Graph labelings derived from models in distributed computing

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Abstract. We discuss eleven well-known basic models of distributed computing: four message-passing models that differ by the (non-)existence of port-numbers and a hierarchy of seven local computations models. In each of these models, we study the computational complexity of the decision problem whether the leader election and/or naming problem can be solved on a given network. It is already known that this problem is solvable in polynomial time for two models and co-NP-complete for another one. Here, we settle the computational complexity for the remaining eight problems by showing co-NP-completeness. The results for six models and the already known co-NP-completeness result follow from a more general result on graph labelings.

1 Introduction

In distributed computing, one can find a wide variety of models of communication. These models reflect different system architectures, different levels of synchronization and different levels of abstraction. In this paper we consider eleven well-known basic models that satisfy the following two underlying assumptions. Firstly, a distributed system is represented by a simple (i.e., without loops or multiple edges), connected, undirected graph. Its vertices represent the processors, and its edges represent direct communication links. Secondly, in a distributed algorithm, all the processors execute the same code to solve some problem and they do not have initial identifiers.

The eleven basic models can be divided into four *message-passing* models [6, 15, 17] and seven *local computations* models [1, 4, 5, 12, 13]. In a message-passing model, processors communicate by sending and receiving messages. In a local computations model, communication between processors is achieved thanks to synchronization (encoded by local relabeling rules) between neighboring processors.

Understanding the computational power of various models enhances our understanding of basic distributed algorithms. For this purpose a number of standard problems in distributed computing are studied. The election problem is one of the paradigms of the theory of distributed computing. In our setting, a distributed algorithm solves the election problem if it always terminates and in the final configuration exactly one processor is marked as *elected* and all the other processors are *non-elected.* Elections constitute a building block of many other distributed algorithms, since the elected vertex can be subsequently used to make centralized decisions. A second important problem in distributed computing is the naming problem. Here, the aim is to arrive at a final configuration where all processors have been assigned unique identities. Again this is an essential prerequisite to many other distributed algorithms that only work correctly under the assumption that all processors can be unambiguously identified. For a survey on distributed algorithms we refer to [14].

OUR RESULTS. Whether the naming or election problem can be solved on a given graph depends on the properties of the considered model. If it is possible to solve the election (naming) problem we call the graph a *solution graph* for the election (naming) problem. It is a natural question to ask how hard it is to check whether a given graph is a solution graph in a certain model. For two models this problem is known to be polynomially solvable [2] and for one model it is co-NP-complete [16]. What about the computational complexity of this problem for the other models? In this paper we solve this question by showing that this decision problem is co-NP-complete for all remaining models.

The paper is organized as follows. In Section 2 we define the necessary graph terminology. To obtain our results we translate known characterizations [1, 4–7, 12, 13, 15, 17] of solution graphs in terms of graph labelings. This is shown in Section 3 for the message-passing models and in Section 4 for the local computations models. In Section 5 we introduce a new kind of labeling that does not correspond to any model of distributed computing but that enables us to present a simpler co-NP-completeness proof for seven basic models including the already known model in [16]. In Section 6 we give the results for the remaining two models.

2 Preliminaries

For graph terminology not defined below we refer to [3]. A *labeling* of a graph $G = (V_G, E_G)$ is a mapping $\ell : V_G \to \{1, 2, 3, \ldots, \}$. For a set $S \subseteq V_G$ we use the shorthand notation $\ell(S)$ to denote the image set of S under ℓ, i.e., $\ell(S) = \{\ell(u) \mid u \in S\}$. A *labeling* ℓ of G is called *proper* if $|\ell(V_G)| < |V_G|$. For any *label* $i \geq 1$, the set $\ell^{-1}(i)$ is equal to $\{u \in V_G \mid \ell(u) = i\}$. The subgraph of G induced by a subset $S \subseteq V_G$ is denoted by $G[S]$. For a label $i \geq 1$ we write $G[i] = G[\ell^{-1}(i)]$. For two labels i, j, we let $G[i, j]$ be the bipartite graph obtained from $G[\ell^{-1}(i) \cup \ell^{-1}(j)]$ by deleting all edges $\{u, v\}$ with $\ell(u) = \ell(v) = i$ or with $\ell(u) = \ell(v) = j$.

For a vertex $u \in V_G$ in a graph $G = (V_G, E_G)$, we denote its *neighborhood* by $N_G(u) = \{v \mid [u, v] \in E_G\}$. A graph is *regular*, if all its vertices have the same number k of neighbors (i.e. are of *degree* $\deg_G(u) = k$), in that case we also say that the graph is k-regular. A graph is *regular bipartite* if it is regular and bipartite. A graph is *semi-regular bipartite* if it is bipartite and the vertices of one class of the bipartition are of degree k and all others are of degree l, in that case we also say that the graph is (k, l)*-regular bipartite*. In our context a *perfect matching* is a $(1, 1)$-regular bipartite graph.

3 Message-passing models

In [15–17], Yamashita and Kameda study four message-passing models. In the *port-to-port* model, each processor can send different messages to different neighbors (by having access to unique *port-numbers* that distinguish between neighbors), and each processor knows the neighbor each receiving message is coming from (again by using the port-numbers). In the *broadcast-to-mailbox* model, port-numbers do not exist. A processor can only send a message to all of its neighbors and all receiving messages arrive in a *mailbox*, so it never knows their senders. The two mixed models are called the *broadcast-to-port* model and the *port-to-mailbox* model. There exists an election (or naming) algorithm for a graph G if and only if the algorithm solves the problem on G whatever the port-numbers are.

In [17], Yamashita and Kameda characterize these four models: a graph G is a solution graph for the election and naming problem in the port-to-port model if and only if G does not have a proper *symmetric regular labeling*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular and contains a perfect matching if its vertices have odd degree, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is regular bipartite.

A graph G is a solution graph for the election and naming problem in the port-to-mailbox model if and only if G does not have a proper *regular labeling*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is regular bipartite.

A graph G is a solution graph for the election and naming problem in the broadcast-to-mailbox and the broadcast-to-port model if and only if G does not have a proper *semi-regular labeling*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is semi-regular bipartite.

In [1, 6], different characterizations for these models are obtained (based on fibrations and coverings of directed graphs). The problem of deciding whether a graph G is a solution graph for the election and naming problem in the port-to-port model is co-NP-complete [16]. On the other hand, in [2], it is shown that the problem of deciding whether a graph G is a solution problem for the election and naming problem is polynomially solvable in the broadcast-to-mailbox and the broadcast-to-port model (by computing the degree refinement of G).

4 Local computations models

In the local computations models, a computation step can be described by the application of some *local relabeling rule* that enables the modification of the states of the different vertices involved in the synchronization. Two local computation models are different in the types of relabeling rules that they allow, see Figure 1. In models (5), (6) and (7) of Figure 1, a computation step involves some synchronization between one vertex and all its neighbors, whereas in models (1), (2), (3) and (4), a computation step involves some synchronization between two neighbors.

Mazurkiewicz [12] characterizes model (7) of Figure 1: a graph G is a solution graph for the election and naming problem if and only if G does not have a proper *perfect-regular coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is empty, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else is a perfect matching.

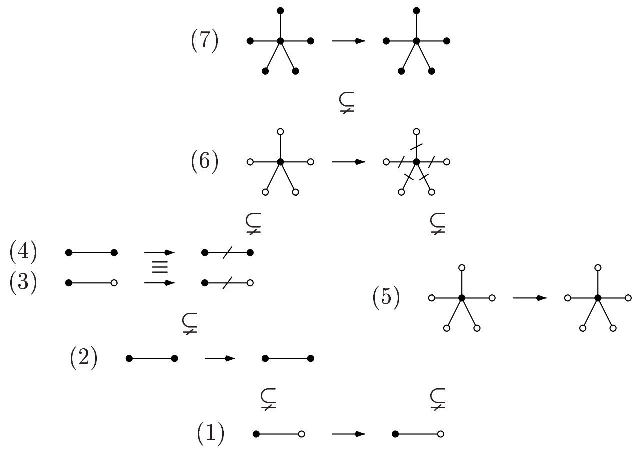

Fig. 1. A hierarchy of local computations models. Labels of black vertices can change when the rule is applied. Labels of white vertices only enable to apply the relabeling rule but do not change. A relabeling rule can modify edge labels only in models (3), (4) and (6). If $r_i \subsetneq r_j$ for rules r_i and r_j then r_j can simulate r_i but not vice versa, i.e., r_j has a greater computational power than r_i. If $r_i \equiv r_j$ then r_i and r_j have the same computational power. Otherwise, r_i and r_j are incomparable.

Boldi et al. [1] characterize model (5) of Figure 1: a graph G is a solution graph for the naming problem if and only if G does not have a proper *semi-regular coloring*, i.e., a proper labeling ℓ of G such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is empty, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is semi-regular bipartite.

[5] characterizes the models (3), (4) and (6) of Figure 1: a graph G is a solution graph for the election and the naming problem in each of these models if and only if G does not have a proper *regular coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is empty, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is regular bipartite.

We note that Mazurkiewicz [13] given an equivalent characterization of model (4) in terms of equivalence relations over vertices and edges. The characterizations for model (6) can also be obtained from [1].

[4] characterizes model (2) of Figure 1: a graph G is a solution graph for the election and naming problem if and only if G does not have a proper *pseudo-regular coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is empty, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else contains a perfect matching.

[7] characterizes model (1) of Figure 1: a graph G is a solution graph for the naming problem if and only if G does not admit any proper *connected coloring*, i.e., a proper labeling ℓ such that

(i) for all $i \in \ell(V_G)$, $G[i]$ is empty, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else has minimum degree one.

We note that the hierarchy in Figure 1 is also reflected by the labelings, e.g., a perfect-regular coloring is also a regular coloring, and so on.

5 Pseudo-regular labelings

We call a labeling ℓ of a graph G a *pseudo-regular labeling* if

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else contains a perfect matching.

In this section we prove that the problem whether a given graph G has a proper pseudo-regular labeling is NP-complete. The following observation is useful.

Observation 1 *Let ℓ be a pseudo-regular labeling of a connected graph G. Then $|\ell^{-1}(i)| = \frac{|V_G|}{|\ell(V_G)|}$ for all $i \in \ell(V_G)$.*

Let $G = (V_G, E_G)$ and $H = (V_H, V_G)$ be two graphs. We write $V_H = \{1, 2, \ldots, |V_H|\}$. For a mapping $f : V_G \to V_H$ and a set $S \subseteq V_G$, we write $f(S) = \{f(u) \mid u \in S\}$. A *graph homomorphism* from G to H is a vertex mapping $f : V_G \to V_H$ satisfying the property that for any edge $[u, v]$ in E_G, we have $[f(u), f(v)]$ in E_H, in other words, $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V_G$. A homomorphism f from G to H that induces a one-to-one mapping on the neighborhood of every vertex is called *locally bijective*, i.e., for all $u \in V_G$ it satisfies $f(N_G(u)) = N_H(f(u))$ and $|N_G(u)| = |N_H(f(u))|$. In that case we write $G \xrightarrow{B} H$, and call the vertices of H *colors* of G. Sometimes, we also say that the labels $\ell(i)$ of a labeling ℓ of G are *colors* of G.

The H-Cover problem asks whether there exists a locally bijective homomorphism from an instance graph G to a fixed graph H. In our NP-completeness proof we use reduction from the K-Cover problem, where K is the graph obtained after deleting an edge in the complete graph K_5 on five vertices. The K-Cover problem is NP-complete [11]. Note that the two non-adjacent vertices have degree three. The other three vertices are adjacent to two vertices of degree three and two vertices of degree four. Then the following observation immediately follows from the definition of a locally bijective homomorphism.

Observation 2 *Let G be a graph with $G \xrightarrow{B} K$. Then $V_G = B_1 \cup B_2$ for two blocks B_1 and B_2 with $|B_1| = 2k$ and $|B_2| = 3k$ for some $k \geq 1$ such that*

- *for all $u \in B_1$, $|N_G(u) \cap B_1| = 0$ and $|N_G(u) \cap B_2| = 3$*
- *for all $u \in B_2$, $|N_G(u) \cap B_1| = 2$ and $|N_G(u) \cap B_2| = 2$.*

Since the conditions in Observation 2 can be checked in polynomial time, we assume without loss of generality that any instance graph G of the K-COVER problem satisfies these conditions.

For our NP-completeness structure we modify an instance graph G of the K-COVER as follows. Let u and v be vertices of G with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. We replace the edge $[u, v]$ by a chain of $q \geq 1$ "diamonds" as described in Figure 2. We call the resulting graph G' a *diamond graph* of G *with respect to* the edge $[u, v]$. For $i = 1, \ldots, q$, the subgraph $D_i = G[\{a_i, b_i, c_i, d_i, e_i\}]$ is called a *diamond* of G'. The next lemma shows among others

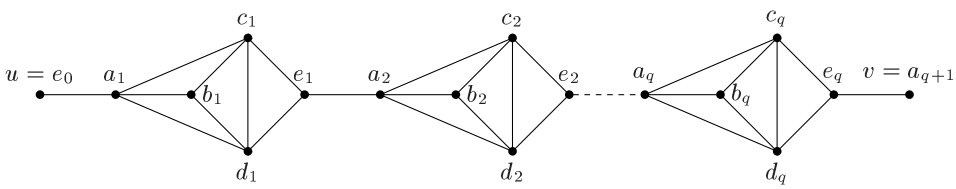

Fig. 2. The chain of q diamonds that replace the edge $[u, v]$.

that a pseudo-regular labeling is injective on the neighborhood of any vertex in a diamond. Its proof involves a case analysis and will be presented in the journal version of our paper.

Lemma 3. *Let G be a graph on $5k$ vertices that contains adjacent vertices u, v with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. Let G' be a diamond graph of G with respect to $[u, v]$ that has diamonds $D_1, \ldots D_q$, where $q > k+2$ and $q+k$ is a prime number. If ℓ is a proper pseudo-regular labeling of G', then $|\ell(V_{D_i})| = 5$ and $\ell(e_{i-1}) \notin \ell(D_i \backslash \{e_i\})$ for all $1 \leq i \leq q$.*

The following lemma is a key result.

Lemma 4. *Let G be a graph on $5k$ vertices that contains adjacent vertices u, v with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. Let G' be a diamond graph of G with respect to $[u, v]$ that has diamonds $D_1, \ldots D_q$, where $q > k+2$ and $q+k$ is a prime number. If ℓ is a proper pseudo-regular labeling of G' then $|\ell(V_{G'})| = 5$.*

Proof. We write $p = q + k$. Then $|V_{G'}| = 5p$ and p is a prime number. Hence we find that $|\ell(V_{G'})| = 5$ or $|\ell(V_{G'})| = p$, due to Observation 1.

Suppose $|\ell(V_{G'})| = p > 5$. By our choice of q, there exist a vertex u in a diamond D_i with the same color as a vertex v in a diamond D_j. By Lemma 3, we may assume that $i < j$. We choose u and v such that there do not exist two vertices in $G[D_i \cup \ldots \cup D_{j-1}]$ having the same color. By Lemma 3, we can write $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, $\ell(d_i) = 4$ and $\ell(e_i) = 5$, and we find that $\ell(e_{i-1}) \notin \{1, 2, 3, 4\}$. If $\ell(e_{i-1}) = 5$, then $\ell(a_{i+1}) = 5$ and consequently $|\ell(V_{G'})| = 5 < p$, so we write $\ell(e_{i-1}) = 6$.

By Observation 2 and the construction of G', every vertex of G has either degree 3 or 4. Note that, for each x in G' with $\ell(x) = 1$ (respectively $\ell(x) = 3$, $\ell(x) = 4$), we have that $\{2, 3, 4, 6\} \subseteq \ell(N_{G'}(x))$ (respectively $\{1, 2, 4, 5\} \subseteq \ell(N_{G'}(x))$, $\{1, 2, 3, 5\} \subseteq \ell(N_{G'}(x))$). Consequently, each vertex x with $\ell(x) \in \{1, 3, 4\}$ has $\deg_{G'}(x) = 4$.

By our choice of D_i and D_j, vertex a_{i+1} belongs to some diamond. By Lemma 3, we know that $|\ell(N_{G'}(a_{i+1}))| = 4$. Then each vertex x with $\ell(x) = \ell(a_{i+1})$ has $\deg_{G'}(x) = 4$.

Suppose now that there exists a vertex y such that $\deg_{G'}(y) = 4$ and $\ell(y) = 2$ (respectively $\ell(y) = 5$). Then $\ell(N_{G'}(y)) = \{1, 3, 4\}$ (respectively $\ell(N_{G'}(y)) = \{3, 4, \ell(a_{i+1})\}$). Then y has three neighbors of degree four and this is not possible due to Observation 2. Consequently, each vertex y with $\ell(y) \in \{2, 5\}$ has $\deg_{G'}(y) = 3$.

We show that $1 \notin \ell(D_j)$. Suppose $\ell(a_j) = 1$. From our choice of D_i and D_j, we know that $\ell(e_{j-1}) \notin \{2, 3, 4\}$. Then $\ell(\{b_j, c_j, d_j\}) = \{2, 3, 4\}$ and $\ell(e_{j-1}) = 6$. Then $\ell(V_G) = \ell(D_i \cup \ldots \cup D_{j-1})$ and since all colors are different on diamonds $D_i, D_{i+1}, \ldots, D_{j-1}$, we find that $p = |\ell(V_G)| = 5(j - i)$. Since p is a prime number not equal to 5, this is not possible. We already know that $1 \notin \ell(\{b_j, e_j\})$ since $\deg_{G'}(b_j) = \deg_{G'}(e_j) = 3$. Suppose $\ell(c_j) = 1$ (respectively $\ell(d_j) = 1$). Then $\ell(d_j) \in \{3, 4\}$ (respectively $\ell(c_j) \in \{3, 4\}$) and $\ell(\{b_j, e_j\}) = \{2, 6\}$. Then a vertex with color in $\{3, 4\}$ is adjacent to a vertex with color 6. This is not possible.

We show that $2 \notin \ell(D_j)$. We already know that the only vertices in D_j that can be mapped to 2 are b_j and e_j in D_j. If $\ell(b_j) = 2$, then $1 \in \ell(\{a_j, c_j, d_j\})$. If $\ell(e_j) = 2$, then either $1 \in \ell(\{c_j, d_j\})$ or $\ell(\{c_j, d_j\}) = \{3, 4\}$ and in the second case $\ell(a_j) = 1$.

We show that $3 \notin \ell(D_j)$. We already know that only vertices a_j, c_j, d_j can be mapped to 3. If $\ell(a_j) = 3$ then 1, which does not occur on D_j, must be the color of $\ell(e_{j-1})$. This is not possible due to our choice of D_i and D_j. In the other two cases we find that $1 \in \ell(D_j)$. By symmetry, we deduce that $4 \notin \ell(D_j)$.

Finally, we show that $5 \notin \ell(D_j)$. We already know that only vertices b_j and e_j can be mapped to 5. In both cases, at least one of the colors $3, 4$ is a color of a vertex in D_j. This finishes the proof of the lemma. □

Lemma 5. *Let G be a graph that contains adjacent vertices u, v with $\deg_G(u) = 3$ and $\deg_G(v) = 4$. Let G' be a diamond graph of G with respect to (u, v). Then $G \xrightarrow{B} K$ if and only if $G' \xrightarrow{B} K$.*

Proof. We denote the vertices of K by $1, 2, 3, 4, 5$ and its edges by $[1, 2]$, $[1, 3]$, $[1, 4]$, $[1, 5]$, $[2, 3]$, $[2, 4]$, $[3, 4]$, $[3, 5]$, $[4, 5]$. Suppose $G \xrightarrow{B} K$. Without loss of generality we assume that u has color 5 and v has color 1. Then we assign color 1 to all a_i, color 2 all b_i, color 3 to all c_i, color 4 to all d_i and color 5 to all e_i.

Suppose $G' \xrightarrow{B} K$. The restriction of any locally bijective homomorphism $f' : V_{G'} \to V_K$ to V_G is a witness for $G \xrightarrow{B} K$. □

Theorem 1. *The problems that ask whether a given graph G allows a proper pseudo-regular coloring, a proper pseudo-regular labeling, a proper regular coloring, a proper regular labeling, a proper symmetric regular labeling, or a proper perfect-regular coloring, respectively, are* NP*-complete.*

Proof. Obviously, all problems are in NP. We use reduction from the NP-complete problem K-Cover [11]. Let G be an instance graph of this problem. By Observation 2, graph G has $5k$ vertices for some $k \geq 1$ and contains adjacent vertices u of degree three and v of degree four. We construct the diamond graph G' with respect to $[u, v]$ that has q diamonds $D_1, \ldots, D_q$, where we chose q such that $q > k + 2$ and $p = q + k$ is a prime number. By Lemma 5 we can consider G' as our instance graph for the K-Cover problem.

Any locally bijective homomorphism is a proper perfect-regular coloring, which is a regular coloring, which is a symmetric regular labeling, which is a regular labeling, which is a pseudo-regular labeling, and any regular coloring is a pseudo-regular coloring, which is a pseudo-regular labeling.

So we are left to show that a proper pseudo-regular labeling of G' implies that $G' \xrightarrow{B} K$. Suppose G' allows a proper pseudo-regular labeling ℓ. By Lemma 3, $|\ell(D_1)| = 5$. Let $\ell(a_1) = 1$, $\ell(b_1) = 2$, $\ell(c_1) = 3$, $\ell(d_1) = 4$ and $\ell(e_1) = 5$. By Lemma 3, $\ell(e_0) \notin \{1, 2, 3, 4\}$. Since $|\ell(V_G)| = 5$ due to Lemma 4, we then find that $\ell(e_0) = 5$. This means that ℓ defines a locally bijective homomorphism from G to K. □

6 Connected colorings and semi-regular colorings

A *hypergraph* $(Q, \mathcal{S})$ is a set $Q = \{q_1, \dots, q_m\}$ together with a set $\mathcal{S} = \{S_1, \dots, S_n\}$ of subsets of Q. A 2-*coloring* of a hypergraph $(Q, \mathcal{S})$ is a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \neq \emptyset$ and $Q_2 \cap S_j \neq \emptyset$ for $1 \leq j \leq n$. In our proofs we use reduction from the following, well-known NP-complete problem (cf. [9]).

Hypergraph 2-Colorability
Instance: A hypergraph $(Q, \mathcal{S})$.
Question: Does $(Q, \mathcal{S})$ have a 2-coloring?

With a hypergraph $(Q, \mathcal{S})$ we associate its *incidence graph* I, which is a bipartite graph on $Q \cup \mathcal{S}$, where $[q, S]$ forms an edge if and only if $q \in S$. From the incidence graph I we act as follows. Let C_k denote a cycle on k vertices. First we make a copy S' for each $S \in \mathcal{S}$. We add edges (S', q) if and only if $q \in S$. Let $\mathcal{S}' = \{S'_1, \dots, S'_n\}$. Then we glue a cycle C_{q_i} isomorphic to a C_{6i-3} in I by vertex q_i for $1 \leq i \leq m$. We add a new vertex v and edges from v to all vertices in $\mathcal{S}$. Finally we glue a cycle C_v isomorphic to C_{6m+3} in I by v. We call the resulting graph I^* the C_3-*minimizer* of $(Q, \mathcal{S})$. See Figure 3 for an example.

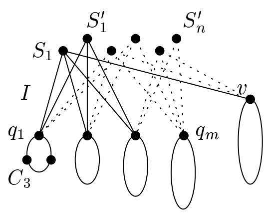

Fig. 3. Example of a C_3-minimizer I^* of a hypergraph $(Q, \mathcal{S})$.

The proof of the following lemma will be included in the journal version.

Lemma 6. *Let I^* be the C_3-minimizer of a hypergraph $(Q, \mathcal{S})$ with $S_j \neq S_k$ for all j, k. If ℓ is a proper connected coloring of I^* then $|\ell(V_{I^*})| = 3$.*

Theorem 2. *The problem that asks whether a given graph G has a proper connected coloring is NP-complete.*

Proof. Obviously, this problem is in NP. We prove NP-completeness by reduction from the Hypergraph 2-Colorability problem. Let $(Q, \mathcal{S})$ be a hypergraph. We assume without loss of generality that $S_j \neq S_k$ for $j \neq k$. We claim that $(Q, \mathcal{S})$ has a 2-coloring if and only if its C_3-minimizer I^* admits a proper connected coloring.

Suppose $(Q, \mathcal{S})$ has a 2-coloring $Q_1 \cup Q_2$. Define $\ell(v) = 1$, $\ell(S) = 2$ for all $S \in \mathcal{S} \cup \mathcal{S}'$, $\ell(q) = 1$ for all $q \in Q_1$ and $\ell(q) = 3$ for all $q \in Q_2$. Finish the coloring in the obvious way.

Suppose I^* has a proper connected coloring ℓ. By Lemma 6 we find $|\ell(V_{I^*})| = 3$. Let $\ell(v) = 1$. Then $\ell(S_j) \in \{2, 3\}$ for all j. If $\ell(S'_j) = 1$ for some j, then S'_j needs a neighbor of color 2 and a neighbor of color 3, both are adjacent to S_j. Hence, $\ell(S'_j) \in \{2, 3\}$ for all j. We define $Q_1 = \{q \in Q \mid \ell(q) = 1\}$ and $Q_2 = Q \backslash Q_1$. Since each S'_j needs at least two neighbors with different colors and at least one neighbor with color 1, the partition $Q_1 \cup Q_2$ is a 2-coloring of $(Q, \mathcal{S})$. □

The proof of Theorem 3 uses arguments of the proofs of Theorem 1 and Theorem 2 but the NP-completeness construction is more involved. We postpone it to the journal version.

Theorem 3. *The problem that asks whether a given graph G has a proper semi-regular coloring is* NP*-complete.*

7 Conclusions

By Theorems 1, 2 and 3 we have determined the computational complexity of the question whether the election and/or naming problem can be solved on a given graph in eleven different models of distributed computing that all have been studied in the literature.

Corollary 1. *It is co-*NP*-complete to decide if on a given graph G we can solve*

(a) the election problem in the models described in Sections 3 and 4 except for the broadcast-to-port model, the broadcast-to-mailbox model and models (1), (5) of Figure 1;

(b) the naming problem in the models described in Sections 3 and 4 except for the broadcast-to-port and broadcast-to-mailbox model.

As a matter of fact the above decision problem is co-NP-complete for the election problem in model (5) as well. We need to modify the corresponding labeling a little. Showing how to do this is postponed to the journal version. For the election problem in model (1) a characterization in terms of a graph labeling is still unknown.

We note that the problem that asks whether a given connected graph G has a proper covering is equivalent to the problem that asks whether $G \xrightarrow{B} H$ for some connected graph H with $|V_H| < |V_G|$. A graph homomorphism f from G to H satisfying $f(N_G(u)) = N_H(f(u))$ for all $u \in V_G$ is called *locally surjective*. If such a homomorphism exists, we write $G \xrightarrow{S} H$. The problem that asks whether a connected graph G has a proper connected coloring is equivalent to the problem that asks whether $G \xrightarrow{S} H$ for some connected graph H with $|V_H| < |V_G|$. Let $\mathcal{C}$ denote the set of connected graphs (up to isomorphism). In [8] it has been proven that $(\mathcal{C}, \xrightarrow{B})$ and $(\mathcal{C}, \xrightarrow{S})$ are partial orders. Theorem 1 and 2 imply that it is co-NP-complete to check whether a graph is minimal in $(\mathcal{C}, \xrightarrow{B})$ and $(\mathcal{C}, \xrightarrow{S})$, respectively. Also the other studied graph labeling problems can easily be formulated as problems that ask whether there exist a homomorphism f, that satisfies a few extra constraints, from a given graph G to a smaller graph H. In the future we will study the relations between these *constrained* homomorphisms more carefully.

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Appendix A

Here is the proof of Lemma 3.

Proof. We write $p = q + k$. Then $|V_{G'}| = 5p$ and p is a prime number. Hence we find that $|\ell(V_{G'})| = 5$ or $|\ell(V_{G'})| = p \geq 5$, due to Observation 1. Let D_i be a diamond. Recall that $u = e_0$ and $v = a_{q+1}$ have been defined. We prove the lemma by a sequence of claims. Let $\ell(a_i) = 1$.

Claim 1. We may assume that $\ell(b_i) = 2$.

We prove this claim as follows. Suppose $\ell(b_i) = 1$. If $\ell(c_i) = \ell(d_i) = 1$, then $|\ell(V_{G'})| = 1 < 5$, which is not possible.

Suppose $\ell(c_i) = 1$. Then $\ell(d_i) \neq 1$. We assume $\ell(d_i) = 2$. Since $G'[1, 2]$ contains a perfect matching, we then find that $\ell(e_{i-1}) = \ell(e_i) = 2$. Then $|\ell(V_{G'})| = 2 < 5$. Hence $\ell(c_i) \neq 1$. Say $\ell(c_i) = 2$.

If $\ell(d_i) = 1$ then we return to the previous case. If $\ell(d_i) = 2$ then $\ell(e_i) = 1$ or $\ell(e_i) = 2$, as otherwise $G'[2, \ell(e_i)]$ does not contain a perfect matching. In both cases, however, $|\ell(V_{G'})| = 2 < 5$. Suppose $\ell(d_i) \notin \{1, 2\}$, say $\ell(d_i) = 3$. If $\ell(e_{i-1}) = 1$ then $G'[1]$ is not regular. If $\ell(e_{i-1}) = i$ with $i \in \{2, 3\}$, then $G'[1, 5-i]$ does not contain a perfect matching. Suppose $\ell(e_{i-1}) \notin \{1, 2, 3\}$, say $\ell(e_{i-1}) = 4$. Then $G'[1, 4]$ does not contain a perfect matching. This proves Claim 1. From now on we assume that $\ell(a_i) = 1$ and $\ell(b_i) = 2$.

Claim 2. We may assume that $\ell(c_i) = 3$.

We prove this claim as follows. Suppose $\ell(c_i) = 1$. Suppose $\ell(d_i) = 1$. Since $G'[1, 2]$ has a perfect matching, $\ell(e_{i-1}) = \ell(e_i) = 2$. Then $|\ell(V_{G'})| = 2 < 5$. Suppose $\ell(d_i) = 2$. Then $G'[2]$ is 1-regular, and hence $\ell(e_i) = 1$. Then $|\ell(V_{G'})| = 2 < 5$. Suppose $\ell(d_i) \notin \{1, 2\}$, say $\ell(d_i) = 3$. If $\ell(e_i) \in \{1, 2, 3\}$, then $|\ell(V_{G'})| = 3 < 5$. We assume without loss of generality that $\ell(e_i) = 4$. Since $G'[1, 4]$ contains a perfect matching, $\ell(e_{i-1}) = 4$. Then $G'[1, 2]$ does not have a perfect matching.

Suppose $\ell(c_i) = 2$. If $\ell(d_i) = 1$ then we return to a previous case. If $\ell(d_i) = 2$, then $G'[1, 2]$ does not contain a perfect matching. Suppose $\ell(d_i) \notin \{1, 2\}$, say $\ell(d_i) = 3$. Since $G'[2, 3]$ has a perfect matching, $\ell(e_i) = 3$. Then $G'[1, 2]$ does not allow a perfect matching. This proves the claim, and from now on we assume that $\ell(a_i) = 1$, $\ell(b_i) = 2$ and $\ell(c_i) = 3$.

Claim 3. We may assume that $\ell(d_i) = 4$.

We prove this claim as follows. If $\ell(d_i) = 1$ or $\ell(d_i) = 2$ then we return to a previous case. Suppose $\ell(d_i) = 3$. Since $G'[2, 3]$ has a perfect matching, $\ell(e_i) = 2$. Then $G'[1, 3]$ does not contain a perfect matching. This proves the claim, and from now on we assume that $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, and $\ell(d_i) = 4$.

Claim 4. We may assume that $\ell(e_i) = 5$.

We prove this claim as follows. Suppose $\ell(e_i) = 1$. Since $G'[1, 2]$ has a perfect matching, $\ell(a_{i+1}) = 2$. Then $|\ell(V_{G'})| = 4 < 5$. Suppose $\ell(e_i) = 2$. Since $G'[1, 2]$ has a perfect matching, $\ell(a_{i+1}) = 1$. Then $G'[2, 3]$ does not have a perfect matching. Suppose $\ell(e_i) = 3$. Since $G'[3, 4]$ has a perfect matching, $\ell(a_{i+1}) = 4$. Then $G'[2, 3]$ does not have a perfect matching. By symmetry $\ell(e_i) \neq 4$ either. This proves the claim, and from now on we assume that $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, $\ell(d_i) = 4$, and $\ell(e_i) = 5$.

To finish the proof of the lemma, we show that e_{i-1} is not mapped to a color in $\{1,2,3,4\}$. If $\ell(e_{i-1}) = 1$, then e_{i-1} must have neighbors colored $1,2,3,4$. This is not possible, since $\deg_{G'}(e_{i-1}) = 3$. If $\ell(e_{i-1}) = 3$, then e_{i-1} must have neighbors colored $1,2,4,5$. This is not possible, since $\deg_{G'}(e_{i-1}) = 3$. By symmetry, e_{i-1} can not be mapped to 4 either. Suppose $\ell(e_{i-1}) = 2$. Then the two neighbors of e_{i-1} outside D_i must be colored with 3 and 4. Then $G'[1,2]$ does not have a perfect matching. $\square$

Appendix B

Here is the proof of Lemma 6.

Proof. Suppose ℓ is a proper connected coloring of I^*. We note that, by definition, two neighbors must be mapped to different colors. We write $\ell(q_1) = 1$. Let the other two vertices of C_{q_1} be s, t with $\ell(s) = 2$ and $\ell(t) = 3$. If q_1 only has neighbors with color 1 or 2, then $\ell(V_{I^*}) = \{1,2,3\}$, and we are done.

Suppose q_1 has a neighbor in $\mathcal{S} \cup \mathcal{S}'$ with a color not in $\{2,3\}$. Then all vertices of I^* mapped to 1 have at least degree three. By a sequence of claims, we show that $|\ell(V_{I^*})| = |V_{I^*}|$. This is then a contradiction with our assumption that ℓ is proper.

Claim 1. Colors $2,3$ are not in $\ell(V_{I^*} \backslash \{s,t\})$.

In order to obtain a contradiction let $\ell(w) = 2$ for some $w \in V_{I^*} \backslash \{s,t\}$. Suppose w is in $V_{C_p} \backslash \{p\}$ for some $p \in Q \cup \{v\}$, then w needs a neighbor with color 1. Recall that such a neighbor must have degree at least three. The only candidate is p. However, w also needs a neighbor with color 3 and this neighbor must be adjacent to a neighbor with color 1. Since $|C_p|$ contains at least six vertices, this is not possible.

Suppose $w = p$ for some $p \in Q \cup \{v\}$. Let x be a neighbor of w on C_p. Then x must have color 1 or 3. The first case is not possible since x has degree $2 < 3$. The second case is not possible, since then x has a (degree-two) neighbor y on C_p with color 1.

Suppose $w = S$ for some $S \in \mathcal{S} \cup \mathcal{S}'$. Then w must have a neighbor p', which is is $Q \cup \{v\}$, with color 3. By symmetry of x and y, we can return to the previous case. This finishes the proof of Claim 1.

Claim 2. For all $p \in Q \cup \{v\}$, $|\ell(V_{C_p})| = |V_{C_p}|$.

For $p = q_1$, this condition is satisfied. In order to obtain a contradiction let $|\ell(V_{C_p})| < |V_{C_p}|$ for some $p \in (Q \backslash \{q_1\}) \cup \{v\}$. We first make the following observation, which can easily be proven by an inductive argument:

Let $a_1, a_2, \ldots, a_k$ be a sequence of different colors from $\ell(V_{I^*})$ such that, for $j = 1 \ldots k$, the subgraph $I^*[a_j, a_{j+1}]$ is not edgeless. Then, for any vertex r with color a_1, there exists a path $P = r_1, s_2, \ldots, r_k$ from $r_1 = r$ to some vertex r_k such that $\ell(r_h) = a_h$ for $h = 1, \ldots, k$.

Now suppose $z \in V_{C_p} \backslash \{p\}$ has color $\ell(p)$. By Claim 1, color 2 is not a color of any vertex in C_p. Since p is a cutvertex of I^*, any path from z with color $a_1 = \ell(p)$ to a vertex with color $a_k = 2$ contains p with color $\ell(p) = a_1$. This is not possible due to the above observation. By the same argument, we deduce that any other color not equal to $\ell(p)$ appears at most twice on C_p.

Suppose $\ell(u_1) = \ell(u_2)$ for some $u_1, u_2 \in V_{C_p} \backslash \{p\}$. By the above observation with $a_1 = \ell(u_1)$ and $a_k = 2$, the path P_1 from u_1 to p not using u_2 and the path P_2 from u_2 to

p not using u_1 must use exactly the same $|V_{P_1}| = |V_{P_2}|$ colors (in exactly the same order). We choose u_1 and u_2 such that P_1 and P_2 are maximal. Let v_1 be the neighbor of u_1 not on P_1, and let v_2 be the neighbor of u_2 not on P_2. If v_1 is not equal to v_2, then v_1 and v_2 must be mapped to the same color. This contradicts the maximality of P_1 and P_2. Hence $v_1 = v_2$, and exactly one color appears once on $C_p \backslash \{p\}$ and all other colors appear twice on $C_p \backslash \{p\}$. This implies that $|C_p|$ is even. This is not possible, since $|C_p|$ is divisible by 3. Hence, we have proven Claim 2.

Claim 3. $\ell(V_{C_p}) \cap \ell(V_{C_q}) = \emptyset$ *for all* $p, q \in Q \cup \{v\}$ *with* $p \neq q$.

Suppose $\ell(V_{C_p}) \cap \ell(V_{C_q}) \neq \emptyset$ for some $p, q \in Q \cup \{v\}$ with $p \neq q$. We assume $p < q$, so C_p contains less vertices than C_q. We note that due to Claim 2, both neighbors of p on C_p have a different color. Suppose these colors are the only colors the neighbors of p have. Then $|\ell(V_{I^*})| = |V_{C_p}|$. This not possible, since the number of different colors on I^* is at least $|V_{C_v}| = 6m + 3 > |V_{C_p}|$, due to Claim 2. So on the neighborhood of p at least three different colors are used. This means that any vertex with color $\ell(p)$ must have degree at least three.

Let a be a common color on C_p and C_q. Suppose a is not equal to $\ell(p)$ already. Then there is a path in C_q from a vertex x with color a to a vertex $y \neq q$ with color $\ell(p)$, because C_q has at least three more vertices than C_p. Since we showed that a vertex with color $\ell(p)$ must have degree at least three, we find that $a = \ell(q) = \ell(p)$ and $\ell(V_{C_p}) \cap \ell(V_{C_q}) = \{a\}$.

Let r_1 be a neighbor of p on C_p and let r_2 be a neighbor of $r_1 \neq p$. Then $\ell(r_1)$ is the color of a vertex in $\mathcal{S} \cup \mathcal{S}'$ and consequently $\ell(r_2) \neq \ell(p)$ is the color of a vertex on $C_{q'}$ for some $q' \in Q \cup \{v\}$. We consider C_p and $C_{q'}$ instead of C_p and C_q, and obtain a contradiction. This proves Claim 3.

By Claim 2 and Claim 3, all vertices in the union of all cycles C_p over $p \in Q \cup \{v\}$ are mapped to different colors. Since any two $S_j, S_k \in \mathcal{S}$ with $j \neq k$ are different subsets of Q, they can not have the same color. The same holds for any two $S'_j, S'_k \in \mathcal{S}'$. Furthermore, all S'_j are not adjacent to v, so $\ell(\mathcal{S}) \cap \ell(\mathcal{S}')$ is empty. Hence we have found that $|\ell(\mathcal{S} \cup \mathcal{S}')| = |\mathcal{S}| + |\mathcal{S}'| = 2n$.

Suppose some $S \in \mathcal{S} \cup \mathcal{S}'$ has the same color as a vertex u of some C_p. Then the colors of the neighbors of u on C_p must appear on the neighbors of S, which lie on some cycle. This violates Claim 2. Hence $|\ell(V_{I^*})| = |V_{I^*}|$ and ℓ is not proper. This finishes the proof of the lemma. □

Appendix C

Here we prove Theorem 3. Obviously, deciding if G admits a proper semi-regular coloring is in NP. To show that the problem is NP-complete, we will use the NP-completeness of the K_4-cover problem [10], where K_4 is the complete graph on four vertices.

Consider a graph G. We may assume that G is a 3-regular graph with $|V_G| = 4q$ and $|E_G| = 6q$ for some $q \geq 0$; otherwise $G \xrightarrow{B} K_4$ is false. Let $E_G = \{e_1, e_2, \ldots, e_m\}$. For each $k \in [1, m]$, we replace the edge e_k by a chain of $k+1$ *multi-diamonds* $D_1(k), \ldots D_{k+1}(k)$ as represented in Figure 4. We denote the resulting graph by G'. The vertices of the chain that replace the edge e_k are

$$\{a_i(k), b_i(k), b'_i(k), c_i(k), c'_i(k), d_i(k), d'_i(k), e_i(k), e'_i(k), f_i(k), f'_i(k), g_i(k) \mid 1 \leq i \leq k+1\}.$$

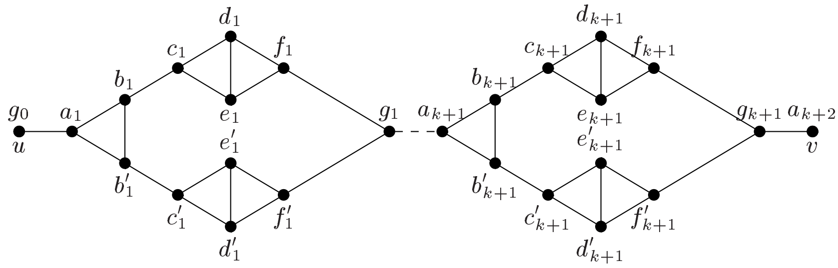

Fig. 4. The chain of $k+1$ multi-diamonds that replace the edge $e_k = [u, v]$.

When no confusion is possible, we will note a_i for $a_i(k)$, etc. The next property is useful.

Lemma 7. *Let ℓ be a semi-regular coloring of G' then for any multi-diamonds $D_i(k), D_j(k')$, $\ell(g_i(k)) = \ell(g_j(k'))$ and $\ell(a_{i+1}(k)) = \ell(a_{j+1}(k'))$ if and only if $\ell(g_{i-1}(k)) = \ell(g_{j-1}(k'))$ and $\ell(a_i(k)) = \ell(a_j(k'))$.*

Proof. Suppose $\ell(g_i(k)) = \ell(g_j(k'))$ and $\ell(a_{i+1}(k)) = \ell(a_{j+1}(k'))$. Then $\ell(\{f_i(k), f_i'(k)\}) = \ell(\{f_j(k'), f_j'(k')\})$. Without loss of generality, we assume $\ell(f_i(k)) = \ell(f_j(k'))$ and $\ell(f_i'(k)) = \ell(f_j'(k'))$. Consequently, $\ell(\{d_i(k), e_i(k)\}) = \ell(\{d_j(k'), e_j(k')\})$ and $\ell(c_i(k)) = \ell(c_j(k'))$. Then $\ell(b_i(k)) = \ell(b_j(k'))$ and by symmetry, $\ell(b_i'(k)) = \ell(b_j'(k'))$. Hence, $\ell(g_{i-1}(k)) = \ell(g_{j-1}(k'))$ and $\ell(a_i(k)) = \ell(a_j(k'))$. In the same way we show the reverse statement. $\square$

By using Lemma 7, we deduce that $G \xrightarrow{B} K_4$ if and only if $G' \xrightarrow{B} K_4$. Any witness for $G' \xrightarrow{B} K_4$ is a proper perfect-regular coloring of G', which is a proper semi-regular coloring of G'. We are left to show that if $G' \xrightarrow{B} K_4$ is not true (we say if G does not *cover* K_4), then G' does not allow a proper semi-regular coloring. For this, we need a few lemmas. In the following one, we show that if G' does not cover K_4 then all the vertices in a multi-diamond have different colors.

Lemma 8. *If G' does not cover K_4, then $|\ell(D_i(k)| = 12$ for any multi-diamond $D_i(k)$ and for any semi-regular coloring ℓ of G'.*

Proof. Note that c_i, d_i, e_i have different colors. Let $\ell(d_i) = 1$, $\ell(e_i) = 2$ and $\ell(c_i) = 3$.

Claim 1. We may assume that $\ell(b_i) = 4$ and $\ell(b_i') \notin \{\ell(d_i'), \ell(e_i')\}$.

Note that $\ell(b_i) \neq 3$. We write $a = \ell(f_i)$ and $b = \ell(g_i)$. If $\ell(b_i) = 1$, then either $\ell(a_i) = 2$ and $\ell(b_i') = a$, or $\ell(a_i) = a$ and $\ell(b_i') = 2$. In the first case, $\ell(c_i') = b$ and then either $\ell(d_i') = \ell(f_i')$, or $\ell(e_i') = \ell(f_i')$, which is impossible. In the second case, $\ell(c_i') = 3$, and then c_i' must have two neighbors colored by 1, but then $\ell(e_i') = \ell(d_i') = 1$. This is impossible. Hence, we can write $\ell(b_i) = 4$. By symmetry, we find that $\ell(b_i) \neq 2$. Then, by symmetry, $\ell(b_i') \notin \{\ell(d_i'), \ell(e_i')\}$.

Claim 2. We may assume that $\ell(f_i) = 5$ and $\ell(c_i') \neq \ell(f_i')$.

Note that $\ell(f_i) \notin \{1, 2\}$. Suppose $\ell(f_i) = \ell(b_i) = 4$. Then $\{\ell(a_i), \ell(b_i')\} = \{1, 2\}$ and $\ell(g_i) = 3$. Consequently, $\ell(V_{G'}) = \{1, 2, 3, 4\}$ and for each $v \in V_{G'}$, $\ell(v) = 1$ (respectively $\ell(v) = 2$, $\ell(v) = 3$, $\ell(v) = 4$), then $\ell(N_{G'}(v)) = \{2, 3, 4\}$ (respectively $\ell(N_{G'}(v)) = \{1, 3, 4\}$,

$\ell(N_{G'}(v)) = \{1,2,4\}$, $\ell(N_{G'}(v)) = \{1,2,3\}$). Then $G' \xrightarrow{B} K_4$, which is impossible. Suppose $\ell(f_i) = \ell(c_i) = 3$. We write $a = \ell(a_i)$ and $b = \ell(b_i')$. Since f_i must have a neighbor labeled by 4, we find $\ell(g_i) = 4$. Consequently, either $\ell(f_i') = a$ or $\ell(f_i') = b$. In the first case, $b = \ell(b_i')$ must belongs to $\{\ell(d_i'), \ell(e_i')\}$ but, by Claim 1, this is not possible. In the second case, either $\ell(c_i') = \ell(d_i')$ or $\ell(c_i') = \ell(e_i')$ but this is not possible. Hence we can write $\ell(f_i) = 5$. By symmetry, we find that $\ell(c_i') \neq \ell(f_i')$.

Claim 3. We may assume that $\ell(a_i) = 6$ *and* $6 \notin \{\ell(b_i'), \ell(c_i'), \ell(d_i'), \ell(e_i'), \ell(f_i')\}$.

We know that $\ell(a_i) \neq 4$. Since a_i has a neighbor labeled by 4 whereas c_i and d_i do not have such a neighbor, we know that $\ell(a_i) \notin \{1,2\}$. Suppose $\ell(a_i) \in \{3,5\}$. Then $\ell(b_i') \in \{1,2\}$; without loss of generality we say that $\ell(b_i') = 1$. But in this case, $\ell(b_i) = 4$ must appear in $\ell(N_{G'}(d_i))$, which is impossible. Hence we can write $\ell(a_i) = 6$. By symmetry, $6 \notin \{\ell(b_i'), \ell(c_i'), \ell(d_i'), \ell(e_i'), \ell(f_i')\}$.

Claim 4. We may assume that $\ell(b_i') = 7$.

We know that $\ell(b_i') \notin \{1,2,3,4,6\}$. Suppose $\ell(b_i') = 5$. Then $\ell(N_{G'}(b_i'))$ must contain $\{1,2,4,6\}$ but this is impossible since $\deg_{G'}(b_i') = 3$. Hence, we can write $\ell(b_i') = 7$.

Claim 5. We may assume that $\ell(c_i') = 8$.

We note that $\ell(c_i') \notin \{1,2,3,4,6,7\}$. Suppose that $\ell(c_i') = 5$. Then $\ell(g_i) = 7$, and $\ell(\{d_i', e_i'\}) = \{1,2\}$. Consequently, $\ell(f_i') = 3$, but this is impossible since $7 \notin \ell(N_{G'}(c_i))$.

Claim 6. We may assume that $\ell(d_i') = 9$ *and* $\ell(e_i') = 10$.

We note that $\ell(d_i') \notin \{1,2,3,4,6,7,8\}$. Suppose that $\ell(d_i') = 5$, then $\ell(e_i') \in \{1,2\}$ but this is impossible since $8 \notin \ell(N_{G'}(d_i)) \cup \ell(N_{G'}(e_i))$. We can write $\ell(d_i') = 9$. By symmetry and since $\ell(e_i') \neq \ell(d_i')$, we find that $\ell(e_i') \notin \{1,2,3,4,5,6,7,8,9\}$.

Claim 7. We may assume that $\ell(f_i') = 11$.

We know that $\ell(f_i') \notin \{1,2,3,4,6,7,8,9,10\}$. Suppose that $\ell(f_i') = 5$. Then $\ell(N_{G'}(f_i'))$ must contain $\{1,2,9,10\}$ but this is impossible since $\deg_{G'}(f_i') = 3$.

Claim 8. We may assume that $\ell(g_i') = 12$.

We know that $\ell(g_i') \notin \{1,2,3,4,5,7,8,9,10,11\}$. Suppose that $\ell(f_i') = 6$. Then $\ell(N_{G'}(g_i'))$ must contain $\{4,5,7,11\}$ but this is impossible since $\deg_{G'}(g_i') = 3$. This ends the proof of the lemma. $\square$

In the following lemma, we show that if G' does not cover K_4, a color that appears on a vertex $g_i(k)$ cannot appear on another multi-diamond $D_j(k')$ elsewhere than in $g_j(k')$.

Lemma 9. *If* G' *does not cover* K_4*, then for any semi-regular coloring* ℓ *of* G'*, for any multi-diamonds* $D_i(k)$ *and* $D_j(k')$*, for each* $u \in D_i(k) \setminus \{g_i(k)\}$*,* $\ell(u) \neq \ell(g_j(k'))$.

Proof. For any vertex $u \in D_i(k) \setminus \{g_i(k)\}$, there exists two vertices $v, w \in N_{G'}(u)$ such that $[v,w] \in E_{G'}$.

Suppose that $\ell(u) = \ell(g_j(k'))$. Then $\{\ell(v), \ell(w)\} \cap \{\ell(f_j(k')), \ell(f_j'(k'))\} \neq \emptyset$; without loss of generality, we say that $\ell(v) = \ell(f_j(k'))$. Then $\ell(w) \in \{\ell(d_j(k')), \ell(e_j(k'))\}$; without loss of generality, we say that $\ell(w) = \ell(d_j(k'))$.

Consequently, $\ell(g_j(k')) = \ell(u) \in \ell(N_{G'}(d_j(k'))) = \{\ell(c_j(k')), \ell(e_j(k')), \ell(f_j(k'))\}$, but this is impossible from Lemma 8. $\square$

In the following lemma, we show that if G' does not cover K_4, a color that appears on a vertex $a_i(k)$ cannot appear on another multi-diamond $D_j(k')$ elsewhere than in $a_j(k')$.

Lemma 10. *If G' does not cover K_4, then for any semi-regular coloring ℓ of G', for any multi-diamonds $D_i(k)$ and $D_j(k')$, for each $u \in D_i(k) \setminus \{a_i(k)\}$, $\ell(u) \neq \ell(a_j(k'))$.*

Proof. From Lemma 8, one can suppose that $\ell(d_j(k')) = 1, \ell(e_j(k')) = 2, \ell(c_j(k')) = 3, \ell(b_j(k')) = 4, \ell(f_j(k')) = 5, \ell(a_j(k')) = 6, \ell(b'_j(k')) = 7, \ell(c'_j(k')) = 8, \ell(d'_j(k')) = 9, \ell(e'_j(k')) = 10, \ell(f'_j(k')) = 11, \ell(g_j(k')) = 12$, as represented on the left of Figure 5. We will also note a and b for $\ell(g_{j-1}(k'))$ and $\ell(a_{j+1}(k'))$.

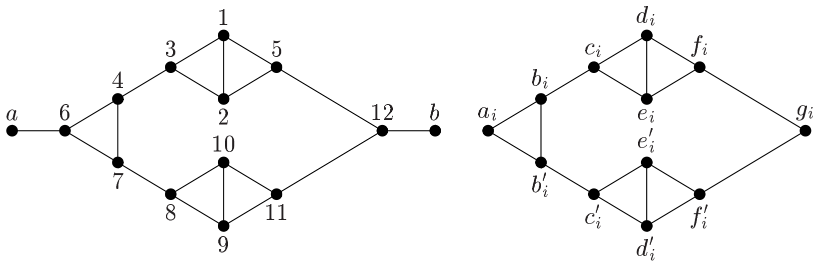

Fig. 5. The two multi-diamonds we consider for the proof of Lemma 10.

We will note a_i for $a_i(k)$, etc. We just have to show that for each $v \in \{b_i, c_i, d_i, f_i, g_i\}$, $\ell(v) \neq 6$. From Lemma 9, we already know that $\ell(g_i) \neq 6$.

Suppose that $\ell(c_i) = 6$ (resp. $\ell(f_i) = 6$). Then $\{\ell(d_i), \ell(e_i)\} \cap \{4, 7\} \neq \emptyset$. Without loss of generality, we say that $\ell(d_i) = 4$. Then c_i (resp. f_i) and d_i must both have a neighbor labeled by 7. From Lemma 8, it implies that $\ell(e_i) = 7$. Since e_i must also have a neighbor labeled by 8, it implies that $\ell(f_i) = 8$ (resp. $\ell(c_i) = 8$), but this is impossible since f_i (resp c_i) cannot have a neighbor labeled by 4.

Suppose that $\ell(d_i) = 6$. Then $\{\ell(c_i), \ell(f_i)\} \cap \{4, 7\} \neq \emptyset$. Without loss of generality, suppose that $\ell(c_i) = 4$ (resp. $\ell(f_i) = 4$). Then c_i (resp. f_i) and d_i must both have a neighbor labeled by 7. From Lemma 8, it implies that $\ell(e_i) = 7$. Since e_i must also have a neighbor labeled by 8, it implies that $\ell(f_i) = 8$ (resp. $\ell(c_i) = 8$), but this is impossible since f_i (resp. c_i) cannot have a neighbor labeled by 6.

Suppose that $\ell(b_i) = 6$. If $j \geq 2$, b_i must have a neighbor labeled by $a = \ell(g_{j-1}(k'))$, but this is impossible from Lemma 9. We will now suppose that $j = 1$. From Lemma 8, $\ell(\{a_i, b'_i\}) = \{4, 7\}$. Without loss of generality, we say that $\ell(a_i) = 7$ and $\ell(b'_i) = 4$. Consequently, $\ell(c'_i) = 3$, $\ell(\{d'_i, e'_i\}) = \{1, 2\}$, $\ell(f'_i) = 5$ and $\ell(g_i) = 12$. Consequently, either $\ell(f_i) = b$, or $\ell(f_i) = 11$. In the first case, since $j = 1$, $b = \ell(a_2(k'))$ and we already know that it is impossible. In the second case, if $\ell(f_i) = 11$, then $\ell(\{d_i, e_i\}) = \{9, 10\}$ and then $\ell(c_i) = 8$, but this is impossible since c_i cannot have a neighbor labeled by 6. ☐

In the following lemma, we show that if G' does not cover K_4, then a vertex u that does not belong to any multi-diamond (i.e. a vertex that was in the graph G) cannot have the same color as a vertex that belongs to a multi-diamond.

Lemma 11. *If G' does not cover K_4, then for any semi-regular coloring ℓ of G', for any multi-diamond $D_i(k)$, for any $v \in D_i(k)$ and for any $u \in V_{G'}$ such that $\forall k, \forall i, u \notin D_i(k)$, we have $\ell(u) \neq \ell(v)$.*

Proof. Consider such a vertex u. In G', for any $u' \in N_{G'}(u)$, there exists k' such that either $u' = a_1(k')$ or $u' = g_{k'+1}(k')$. Consider any vertex v of any multi-diamond $D_i(k)$. There exists $v' \in N_{G'}(v)$ such that $v' \in D_i(k) \setminus \{a_i(k), g_i(k)\}$. If $\ell(u) = \ell(v)$, then there exists a vertex $u' \in N_{G'}(u)$ such that $\ell(v') = \ell(u')$, but this is impossible from Lemmas 9 and 10. □

In the following lemma, we show that if G' does not cover K_4, any semi-regular coloring ℓ of G' is a perfect-regular coloring of G'.

Lemma 12. *If G' does not cover K_4, for any semi-regular coloring ℓ of G', for each vertex $v \in V_{G'}$, $|\ell(N_{G'}(v))| = |N_{G'}(v)|$.*

Proof. We first consider vertices that belong to some multi-diamond. Consider a multi-diamond $D_i(k)$ for some i, k. From Lemma 8, we already know that $|\ell(N_{G'}(v))| = |N_{G'}(v)|$ if $v \notin \{a_i(k), g_i(k)\}$. From Lemmas 9, 10 and 11, we also know that $|\ell(N_{G'}(v))| = |N_{G'}(v)|$ if $v \in \{a_i(k), g_i(k)\}$.

We now consider a vertex u that does not belong to any multi-diamond $D_i(k)$. Suppose that there exist two distinct vertices $v, v' \in N_{G'}(u)$ such that $\ell(v) = \ell(v')$. From Lemmas 9 and 10, we know that either $v = a_1(k)$ and $v' = a_1(k')$ or $v = g_{k+1}(k)$ and $v' = g_{k'+1}(k')$ for some k, k'. By construction of G', we know that $k \neq k'$; without loss of generality, we say that $k < k'$. If we apply Lemma 7 $k+1$ times, then $\ell(a_{k+2}(k)) = \ell(a_{k+2}(k')$ (respectively $\ell(g_0(k)) = \ell(g_{k'-k}(k'))$ for the second case) but from Lemma 11, this is impossible since $a_{k+2}(k')$ (respectively $g_{k'-k}(k')$) belongs to some multi-diamond but $a_{k+2}(k)$ (respectively $g_0(k)$) does not. □

In the following lemma, we show that if G' does not cover K_4, any semi-regular coloring of G' needs $|V_{G'}|$ colors.

Lemma 13. *If G' does not cover K_4, any semi-regular coloring ℓ of G' is not proper.*

Proof. Consider a vertex u that does not belong to any multi-diamond $D_i(k)$. Suppose that there exists $u' \in V_{G'}$ such that $\ell(u) = \ell(u')$. From Lemma 11, we already know that u' does not either belong to any multi-diamond.

There exists v in $N_{G'}(u)$ and v' in $N_{G'}(u')$ such that $\ell(v) = \ell(v')$. From Lemmas 9 and 10, we know that either $v = a_1(k)$ and $v' = a_1(k')$ or $v = g_{k+1}(k)$ and $v' = g_{k'+1}(k')$ for some k, k'. By construction of G', we know that $k \neq k'$ and then with the same proof as for Lemma 12, one can show that there is a contradiction.

Consequently, $|\ell^{-1}(\ell(u))| = 1$. From Lemma 12 it is easy to see that ℓ is a pseudo-regular labeling. Then we know from Observation 1 that for any vertex $v \in V_{G'}$, $|\ell^{-1}(\ell(v))| = 1$.

Consequently $|\ell(V_{G'})| = |V_{G'}|$ and ℓ is not a proper semi-regular coloring. □

Summarizing, $G \xrightarrow{B} K_4$ if and only if $G' \xrightarrow{B} K_4$ if and only if G' allows a proper semi-regular coloring. Therefore, we have proven Theorem 3.