-neighbors of  $v_1$ , consequently it needs three predecessors which can be selected among cells  $v_2 = (i + 2, j + 1)$ ,  $v_3 = (i + 2, j + 2)$ ,  $a = (i + 1, j + 2)$  and  $b = (i, j + 2)$ . However, if both cells  $a$  and  $b$  are selected as the predecessors of  $v_1$ , then  $u_1$  would have four in-neighbors from the opposite firing graph, then it would need five predecessors which is not possible. Hence, only one of  $a$  and  $b$  can be a predecessor of  $v_1$ , while the other two have to be the cells  $v_2$  and  $v_3$ . Now considering that cells  $v_0$ ,  $v_1$  and  $a$  (resp.  $b$ ) are in-neighbors of  $u_1$ , it needs four predecessors in order not to topple in the other firing graph, which can only be cells  $b$  (resp.  $a$ ),  $u_2 = (i - 1, j + 2)$ ,  $u_3 = (i - 1, j + 1)$ , and  $u_4 = (i - 1, j)$ . Since  $b$  (resp.  $a$ ) is an in-neighbor of  $v_1$ , this latter needs an extra predecessor, which is not possible by our previous considerations.

**Rightward crossing.** It is, up to rotation, the same proof.  $\square$





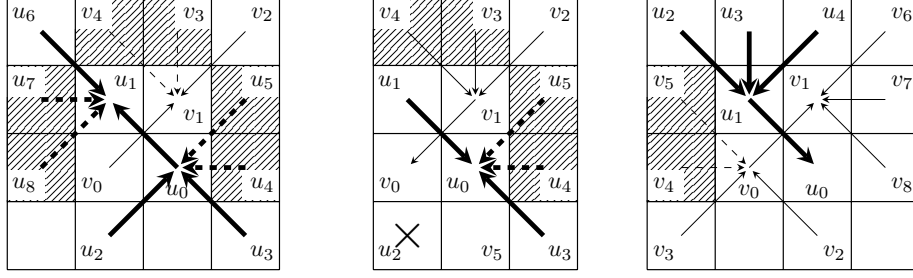
**Theorem 5.** *Neighborhood 127 does not admit a crossover gate.*

*Proof.* The four cells given by Lemma 2 can be oriented as an upward, downward, rightward or leftward crossing.

**Upward crossing.** For a better understanding of this proof, see Figure 8a. Since  $v_0$  and  $v_1$  are in-neighbors of  $u_0$ , it needs three predecessors which can be taken among cells  $u_2 = (i, j - 1)$ ,  $u_3 = (i + 2, j - 1)$ ,  $u_4 = (i + 2, j)$  and  $u_5 = (i + 2, j + 1)$ . Let us prove that cells  $u_4$  and  $u_5$  cannot both be predecessors of  $u_0$ . If we pick both  $u_4$  and  $u_5$ , since they are in-neighbors of  $v_1$ , then  $v_1$  would need four predecessors (because  $u_1$  is also an in-neighbor of  $v_1$ ). One of them is  $v_0$  by the crossing constraint (Lemma 2), while the three other ones must be cells  $v_2 = (i + 2, j + 2)$ ,  $v_3 = (i + 1, j + 2)$  and  $v_4 = (i, j + 2)$ . In turn, since  $v_1$ ,  $v_3$  and  $v_4$  are in-neighbors of  $u_1$ , this latter would need four predecessors. One of them is  $u_0$  by Lemma 2, while the other ones can only be cells  $u_6 = (i - 1, j + 2)$ ,  $u_7 = (i - 1, j + 1)$  and  $u_8 = (i - 1, j)$ . Considering that  $u_0$ ,  $u_1$ ,  $u_7$  and  $u_8$  are in-neighbors of  $v_0$ , then it would need five predecessors, which is not possible. In simple words, cells  $u_4$  and  $u_5$  cannot be predecessors of  $u_0$  simultaneously, because it causes a ripple effect which lets  $v_0$  without enough predecessors.

As a consequence, both  $u_2$  and  $u_3$  must be predecessors of  $u_0$ , while the third one has to be selected between  $u_4$  and  $u_5$ . For the purpose of this proof, it does not matter if we choose  $u_4$  or  $u_5$ , since both are in-neighbors of  $v_1$ . Then, without loss of generality, let us pick cell  $u_4$  as the third predecessor of  $u_0$ . In this way,  $v_1$  needs only three predecessors. The predecessors of  $v_1$  can be chosen among cells  $u_5$ ,  $v_2$ ,  $v_3$  and  $v_4$ . However, if we choose  $u_5$ , then  $u_0$  would need an extra predecessor, which is impossible. On the other hand, if we choose  $v_3$  and  $v_4$ , it will again cause a ripple effect which lets cell  $v_0$  without enough predecessors. Hence, similarly as before,  $v_2$  must be one of the predecessors of  $v_1$ , while the other one has to be chosen between  $v_3$  and  $v_4$ . Without loss of generality (since they are both in-neighbors of  $u_1$ ), we choose  $v_3$ . Let us take a look at cell  $u_1$ . Since  $v_1$  and  $v_3$  are its in-neighbors, it needs three predecessors. We already know that one of them is  $u_0$ , while the other two have to be selected among cells  $v_4$ ,  $u_6$ ,  $u_7$  and  $u_8$ . Nevertheless, if  $v_4$  is one of them, cell  $v_1$  would need an extra predecessor, which is not possible. Furthermore, if both  $u_7$  and  $u_8$  are predecessors of  $u_1$ , then  $v_0$  would need five predecessors, which we already know is not possible. In other words, one of the predecessors of  $u_1$  must be cell  $u_6$ , while the second one has to be selected between  $u_7$  and  $u_8$ . Again, for purpose of this proof, it does not matter which one we choose, so let us consider that it is  $u_7$ . The latter makes  $v_0$  to have three in-neighbors from the opposite firing graph (cells  $u_0$ ,  $u_1$  and  $u_7$ ), thus it needs four predecessors which is not possible.

**Downward crossing.** We will use the same cell names as in the previous case. First, let us take a look at cell  $u_0$ . Since  $v_0$  and  $v_1$  are its in-neighbors, it needs three predecessors. One of them is  $u_1$  by Lemma 2, while the other two have to be selected among cells  $u_2$ ,  $u_3$ ,  $u_4$  and  $u_5$ . Let us assume  $u_2$  is not a predecessor of  $u_0$ , *i.e.*, it does not belong to the firing graph of  $u$  (see Figure 8b). Note that if both cells  $u_4$  and  $u_5$  are in-neighbors of  $u_0$ , then the cell  $v_1$  would have three in-neighbors from the opposite firing graph, *i.e.*, it would need four predecessors, which is not possible. Therefore,  $u_3$  must be one of the predecessors of  $u_0$ , while the other one has to be chosen between  $u_4$  and  $u_5$ . For the purpose of this proof, it does not matter which one



(a) Upward crossing. Pairs of cells cannot belong to the same firing graph:  $u_4$  with  $u_5$ ,  $v_3$  with  $v_4$ , and  $u_7$  with  $u_8$ .  
(b) Downward crossing. Assuming  $u_2$  does not belong to the firing graph of  $u$ .  
(c) Rightward crossing. Only one of  $v_4$  and  $v_5$  can be a predecessor of  $v_0$ .

**Fig. 8:** Proof of Theorem 5 for neighborhood 127.

we choose, so, without loss of generality, let us consider that it is  $u_4$ . Now, considering that cells  $u_1$  and  $u_4$  are in-neighbors of  $v_1$ , it needs three predecessors, which have to be cells  $v_2$ ,  $v_3$  and  $v_4$ . Since  $v_1$ ,  $v_3$  and  $v_4$  are also in-neighbors of  $u_1$ , this latter needs four predecessors, which is impossible.

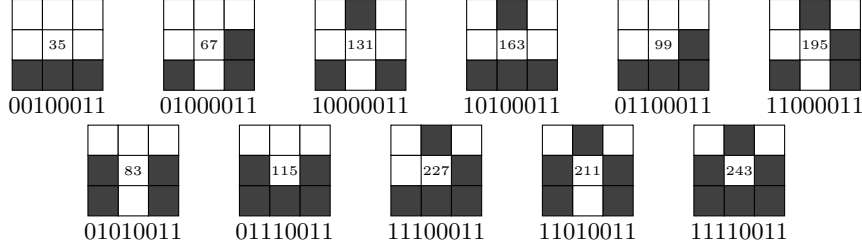
As a consequence, cell  $u_2$  must be a predecessor of  $u_0$ . By symmetry, it also implies that cell  $v_5 = (i, j - 1)$  must be a predecessor of  $v_0$ . However, it would generate an upward crossing with arcs  $(u_2, u_0)$  and  $(v_5, v_0)$ , which we already proved is not allowed in a crossover gate for neighborhood 127.

**Rightward crossing.** In this case, we will not use the same cell numbering as in the previous case. An illustration of this proof is depicted on Figure 8c. Let us focus on cell  $v_0$ . Since  $u_0$  and  $u_1$  are its in-neighbors, it needs three predecessors, which can be chosen among cells  $v_2 = (i + 1, j - 1)$ ,  $v_3 = (i - 1, j - 1)$ ,  $v_4 = (i - 1, j)$  and  $v_5 = (i - 1, j + 1)$ . However, if both cells  $v_4$  and  $v_5$  are its predecessors, since they are also in-neighbors of  $u_1$ , then  $u_1$  would need four predecessors ( $v_1$  is also a predecessor of  $u_1$ ), which is not possible. It means that both cells  $v_2$  and  $v_3$  must be predecessors of  $v_0$ , while the third one has to be chosen between cells  $v_4$  and  $v_5$ . For the purpose of this proof, it does not matter which one we pick. Without loss of generality, let us consider  $v_4$  as a predecessor of  $v_0$ . Since  $v_1$  and  $v_4$  are in-neighbors of  $u_1$ , it needs three predecessors which can be selected among cells  $v_5$ ,  $u_2 = (i - 1, j + 2)$ ,  $u_3 = (i, j + 2)$  and  $u_4 = (i + 1, j + 2)$ . Nevertheless, selecting  $v_5$  would make  $v_0$  to need an extra predecessor, which is impossible. In other words, the predecessors of  $u_1$  must be cells  $u_2$ ,  $u_3$  and  $u_4$ . Note that  $u_1$ ,  $u_3$  and  $u_4$  are also in-neighbors of  $v_1$ , *i.e.*, cell  $v_1$  needs four predecessors. One of them is  $v_0$  by Lemma 2, while the other three have to be cells  $v_6 = (i + 2, j + 2)$ ,  $v_7 = (i + 2, j + 1)$  and  $v_8 = (i + 2, j)$ . Considering that  $v_0$ ,  $v_1$ ,  $v_7$  and  $v_8$  are all in-neighbors of  $u_8$ , then this latter needs five predecessors, which is not possible.

**Leftward crossing.** Considering the symmetry of neighborhood 127, the proof for this case is analogous to the rightward crossing.  $\square$

### 3.5 Crossover impossibility: two diagonals

Neighborhoods studied in this subsection:



Let us now study the neighborhoods having only two cells in the diagonals, numbered 35, 67, 131, 163, 99, 195, 83, 115, 227, 211 and 243. In all these cases, it is enough to check whether a downward crossing is possible, because the two cells in the diagonals are on positions south-east and south-west (up to the equivalences presented in Table 1). The arguments are adapted to each neighborhood, they rely on short case analysis leading either to a direct contradiction, or by induction to the fact that a crossover gate would be of infinite size.

**Theorem 6.** *Neighborhoods 35 and 67 do not admit a crossover gate.*

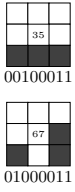
*Proof.* The proofs for the neighborhoods 35 and 67 are very similar. According to the definitions of these neighborhoods, the four cells given by Lemma 2 must be oriented downward, as depicted on Figure 9a.

One of the cells  $v_0$  or  $v_1$  is an in-neighbor of  $u_0$ , hence  $u_0$  requires another predecessor (otherwise it also belongs to the other firing graph), which must be at position  $u_2 = (i+2, j+1)$ . There is a horizontal pattern  $u_1, v_1, u_2$  of three adjacent cells, alternating between the two firing graphs. We will now show that in both neighborhoods 35 and 67, it implies that there is another such pattern on the row just above. Since the crossover gate must eventually connect the in-neighbors of these two extremities (e.g.  $u_1$  and  $u_2$ ) to a unique cell on the border, we conclude that the crossover gate cannot have a finite size, which is impossible.

Consider neighborhood 35, and the three potential predecessors  $v_2$  of cell  $v_1$ . If  $v_2 = (i+1, j+2)$  is just above  $v_1$ , then it is an in-neighbor of both  $u_1$  and  $u_2$ , as a consequence each of them must have two predecessors. Now observe (see Figure 9b) that any choice of these two predecessors for each lead to the fact that  $v_1$  belongs to both firing graphs, hence contradicting Lemma 1. Therefore  $v_2 = (i, j+2)$  or  $v_2 = (i+2, j+2)$ . The two cases are symmetric and we consider  $v_2 = (i+2, j+2)$  (see Figure 9c). Since  $v_2$  is an in-neighbor of  $u_2$ , cell  $u_2$  must have two predecessors, which can only be cells  $u_3 = (i+1, j+2)$  and  $u_4 = (i+3, j+2)$ . Cells  $u_3, v_2, u_4$  form a pattern leading to the announced contradiction.

Consider neighborhood 65. Cell  $u_1$  is an in-neighbor of  $v_1$ , therefore  $v_1$  must have two predecessors, including  $v_2 = (i+2, j+2)$  (see Figure 9c). Since  $v_3$  is an in-neighbor of  $u_2$ , cell  $u_2$  must have two predecessors, which can only be cells  $u_3 = (i+1, j+2)$  and  $u_4 = (i+4, j+2)$ . Cells  $u_3, v_3, u_4$  form a pattern leading to the announced contradiction.  $\square$

**Theorem 7.** *Neighborhood 131 does not admit a crossover gate.*



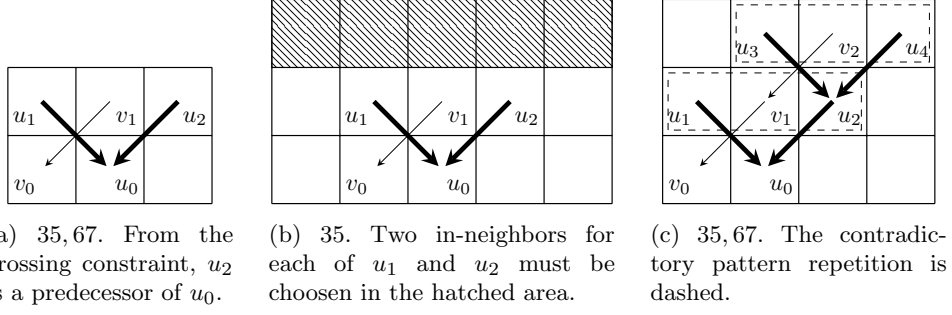


**Fig. 9:** Proof of Theorem 6 for neighborhoods 35 and 67.

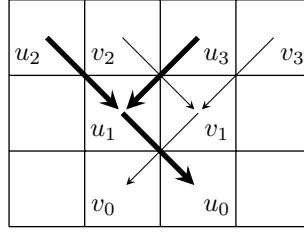


**Fig. 10:** Proof of Theorem 7 for neighborhood 131.

*Proof.* The four cells given by Lemma 2 must be oriented downward, as depicted on Figure 10. Cell  $v_0$  is an in-neighbor of  $u_1$ , therefore  $u_1$  requires two predecessors which can only be cells  $u_2 = (i-1, j+2)$  and  $u_3 = (i+1, j+2)$ . Similarly, in the other firing graph, cells  $v_2 = (i, j+2)$  and  $v_3 = (i+2, j+2)$  must be predecessors of  $v_1$ . This generates another downward crossing just above the original one. Let us assume that the height of the gate is  $N$ . The preceding argument can be repeated  $N+1$  times, contradicting the assumption about the gate size.  $\square$

**Theorem 8.** *Neighborhood 163 does not admit a crossover gate.*



*Proof.* The four cells given by Lemma 2 must be oriented downward, as depicted on Figure 11. Let us focus on node  $u_1$ . Since  $v_0$  is an in-neighbor of it,  $u_1$  needs at least two predecessors. The available cells are  $a = (i-1, j+1)$ ,  $b = (i, j+1)$  and  $c = (i+1, j+1)$ . This means that at least one of the predecessors is cell  $b$  or  $c$ . However, it would imply that node  $v_1$  has three or four predecessors (because  $u_0$  is also an in-neighbor of  $v_1$ ), which is impossible: not enough cells are available in the in-neighborhood of  $v_1$ .  $\square$

**Theorem 9.** *Neighborhoods 99 and 195 do not admit a crossover gate.*



*Proof.* For neighborhood 99, cell  $u_0$  needs three predecessors, but there are only two available. The same thing happens with cell  $v_1$  in the case of the neighborhood 195.  $\square$



**Theorem 10.** *Neighborhood 83 does not admit a crossover gate.*



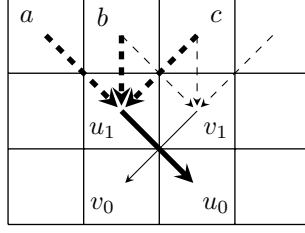
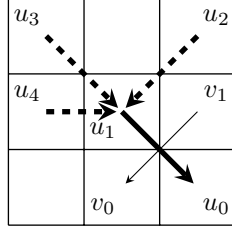
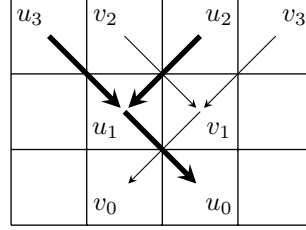


**Fig. 11:** Proof of Theorem 8 for neighborhood 163. The dashed arrows represent the possible predecessors of cells  $u_1$  and  $v_1$ .



(a) Available predecessors of cell  $u_1$ .



(b) Cells  $u_2$  and  $u_3$  (resp.  $v_2$  and  $v_3$ ) must be predecessors of  $u_1$  (resp.  $v_1$ ).

**Fig. 12:** Proof of Theorem 10 for neighborhood 83.

*Proof.* The four cells given by Lemma 2 must be oriented downward. Cell  $v_1$  is an in-neighbor of  $u_1$ , therefore  $u_1$  needs two predecessors in its firing graph. There are three available cells:  $u_2 = (i + 1, j + 2)$ ,  $u_3 = (i - 1, j + 2)$  and  $u_4 = (i - 1, j + 1)$  (see Figure 12a). However, if  $u_4$  is a predecessor of  $u_1$ , then cell  $v_0$  will have two in-neighbors in the opposite firing graph ( $u_0$  and  $u_4$ ), and it would need three predecessors, which is not possible. As a consequence, the only possible predecessors for  $u_1$  are the cells  $u_2$  and  $u_3$ . By the symmetry of neighborhood 83, the same reasoning applies to obtain the predecessors of  $v_1$ , which can only be the cells  $v_2 = (i, j + 2)$  and  $v_3 = (i + 2, j + 2)$  (see Figure 12b). The preceding argument can be repeated  $N + 1$  times, contradicting the assumption about the gate size.  $\square$



**Theorem 11.** *Neighborhood 115 does not admit a crossover gate.*

*Proof.* The four cells given by Lemma 2 must be oriented downward, as depicted on Figure 13. Cells  $v_0$  and  $v_1$  are in-neighbors of  $u_0$ , hence cell  $u_0$  needs at least three predecessors in the firing graph, which can only be cells  $u_1$ ,  $u_2 = (i + 2, j + 1)$  and  $u_3 = (i + 2, j)$ . Since  $u_1$  and  $u_2$  are in-neighbors of  $v_1$ , cell  $v_1$  needs three predecessors, and the only cells available for this are the three cells above it:  $v_2 = (i, j + 2)$ ,  $v_3 = (i + 1, j + 2)$ ,  $v_4 = (i + 2, j + 2)$ . In turn, this causes cell  $u_2$  to need three predecessors too, but it does not have enough available in-neighbors.  $\square$



**Theorem 12.** *Neighborhood 227 does not admit a crossover gate.*

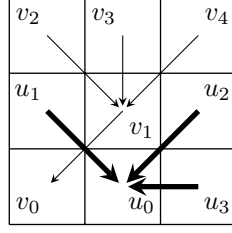


**Fig. 13:** Proof of Theorem 11 for neighborhood 115.

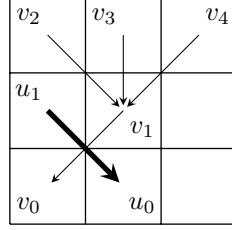


**Fig. 14:** Proof of Theorem 12 for neighborhood 227.

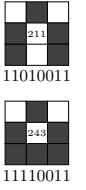
*Proof.* The four cells given by Lemma 2 must be oriented downward, as depicted on Figure 14. Both cells  $u_0$  and  $u_1$  are in-neighbors of cell  $v_1$ , consequently cell  $v_1$  needs at least three predecessors, which can only be cells  $v_2 = (i, j + 2)$ ,  $v_3 = (i + 1, j + 2)$  and  $v_4 = (i + 2, j + 2)$ . Since  $v_2$  and  $v_3$  are in-neighbors of  $u_1$ , the latter must have three predecessors, which is not possible.  $\square$

**Theorem 13.** *Neighborhoods 211 and 243 do not admit a crossover gate.*

*Proof.* The four cells given by Lemma 2 must be oriented downward, as depicted on the two cases of Figure 15. Let us focus on cell  $u_1$ . Since  $v_0$  and  $v_1$  are both in-neighbors of  $u_1$ , it needs at least three predecessors.

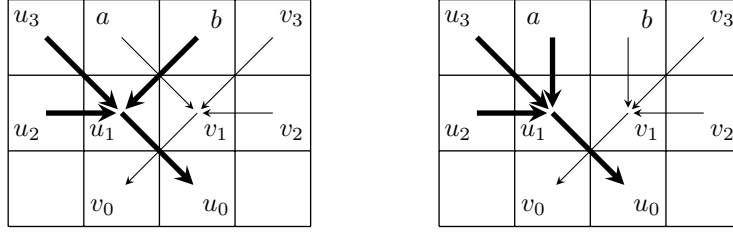
For neighborhood 211 these three predecessors of  $u_1$  can only be cells  $u_2 = (i - 1, j + 1)$ ,  $u_3 = (i - 1, j + 2)$  and  $a = (i + 1, j + 2)$  (see Figure 15a). By the symmetry of neighborhood 211, the same reasoning applies to  $v_1$ , and its only three available predecessors are  $v_2 = (i + 2, j + 1)$ ,  $v_3 = (i + 2, j + 2)$  and  $b = (i, j + 2)$ . Now, observe that cells  $v_1$ ,  $v_2$  and  $b$  are all in-neighbors of  $a$ . Cell  $a$  therefore requires four predecessors, which is not possible.

In the case of the neighborhood 243 (see Figure 15b), the three predecessors of cell  $u_1$  can be selected among cells  $u_2$ ,  $u_3$ ,  $a$  and  $b$ . Hence at least one of  $a$  and  $b$  must be chosen. Since  $a$  and  $b$  are both in-neighbors of  $v_1$ , the latter needs at least four predecessors (because  $u_0$  and  $u_1$  are also in-neighbors of  $v_1$ ), which is again not possible.  $\square$



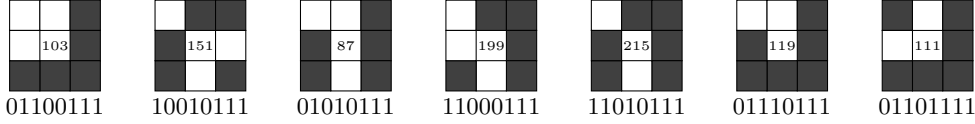
### 3.6 Crossover impossibility: three or four diagonals

Neighborhoods studied in this subsection:



(a) 211, 243: Cell  $a$  (resp.  $b$ ) is a predecessor of  $v_1$  (resp.  $u_1$ ). (b) 243: Cell  $a$  (resp.  $b$ ) is a predecessor of  $u_1$  (resp.  $v_1$ ).

**Fig. 15:** Proof of Theorem 13 for neighborhoods 211 and 243.



In this subsection we study neighborhoods having three or four diagonal cells, hence for which the crossing constraint (Lemma 2) has two to four possible orientations. We continue to present proofs by contradiction, following case analysis. Neighborhood 111 requires a somewhat long development. For neighborhoods 95 and 39, we present a different approach in the next subsection.



**Theorem 14.** *Neighborhood 103 does not admit a crossover gate.*

*Proof.* The four cells given by Lemma 2 can have two orientations: downward or rightward crossing.

**Downward crossing.** See Figure 16. First notice that  $v_0$  and  $v_1$  are in-neighbors of cell  $u_0$ , hence it needs two extra predecessors, which can only be cells  $u_2 = (i, j - 1)$  and  $u_3 = (i + 2, j + 1)$ . At the same time,  $u_1$  is an in-neighbor of cell  $v_1$ , hence it needs two predecessors, that have to be selected among cells  $v_2 = (i, j + 2)$ ,  $v_3 = (i + 1, j + 2)$  and  $v_4 = (i + 2, j + 2)$ . Consequently, at least one of  $v_2$  or  $v_3$  must be a predecessor of  $v_1$ . The latter means that cell  $u_1$  also needs two predecessors, which can in turn be chosen among cells  $v_2$ ,  $v_3$ ,  $u_4 = (i - 1, j)$ ,  $u_5 = (i - 1, j + 1)$  and  $u_6 = (i - 1, j + 2)$ . However, choosing  $v_2$  or  $v_3$  makes  $v_1$  need another predecessor, but it already ran out of possibilities. Then the remaining cells are  $u_4$ ,  $u_5$  and  $u_6$ , and at least one of  $u_4$  and  $u_5$  must be a predecessor of  $u_1$ . If we chose  $u_4$  (resp.  $u_5$ ), then the cell  $v_0$  has two in-neighbors from the opposite firing graph, and it needs three predecessors, which can only be cells  $v_1$ ,  $v_5$  and  $u_5$  (resp.  $u_4$ ). This implies that  $u_1$  needs an additional predecessor, but the only remaining available cells are  $v_2$  or  $v_3$ , which we already showed cannot be selected as predecessors of  $u_1$ .



**Rightward crossing.** Up to rotation, the rightward crossing has the same proof of impossibility as the downward crossing.  $\square$

**Theorem 15.** *Neighborhood 151 does not admit a crossover gate.*

*Proof.* **Downward crossing.** Let us place the four cells given by Lemma 2 as depicted in Figure 17a. First of all, notice that cell  $u_1$  has  $v_0$  and  $v_1$  as in-neighbors, hence it



has to be one of the predecessors of  $v_0$ , while the second one has to be chosen between  $v_3$  and  $v_4$ . For the purpose of this proof, there is no difference if we choose  $v_3$  or  $v_4$ , then, without loss of generality, let us pick  $v_3$ . Now,  $u_1$  has two in-neighbors from the opposite firing graph:  $v_1$  and  $v_3$ , and consequently it needs three predecessors that can only be cells  $v_4$ ,  $u_2 = (i - 1, j + 2)$  and  $u_3 = (i + 1, j + 2)$ . However, since  $v_4$  is an in-neighbor of  $v_0$ , it follows that  $v_0$  needs an extra predecessor, which is not possible.

**Downward crossing.** For this case, we use the same cell numbering as before. This proof is depicted in Figure 18b. First of all, let us prove that cells  $u_2$  and  $u_3$  have to be predecessors of  $u_1$ . Initially,  $u_1$  has only one in-neighbor from the opposite firing graph, which is cell  $v_1$ , then it needs two predecessors. As we want to prove that both  $u_2$  and  $u_3$  have to be predecessors of  $u_1$ , let us show what happens if we pick one of  $v_3$  or  $v_4$ . Considering that  $v_3$  (resp.  $v_4$ ) and  $u_0$  are in-neighbors of  $v_0$ , the latter therefore needs three predecessors. One of them is  $v_1$  by Lemma 2, while the other two can only be cells  $v_2$  and  $v_4$  (resp.  $v_3$ ). Note that  $v_4$  (resp.  $v_3$ ) is also an in-neighbor of  $u_1$ , hence now it needs an extra predecessor. In other words, selecting one of  $v_3$  and  $v_4$  as a predecessors of  $u_1$ , just makes  $u_1$  need an extra predecessor. Subsequently, we can ensure that  $u_2$  and  $u_3$  are predecessors of  $u_1$  since there are the only remaining in-neighbors.

Now, we prove that no matter what predecessors are chosen, the following happens (see Figure 18c): the initial cross implies either another cross, or a given pattern  $P$ ; then in turn the pattern  $P$  implies either another cross, or another pattern  $P$ . Inasmuch as the crossover gate has to be of finite size, this leads to contradiction. Since  $u_1$  is an in-neighbor of  $v_1$ , cell  $v_1$  needs two predecessors. Its available in-neighbors are cells  $v' = (i, j + 2)$ ,  $v_2 = (i + 2, j + 1)$  and  $v_3 = (i + 2, j + 2)$ . However, if  $v'$  is a predecessor, then it would be generating another cross above the original one. Therefore, the only option left is to consider cells  $v_2$  and  $v_3$  as the predecessors of  $v_1$ . This is the first appearance of the pattern, which consists in cells  $u_2$ ,  $v'$ ,  $u_1$ ,  $v_1$  and  $v_2$ . Considering that  $v_3$  is an in-neighbor of  $u_3$ , then  $u_3$  needs two predecessors, which can only be cells  $u_4 = (i, j + 3)$  and  $u_5 = (i + 2, j + 3)$  (cell  $v'$  cannot be chosen since it makes  $v_1$  to exhaust its available predecessors). Since in turn  $u_3$  is an in-neighbor of  $v_3$ , cell  $v_3$  needs two predecessors which can be chosen among cells  $v'' = (i + 1, j + 3)$ ,  $v_4 = (i + 3, j + 2)$  and  $v_5 = (i + 3, j + 3)$ . If  $v''$  is chosen, then it generates another cross, therefore it is necessary to select  $v_4$  and  $v_5$ . Here the pattern appears again, formed by cells  $u_4$ ,  $v''$ ,  $u_3$ ,  $v_3$  and  $v_4$ .  $\square$



**Theorem 17.** *Neighborhood 199 does not admit a crossover gate.*

*Proof.* **Downward crossing.** Cells  $u_0$  and  $u_1$  are in-neighbors of  $v_1$ , i.e.,  $v_1$  needs three predecessors, but it does not have enough available in-neighbors. Therefore a downward crossing is impossible.

**Rightward crossing.** The cells and arcs used for this proof are depicted in Figure 19. Cells  $u_0$  and  $u_1$  are in-neighbors of  $v_1$ , therefore it needs three predecessors which can only be cells  $v_0$ ,  $v_2 = (i + 2, j + 2)$  and  $v_3 = (i, j + 2)$ . On the other hand, cell  $v_0$  is an in-neighbor of  $u_1$ , hence  $u_1$  needs two predecessors. The in-neighbors available are cells  $u' = (i + 1, j + 2)$ ,  $u_2 = (i - 1, j + 2)$ ,  $u_3 = (i - 1, j + 1)$ ,  $u_4 = (i - 1, j)$ . If

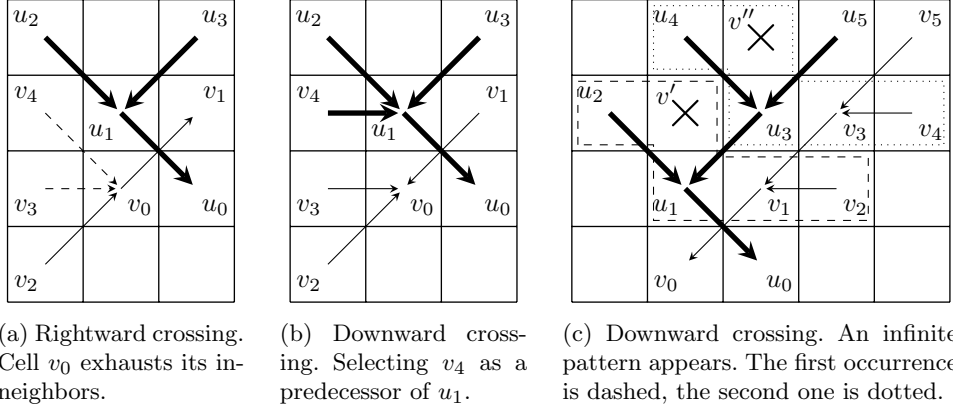


**Fig. 18:** Proof of Theorem 16 for neighborhood 87.

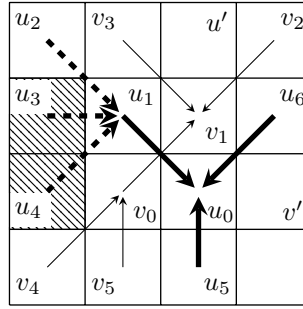


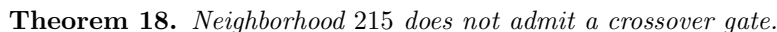
**Fig. 19:** Proof of Theorem 17 for neighborhood 199. Dashed arrows show the possible predecessors of  $u_1$ , however only one from the hatched area can be selected.

we choose  $u'$  as one of the predecessors, we would be generating a downward crossing with arcs  $(u', u_1)$  and  $(v_3, v_1)$ , but this is impossible for neighborhood 199 by the previous case. Additionally, if we choose both  $u_3$  and  $u_4$ , then cell  $v_0$  would need three predecessors which is not possible. In other words, one of the predecessors of  $u_1$  has to be  $u_2$ , while the other one has to be either  $u_3$  or  $u_4$ . Since  $u_3$  and  $u_4$  are in-neighbors of  $v_0$ , it needs two predecessors and  $v_4 = (i-1, j-1)$  together with  $v_5 = (i, j-1)$  are the only in-neighbors available. Cells  $u_3$  and  $u_4$  cannot be predecessors of  $v_0$ , because it would follow that  $u_1$  needs an extra predecessor, which can only be contradictory by the preceding case analysis.

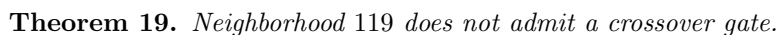
Now let us focus on cell  $u_0$ . It has two in-neighbors from the opposite firing graph, which are cells  $v_0$  and  $v_5$ , and consequently it needs three predecessors, which can only be cells  $u_1$ ,  $u_5 = (i+1, j-1)$  and  $u_6 = (i+2, j+1)$ . Remark that we can consider without loss of generality that  $v_1$  has a successor. We deduce that the successor of  $v_1$  can only be  $v' = (i+2, j)$ . However this creates a downward crossing with arcs  $(v_1, v')$  and  $(u_6, u_0)$ , which is impossible by the previous case.  $\square$



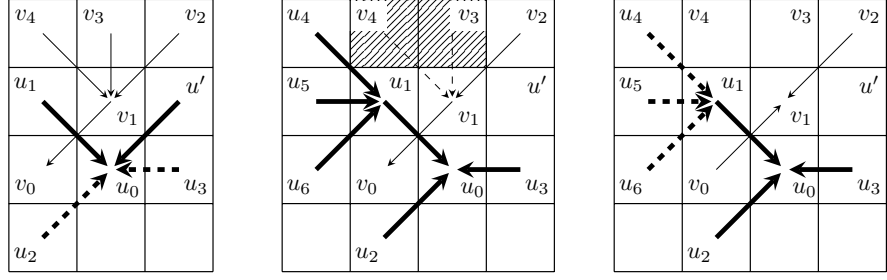
**Fig. 20:** Proof of Theorem 18 for neighborhood 215.



**Rightward crossing.** The cells for this proof are depicted in Figure 20b. Analogously to the rightward crossing,  $u_2$  and  $u_3$  have to be two of the three predecessors of  $u_1$ , and the third one has to be selected between  $u_4$  and  $u_5$ . Now let us take a look at cell  $v_0$ . It has two in-neighbors from the opposite firing graph: the first one is either  $u_4$  or  $u_5$ , and the other is  $u_0$ . Consequently, cell  $v_0$  needs three predecessors, which is not possible.  $\square$



We deduce that we have to pick cells  $u_2$  and  $u_3$  as predecessors of  $u_0$  (see Figure 21b). In that case,  $v_1$  needs only two predecessors. Since  $v_3$  and  $v_4$  cannot simultaneously be predecessors of  $v_1$  (otherwise  $u_1$  would need four predecessors again), only one of them has to be selected, while the second one must be cell  $v_2$ . Note that



(a) Cell  $u'$  cannot be chosen as a predecessor of  $u_0$ .

(b) Downward crossing. Cell  $v_0$  does not have enough in-neighbors.

(c) Rightward crossing. Cell  $v_0$  does not have enough in-neighbors.

**Fig. 21:** Proof of Theorem 19 for neighborhood 119.

cell  $u'$  cannot be selected as a predecessor of  $v_1$ , since it would make  $u_0$  to need an extra predecessor, which is not possible. Now cell  $u_1$  needs three predecessors, which can only be cells  $u_4 = (i - 1, j + 2)$ ,  $u_5 = (i - 1, j + 1)$  and  $u_6 = (i - 1, j)$ . Note that  $v_3$  and  $v_4$  cannot be selected as predecessors of  $u_1$ , because it would make cell  $v_1$  need an extra predecessor, nevertheless there are no more available in-neighbors. Since  $u_0$ ,  $u_1$ ,  $u_5$  and  $u_6$  are all in-neighbors of  $v_0$ , cell  $c_0$  needs five predecessors, which is not possible.

**Rightward crossing.** The proof for this case is depicted in Figure 21c. Analogously to the downward crossing, cell  $u_0$  needs two more predecessors and  $u'$  cannot be chosen as one of them by the exact same reasoning. As a consequence, we have to take cells  $u_1$ ,  $u_2$  and  $u_3$  as predecessors of  $u_0$ . Now cell  $v_1$  needs two predecessors, one of them is  $v_0$  while the other can be chosen among cells  $v_2$ ,  $v_3$  and  $v_4$ . Let us study these cases separately.

If we choose  $v_3$  (resp.  $v_4$ ), then  $u_1$  needs three predecessors as both  $v_1$  and  $v_3$  (resp.  $v_4$ ) are its in-neighbors. These three predecessors have to be chosen among the cells  $v_4$  (resp.  $v_3$ ),  $u_4$ ,  $u_5$  and  $u_6$ . The latter implies that at least one of  $u_5$  and  $u_6$  is a predecessor of  $u_1$ . Since  $u_0$ ,  $u_5$  and  $u_6$  are in-neighbors of  $v_0$ , then it needs at least three predecessors which is not possible.

On the other hand, if we choose  $v_2$  then  $u_1$  needs only two predecessors (as  $v_1$  is its in-neighbor). These two predecessors have to be taken from cells  $v_3$ ,  $v_4$ ,  $u_4$ ,  $u_5$  and  $u_6$ . Note that if we take  $v_3$  (resp.  $v_4$ ), then cell  $v_1$  would need an extra predecessor, which can only be cell  $v_4$  (resp.  $v_3$ ), what in turn would make  $u_1$  to need an extra predecessor. In other terms, selecting  $v_3$  or  $v_4$  is optional in this case and, without loss of generality, we will consider only cells  $u_4$ ,  $u_5$  and  $u_6$ , as possible predecessors for  $u_1$ . The latter implies that at least one of  $u_5$  and  $u_6$  is a predecessor of  $u_1$ . Since  $u_0$ ,  $u_5$  and  $u_6$  are in-neighbors of  $v_0$ , then it needs at least three predecessors which is not possible.  $\square$

**Theorem 20.** *Neighborhood 111 does not admit a crossover gate.*



*Proof. Upward crossing.* The cells for this proof are depicted in Figure 22a. First of all, notice that cell  $u_0$  needs three predecessors. They can only be cells  $u_2 = (i, j - 1)$ ,  $u_3 = (i + 2, j - 1)$  and  $u_4 = (i + 2, j + 1)$ . On the other hand,  $v_0$  needs two predecessors, which can be selected among cells  $v_2 = (i - 1, j + 1)$ ,  $v_3 = (i - 1, j)$ ,  $v_4 = (i - 1, j - 1)$  and  $v' = (i + 1, j - 1)$ . We will prove that  $v'$  has to be a predecessor of  $v_0$ , which creates another upward crossing just below the original one, with arcs  $(v', v_0)$  and  $(u_2, u_0)$ . Assuming that the height of the gate is  $N$ , the preceding argument can be repeated  $N + 1$  times, contradicting the assumption about the gate size. We prove the latter by contradiction, assuming  $v'$  is not a predecessor of  $v_0$ . In such a case, both predecessors of  $v_0$  have to be chosen among  $v_2$ ,  $v_3$  and  $v_4$ . Note that if we choose  $v_3$  and  $v_4$ , they make  $u_2$  to need four predecessors, which is not possible. In other words, one predecessor of  $v_0$  must be  $v_2$ , while the second predecessor is  $v_3$  or  $v_4$ . If we choose  $v_3$  (resp.  $v_4$ ), then cell  $u_2$  needs three predecessors. The predecessors of  $u_2$  can only be cells  $v_4$  (resp.  $v_3$ ),  $u_5 = (i - 1, j - 2)$  and  $u_6 = (i + 1, j - 2)$ . Considering that  $v_4$  (resp.  $v_3$ ) is an in-neighbor of  $v_0$ , then  $v_0$  needs an extra predecessor and the only remaining in-neighbor of  $v_0$  is  $v'$ .

**Leftward crossing.** This case is symmetric to the upward crossing.

**Downward crossing.** We rename the cells for this case, starting with  $u_0, u_1, v_0, v_1$  given by Lemma 2 as depicted in Figure 22b. Both  $v_0$  and  $v_1$  are in-neighbors of  $u_0$ , hence it needs three predecessors. One of them is  $u_1$ , while the other two have to be chosen among cells  $u_2 = (i, j - 1)$ ,  $u_3 = (i + 2, j - 1)$  and  $u' = (i + 2, j + 1)$ . We first proceed by contradiction, assuming  $u'$  is a predecessor of  $u_0$ , in order to prove that this is impossible (see Figure 22b). Let us consider cell  $v_1$ . It has one in-neighbor from the opposite firing graph (cell  $u_1$ ), *i.e.*, it needs two predecessors which can be selected among cells  $v' = (i + 2, j)$ ,  $v_2 = (i + 2, j + 2)$ ,  $v_3 = (i + 1, j + 2)$  and  $v_4 = (i, j + 2)$ . We can immediately discard cell  $v'$  since it creates a leftward crossing with arcs  $(v', v_1)$  and  $(u', u_0)$ , which we already proved impossible. Moreover, cells  $v_2$  and  $v_3$  cannot both be predecessors of  $v_1$ . This is because they are, together with  $v_1$ , in-neighbors of  $u'$ , making it to need four predecessors, which is not possible. It follows that one of the in-neighbors of  $v_1$  must be  $v_4$ , while the second one can be chosen between  $v_2$  and  $v_3$ . If  $v_2$  (resp.  $v_3$ ) is chosen, then cell  $u'$  needs three predecessors that can only be cells  $u'' = (i + 3, j)$ ,  $u''' = (i + 3, j + 2)$  and  $v_3$  (resp.  $v_2$ ). Since  $v_3$  (resp.  $v_2$ ) is also an in-neighbor of  $v_1$ , cell  $v_1$  needs an extra predecessor and the only available in-neighbor is  $v'$  which, as we already mentioned, generates a leftward crossing. With this we proved that  $u'$  cannot be a predecessor of  $u_0$ .

The latter implies that the predecessors of  $u_0$  are  $u_2$  and  $u_3$ , so let us continue the reasoning with this hypothesis (see Figure 22c). Apart from  $u_2$  and  $u_3$ , we rename again the cells for a better understanding of the proof. Note that  $u_1$  is an in-neighbor of  $v_0$ , hence  $v_0$  needs an extra predecessor which can be selected among cells  $v_2 = (i + 1, j - 1)$ ,  $v_3 = (i - 1, j + 1)$ ,  $v_4 = (i - 1, j)$  and  $v_5 = (i - 1, j)$ . We now prove that none of them can be the extra predecessor of  $v_0$ , thus the downward crossing is not possible. Let us proceed by cases.

**Case  $v_2$ .** It generates an upward crossing with arcs  $(v_2, v_0)$  and  $(u_2, u_0)$ , which has already been proven impossible.

**Case  $v_3$ .** Since  $v_3$  is an in-neighbor of  $u_1$ , cell  $u_1$  needs two predecessors which can be selected among cells  $v_4$ ,  $u_4 = (i-1, j+2)$ ,  $u_5 = (i, j+2)$  and  $u_6 = (i+1, j+2)$ . We again proceed by exhaustion.

First, let us prove that  $v_4$  cannot be a predecessor of  $u_1$  (see Figure 22d). In such a case, cell  $v_0$  needs an extra predecessor which can only be cell  $v_5$  (selecting  $v_2$  generates an upward crossing). Since  $v_0$  and  $v_5$  are in-neighbors of  $u_2$ , it needs three predecessors and the only remaining in-neighbors are the cells  $u_4$ ,  $u_7 = (i-1, j-2)$  and  $u_8 = (i+1, j-2)$ . At the same time, since  $v_4$  is an in-neighbor of  $v_5$ , cell  $v_5$  requires two predecessors which can be selected among cells  $v_6 = (i-2, j)$ ,  $v_7 = (i-2, j-1)$ ,  $v_8 = (i-2, j-2)$  and  $v_9 = (i, j-2)$ . Selecting  $v_9$  creates an upward crossing with arcs  $(v_9, v_5)$  and  $(u_7, u_2)$ , thus we can discard it. On the other hand, selecting (at least) one of  $v_6$  and  $v_7$  (which is mandatory) makes  $v_4$  to exhaust its possible predecessors (note that  $v_3$  is also an in-neighbor  $v_4$ ). We conclude that  $v_4$  cannot be a predecessor of  $u_1$ .

Now, let us prove that choosing  $u_6$  as predecessor of  $u_1$  also leads to a contradiction (see Figure 22e). Since both  $u_1$  and  $u_6$  are in-neighbors of  $v_1$ , it needs three predecessors that can only be the cells  $u_5$ ,  $v_{10} = (i+2, j)$  and  $v_{11} = (i+2, j+2)$ . This creates a new downwards crossing with arcs  $(u_5, v_1)$  and  $(u_6, u_1)$ . Observe that the arc  $(v_{11}, v_1)$  also leads to a contradiction since  $v_{11}$  cannot be a predecessor of  $v_1$ , for the same reason that cell  $u' = (i+2, j+1)$  cannot be a predecessor of  $u_0$  (Figure 22b). Therefore, cell  $u_6$  cannot be a predecessor of  $u_1$ .

Given that none of  $v_4$  and  $u_6$  can be predecessor of  $u_1$ , we are forced to assume that  $u_4$  and  $u_5$  are the only two predecessors of  $u_1$ . We now prove that a pattern is repeated infinitely within the crossing gate, which is impossible (see Figure 22f). Since both  $u_4$  and  $u_5$  are in-neighbors of  $v_3$ , it needs three predecessors which can only be cells  $v_6$ ,  $v_{12} = (i-2, j+1)$  and  $v_{13} = (i-2, j+2)$ . Consequently, since  $v_{12}$  and  $v_{13}$  are in-neighbors of  $u_4$ , it needs three predecessors which can only be cells  $u_9 = (i-2, j+3)$ ,  $u_{10} = (i-1, j+3)$  and  $u_{11} = (i, j+3)$ . This is the first occurrence of the pattern, which is composed of the cells  $v_6$ ,  $v_{12}$ ,  $v_{13}$ ,  $u_9$ ,  $u_{10}$  and  $u_{11}$  (dashed on Figure 22f). Following with the reasoning, note that cells  $u_9$  and  $u_{10}$  are both in-neighbors of  $v_{13}$ , which means that it needs three predecessors. These predecessors can only be cells  $v_{14} = (i-3, j+1)$ ,  $v_{15} = (i-3, j+2)$  and  $v_{16} = (i-3, j+3)$ . Both cells  $v_{15}$  and  $v_{16}$  are in-neighbors of  $u_9$ , *i.e.*, it needs three predecessors which can only be cells  $u_{12} = (i-3, j+4)$ ,  $u_{13} = (i-2, j+4)$  and  $u_{14} = (i-1, j+4)$ . This creates the second occurrence of the pattern which consists in the cells  $v_{14}$ ,  $v_{15}$ ,  $v_{16}$ ,  $u_{12}$ ,  $u_{13}$  and  $u_{14}$ . Let us assume that the height of the gate is  $N$ . The preceding argument can be repeated  $N+1$ , contradicting the assumption about the gate size (each pattern repetition necessitates one more unit of height). We conclude that  $u_4$  and  $u_5$  cannot be the predecessors of  $u_1$ , which in turn implies that  $v_3$  cannot be the predecessor of  $v_0$ .

**Case  $v_4$ .** This case is depicted in Figure 22g. Note that the cells  $v_0$  and  $v_4$  are in-neighbors of  $u_2$ , *i.e.*, it needs three predecessors, which can only be cells  $v_5$ ,  $u_7$  and  $u_8$ . On the other hand, cells  $u_1$  and  $u_2$  are in-neighbors of  $v_4$ , hence it needs three predecessors which can be selected among cells  $v_3$ ,  $v_6$ ,  $v_7$  and  $v_{12}$ . There are two important facts to note at this point: first that cells  $v_0$ ,  $v_4$ ,  $v_6$  and  $v_7$  are all in-neighbors of  $v_5$ , and second that since  $v_4$  needs three predecessors and there are four

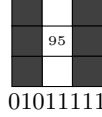
options, at least one of  $v_6$  and  $v_7$  has to be chosen. It follows that cell  $v_5$  needs at least four predecessors, which is not possible.

**Case  $v_5$ .** Cells  $v_0$  and  $v_5$  are in-neighbors of  $u_2$ , thus it needs three predecessors, which can only be cells  $v_4$ ,  $u_7$  and  $u_8$ . Since  $v_4$  is an in-neighbor of  $v_0$ , cell  $v_0$  needs an extra predecessor, however we have already exhausted all the other possibilities in the previous cases.

**Righward crossing.** This case is symmetric to the downward crossing. □

### 3.7 Crossover impossibility: escape cells and timestamps

Neighborhood studied in this subsection:



For the proofs that crossover gates do not exist in neighborhoods 95 and 39, we will strengthen the hypothesis on the crossing constraint given by Lemma 2 by considering the *first* crossing constraint occurring in one of the two firing graphs. This temporal minimality is formalized as follows.

**Definition 5** (Escape cell). Given a crossover gate  $g$  with firing graphs  $G_{ns}^g$  and  $G_{we}^g$ , let us call *crossing edge* of  $G_{ns}^g$  (resp.  $G_{we}^g$ ) an arc  $(u', u) \in A_{ns}$  (resp.  $\in A_{we}$ ) that intersects an arc  $(v', v) \in A_{we}$  (resp.  $A_{ns}$ ). In this case  $u$  and  $v$  are called *escape cells*.

**Definition 6** (Timestamp). Given a crossover gate  $g$  with firing graphs  $G_{ns}^g$  and  $G_{we}^g$ , we associate to each cell of  $V_{ns}$  and  $V_{we}$  a timestamp  $t \in \mathbb{N}$  defined as the one plus the maximum timestamp among its predecessors, starting with timestamp 1 for cells  $n$  and  $w$ .

Timestamps exactly correspond to the temporality of topplings within a crossover gate. Note that a firing graph  $G_{ns}^g$  (resp.  $G_{we}^g$ ) can have several escape cells with the same timestamp, and we will consider any of the minimum ones. Intuitively, this extra hypothesis of minimality will be helpful in reaching contradictions: in some cases a predecessor of the minimum escape cell under consideration will be surrounded by cells from the other firing graph, leaving no choice but to have another crossing edge on the path from  $n$  (resp.  $w$ ) to it, hence another escape cell with a strictly smaller timestamp, contradicting the assumption on its minimality.

We formalize this intuition into the following lemma, but first define a notion of path partitioning the crossover gate (joining the two extremities of a firing graph).

**Definition 7** (Dividing path). In a firing graph, a *dividing path* is a directed path from  $n$  to  $s$  (in  $G_{ns}$ ), or a directed path from  $w$  to  $e$  (in  $G_{we}$ ). Such a path divides the crossover gate into two parts.

Without loss of generality, we can assume that each cell (apart from the starting and ending cells of the firing graph) from both firing graphs has at least one predecessor and at least one successor, otherwise it can be removed from the firing graph (by setting its sand content to 0) whilst the configuration is still a crossover gate. Observe that with this assumption, any directed path can be extended into a dividing path.

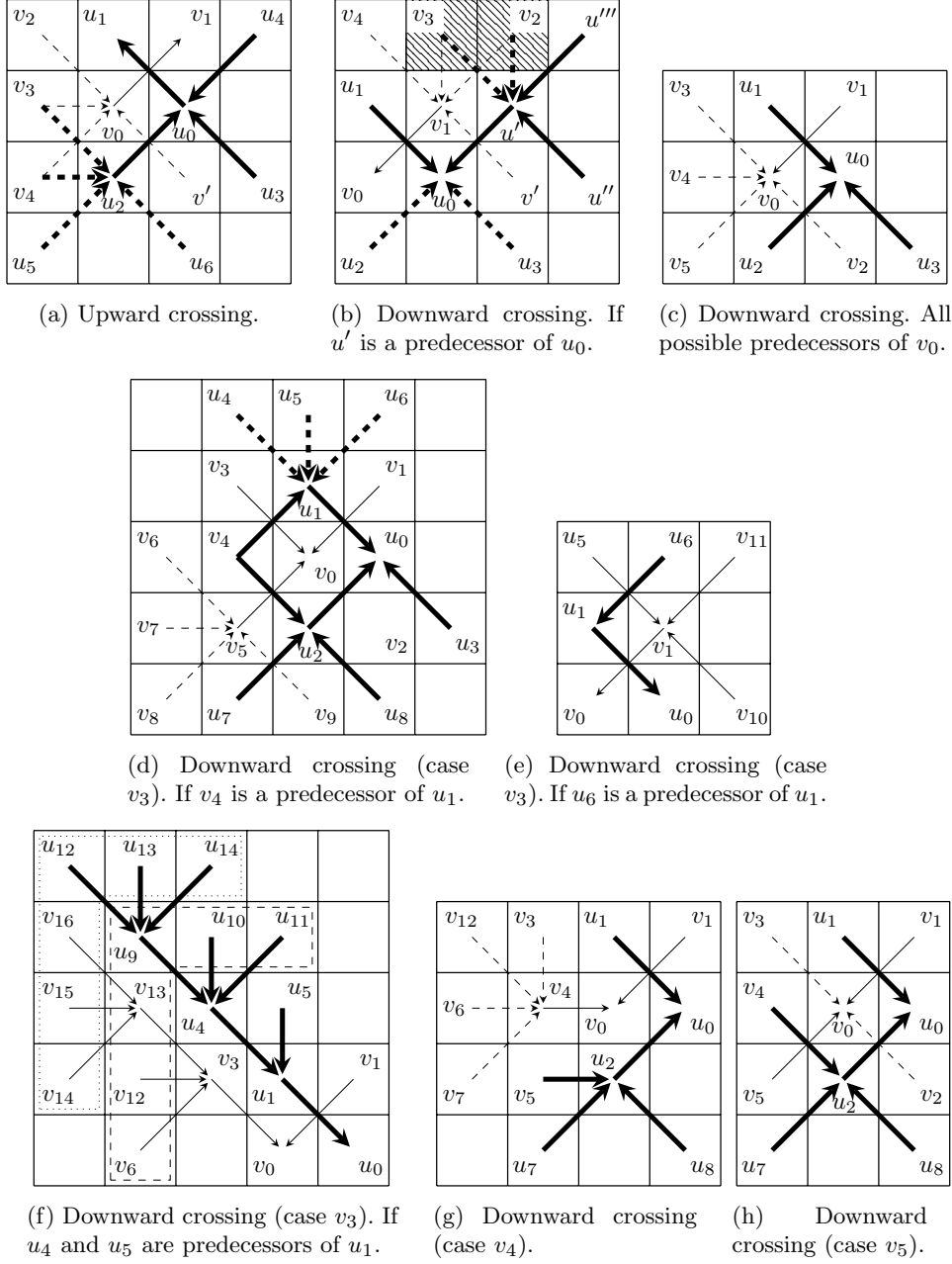


**Fig. 22:** Proof of Theorem 20 for neighborhood 111.

**Lemma 3.** *Given the pair of firing graphs of a crossover gate, if cell  $u$  has two ancestors  $u', u''$ , such that in the other firing graph there is a dividing path with  $u'$*

and  $u''$  lying in different parts, then  $u$  has an ancestor distinct from itself which is an escape cell.

*Proof.* Cells  $u'$  and  $u''$  must have an ancestor in common (the starting cell of their firing graph), which implies that the corresponding path (linking  $u'$  and  $u''$ ) intersects the dividing path from the other firing graph. Since we consider only ancestors of  $u'$  and  $u''$ , cell  $u$  must have a strict ancestor (not itself) which is an escape cell.  $\square$

We also derive the following corollary, because the intersection of the (directed) dividing path with the ancestors of  $u$  (more precisely with some path from its starting cell to  $u$ ) must happen ahead of the predecessors of  $u$ .

**Corollary 2.** *In Lemma 3, if the dividing path intersects an arc whose head is cell  $u$ , then it has another intersection with the other firing graph, which happens strictly earlier on the dividing path.*

**Theorem 21.** *Neighborhood 95 does not admit a crossover gate.*



*Proof.* Let  $u_1 = (i, j)$  be a minimum escape cell of one of the two firing graphs. It is important to mention that  $(u_0, u_1)$  is a crossing edge, *i.e.* it intersects an arc of the opposite firing graph. The two endpoints of this latter, denoted  $v_0$  and  $v_1$ , are fixed by  $(u_0, u_1)$ , but we will ignore its precise direction (which cell is the head or tail does not matter).

According to Lemma 2, there are four possible orientations for the arc  $(u_0, u_1)$ . However, they are all equivalent by symmetry. Therefore, without loss of generality, we consider only the case with  $u_0 = (i + 1, j - 1)$ ,  $v_0 = (i, j - 1)$  and  $v_1 = (i + 1, j)$  (see Figure 23).

Since  $v_1$  is an in-neighbor of  $u_1$ , then  $u_1$  requires an extra predecessor. This predecessor can be one of the following cells:  $w_2 = (i - 1, j - 1)$ ,  $w_4 = (i - 1, j)$ ,  $w_6 = (i - 1, j + 1)$  or  $w_8 = (i + 1, j + 1)$ . We will prove that these four cases lead to contradictions.

**Case  $w_4$ .** If  $w_4$  is a predecessor of  $u_1$ , then cells  $w_4$  and  $u_0$  are both predecessors of  $u_1$ . Furthermore, the edge between  $v_0$  and  $v_1$  can be extended into a dividing path, with  $w_4$  and  $u_0$  on different sides. Lemma 3 applies, contradicting the temporal minimality of escape cell  $u_1$ .

**Case  $w_6$ .** If  $w_6$  is a predecessor of  $u_1$ , then the situation is analogous to the previous case with  $w_6$  playing the role of  $w_4$  (on the opposite side of  $u_0$ ).

**Case  $w_2$ .** If the extra predecessor of  $u_1$  is  $w_2$ , then  $v_0$  needs three predecessors, since both  $u_0$  and  $w_2$  are its in-neighbors. Considering that  $v_0$  also needs a successor, then the following cells are in its firing graph:  $v_1$ ,  $w_4$ ,  $w_0 = (i - 1, j - 2)$  and  $w_1 = (i + 1, j - 2)$ . For any choice of the successor among them, a directed path can be extended into a dividing path with  $w_2$  and  $u_0$  on different sides. It follows again by Lemma 3 that there is an escape cell with a strictly smaller timestamp than  $u_1$ , a contradiction.

**Case  $w_8$ .** Finally, if  $w_8$  is the extra predecessor of  $u_1$ , then cell  $v_1$  has an in-neighbor ( $u_1$ ) from the other firing graph hence it needs at least two predecessors, and still at least one successor. For any choice of these three relatives of  $v_1$  (cell  $v_0$  and two more, chosen among cells  $w_7 = (i, j + 1)$ ,  $w_3 = (i + 2, j - 1)$ ,  $w_5 = (i + 2, j)$  or  $w_9 = (i + 2, j + 1)$ ), it creates a directed path that can be extended into a dividing path with  $u_0$  and  $w_8$  on different sides. By Lemma 3 we reach again a contradiction.  $\square$

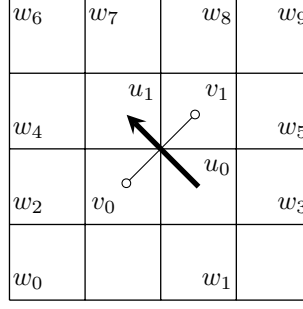
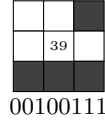


**Fig. 23:** Proof of Theorem 21 for neighborhood 95.

### 3.8 Crossover impossibility: almost crossing signals

Neighborhood studied in this subsection:



The study of neighborhood 39 requires much more developments, and the impossibility of crossover consists in a much longer proof split in multiple intermediate results. The intuitive reason for this is that neighborhood 39 almost has a crossover gate. This behavior is illustrated on Figure 24, which presents a situation where two disjoint firing graphs (none of the vertices from one firing graph is toppled by the other firing graph) almost cross each other. They are intricated, and this intrication can be continued up to an arbitrary length. We eventually prove that, despite of this construction where the two firing graphs interact in a non trivial way, a crossover gate is impossible in neighborhood 39 (Theorem 22).

For the proof, we first identify two forbidden patterns (Lemma 4; even though the forbidden pattern 2 will not be necessary, we still present it). Then the proof proceeds again by contradiction (assuming a crossover gate exists), with a large case analysis split into multiple lemmas. Each case is discard either (1) by encountering a forbidden pattern, or (2) because the crossover gate would be of infinite size, or (3) when it contradicts the temporal minimality of the escape cell under consideration.

**Lemma 4** (Forbidden patterns). *The firing graphs  $G_{ns}^g$  and  $G_{we}^g$  of a crossover gate  $g$  for neighborhood 39 cannot have any of the two induced pairs of subgraphs  $G_{F_i} = (V_{F_i}, A_{F_i})$  and  $G'_{F_i} = (V'_{F_i}, A'_{F_i})$  at any position  $p, q \in \mathbb{Z}$  for  $i \in \{1, 2\}$ , where (see Figure 25):*

**Forbidden pattern 1.**

$$V_{F_1} = \{(p, q), (p, q - 2), (p + 1, q - 1)\} \text{ and } A_{F_1} = \{((p, q - 2), (p + 1, q - 1))\}, \\ V'_{F_1} = \{(p, q - 1), (p + 1, q)\} \text{ and } A'_{F_1} = \{((p, q - 1), (p + 1, q))\}.$$

**Forbidden pattern 2.**

$$V_{F_2} = \{(p, q), (p - 1, q - 1), (p - 2, q), (p - 3, q - 1)\} \text{ and } A_{F_2} = \{((p, q), (p - 1, q - 1)), ((p - 2, q), (p - 1, q - 1)), ((p - 2, q), (p - 3, q - 1))\}, \\ V'_{F_2} = \{(p - 1, q), (p - 2, q - 1)\} \text{ and } A'_{F_2} = \{((p - 1, q), (p - 2, q - 1))\}.$$



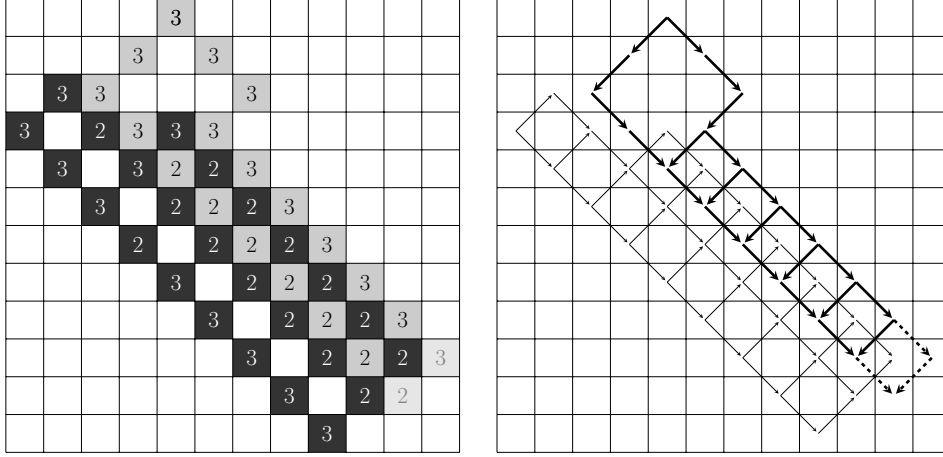


**Fig. 24:** Almost crossover gate for 39. The pattern can be repeated up to an arbitrary length, while preserving the fact that both firing graphs are disjoint. However, the two graphs do not eventually cross each other, as this is proven impossible in Theorem 22.

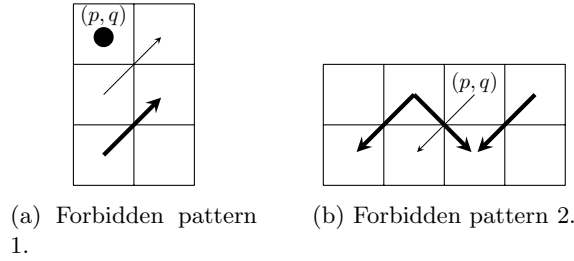


**Fig. 25:** Forbidden patterns for neighborhood 39 (Lemma 4).

*Proof.* We show that each of these two subgraphs, if present once, must be repeated endlessly, hence contradicting the finiteness of  $g$ .

**Forbidden pattern 1.** See Figure 26a. In such a case, cell  $(p, q - 2)$  needs two predecessors which can only be the cells  $(p - 1, q - 1)$  and  $(p - 1, q - 3)$ . Something similar occurs to  $(p, q - 1)$ , it needs two predecessors and the only options are cells  $(p - 1, q)$  and  $(p - 1, q - 2)$ . This repeats the original pattern shifted by  $(-1, -1)$ .

**Forbidden pattern 2.** Cell  $(p, q)$  needs to “come” from somewhere, *i.e.*, it needs a predecessor. If cell  $(p - 1, q + 1)$  is its predecessor (Figure 26b), then cell  $(p - 1, q)$  needs two predecessors which can only be cells  $(p - 2, q + 1)$  and  $(p, q + 1)$ . Since  $(p, q + 1)$  is an in-neighbor of  $(p, q)$ , it needs an extra predecessor that can only be the cell  $(p + 1, q + 1)$ , which causes  $(p + 1, q)$  to need two predecessors and the only remaining options are  $(p, q + 1)$  and  $(p + 2, q + 1)$ . The latter generates a repetition of the original pattern shifted by  $(1, 1)$ . The exact same pattern is obtained if  $(p + 1, q + 1)$  is initially considered as the predecessor of  $(p, q)$ .

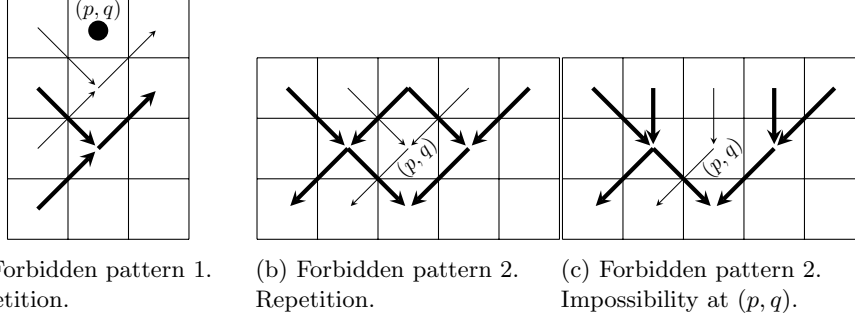


**Fig. 26:** Proof of Lemma 4 for neighborhood 39.

On the other hand, if  $(p, q+1)$  is considered as the predecessor of  $(p, q)$  (Figure 26c), then both cells  $(p-1, q)$  and  $(p+1, q)$  need two predecessors each. The only options for cell  $(p-1, q)$  are cells  $(p-2, q+1)$  and  $(p-1, q+1)$ , while for  $(p+1, q)$  the only options are  $(p+1, q+1)$  and  $(p+2, q+1)$ . Since both cells  $(p-1, q+1)$  and  $(p+1, q+1)$  are in-neighbors of  $(p, q)$ , cell  $(p, q)$  needs two extra predecessors, which is not possible.  $\square$

We will consider a minimum escape cell, and number again the cells as given by Lemma 2, that is with  $v_0 = (i, j)$ ,  $v_1 = (i+1, j+1)$ ,  $u_0 = (i+1, j)$  and  $u_1 = (i, j+1)$ . For neighborhood 39, up to symmetries there are four ways that a cell can be a minimum escape cell, which correspond to the following scenarios:

1. rightward crossing where  $u_0$  is a minimum escape cell,
2. rightward crossing where  $v_1$  is a minimum escape cell,
3. downward crossing where  $u_0$  is a minimum escape cell,
4. downward crossing where  $v_0$  is a minimum escape cell.

Scenarios 1 and 4 are the most difficult to handle, and in Lemma 5 we prove that they are actually equivalent, in the sense that the existence of one implies the existence of the other. Then a succession of tedious studies will lead to Lemma 8, stating that scenario 1 is impossible (and so is scenario 4). Lemma 9 proves that scenario 3 is impossible, which is fairly straightforward. At this point, given that only scenario 2 is possible in *both* firing graphs, a case analysis on the orientation of the crossover gate (regarding the cardinal directions north, east, south, west) will allow to conclude that a crossover gate is impossible for neighborhood 39. This last step of the proof (Theorem 22) exploits new arguments, using the fact that neighborhood 39 cannot transmit grains in the north-west direction.

**Lemma 5.** *The existence of a crossover gate  $g$  for neighborhood 39 with a rightward (resp. downward) crossing such that  $u_0$  (resp.  $v_0$ ) is a minimum escape cell, implies that  $g$  also contains a downward (resp. rightward) crossing with arcs  $(v_1, (i+2, j))$  and  $((i+2, j+1), u_0)$  (resp.  $(u_1, (i-1, j))$  and  $((i-1, j+1), v_0)$ ) (see Figure 27).*



*Proof.* We prove the two respective implications of the statement one after the other.

**Rightward crossing  $\implies$  downward crossing.** For the rightward crossing (Figure 28a), cell  $u_0$  needs two predecessors, one of them is  $u_1$  by our hypothesis, while

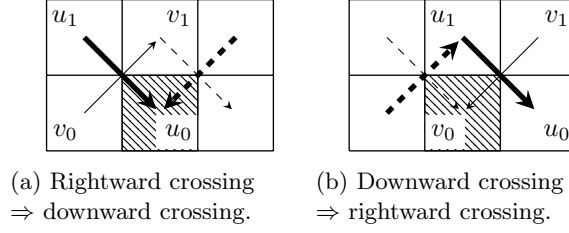


**Fig. 27:** Statement of Lemma 5 for neighborhood 39. Minimum escape cell is dashed.

the second one can be chose among  $u' = (i, j - 1)$  and  $u_2 = (i + 2, j + 1)$ . However, choosing  $u'$  creates a forbidden pattern 1 (Lemma 4), hence the second predecessor of  $u_0$  must be cell  $u_2$ . Now we want to prove that  $v_1$  can only have  $v_2 = (i + 2, j)$  as its successor. The other possibily is  $v_3 = (i + 2, j + 2)$ , but it would create a dividing path with  $u_1$  and  $u_2$  on different parts (recall that any directed path can be extended to a dividing path, and firing graphs are acyclic), contradicting by Lemma 3 the minimality of  $u_0$ .

**Downward crossing  $\Rightarrow$  rightward crossing.** For a downward crossing where  $v_0$  is a minimum escape cell, cell  $u_0$  also needs an extra predecessor, which can be either be cell  $u' = (i + 2, j + 1)$  or cell  $u_2 = (i, j - 1)$  (Figure 28b). However, note that choosing  $u'$  would let it be on the other side (compared to  $u_1$ ) of any dividing path going through arc  $(v_1, v_0)$ , hence contradicting the minimality of  $v_0$  (by Corollary 2 on the dividing path). The latter means that  $u_2$  must be the extra predecessor of  $u_0$ . Since  $v_0$  is an in-neighbor of  $u_2$ ,  $u_2$  needs two predecessors that can only be the cells  $u_3 = (i - 1, j - 2)$  and  $u_4 = (i - 1, j)$ . The next step consists in choosing a predecessor for cell  $v_0$ . The possibilities are cells  $v' = (i - 1, j - 1)$  and  $v_2 = (i - 1, j + 1)$ . Nevertheless, choosing  $v'$  creates a forbidden pattern 1 (Lemma 4), hence  $v_2$  is the only option.

Now we prove that the arc  $(u_4, u_1)$  is mandatory. To do so, we analyse all cases over the possible predecessors of cell  $u_1$ . There are four possibilities:  $u_4, u_5 = (i - 1, j + 2)$ ,  $u_6 = (i, j + 2)$  and  $u_7 = (i + 1, j + 2)$ .

**Case  $u_7$**  (Figure 28c). Since  $u_7$  is an in-neighbor of  $v_1$ , cell  $v_1$  needs two predecessors that can only be the cells  $v_3 = (i + 1, j + 2)$  and  $u_6$ . However, this turns  $v_1$  into an escape cell that topples strictly before  $v_0$ .

**Case  $u_6$**  (Figure 28d). Considering that  $u_6$  is an in-neighbor of  $v_1$ , then  $v_1$  needs two predecessors that can only be the cells  $v_3$  and  $u_7$ . Since  $u_7$  is also an in-neighbor of  $u_1$ , then  $u_1$  needs an extra predecessor that can be selected among  $u_4$  and  $u_5$ . If it is  $u_4$ , then we are done. Hence, let us consider  $u_5$ . Since both  $u_5$  and  $u_6$  are in-neighbors of  $v_2$ , cell  $v_2$  needs three predecessors, which is not possible.

**Case  $u_5$**  (Figure 28e). Cell  $v_2$  is an in-neighbor of  $u_4$ , therefore  $u_4$  needs two predecessors that have to be chosen among  $u_1, u_8 = (i - 2, j - 1)$  and  $u_9 = (i - 2, j + 1)$ . Since it needs two predecessors, we are forced to pick at least one of  $u_8$  and  $u_9$ . However, they both trap cell  $v_2$  into the hypothesis of Corollary 2, which means there is another escape cell that topples before  $v_0$ .

**Case  $u_4$**  (Figure 28f). It is the only possible option. This generates a rightward crossing with arcs  $(u_4, u_1)$  and  $(v_2, v_0)$ , as announced.  $\square$

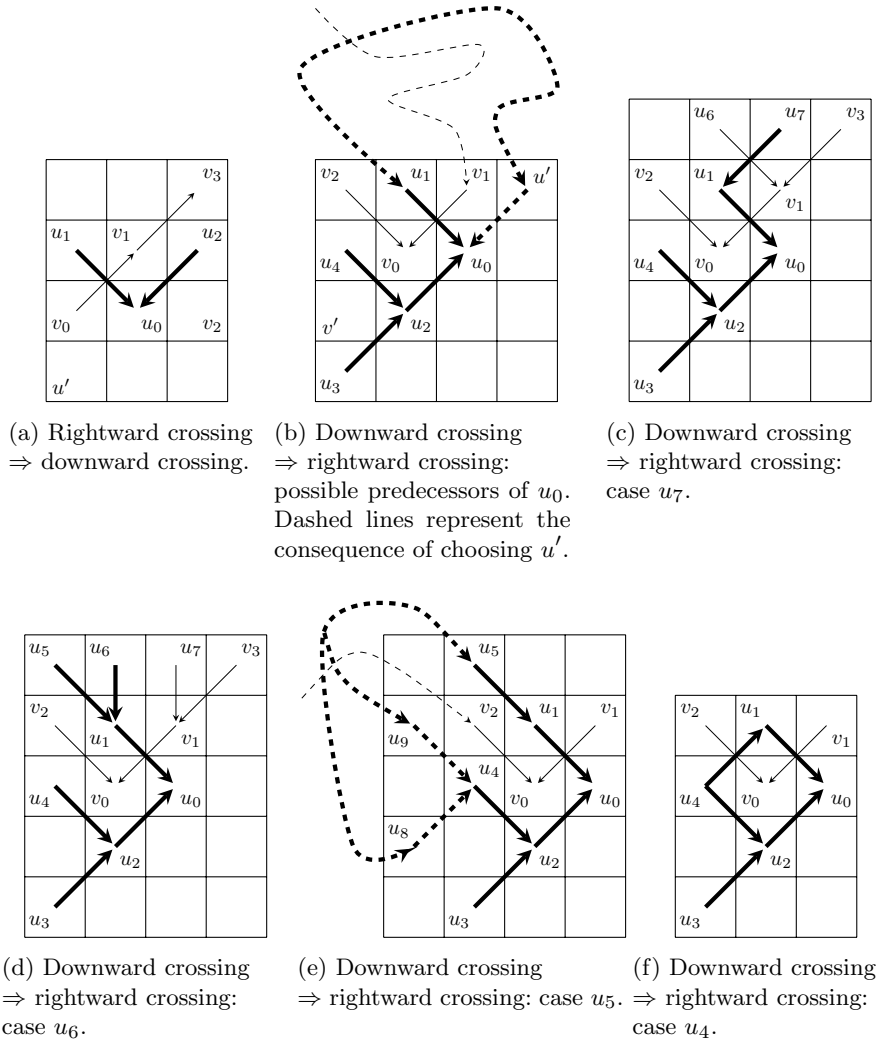


**Fig. 28:** Proof of Lemma 5 for neighborhood 39.

Next is a technical result (Lemma 6) for proving the existence of one arc in a precise situation. It will be an ingredient in the subsequent reasoning (Lemma 7), which aims at giving a large setup for the firing graphs in the case of scenario 1. Then Lemma 8 will conclude that scenario 1 is impossible.

**Lemma 6.** *If there exist a crossover gate  $g$  for neighborhood 39 with firing graphs  $G_{ns}^g$  and  $G_{we}^g$ , such that the following arcs belong to  $G_{ns}^g$  (resp.  $G_{we}^g$ ) (see Figure 29a):  $((0, 0), (1, -1))$ ,  $((2, 0), (1, -1))$ , and the following arcs belong to  $G_{we}^g$  (resp.  $G_{ns}^g$ ):  $((0, -1), (1, -2))$ ,  $((0, -3), (1, -2))$ ,  $((1, -2), (2, -1))$ ,  $((1, 0), (2, -1))$ ,*



then the arc  $((1, -1), (2, -2))$  belongs to  $G_{ns}^g$  (resp.  $G_{we}^g$ ).  
Any translation of the statement holds as well.

*Proof.* If the first part of the statement holds, then it is obvious that it holds up to translation by any vector  $(p, q) \in \mathbb{Z}^2$ , since we have no assumption peculiar to any absolute position within the  $\mathbb{Z}^2$  grid.

The hypothesis are depicted on Figure 29a, and our goal is to prove that the dashed arc  $(1, -1), (2, -2)$  is enforced. Note that the cell  $(1, -1)$  needs a successor that can be either  $(2, -2)$  or  $(0, -2)$ . If it is  $(2, -2)$  we are done, so let us study the other case. If  $(0, -2)$  is the successor of  $(1, -1)$ , then the cell  $(0, -3)$  needs two predecessors that can only be the cells  $(-1, -2)$  and  $(-1, -4)$ . Also note that  $(0, -1)$  is an in-neighbor of  $(0, -2)$ , then cell  $(0, -2)$  needs an extra predecessor that has to be chosen among cells  $(-1, -1)$  and  $(-1, -3)$ . However, it cannot be the cell  $(-1, -3)$  because it generates the forbidden pattern 1 (Lemma 4), then the only option is the cell  $(-1, -1)$ . Observe that the latter reasoning generates a repetition of the initial pattern, shifting by vector  $(-1, -1)$  (Figure 29b).

There is now a choice on the successor of cell  $(0, -2)$ , that can be either  $(1, -3)$  or  $(-1, -3)$ . If  $(1, -3)$  is chosen, then the reasoning generates another repetition of the initial pattern, shifted by vector  $(-2, -2)$ . This process can be repeated an arbitrary number of times (but a finite number, because a crossover gate is finite), until the successor of cell  $(-k+1, -k-1)$  for some  $k \in \mathbb{N}$  is chosen to be cell  $(-k+2, -k-2)$  (see Figure 29c).

In that case,  $(-k+2, -k-2)$  needs an extra predecessor since the cell  $(-k+2, -k-1)$  is its in-neighbor. There are two options: cell  $(1-k, -k-3)$ , which generates the forbidden pattern 1 (Lemma 4) hence we cannot choose it, and cell  $(-k+3, -k-1)$ . Now observe that cell  $(-k+3, -k-1)$  has cell  $(3-k, -k)$  as an in-neighbor, therefore it needs two predecessors which can only be cells  $(-k+2, -k)$  and  $(-k+4, -k)$ . Moreover, the same reasoning applies inductively to all the cells:

$$(-k+3, -k-1), (-k+4, -k), (-k+5, -k+1), \dots, (1, -3), (2, -2).$$

This is depicted in Figure 29d. Eventually, cell  $(2, -2)$  needs two predecessors which can only be cells  $(1, -1)$  and  $(3, -1)$ , i.e., the arc  $((1, -1), (2, -2))$  is enforced.  $\square$

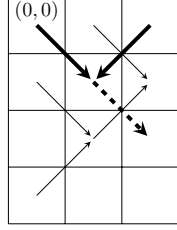
The next lemma settles a large part of the architecture of firing graphs in scenario 1.

**Lemma 7.** *If there exists a crossover gate for neighborhood 39 with a rightwards crossing such that  $u_0$  is a minimum escape cell (scenario 1), then the firing graphs for this crossover gate has the following nodes (see Figure 30a):  $u_0, u_1, u_2 = (i+2, j+1)$ ,  $u_3 = (i+3, j+2)$ ,  $u_4 = (i+2, j-1)$ ,  $v_0, v_1, v_2 = (i-1, j+1)$ ,  $v_3 = (i-1, j-1)$ ,  $v_4 = (i, j-2)$ ,  $v_5 = (i+1, j-1)$ ,  $v_6 = (i+2, j)$ , and arcs:  $(u_1, u_0)$ ,  $(u_0, u_4)$ ,  $(u_2, u_0)$ ,  $(u_3, u_2)$ ,  $(v_2, v_0)$ ,  $(v_3, v_0)$ ,  $(v_0, v_1)$ ,  $(v_0, v_5)$ ,  $(v_4, v_5)$ ,  $(v_1, v_6)$ ,  $(v_5, v_6)$ .*

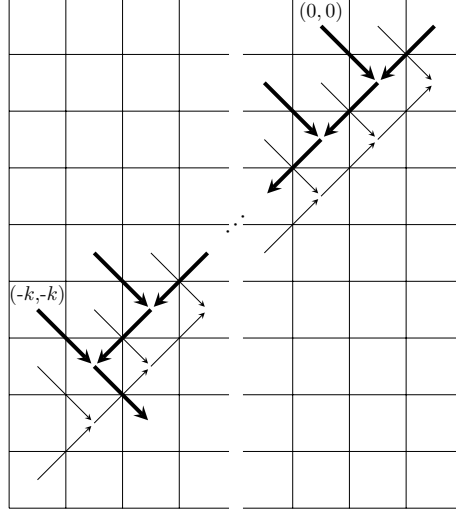
*Proof.* The goal of this proof are the firings graphs depicted in Figure 30a. Starting from the hypothesis that  $u_0$  is a minimum escape cell within a rightward crossing, we proceed by proving that all the arcs stated in this Lemma are mandatory.

**Arcs  $(u_2, u_0)$  and  $(v_1, v_6)$ .** They follow by Lemma 5.

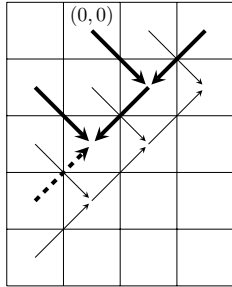




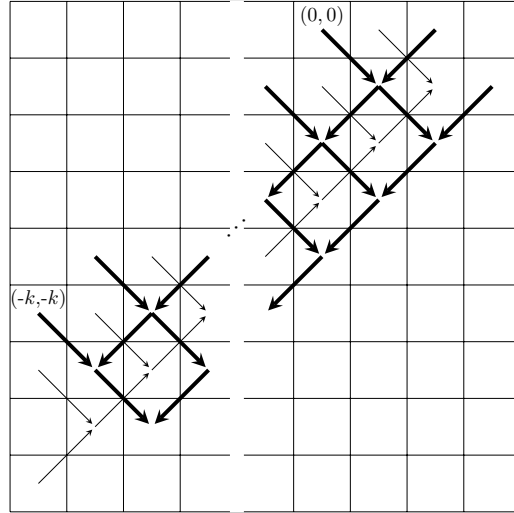
(a) Initial pattern with plain arcs, and the enforced arc dashed.



(c) The pattern repeated  $k$  times.



(b) Choosing  $(0, -2)$  as a successor of  $(1, -1)$ . The dashed arc generates a forbidden pattern.



(d) Cell  $(2, -2)$  requires  $(1, -1)$  as a predecessor.

**Fig. 29:** Proof of Lemma 6 for neighborhood 39.

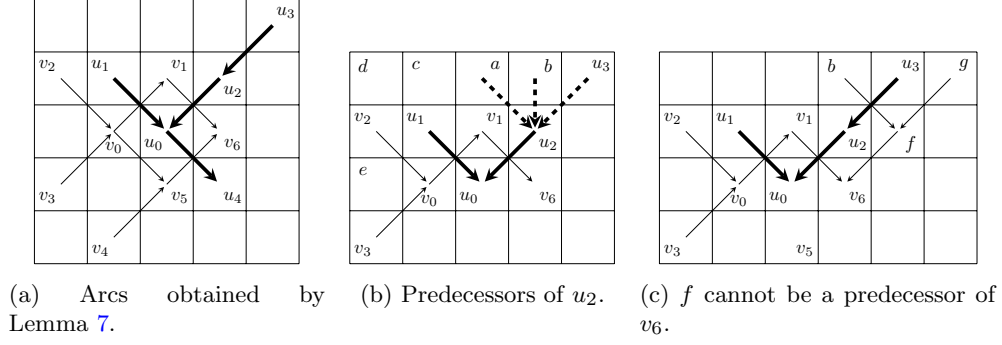


**Fig. 30:** Proof of Lemma 7 for neighborhood 39.

**Arcs  $(v_2, v_0)$  and  $(v_3, v_0)$ .** Cell  $v_0$  has  $u_1$  as an in-neighbor, hence it needs two predecessors which can only be cells  $v_2$  and  $v_3$ .

**Arc  $(u_3, u_2)$ .** Let us check the possible predecessors of cell  $u_2$  (see Figure 30b), which can be  $a = (i + 1, j + 2)$ ,  $b = (i + 2, j + 2)$  or  $u_3$ . If  $a$  is chosen, considering that it is an in-neighbor of  $v_1$ , then  $v_1$  needs an extra predecessor which can be either  $c = (i, j + 2)$  or  $b$ . If  $b$  is chosen, then  $u_2$  would turn into an escape cell that topples before  $u_0$ , contradicting its temporal minimality as an escape cell. On the other hand, if  $c$  is chosen as the extra predecessor of  $v_1$ , then  $u_1$  would need two predecessors that have to be chosen among the cells  $a$ ,  $d = (i - 1, j + 2)$  and  $e = (i - 1, j)$ . However, choosing  $e$  generates the forbidden pattern 1 (Lemma 4), consequently  $a$  and  $d$  have to be the predecessors of  $u_1$ . Nevertheless, since  $a$  is an in-neighbor of  $v_1$ , cell  $v_1$  needs an extra predecessor that can only be the cell  $b$ , but we already proved this to be contradictory.

In the case that  $b$  is chosen as the predecessor of  $u_2$ , then  $v_1$  needs an extra predecessor that can be chosen between  $a$  and  $c$ . If we chose  $a$ , since it is an in-neighbor of  $u_2$ , then  $u_2$  needs an extra predecessors, which can only be the cell  $u_3$ , and we are done. The remaining option is choosing  $c$  as predecessor of  $v_1$ . In such a case,  $u_1$  would need two predecessors among cells  $a$ ,  $d$  and  $e$ . Choosing cell  $e$  generates the forbidden pattern 1 (Lemma 4), then  $a$  and  $d$  have to be the predecessors of  $u_1$ . Nevertheless, since  $a$  is an in-neighbor of  $v_1$ , cell  $v_1$  needs an extra predecessor, but there are no more available in-neighbors.

Finally, the only remaining possible predecessor of  $u_2$  is  $u_3$ .

**Arc  $(v_5, v_6)$ .** Since  $u_2$  is an in-neighbor of  $v_6$ , then  $v_6$  needs an extra predecessor, that can be either  $f = (i + 3, j + 1)$  or  $v_5$  (Figure 30c). If  $f$  is chosen then it would need two predecessors, this is because  $u_3$  is an in-neighbor of  $f$ . The only possible predecessors for  $f$  are the cells  $b$  and  $g = (i + 4, j + 2)$ . However, this would turn  $u_2$  into an escape cell that topples strictly before  $u_0$ . Therefore, the extra predecessor of  $v_6$  has to be  $v_5$ .

**Arcs  $(v_0, v_5)$  and  $(v_4, v_5)$ .** Since  $u_0$  is an in-neighbor of  $v_5$ , cell  $v_5$  needs two predecessors which can only be the cells  $v_0$  and  $v_4$ .

**Arc  $(u_0, u_4)$ .** It is obtained by Lemma 6.  $\square$

Eventually, scenario 1 is proven to be impossible.

**Lemma 8.** *There does not exist a crossover gate  $g$  for neighborhood 39 with a rightward crossing such that  $u_0$  is a minimum escape cell (scenario 1).*

*Proof.* We proceed by contradiction, assuming that there exists a crossover gate  $g$  such that  $u_0$  is a minimum escape cell within a rightward crossing. We use the firing graphs obtained in Lemma 7 (Figure 30a) as a starting point. We prove by induction that this pattern requires a crossover gate of infinite size, which is absurd. The induction relies on shifting the pattern by vector  $(1, -1)$  (except the arc from  $u_3$  to  $u_2$ ). We prove the first pattern repetition as a base case, and then prove that it would repeat indefinitely, because the reasoning used after this first repetition does not require the presence of  $u_3$  anymore. Let us assume that the height of gate  $g$  is  $N$ .

**Base case.** Cell  $v_6$  is an in-neighbor of  $u_4$ , then  $u_4$  needs an extra predecessor. There are two options:  $u' = (i + 1, j - 2)$  which generates the forbidden pattern 1 (Lemma 4), and  $u_5 = (i + 3, j)$  which must therefore be chosen (see Figure 31a).

Now let us study the successor of  $v_6$ . There are two possible out-neighbors to select. One of them is  $v' = (i + 3, j + 1)$  and the other one is  $v_7 = (i + 3, j - 1)$ . We prove that choosing  $v'$  is not possible (see Figure 31b). If  $v'$  is the successor of  $v_6$ , since  $u_3$  is its in-neighbor then it needs an extra predecessor than can be either  $v'' = (i + 2, j + 2)$  or  $v''' = (i + 4, j + 2)$ . However,  $v''$  cannot be chosen because it would turn  $u_2$  into an escape cell that topples before  $u_0$ . Cell  $v'''$  cannot be selected either, since it traps the cell  $u_1$ , meaning there is another escape cell that topples before  $u_0$  (Corollary 2 applies, considering the ancestors  $v_2$  and  $v'''$  of  $v'$ , and the dividing path going through  $u_1, u_0, u_4$ ). The latter means that the successor of  $v_6$  has to be  $v_7$  (Figure 31c).

Note that  $u_5$  is an in-neighbor of  $v_7$ , then  $v_7$  needs an extra predecessor. This predecessor can be either  $v'''' = (i + 4, j)$  or  $v_8 = (i + 2, j - 1)$ . Analogously to the previous reasoning, choosing  $v''''$  traps the cell  $u_1$  (Figure 31d), therefore  $v_8$  has to be selected.

The next step is to consider cell  $v_8$ . It needs two predecessors, since  $u_4$  is its in-neighbor. The only options are  $v_5$  and  $v_9 = (i + 1, j - 3)$ .

Finally, consider the cell  $u_6 = (i + 3, j - 2)$ , it can be added as the successor of  $u_4$  by Lemma 6. The obtained situation is depicted on Figure 31e.

**Induction step.** Assume the pattern has already been repeated  $k \in \mathbb{N}$  times, we will prove that it must repeated once more. We got the cells (see Figure 32a):  $\tilde{u}_0 = (i + k + 1, j - k)$ ,  $\tilde{u}_1 = (i + k + 2, j - k + 1)$ ,  $\tilde{u}_2 = (i + k + 2, j - k - 1)$ ,  $\tilde{v}_0 = (i + k, j - k)$ ,  $\tilde{v}_1 = (i + k + 1, j - k + 1)$ ,  $\tilde{v}_2 = (i + k, j - k - 2)$ ,  $\tilde{v}_3 = (i + k + 1, j - k - 1)$ ,  $\tilde{v}_4 = (i + k + 2, j - k)$ , and the arcs:  $(\tilde{u}_0, \tilde{u}_2)$ ,  $(\tilde{u}_1, \tilde{u}_0)$ ,  $(\tilde{v}_0, \tilde{v}_3)$ ,  $(\tilde{v}_2, \tilde{v}_3)$ ,  $(\tilde{v}_1, \tilde{v}_4)$ ,  $(\tilde{v}_3, \tilde{v}_4)$ .

Note that cell  $\tilde{v}_4$  is an in-neighbor of  $\tilde{u}_2$ , then  $\tilde{u}_2$  needs an extra predecessor. There are two options, however, one of them, the cell  $\tilde{u}' = (i + k + 1, j - k - 2)$ , generates the forbidden pattern 1 (Lemma 4), consequently the only valid option is cell  $\tilde{u}_3 = (i + k + 3, j - k)$  (Figure 32b).

Now let us analyse the possible successors of  $\tilde{v}_4$ . There are two possibilities: cells  $\tilde{v}_5 = (i + k + 3, j - k - 1)$  and  $\tilde{v}_6 = (i + k + 3, j - k + 1)$ . We prove that choosing  $\tilde{v}_6$  implies that the arc  $(\tilde{v}_4, \tilde{v}_5)$  also exists (Figure 32c). If  $\tilde{v}_6$  is the successor of  $\tilde{v}_4$  we can assume that  $\tilde{v}_4$  is the only cell in the path  $p = (v_1, v_6, v_7, \dots, \tilde{v}_1, \tilde{v}_4)$  that has its north-east out-neighbor as a successor. Also, if  $\tilde{v}_6$  is the successor of  $\tilde{v}_4$ , then  $\tilde{u}_3$  needs two predecessors (since  $\tilde{v}_6$  is its in-neighbor). There are only two options: cells  $\tilde{u}_1$  and

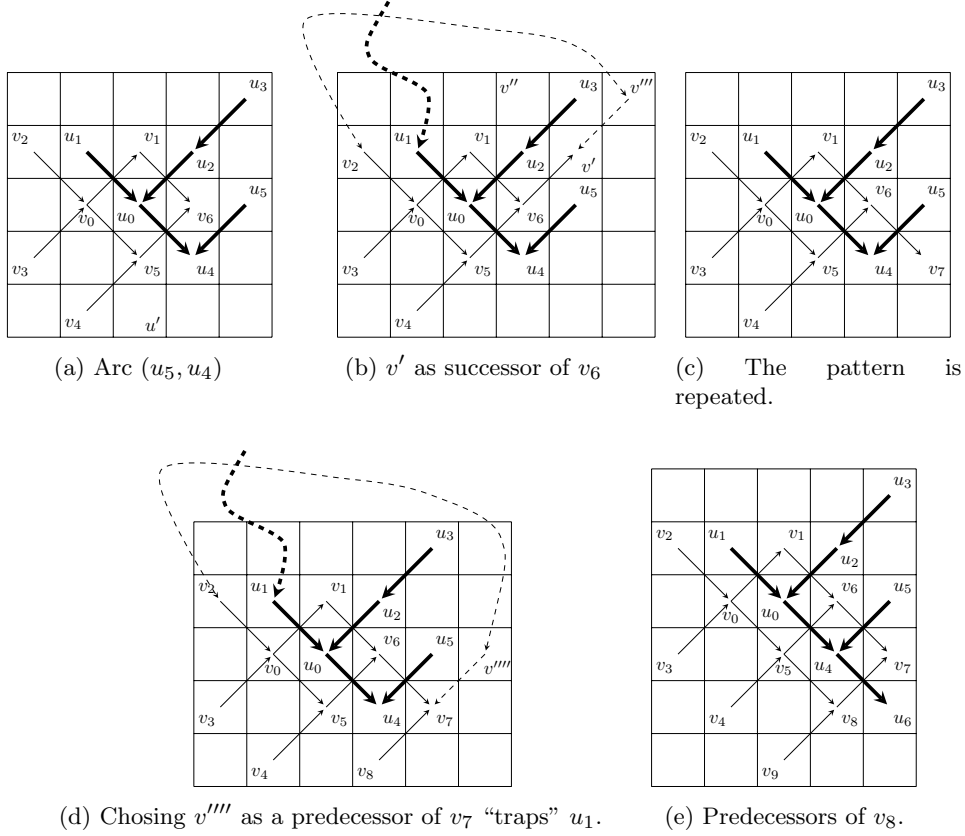


**Fig. 31:** Base case for Lemma 8 for neighborhood 39.

$\tilde{u}_4 = (i + k + 4, j - k + 1)$ . Now, the same reasoning that was applied over  $\tilde{v}_4$  can be applied over  $\tilde{v}_6$ . Moreover, it can be applied inductively. Let assume it has been applied  $\tilde{k} \in \mathbb{N}$  times. The latter means that we have the following cells (see Figure 32c):  $\hat{v}_0 = (k + \tilde{k} + 2, -k + \tilde{k})$ ,  $\hat{v}_1 = (k + \tilde{k} + 1, -k + \tilde{k} - 1)$ ,  $\hat{u}_0 = (k + \tilde{k} + 2, -k + \tilde{k} - 1)$ ,  $\hat{u}_1 = (k + \tilde{k} + 1, -k + \tilde{k})$ ,  $\hat{u}_2 = (k + \tilde{k} + 3, -k + \tilde{k})$ , and arcs:  $(\hat{v}_1, \hat{v}_0)$ ,  $(\hat{u}_1, \hat{u}_0)$ ,  $(\hat{u}_2, \hat{u}_0)$ .

Now  $\hat{v}_0$  needs a successor that can be either  $\hat{v}' = (k + \tilde{k} + 3, -k + \tilde{k} + 1)$ , repeating the pattern once again, or  $\hat{v}_2 = (k + \tilde{k} + 3, -k + \tilde{k} - 1)$ . As choosing  $\hat{v}'$  just repeats the pattern, let us focus on  $\hat{v}_2$ . Since  $\hat{u}_2$  is an in-neighbor of  $\hat{v}_2$ , it needs an extra predecessor. There are two options, the cells  $\hat{v}'' = (k + \tilde{k} + 4, -k + \tilde{k})$  and  $\hat{v}_3 = (k + \tilde{k} + 2, -k + \tilde{k} - 2)$  (Figure 32d). However, choosing cell  $\hat{v}''$  traps cell  $u_1$  (as depicted in Figure 32d), which would contradict the minimality of escape cell  $u_0$ . One could think, in order to avoid contradicting the minimality of  $u_0$ , that one of the cell of the path  $p$  can be an ancestor of  $\hat{v}''$ , however, as we mentioned before, we assume that  $v_4$  is the only cell in  $p$  that has its north-east out-neighbor as a successor, and  $v_4$  is clearly not an ancestor of  $\hat{v}''$ . In other words,  $v_2$  and  $\hat{v}''$  have an ancestor in common that is above both of them, then, we can extend the arc  $(u_1, u_0)$  into a dividing path with  $v_2$  and  $\hat{v}''$

on different sides, contradicting the minimality of escape cell  $u_0$ . Therefore, choosing  $\hat{v}_3$  is the only remaining option. Since  $\hat{u}_0$  is an in-neighbor of  $\hat{v}_3$ , cell  $\hat{v}_3$  needs two predecessors, that can only be the cells  $\hat{v}_1$  and  $\hat{v}_4 = (k + \tilde{k} + 1, -k + \tilde{k} - 3)$ . Moreover, all the following cells need two predecessors and they only have the north-east and south-east in-neighbors available (Figure 32e):

$$\begin{aligned} \hat{v}_3 = & (k + \tilde{k} + 2, -k + \tilde{k} - 2), (k + \tilde{k} + 1, -k + \tilde{k} - 3), (k + \tilde{k}, -k + \tilde{k} - 4), \\ & (k + \tilde{k} - 1, -k + \tilde{k} - 5), \dots, (i + k + 4, j - k), \tilde{v}_5 = (i + k + 3, j - k - 1). \end{aligned}$$

In particular, the two predecessors of  $\tilde{v}_5$  must be the cells  $\tilde{v}_4$  and  $\tilde{v}_7 = (i + k + 2, j - k - 2)$ , *i.e.*, the arc  $(\tilde{v}_4, \tilde{v}_5)$  is mandatory.

Now, note that  $\tilde{u}_2$  is an in-neighbor of  $\tilde{v}_7$ , then  $\tilde{v}_7$  needs two predecessors that can only be the cells  $\tilde{v}_3$  and  $\tilde{v}_8 = (i + k + 1, j - k - 3)$  (Figure 32f). Finally, by Lemma 6, cell  $\tilde{u}_4 = (i + k + 3, j - k - 2)$  is a successor of cell  $\tilde{u}_2$ , which completes the  $(k + 1)$ -th repetition of the pattern for the induction.

Repeating this argument  $N + 1$  times contradicts that  $g$  has height  $N$ .  $\square$

We deduce that scenario 4 is also impossible.

**Corollary 3.** *There does not exist a crossover gate  $g$  for neighborhood 39 with a downward crossing such that  $v_0$  is a minimum escape cell (scenario 4).*



*Proof.* Direct consequence of Lemmas 5 (equivalence between scenarios 1 and 4) and 8 (impossibility of scenario 1).  $\square$

Let us now prove that scenario 3 is impossible, which is much simpler to establish.

**Lemma 9.** *There does not exist a crossover gate  $g$  for neighborhood 39 with a downward crossing such that  $u_0$  is a minimum escape cell (scenario 3).*



*Proof.* Note that  $u_1$  is an in-neighbor of  $v_0$ , then  $v_0$  needs an extra predecessor. The only two options are cells  $v_2 = (i - 1, j + 1)$  and  $v_3 = (i - 1, j - 1)$ . However, both of them trap the arc  $(u_1, u_0)$  into a dividing path where  $v_1$  and  $v_2$  (or  $v_3$ ) are on opposite sides (see Figure 33). By Corollary 2, there is an escape cell that topples strictly before  $u_0$  and contradicts its minimality.  $\square$

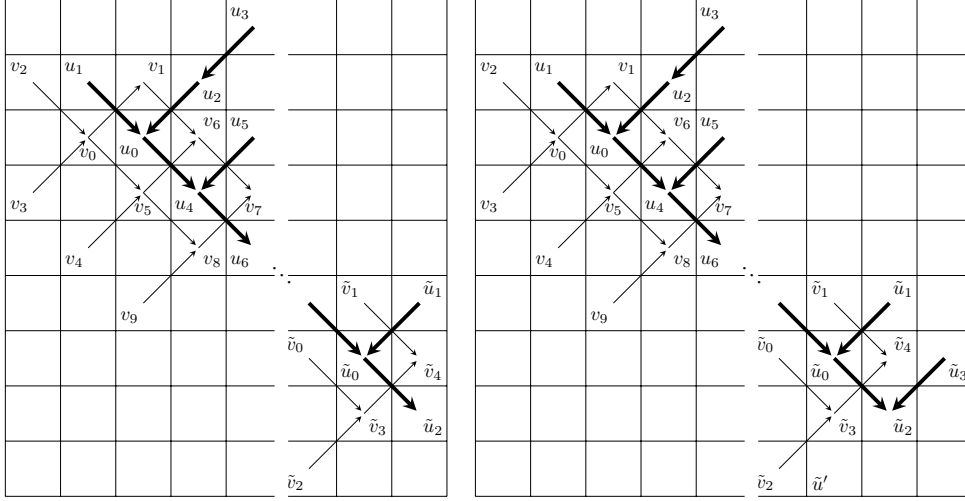
We now turn to the final reasoning for the impossibility of a crossover gate in neighborhood 39. According to the previous results, minimum escape cells from *both* firing graphs must be in scenario 2 (rightward crossing where  $v_1$  is a minimum escape cell). The following lemma expresses the fact that the descendant relationship in neighborhood 39 cannot go in the north-west direction.

**Lemma 10.** *For neighborhood 39, if cell  $(i', j')$  is a descendant of cell  $(i, j)$ , then:*

$$j - i \geq j' - i'.$$

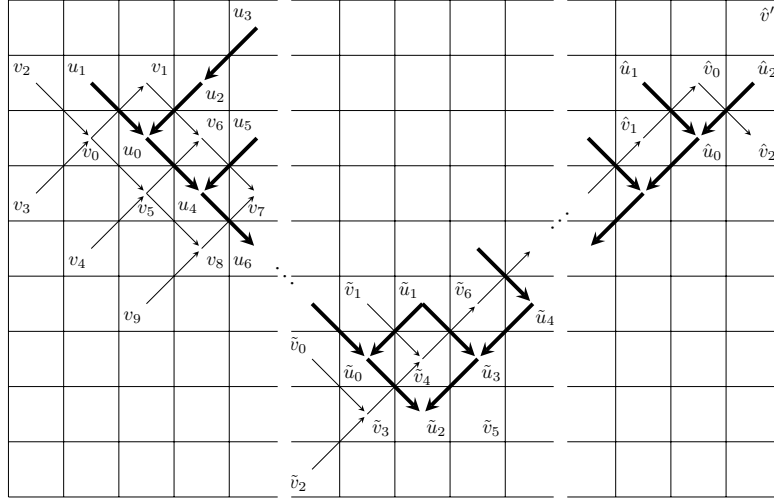
*Proof.* Considering that the neighborhood 39 is defined as:

$$\mathcal{N}_{39} = \{(1, 1), (1, -1), (0, -1), (-1, -1)\}$$



(a) Induction hypothesis

(b)  $\tilde{u}_3$  has to be the extra predecessor of  $\tilde{u}_2$ .



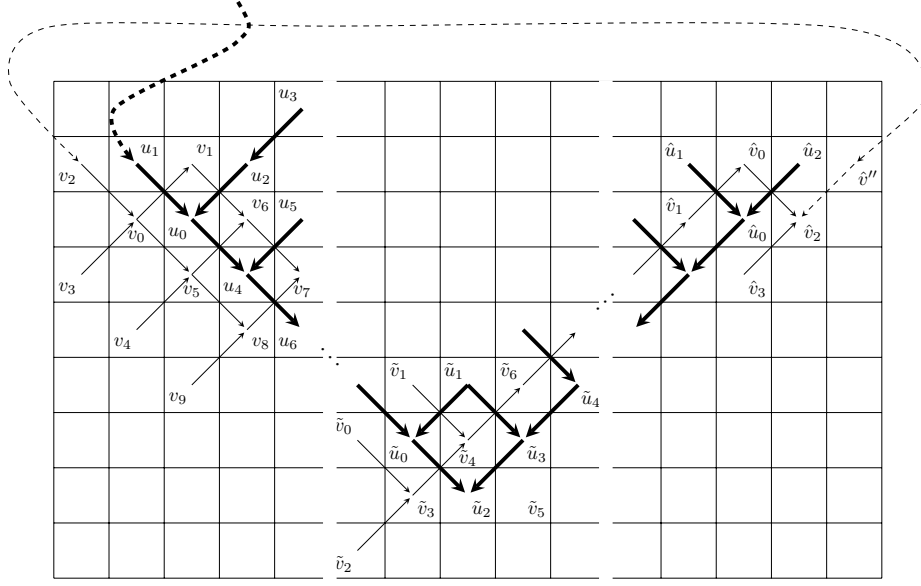
(c)  $\tilde{v}_6$  as successor of  $\tilde{v}_4$ .

**Fig. 32:** Induction step for Lemma 8 for neighborhood 39.

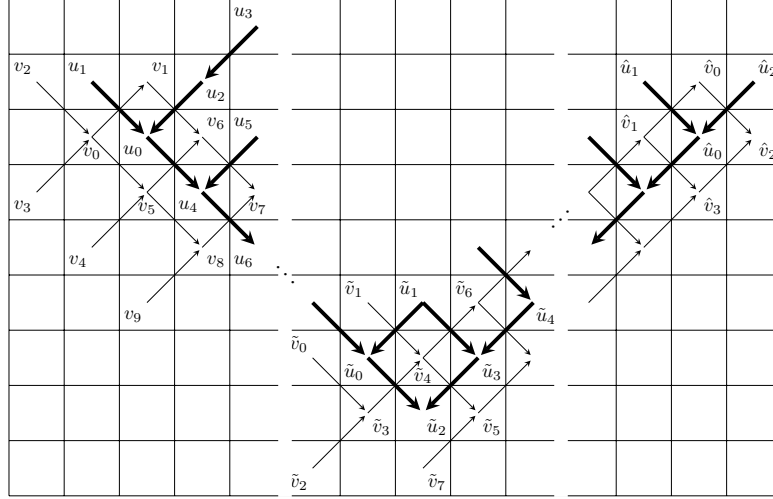
the out-neighbors (potential descendants) of a cell  $(i, j)$  are  $(i + 1, j + 1)$ ,  $(i + 1, j - 1)$ ,  $(i, j - 1)$  and  $(i - 1, j - 1)$ . All of them fulfil the condition, and the lemma follows by transitivity of the order  $\geq$  on integers (recall that the descendent relationship is the transitive closure of the successor relationship).  $\square$

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**Theorem 22.** *Neighborhood 39 does not admit a crossover gate.*



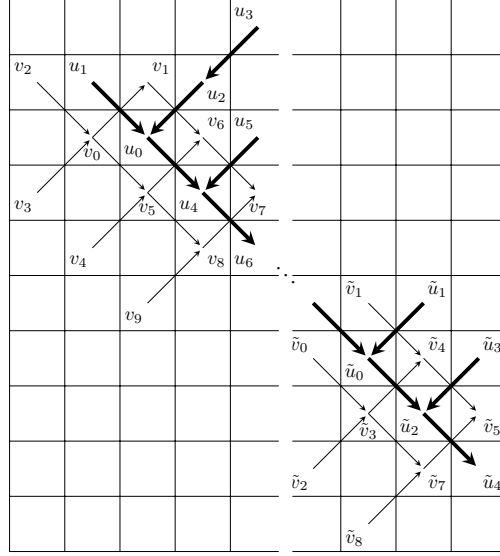
(d) Possible predecessors of  $\hat{v}_2$ .



(e) The arc  $(\tilde{v}_4, \tilde{v}_5)$  is mandatory.

**Fig. 32:** Induction step for Lemma 8 for neighborhood 39.

*Proof.* Considering Corollary 3, Lemma 8 and Lemma 9, if there exists a crossover gate  $g$  with firing graphs  $G_a^g = (V_a, A_a)$  and  $G_b^g = (V_b, A_b)$  for the neighborhood 39, then any minimum escape cell  $(i_a, j_a)$  of  $G_a^g$  (resp.  $(i_b, j_b)$  of  $G_b^g$ ) must appear in a rightward crossing and be toppled by the predecessor  $(i_a - 1, j_a - 1)$  (resp.  $(i_b - 1, j_b - 1)$ ). This



(f) Repetition of the pattern.

**Fig. 32:** Induction step for Lemma 8 for neighborhood 39.

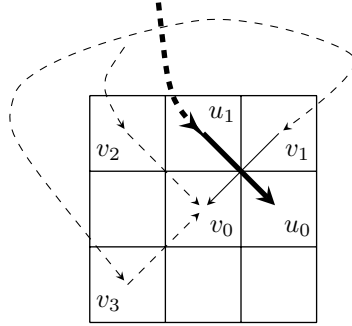


**Fig. 33:** Proof of Lemma 9.

corresponds to scenario 2, which is the situation in *both* firing graphs, since scenarios 1, 3 and 4 in *any* of the two firing graphs has been proven to be impossible.

Notice that the rightward crossing having minimum escape cell  $(i_a, j_a)$  consists in the arcs:

$$((i_a - 1, j_a - 1), (i_a, j_a)) \in A_a \text{ and } ((i_a - 1, j_a), (i_a, j_a - 1)) \in A_b,$$

while the rightward crossing having minimum escape cell  $(i_b, j_b)$  consists in the arcs:

$$((i_b - 1, j_b - 1), (i_b, j_b)) \in A_b \text{ and } ((i_b - 1, j_b), (i_b, j_b - 1)) \in A_a.$$

The orientation of the crossing with a minimum escape cell has been fixed (rightward in scenario 2), therefore it is necessary to consider all global orientations of the crossover gate (in terms of its cardinal endpoints). We study the following four scenarios, which exhaust all possibilities (up to exchange of the two firing graphs).

- 2.1  $G_a^g$  goes from north to south and  $G_b^g$  goes from west to east,
- 2.2  $G_a^g$  goes from north to south and  $G_b^g$  goes from east to west,
- 2.3  $G_a^g$  goes from south to north and  $G_b^g$  goes from west to east,
- 2.4  $G_a^g$  goes from south to north and  $G_b^g$  goes from east to west.

**Scenario 2.1** (Figure 34a). Consider the rightward crossing with minimum escape cell  $(i_a, j_a)$  in  $G_a^g$ . Firing graph  $G_a^g$  must have a path  $p_a$  from the north border to the cell  $(i_a - 1, j_a)$ , and firing graph  $G_b^g$  must have a path  $p_b$  from the west border to the cell  $(i_a - 1, j_a - 1)$ . In this case, either the path  $p_b$  intersects the path  $p_a$  (contradicting the minimality of escape cell  $(i_a, j_a)$  in  $G_a^g$ ), or the path  $p_b$  contains a cell  $(i, j)$  such that  $j - i < j_a - i_a$  (under the dotted diagonal), which cannot reach cell  $(i_a - 1, j_a - 1)$  by Lemma 10 (another contradiction). Consequently this scenario is impossible.

**Scenario 2.2** (Figure 34b). Consider the rightward crossing with minimum escape cell  $(i_b, j_b)$  in  $G_b^g$ . Firing graph  $G_a^g$  must have a path  $p_a$  from the north border to the cell  $(i_b - 1, j_b)$ , and firing graph  $G_b^g$  must have a path  $p_b$  from the east border to the cell  $(i_b - 1, j_b - 1)$ . In this case, either the path  $p_b$  intersects the path  $p_a$  (contradicting the minimality of escape cell  $(i_b, j_b)$  in  $G_b^g$ ), or the path  $p_b$  contains a cell  $(i, j)$  such that  $j - i < j_b - i_b$  (under the dotted diagonal), which cannot reach cell  $(i_b - 1, j_b - 1)$  by Lemma 10 (another contradiction). Consequently this scenario is impossible.

**Scenario 2.3** (Figure 34c). Consider the rightward crossing with minimum escape cell  $(i_b, j_b)$  in  $G_b^g$ . Firing graph  $G_a^g$  must have a path  $p_a$  from the south border to the cell  $(i_b - 1, j_b)$ , and firing graph  $G_b^g$  must have a path  $p_b$  from the west border to the cell  $(i_b - 1, j_b - 1)$ . In this case, either the path  $p_b$  intersects the path  $p_a$  (contradicting the minimality of escape cell  $(i_b, j_b)$  in  $G_b^g$ ), or the path  $p_b$  contains a cell  $(i, j)$  such that  $j - i < j_b - i_b$  (under the dotted diagonal), which cannot reach cell  $(i_b - 1, j_b - 1)$  by Lemma 10 (another contradiction). Consequently this scenario is impossible.

**Scenario 2.4** (Figure 34d). Consider the rightward crossing with minimum escape cell  $(i_a, j_a)$  in  $G_a^g$ . Firing graph  $G_a^g$  must have a path  $p_a$  from the south border to the cell  $(i_a - 1, j_a - 1)$ , and firing graph  $G_b^g$  must have a path  $p_b$  from the east border to the cell  $(i_a - 1, j_a)$ . We proceed by case analysis on the position of cell  $(i_a - 1, j_a - 1)$ . Let  $m$  be the size of  $g$ , and assume without loss of generality that its support is the square of cells  $(i, j)$  verifying  $0 \leq i \leq m - 1$  and  $0 \leq j \leq m - 1$ .

If cell  $(i_a - 1, j_a - 1)$  is such that  $j_a - 1 > i_a - 1$  (it is above the diagonal going from  $(0, 0)$  to  $(m - 1, m - 1)$ ), then the starting cell  $s = (i_s, 0)$  of  $G_a^g$  has to fulfil that  $i_s < 0$  in order to have  $(i_a - 1, j_a - 1)$  as a descendant (by Lemma 10), which is a contradiction because it means that it is outside of the crossover gate  $g$ .

On the other hand, if  $(i_a - 1, j_a - 1)$  is such that  $j_a - 1 < i_a - 1$ , then the ending cell  $n = (i_n, m - 1)$  of  $G_a^g$  has to fulfil that  $i_n > m - 1$  in order to have  $(i_a, j_a)$  as an

ancestor (by Lemma 10), which is a contradiction because it means that it is outside of  $g$ .

Finally, if  $(i_a - 1, j_a - 1)$  is such that  $j_a - 1 = i_a - 1$ , then the starting cell  $e = (m - 1, i_e)$  of  $G_g^b$  has to fulfil that  $i_e > m - 1$  in order to have  $(i_a - 1, j_a)$  as a descendant (by Lemma 10), which is a contradiction because it means that it is outside of  $g$ .  $\square$

## 4 Conclusions and perspectives

### *Crossover gate and prediction problem*

We have performed a systematic study of crossover possibilities for sandpiles on two-dimensional grids  $\mathbb{Z}^2$  within Moore neighborhood, and have encountered cases with is a P-complete prediction problem. The correspondance between the existence of a crossover gate and the P-completeness of  $\mathcal{N}$ -**PRED** is sharp. In Section 3 we have established that, among the 255 subsets of Moore neighborhood (the results are summarized on Table 2):

- 22 do not span  $\mathbb{Z}^2$  and are therefore not relevant to our study,
- 12 admit a crossover gate and have a P-complete prediction problem  $\mathcal{N}$ -**PRED**,
- 99 have a planar sandpile structure and therefore do not admit a crossover gate,
- the 123 remaining neighborhoods (including Moore) do not admit a crossover gate.

We can observe that crossover impossibility is not inherited when considering the subsets of a given neighborhood. The conditions for the possibility or impossibility of crossover may be hard to establish (*e.g.* neighborhood 39 occupies 16 pages). This sheds some more light on the subtlety of embedding computation within sandpiles.

Let us recall that the crossover impossibility is also subject to the radius of the neighborhood. For example it is known by [13] that von Neumann of any radius  $r \geq 2$  admits a crossover gate and therefore has a P-complete  $\mathcal{N}$ -**PRED** problem. It is also known by [23] that there exists a radius  $r_0 \in \mathbb{N}_+$  such that Moore of any radius  $r \geq r_0$  admits a crossover gate.

In a broader perspective, given a neighborhood, it is not known yet whether an algorithmic procedure exists to decide the existence of a crossover gate (the problem is semi-decidable). It is also not known whether there are infinitely many neighborhoods (among those spanning  $\mathbb{Z}^2$ ) that do not admit a crossover gate. Indeed, it may be the case that a bound  $B \in \mathbb{N}$  exists such that every neighborhood  $\mathcal{N}$  of size  $|\mathcal{N}| \geq B$  admit a crossover gate.

The impossibility of a crossover gate means that Banks approach fails at proving P-completeness by reduction from monotone circuit value problem (**MCVP**). Nevertheless, it leave open the possibility of a completely different approach of signal encoding in sandpile dynamics, which is still to be discovered [20]. Neither does it hints at an obvious **NC** algorithm. Indeed, a naive reduction to monotone planar circuit value problem (**MPCVP**) fails because it requires to encode the integer value of the sand content at each cell on multiple bits (at least 3 of them for values 0, 1, 2, 3 and toppling state), so each undirected edge of the sandpile graph would induce a

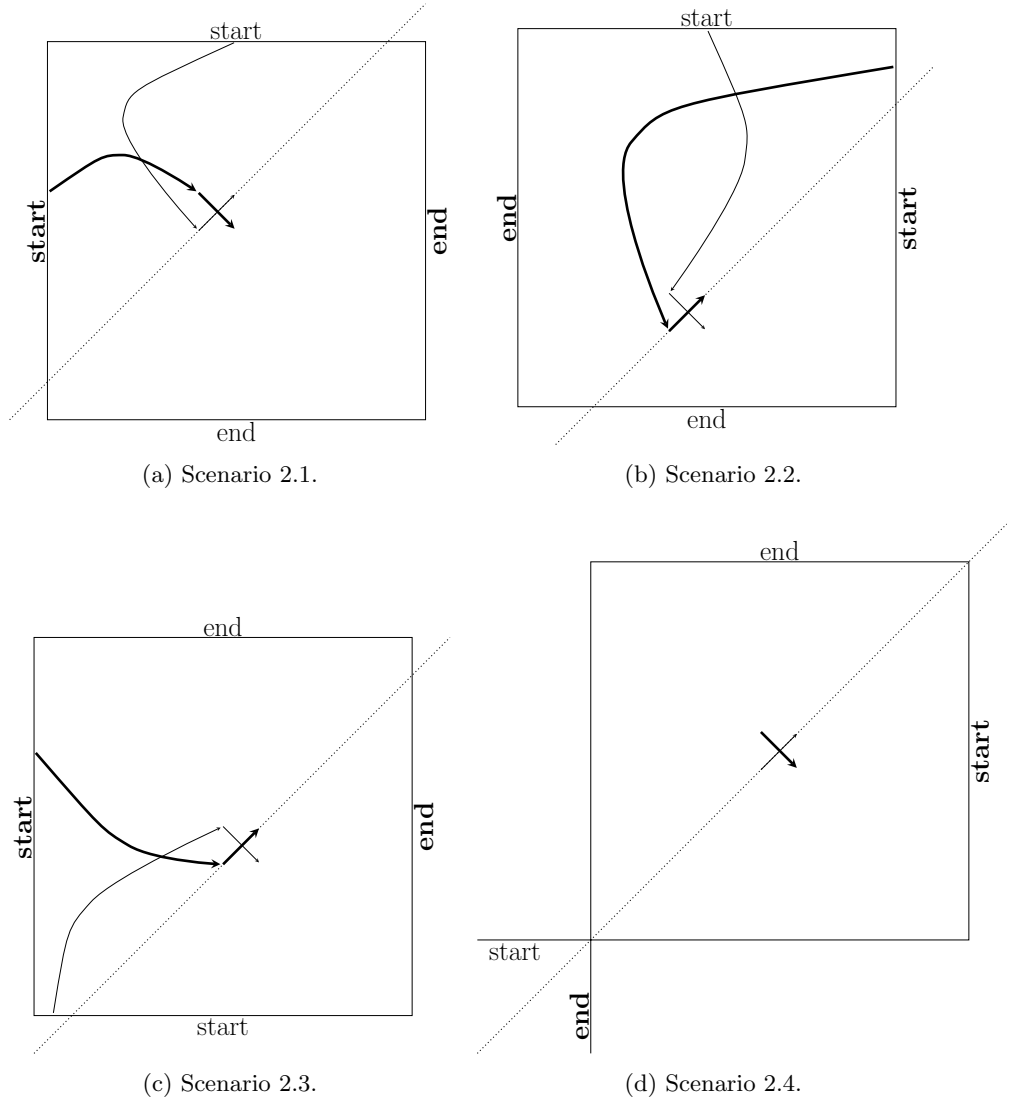


**Fig. 34:** Proof of Theorem 22 for neighborhood 39. Thin lines represent  $G_a^g$ , and thick lines represent  $G_b^g$ . Cells on or under the dotted diagonals cannot have descendants over it by Lemma 10.

non-planar  $K_{3,3}$  (complete bipartite graph of two times 3 vertices). There is no straightforward way of guessing the direction of these edges (in order to make it acyclic and reduce the number of arcs), which would itself require to predict the effective causality among neighboring cells during the dynamics of sandpiles.

We also leave open the graals on von Neumann and Moore.

**Open.** Are  $\mathcal{N}_m$ -**PRED** and  $\mathcal{N}_m$ -**PRED** in NC? P-complete? Neither?

The first alternative requires a finer understanding of the fractal structure of sand-pile dynamics, which is roughly characterized in [28, 29] in terms of Apolonian circle packing. The second alternative would be a great advance in the understanding of how complex two-dimensional von Neumann and Moore neighborhoods are, in an algorithmically effective sense. The third alternative could correspond to an intermediate class, whose existence is proven in [30]. Let us also remark the two problems may not have the same complexity.

### ***Non-uniform sandpiles***

Alternative approaches can be explored by relaxing the definition of the conventional sandpile model. We propose two potential directions, depicted in Figure 35, for which we present the respective crossover gates. On the left hand side, we show an example of a neighborhood of size 3 (neighborhood 67 following our encoding) with a non-uniform distribution of sand grains. Upon reaching 5 sand grains, a cell topples and distributes 1 grain to its east neighbor, 2 grains to its south-east neighbor and 2 grains to its south-west neighbor. Defining all the necessary circuitry for applying the Banks approach for this neighborhood is not particularly challenging, therefore one can prove that this setup has a P-complete prediction problem. On the other side, the illustration on the right introduces a sandpile model where cells may have one of three different neighborhoods. Here the neighborhoods are of size 1 (neighborhoods 1, 2 and 64) then the toppling threshold is also 1. Although the combination of these three neighborhoods allows to design a crossover gate, we conjecture that it is not possible to conceive an *and* gate, leading to the failure of Banks approach.

It is worth noting that the neighborhoods 1, 2, and 64 can be considered as a decomposition of neighborhood 67 or, similarly, one can say that neighborhood 67 can be composed with neighborhoods 1, 2, and 64. This observation, coupled with the existence of a crossover gate for both previous examples, may prompt to exploring how this notion of composition and decomposition could give insights into whether a sandpile model admits a crossover gate.

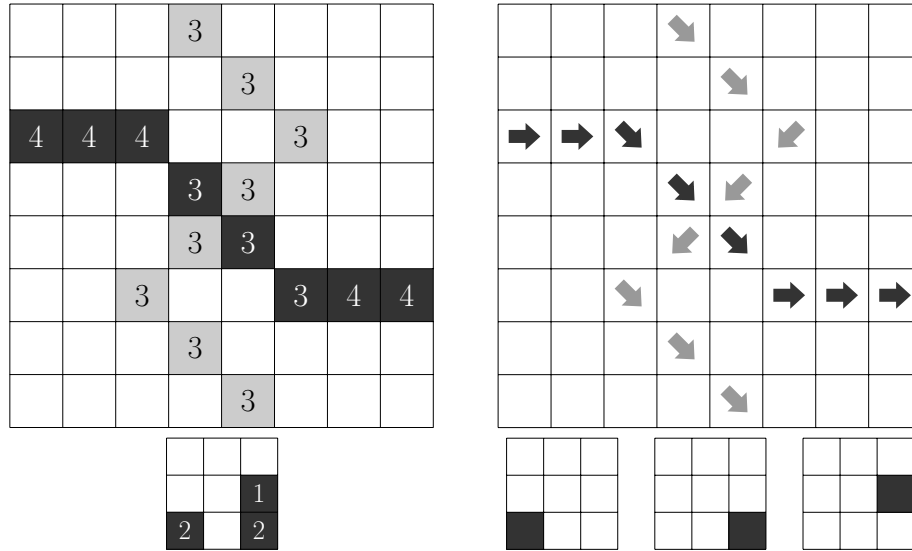


**Fig. 35:** Left: Example of a crossover gate using a non-uniform distribution of grains. Note that in this case the threshold is  $\theta = 5$ . Right: Example of a crossover for a non-uniform distribution of neighborhood using three neighborhoods of size 1. Here the threshold is  $\theta = 1$  and the arrows point at the out-neighbor of the cell. The blank cells can have any of the three neighborhoods. All the cells have 0 grains.

## Declarations

### Ethical Approval

Not applicable.

### Competing interests

Not applicable.

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### Availability of data and materials

Data Availability Statement: No Data associated in the manuscript.

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