Graph Labelings Derived from Models in Distributed Computing: A Complete Complexity Classification*

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We discuss 11 known basic models of distributed computing: four message-passing models that differ by the (non)existence of port-numbers and a hierarchy of seven local computations models. In each of these models, we study the computational complexity of the decision problems if the leader election and if the naming problem can be solved on a given network. It is already known that these two decision problems are solvable in polynomial time for two models and are co-NP-complete for another one. Here, we settle the computational complexity for both problems in the remaining eight models by showing that they are co-NP-complete. We do this by translating each problem into a graph labeling problem. By using this technique, we also obtain an alternative proof for the already known co-NP-completeness result. In the second part of our article, we completely classify the computational complexity of all the corresponding graph labeling problems, i.e., for every fixed integer $k \geq 1$ we determine the complexity of the problem that asks whether a given graph allows a certain graph labeling that uses at most k labels. We also explain the close relationship of these labelings to graph homomorphisms that satisfy some further (global or local) constraints. This yields a new class of "constrained" graph homomorphisms that include the already known locally constrained graph homomorphisms. © 2011 Wiley Periodicals, Inc. NETWORKS, Vol. 000(00), 000–000 2011

1. INTRODUCTION

In distributed computing, one can find a wide variety of models of communication. These models reflect different system architectures, different levels of synchronization and different levels of abstraction. In this article, we consider 11 known basic models that satisfy the following two underlying assumptions. First, a distributed system is represented by a simple (i.e., without loops or multiple edges), connected, undirected graph. Its vertices represent the processors and its edges represent direct communication links. Second, the distributed systems we consider are anonymous, i.e., all the processors execute the same code to solve some problem and they do not have initial identifiers.

The 11 basic models can be divided into four message-passing models [9, 26, 28] and seven local computations models [3, 6, 8, 23, 24]. In a message-passing model, processors communicate by sending and receiving messages. In a local computations model, a computation step (encoded by a local relabeling rule) involves neighboring processors that synchronize, exchange information and modify their states.

Understanding the computational power of various models enhances our understanding of distributed algorithms. For this purpose, a number of standard problems in distributed computing are studied. The election problem is one of the paradigms of the theory of distributed computing. In our setting, a distributed algorithm solves the election problem if it always terminates and in the final configuration exactly one processor is marked as elected and all the other processors are marked as nonelected. Elections constitute a building block of many other distributed algorithms, since the elected vertex can be subsequently used to make centralized decisions. A second important problem in distributed computing is the naming problem. Here, the aim is to arrive at a final configuration where all processors have been assigned unique identities. Again this is an essential prerequisite to many other distributed algorithms that only work correctly under

the assumption that all processors can be unambiguously identified. As examples we mention algorithms for spanning tree construction, termination detection, network topology recognition, consensus, and mutual exclusion. For a survey on distributed algorithms, we refer to the reference book of Tel [25].

Whether the naming or election problem can be solved on a given graph depends on the properties of the considered model. If it is possible to solve the election (naming) problem, we call the graph a solution graph for the election (naming) problem. We note that in many (but not all) models the election and naming problem are equivalent in the sense that a solution graph for the election problem is a solution graph for the naming problem and vice versa. We will give details in Sections 4.1 and 4.2. The following computational complexity question comes immediately to mind and is the first question that we study in this article.

How hard is it to check whether a given graph is a solution graph for the election or naming problem in a certain model of distributed computing?

For two models, this problem is known to be polynomial-time solvable [4], and for one model, it is co-NP-complete [27]. What about the computational complexity of this problem for the other models? We solve this question by showing that this decision problem is co-NP-complete for both the election and naming problem in all remaining models. To obtain our results, we make use of known characterizations [3, 6, 8–10, 23, 24, 26, 28] of solution graphs. Almost all of these characterizations are expressed in terms of graph homomorphisms that satisfy certain local constraints. Some of these locally constrained homomorphisms are well studied in the literature; see, e.g., [1, 14–17, 20–22]. For several models, however, these homomorphisms are not defined on simple graphs, but on graphs that can have multiple edges, or that are directed graphs. To have a more understandable presentation of the links between the different characterizations and to unify our proofs as much as possible, we choose to express the characterizations in terms of graph labelings. This enables us to use simple undirected graphs only.

In Section 2, we will give precise definitions of the graph labelings we use. It is a natural question to ask how hard it is to check whether a given graph allows a certain graph labeling that uses at most k labels for some fixed integer k. As a byproduct of our proof, we can immediately answer this question for almost all values of k. To give a full answer to this question, our second main result completely classifies the computational complexity of this problem for all graph labelings that we consider in this article.

2. TERMINOLOGY

Throughout the article, we consider undirected graphs that have no self-loops and no multiple edges with one exception, namely in Section 5. In that section, we speak of input and pattern graphs, and there we show that it makes sense to allow pattern graphs to have self-loops (but no multiple edges). For graph terminology not defined below we refer to [5].
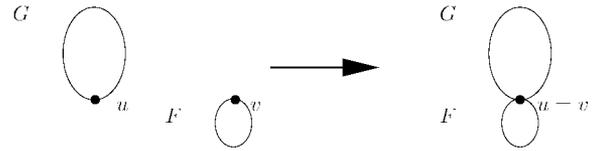
FIG. 1.   A graph F that is v-glued to u in G.

Let $G = (V_G, E_G)$ denote a graph with vertex set V_G and edge set E_G. For $U \subseteq V_G$, the graph $G[U] = (U, \{(u, v) \in E_G \mid u, v \in U\})$ is called the subgraph of G induced by U. For a vertex $u \in V_G$, we denote its neighborhood by $N_G(u) = \{v \mid (u, v) \in E_G\}$ and its degree by $\deg_G(u) = |N_G(u)|$.

Let F and G be two disjoint graphs. Let u be a vertex in G, and let v be a vertex in F. We say that F is v-glued to u in G (or equivalently that G is u-glued to v in F) if we have identified v with u. If no confusion is possible we simply write that F is glued to u in G. We use this notion in Sections 6 and 7. See Figure 1 for an example.

A graph is regular if all its vertices have the same degree p. In that case, we also say that the graph is p-regular. A graph is bipartite if its vertices can be partitioned into two sets A and B such that each edge has one of its endpoints in A and the other one in B. A graph is regular bipartite if it is regular and bipartite. A graph is semiregular bipartite if it is bipartite and the vertices of one class of the bipartition are of degree p and all others are of degree q. In that case, we also say that the graph is (p, q)-regular bipartite. In our context, a perfect matching is a $(1, 1)$-regular bipartite graph. A star is a bipartite graph in which one vertex is adjacent to all the other vertices.

2.1. Graph Labelings and Colorings

A labeling of a graph G is a mapping $\ell : V_G \rightarrow \{1, 2, 3, \ldots\}$ that assigns each vertex $u \in V_G$ a label $\ell(u)$. For a set $U \subseteq V_G$, we use the shorthand notation $\ell(U)$ to denote the image set of U under ℓ, i.e., $\ell(U) = \{\ell(u) \mid u \in U\}$. A labeling ℓ of G is called proper if $|\ell(V_G)| < |V_G|$. For $i \in \ell(V_G)$, we write $\ell^{-1}(i) = \{u \in V_G \mid \ell(u) = i\}$. If no confusion is possible, we use the following shorthand notations. For $i \in \ell(V_G)$, we write $G[i] = G[\ell^{-1}(i)]$. For $i, j \in \ell(V_G)$ with $i \neq j$, we let $G[i, j]$ denote the bipartite graph obtained from $G[\ell^{-1}(i) \cup \ell^{-1}(j)]$ by deleting all edges (u, v) with either $\ell(u) = \ell(v) = i$ or $\ell(u) = \ell(v) = j$.

Let ℓ be a labeling of a graph G. We say that ℓ is a perfect-regular labeling of G if

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless or else is a perfect matching, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i, j]$ is edgeless or else is a perfect matching.

We say that ℓ is a symmetric regular labeling of G if

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular and in the case that its vertices have odd degree it contains a perfect matching, and
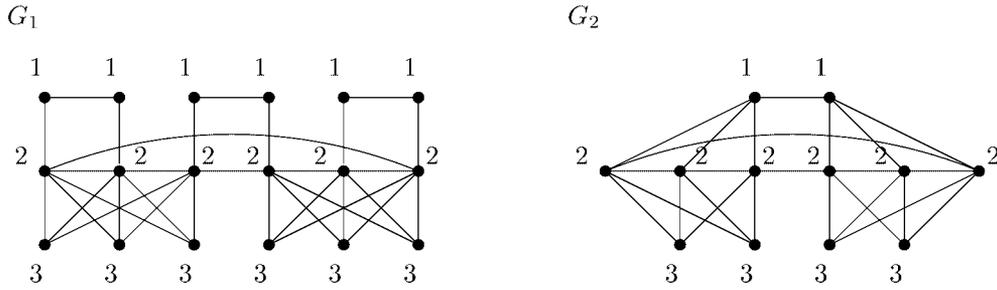
FIG. 2. Graph G_1 with a (symmetric) regular labeling and graph G_2 with a semiregular labeling.

(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i,j]$ is regular bipartite.

We say that ℓ is a (semi)regular labeling of G if

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i,j]$ is (semi)regular bipartite.

We say that ℓ is a pseudo-regular labeling of G if

(i) for all $i \in \ell(V_G)$, $G[i]$ is regular, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i,j]$ is edgeless or else contains a perfect matching.

We say that ℓ is a connected labeling of G if

(i) for all $i \in \ell(V_G)$, $G[i]$ is edgeless or else has minimum degree at least one, and
(ii) for all $i, j \in \ell(V_G)$ with $i \neq j$, $G[i,j]$ is edgeless or else has minimum degree at least one.

See the left hand side of Figure 2 for an example of a graph G_1 with a proper symmetric regular labeling ℓ that uses three labels such that $G_1[1]$ is 1-regular (with a perfect matching), $G_1[2]$ is 2-regular, $G_1[3]$ is 0-regular, $G_1[1,2]$ is $(1,1)$-regular bipartite, $G_1[1,3]$ is $(0,0)$-regular bipartite and $G_1[2,3]$ is $(3,3)$-regular bipartite. See the right hand side of Figure 2 for an example of a graph G_2 with a proper semiregular labeling ℓ that is not a regular labeling, because it uses three labels such that $G_2[1]$ is 1-regular, $G_2[2]$ is 2-regular, $G_2[3]$ is 0-regular, $G_2[1,2]$ is $(3,1)$-regular bipartite, $G_2[1,3]$ is $(0,0)$-regular bipartite, and $G_2[2,3]$ is $(2,3)$-regular bipartite. See Figure 3 for an example of a graph that has a proper pseudo-regular labeling.

We call a labeling ℓ of a graph G a coloring of G if $\ell(u) \neq \ell(v)$ for any edge $(u, v) \in E_G$. Therefore, we sometimes call the label $\ell(u)$ of a vertex u the color of u. If G can be colored with k colors, then G is said to be k-colorable.

The definition of a coloring ℓ of a graph G is equivalent to saying that ℓ is a labeling of G such that $G[i]$ is edgeless for all $i \in \ell(V_G)$. When we replace condition (i) in each of the six label definitions by the condition that $G[i]$ must be edgeless for all $i \in \ell(V_G)$, we obtain a perfect-regular, symmetric regular, (semi)regular, pseudo-regular, and connected coloring, respectively. Because a symmetric regular coloring is a regular coloring, and vice versa, we will not use the

notion of a symmetric regular coloring in the remainder of the article. We make the following observation. Note that the reverse implications in the statements of this observation do not hold.

Observation 1. *Every perfect-regular labeling is a symmetric regular labeling. Every symmetric regular labeling is a regular labeling. Every regular labeling is a semiregular labeling and a pseudo-regular labeling. Every semiregular labeling and every pseudo-regular labeling is a connected labeling. Every coloring is a labeling.*

2.2. Graph Homomorphisms

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. In the context of vertex mappings from V_G to V_H, we will always denote the vertices of H by $V_H = \{1, 2, \ldots, |V_H|\}$. In this way, any vertex mapping $f : V_G \to V_H$ is a labeling of G. For a set $U \subseteq V_G$, we write $f(U) = \{f(u) \mid u \in U\}$.

A homomorphism from a graph G to a graph H is a vertex mapping $f : V_G \to V_H$ satisfying the property that for any edge $(u, v) \in E_G$, we have $(f(u), f(v))$ in E_H, in other words, $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V_G$. If there exists a homomorphism from G to H then we write $G \to H$ to express this. Note that this notion generalizes graph colorings; there is a homomorphism from a graph G to K_k (the complete graph on k vertices) if and only if G is k-colorable.

A homomorphism f from G to H is called locally bijective if it induces a one-to-one mapping on the neighborhood of every vertex, i.e., for all $u \in V_G$ it satisfies $f(N_G(u)) = N_H(f(u))$ and $|N_G(u)| = |N_H(f(u))|$. In that case we write $G \xrightarrow{B} H$ instead of $G \to H$ and say that G covers H.
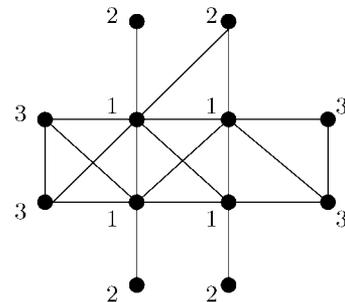
FIG. 3. A graph G with a pseudo-regular labeling.

In Figure 4, we give an example of two graphs G and H with $G \xrightarrow{B} H$. Observe that the left hand side of Figure 4 also serves as an example of a graph with a perfect-regular coloring. This observation can be generalized.

Observation 2. *A graph G has a perfect-regular coloring f if and only if f is a locally bijective homomorphism from G to some graph H.*

A homomorphism f from G to H that induces a surjective mapping on the neighborhood of every vertex is called locally surjective, i.e., for all $u \in V_G$ it satisfies $f(N_G(u)) = N_H(f(u))$. In that case we write $G \xrightarrow{S} H$ instead of $G \rightarrow H$. Note that any locally bijective homomorphism from a graph G to a graph H is locally surjective, i.e., we have that $G \xrightarrow{B} H$ implies $G \xrightarrow{S} H$.

In Figure 5, we give an example of two graphs G and H with $G \xrightarrow{S} H$. Observe that the left hand side of Figure 5 also serves as an example of a graph with a connected coloring. Just as Observation 2, this observation can be generalized.

Observation 3. *A graph G has a connected coloring f if and only if f is a locally surjective homomorphism from G to some graph H.*

2.3. Equitable Partitions

We call a square integer matrix M of order k a degree matrix of a graph G and write $G \xrightarrow{B} M$ if there is a so-called equitable partition of V_G, i.e., a partition of V_G into blocks $\mathcal{B} = B_1, \ldots, B_k$ that, for every i and $u \in B_i$, satisfies:

$$\forall j : |N_G(u) \cap B_j| = m_{i,j}, \qquad (1)$$

and we observe that for all $1 \leq i, j \leq k$, the number of edges between two blocks B_i and B_j is equal to

$$m_{i,j}|B_i| = m_{j,i}|B_j|. \qquad (2)$$

As an example consider the matrix

$$M = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}.$$

The matrix M is a degree matrix of any (p, q)-regular bipartite graph. A graph G can allow several degree matrices. An adjacency matrix is a largest one. The smallest one (up to
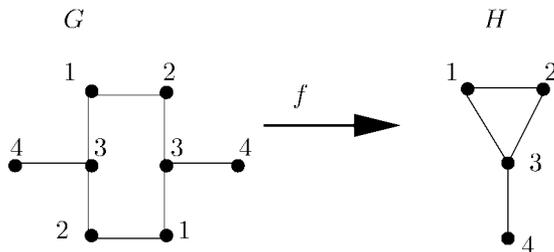
FIG. 4. Two graphs G and H with witness f for $G \xrightarrow{B} H$.
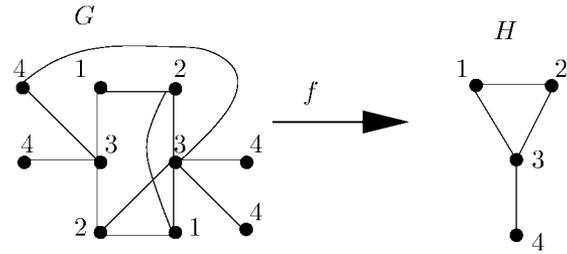
FIG. 5. Two graphs G and H with witness f for $G \xrightarrow{S} H$.

a unique ordering) is called the degree refinement matrix drm(G) of G. For example, if $p = q$ then M is not a degree refinement matrix, because the smaller matrix $M' = (p)$ is a degree matrix of any p-regular graph, and consequently, of any (p, p)-regular bipartite graph. If $p \neq q$, then M is a degree refinement matrix.

3. MAIN RESULTS

We call the six labelings and five colorings defined in Section 2.1 constrained. We define the following two generic problems. In the second problem, k denotes a fixed integer, i.e., not part of the input.

> Proper constrained labeling
> *Instance:* a graph G.
> *Question:* does G have a proper constrained labeling?

> Constrained k-labeling
> *Instance:* a graph G.
> *Question:* does G have a constrained labeling with at most k labels?

For each particular problem, we specify the type of constrained conditions and whether we deal with labelings or colorings. This leads to 11 problems of each of the two kinds.

Here is our first main result stating NP-completeness of the following nine problems. We prove it in Section 6.

Theorem 4. *The following problems are NP-complete:*

- *Proper symmetric regular labeling*
- *Proper perfect-regular labeling*
- *Proper regular labeling*
- *Proper pseudo-regular labeling*
- *Proper perfect-regular coloring*
- *Proper regular coloring*
- *Proper semiregular coloring*
- *Proper pseudo-regular coloring*
- *Proper connected coloring.*

We observe that proper connected labeling is polynomial-time solvable. This can be seen as follows. A graph G has a connected labeling with one color if and only if G is either edgeless, or has no isolated vertices. Otherwise, the labeling that assigns each isolated vertex of G label 1 and each other vertex label 2 is a connected labeling. Such labelings are proper except if G has only one vertex.

We also note that not all NP-completeness results in Theorem 4 were expected in advance due to polynomial-time result for the proper semiregular labeling problem by Boldi and Vigna [4]. They observed that every equitable partition $\mathcal{B} = B_1, \ldots, B_k$ of the vertex set of a graph G corresponds to a semiregular labeling ℓ with $\ell(u) = i$ for $u \in B_i$, and that every semiregular labeling ℓ of G corresponds to an equitable partition with blocks $\ell^{-1}(1), \ldots, \ell^{-1}(|\ell(V_G)|)$. This means that indeed proper semiregular labeling is polynomial-time solvable by using the polynomial-time algorithm of Angluin [2] for computing the degree refinement matrix of a graph; graph G has a proper semiregular labeling using at most $k < n$ colors if and only if drm(G) has size at most k.

Here is our second main result that is complementary to the first one and gives dichotomy results for each constrained k-labeling problem. The proof for the semiregular k-labeling and connected k-labeling problem follows directly from the discussion above. We prove the remaining statements in Section 7.

Theorem 5. *The computational complexity of the following problems can be classified:*

(a) Semiregular k-labeling is polynomial-time solvable for $k \geq 1$.

(b) Perfect regular k-coloring is polynomial-time solvable for $k \leq 3$ and NP-complete for $k \geq 4$.

(c) The following problems are polynomial-time solvable for $k \leq 2$ and NP-complete for $k \geq 3$:
  - *Symmetric regular k-labeling*
  - *perfect-regular k-labeling*
  - *regular k-labeling*
  - *regular k-coloring*
  - *semi-regular k-coloring*
  - *pseudo-regular coloring*
  - *connected k-coloring*

(d) pseudo-regular k-labeling is polynomial-time solvable for $k = 1$ and NP-complete for $k \geq 2$.

(e) Connected k-labeling is polynomial-time solvable for $k \geq 1$.

4. ALGORITHMIC CONSEQUENCES FOR THE ELECTION AND NAMING PROBLEM

We apply Theorem 4 to determine the computational complexity of the problems that ask whether a given graph is a solution graph for the election or naming problem, respectively, in eight basic models in distributed computing. As mentioned in Section 1, these eight models belong to a larger group of 11 models, which can be further divided into four message-passing models [9, 26, 28] and seven local computations models [3, 6, 8, 23, 24]. We will discuss these two types of models separately. We also explain the three models for which this recognition problem has been solved already.

4.1. Message-Passing Models

Yamashita and Kameda [26–28] study four message-passing models. In the port-to-port model, each processor

can send different messages to different neighbors (by having access to unique port-numbers that distinguish between neighbors), and each processor knows the neighbor each receiving message is coming from (again by using the port-numbers). In the broadcast-to-mailbox model, port-numbers do not exist. A processor can only send a message to all of its neighbors and all receiving messages arrive in a mailbox, so it never knows their senders. The two mixed models are called the port-to-mailbox model and the broadcast-to-port model. There exists an election (naming) algorithm for a graph G if and only if the algorithm solves the election (naming) problem on G whatever the port-numbers are.

In [28], Yamashita and Kameda characterize these four models for both the election and naming problems. We note that their models are equivalent to the synchronous versions of the models of Boldi et al. [3] who characterize solution graphs for election using fibrations (for the broadcast-to-port and broadcast-to-mailbox models), coverings (for the port-to-mailbox model), and symmetric coverings (for the port-to-port model) of directed graphs with self-loops and multiple arcs. We use the characterizations of Yamashita and Kameda, because they can be expressed by considering only simple graphs. Below we present them.

Theorem 6 ([28])**.** *The following characterizations of solution graphs for the election and naming problem exist:*

  - *A graph G is a solution graph for the election and naming problem in the port-to-port model if and only if G has no proper symmetric regular labeling.*
  - *A graph G is a solution graph for the election and naming problem in the port-to-mailbox model if and only if G has no proper regular labeling.*
  - *A graph G is a solution graph for the naming problem in the broadcast-to-mailbox and the broadcast-to-port model if and only if G has no proper semiregular labeling.*

    A graph G is a solution graph for the election problem in the broadcast-to-mailbox and the broadcast-to-port model if and only if G has no proper semiregular labeling ℓ such that for all $i \in \ell(V_G)$, $|\ell^{-1}(i)| \geq 2$.

Yamashita and Kameda [27] prove the following co-NP-completeness result for one of the four message-passing models. Note that this result also follows from Theorems 4 and 6.

Theorem 7 ([27])**.** *The problems of deciding whether a graph G is a solution graph for the election and naming problem, respectively, are co-NP-complete for the port-to-port model.*

On the other hand, Theorem 6 and the aforementioned polynomial-time result of Boldi and Vigna [4] for proper semi-regular labeling immediately imply that deciding if a given graph is a solution graph for the naming problem is polynomial-time solvable for the broadcast-to-mailbox and the broadcast-to-port model. For the election problem in these two models, they use a similar degree refinement technique to show polynomial-time solvability.
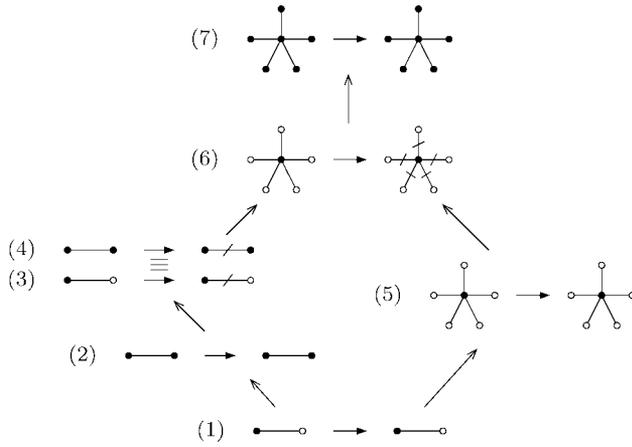
FIG. 6. A hierarchy of local computations models described by the different kinds of relabeling rules they use.

Theorem 8 ([4]). *The problems of deciding whether a graph G is a solution graph for the election and naming problem, respectively, are polynomial-time solvable for the broadcast-to-mailbox and the broadcast-to-port model.*

For the remaining message-passing model, we apply Theorems 4 and 6.

Theorem 9. *The problems of deciding whether a graph G is a solution graph for the election and naming problem, respectively, are co-NP-complete for the port-to-mailbox model.*

4.2. Local Computations Models

In the seven local computations models, a computation step can be described by the application of some local relabeling rule that enables the modification of the states of the different vertices involved in the synchronization. Two local computation models are different in the types of local relabeling rules that they allow. This way some models have a greater computational power than others. Figure 6, which we explain below, displays a hierarchy of the seven local computation models which are numbered from (1) to (7).

The differences between the seven models are as follows. In models (1)–(4), a computation step occurs on an edge, i.e., it involves some synchronization between two neighbors. In models (5)–(7), a computation step occurs on a star, i.e., it involves some synchronization between one vertex and all its neighbors. All models are asynchronous in the sense that not all processors have to be involved in each computation step. In models (1)–(4), two computation steps can occur concurrently if they occur on nonoverlapping edges (i.e., the end vertices of these edges are different). In models (5)–(7), two computation steps can occur concurrently if they occur on stars that do not share any vertex. Labels of black vertices in Figure 6 can change when the local relabeling rule is applied. Labels of white vertices only enable one to apply the rule but

do not change. Only in models (3), (4), and (6) do edges have labels too, and in these three models a rule can modify edge labels as well. Observe that in all seven models these vertex (and edge) labels are only used to encode the state of the processors and have nothing to do with the graph labelings defined in the remainder of this section.

In all these models, one usually speaks about interleaved computations, as any execution can be seen as an execution in which at each step there is exactly one active edge (for models (1)–(4)) or one active star (for models (5)–(7)) wherever a rule is applied. There is no canonical way to define synchronous computations in these models, except for model (5). For this model, Boldi et al. [3] have considered both interleaved computations and synchronous computations. In the latter case, model (5) becomes equivalent to the broadcast-to-mailbox model.

The model hierarchy is displayed as follows. We write $(i) \to (j)$ for two models (i) and (j) if (j) can simulate (i) but not vice versa. This means that (j) has a greater computational power than (i); this relation is transitive. We write $(i) \equiv (j)$ if (i) and (j) have the same computational power. The computational power of model (5) is incomparable with the power of models (2), (3), and (4). Proving this hierarchy is nontrivial, and we refer to [7, 8] for more details.

Mazurkiewicz [23] characterizes solution graphs for model (7).

Theorem 10 ([23]). *A graph G is a solution graph for the election and naming problem in model (7) if and only if G has no proper perfect-regular coloring.*

Chalopin and Métivier [8] characterize solution graphs for models (3), (4), and (6).

Theorem 11 ([8]). *A graph G is a solution graph for the election and the naming problem in models (3), (4), and (6) if and only if G has no proper regular coloring.*

We note that the characterization in Theorem 11 for model (6) can also be obtained from the work of Boldi et al. [3]. Furthermore, the characterization in Theorem 11 also holds for solution graphs for the election and naming problem in the model considered by Angluin in her seminal paper [2] and for the synchronous message-passing model, where a communication between processors requires a synchronization between the sender and the receiver (see [25], p. 47). We also note that Mazurkiewicz [24] has given an equivalent characterization of solution graphs for the election and naming problem in model (4) in terms of equivalence relations over vertices and edges.

Boldi et al. [3] characterize solution graphs for model (5) (this is the interleaved version of their model).

Theorem 12 ([3]). *A graph G is a solution graph for the naming problem in model (5) if and only if G has no proper semiregular coloring. A graph G is a solution graph for the election problem in model (5) if and only if G has no proper*

semiregular coloring ℓ such that $|\ell^{-1}(i)| \geq 2$ for all $i \in \ell(V_G)$.

Chalopin [6] characterizes solution graphs for model (2).

Theorem 13 ([6]). *A graph G is a solution graph for the election and naming problem in model (2) if and only if G has no proper pseudo-regular coloring.*

Chalopin et al. [10] consider the naming and election problem in model (1). They characterize solution graphs for the naming problem in this model.

Theorem 14 ([10]). *A graph G is a solution graph for the naming problem in model (1) if and only if G has no proper connected coloring.*

The same authors [10] also give a characterization of graphs that admit an election algorithm in model (1). They do this in terms of locally surjective homomorphisms, which are closely related to connected colorings, as explained in Observation 3.

Theorem 15 ([10]). *A connected graph G is a solution graph for the election problem in model (1) if and only if the following two conditions are both false:*

1. *There exists a connected graph H such that $G \xrightarrow{S} H$ and such that, for any vertex $v \in V_H$, there exists a subgraph $G(v)$ of G and a locally surjective homomorphism φ from $G(v)$ to H with $|\varphi^{-1}(v)| > 1$.*
2. *There exist two connected graphs H_1 and H_2 with two disjoint subgraphs G_1 and G_2 of G such that $G \xrightarrow{S} H_1$, $G \xrightarrow{S} H_2$, $G_1 \xrightarrow{S} H_1$, and $G_2 \xrightarrow{S} H_2$.*

The characterization in Theorem 15 is useful, because it shows co-NP-membership of the problem of testing whether a given connected graph is a solution graph for the election problem in model (1); note that a certificate for checking conditions 1 and 2 in Theorem 14 has length bounded by a polynomial in $|V_G|$, because a connected graph allows no locally surjective homomorphism to a larger connected graph [16].

A characterization of graphs that admits an election algorithm in model (1) that can be expressed in terms of graphs labelings is not known. Alternatively, the authors of [10] give the following conditions on solution graphs for the election problem in model (1).

Theorem 16 ([10]). *Let G be a graph. Then the following two conditions are valid in model (1).*

(i) *If G has no proper connected coloring, then G is a solution graph for the election problem.*
(ii) *If G has a proper connected coloring ℓ with $|\ell^{-1}(i)| \geq 2$ for all $i \in \ell(G)$, then G is not a solution graph for the election problem.*

We note that the hierarchy in Figure 6 is partially reflected by the relationships between the different labelings as stated in Observation 1: a perfect-regular coloring (model (7)) is also a regular coloring (models (3),(4),(6)). A regular coloring is both a semiregular coloring (model (5)) and a pseudo-regular coloring (model (2)). Both a semiregular coloring and a pseudo-regular coloring are connected colorings (model (1)).

From Theorems 10–16, we note that eight of the 11 constrained labelings and colorings correspond to models of distributed computing; the three exceptions are perfect-regular, connected and pseudo-regular labelings. Combining Theorem 4 with Theorems 10–14, we immediately obtain the following result for the naming problem in models (1)–(7) and for the election problem in models (2), (3), (4), (6), and (7). For the election problem in models (1) and (5), a bit more work is required to obtain co-NP-completeness as stated in Theorem 17; see Remark 24 in Section 6.2.3 and Remark 30 in Section 6.3.3, respectively.

Theorem 17. *The problems of deciding whether a graph G is a solution graph for the election and naming problem, respectively, are co-NP-complete for models (1)–(7).*

5. FUTURE RESEARCH

In this section, we propose the new framework of constrained homomorphisms. If there exists a homomorphism from a graph G to a graph H, then we call G the input graph and H the pattern graph. Recall that we denote the vertices of H by $1, \ldots, |V_H|$. We always assume that input graphs are without multiple edges and self-loops. Pattern graphs do not contain multiple edges either. However, to describe situations in which vertices with the same image may be adjacent, pattern graphs can contain self-loops. A homomorphism f from G to H is constrained if

(i) for all $(i, i) \in E_H$, $G[i]$ satisfies certain conditions, and
(ii) for all $(i, j) \in E_H$ with $i \neq j$, $G[i, j]$ satisfies certain conditions.

We name this framework Conditions on Homomorphisms Imposed by Preimage Subgraphs (CHIPS) and call conditions (i) and (ii) the CHIPS conditions (examples of such conditions will be given later). The generic version of the corresponding decision problem is defined as follows:

H-constrained homomorphism
Instance: a graph G.
Question: Is $G \rightarrow H$ true under CHIPS conditions?

The computational complexity of H-constrained homomorphism depends on H and the type of CHIPS conditions that are under consideration; both H and these conditions are prespecified, i.e., not part of the input. For future research, the following two research questions are interesting:

1. the computational complexity classification of the problem that asks if there exists a constrained homomorphism of a given kind from a given graph G to a fixed graph H;

2. the relationships between all these kinds of constrained homomorphisms.

We end this section with a few examples of constrained homomorphisms already studied in the literature.

Example 1. *Locally constrained homomorphisms.*

Let f be a homomorphism from a graph G to a graph H. We already defined in Section 2.2 when f is locally bijective or surjective. We say that f is locally injective [15] if $|N_G(u)| = |N_H(f(u))|$ holds for all $u \in V_G$. Using CHIPS conditions we can equivalently say that a homomorphism f from G to H is locally injective, locally bijective, or locally surjective if

(i) for all $(i, i) \in E_H$, $G[i]$ has maximum degree at most one, minimum and maximum degree one, or minimum degree at least one, respectively;

(ii) for all $(i, j) \in E_H$ with $i \neq j$, $G[i, j]$ has maximum degree at most one, minimum and maximum degree one, or minimum degree at least one, respectively.

The H-constrained homomorphism problem is called H-cover for the locally bijective constraint, H-partial cover for the locally injective constraint, and H-role assignment for the locally surjective constraint. The complexity classification of the first two problems has been open for many years and is still far from being solved, although many partial results have been obtained; see e.g. [15, 21, 22] for infinite classes of polynomial and NP-complete cases. Contrary to the locally bijective and surjective variants, Fiala and Paulusma [16] obtained a dichotomy theorem for the H-role assignment problem.

Denoting the set of connected simple graphs by $\mathcal{C}$, Fiala et al. [17] showed that $(\mathcal{C}, \xrightarrow{B})$, $(\mathcal{C}, \xrightarrow{I})$ and $(\mathcal{C}, \xrightarrow{S})$ are partial orders with $(\mathcal{C}, \xrightarrow{B}) = (\mathcal{C}, \xrightarrow{I}) \cap (\mathcal{C}, \xrightarrow{S})$. From the proof of Theorem 4 for proper perfect-regular coloring and proper connected coloring, we deduce that the two problems of deciding whether a given graph is minimal in $(\mathcal{C}, \xrightarrow{B})$ or $(\mathcal{C}, \xrightarrow{S})$, respectively, is co-NP-complete. Since $G \xrightarrow{I} H$ for all supergraphs H, there are no minimal elements in the order $(\mathcal{C}, \xrightarrow{I})$.

Example 2. *Pseudo-regular homomorphisms.*

We say that a homomorphism f from a graph G to a graph H is a pseudo-covering of G if f satisfies CHIPS conditions

(i) for all $(i, i) \in E_H$, $G[i]$ is regular, and

(ii) for all $(i, j) \in E_H$ with $i \neq j$, $G[i, j]$ is edgeless or else contains a perfect matching.

Pseudo-coverings correspond to pseudo-regular colorings if pattern graphs are required to be simple, and to pseudo-regular labelings otherwise. We initiated a complexity study for the corresponding decision problem, called H-pseudo-cover, and obtained some partial results [11].
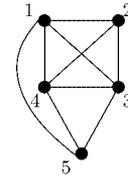
FIG. 7. The graph K.

6. THE PROOF OF THEOREM 4

We split the proof of Theorem 4 into three different parts. In Section 6.1, we prove that proper constrained labeling is NP-complete when we consider symmetric regular labelings, perfect-regular labelings, regular labelings, pseudo-regular labeling, perfect-regular colorings, regular colorings, or pseudo-regular coloring, respectively. Note that these four labelings and three colorings are all pseudo-regular labelings. In Section 6.2, we prove that proper connected coloring is NP-complete. Finally, in Section 6.3, we prove that proper semi-regular coloring is NP-complete.

6.1. All Constrained Labelings that are Pseudo-Regular

In our proofs, we will frequently make use of the following observation.

Observation 18. *Let ℓ be a pseudo-regular labeling of a connected graph G. Then $|\ell^{-1}(i)| = \frac{|V_G|}{|\ell(V_G)|}$ for all $i \in \ell(V_G)$.*

Proof. Let ℓ be a pseudo-regular labeling of a connected graph G. If $|\ell(V_G)| = 1$ then the statement of the observation is true. Suppose $|\ell(V_G)| = k \geq 2$. Because G is connected, the graph H with $V_H = \{1, \ldots, k\}$ and

$E_H = \{ij \mid i \neq j$ and there exists an edge $(u, v) \in E_G$ with

$$\ell(u) = i \text{ and } \ell(v) = j\}$$

is connected, and because $k \geq 2$, it contains at least two vertices. Let j be a neighbor of a vertex $i \in H$. By the definitions of H and ℓ, the subgraph $G[i, j]$ contains a perfect matching. Then, because $G[i, j]$ is bipartite, $|\ell^{-1}(i)| = |\ell^{-1}(j)|$ holds. Because H is connected, we then find that $|\ell^{-1}(i)| = |\ell^{-1}(j)|$ for *any* $i, j \in V_H$ with $i \neq j$. Hence the statement of the observation is true. ∎

6.1.1. The Gadget. Recall that the H-cover problem asks whether there exists a locally bijective homomorphism from an instance graph G to a fixed graph H, i.e., H is not part of the input. In our NP-completeness proof, we use a reduction from the K-cover problem, where K is the graph obtained after deleting an edge in the complete graph K_5 on five vertices; see Figure 7.

The K-cover problem is NP-complete for connected graphs [22]. Note that the two nonadjacent vertices $2, 5$ of K have degree three. The other three vertices, $1, 3, 4$, of K
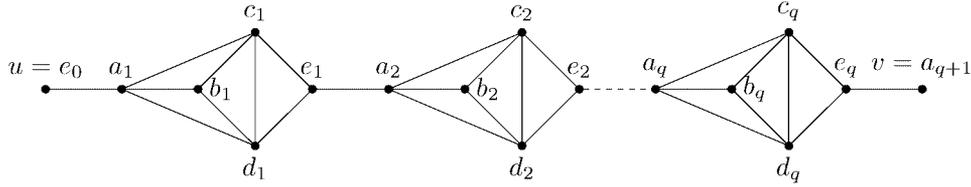
FIG. 8. The chain of q diamonds for diamond pair (u, v).

are adjacent to the two vertices of degree three and to two vertices of degree four. Hence $K \xrightarrow{B} M$ with

$$M = \begin{pmatrix} 0 & 3 \\ 2 & 2 \end{pmatrix}.$$

Let G be a graph with $G \xrightarrow{B} K$. By transitivity, we obtain $G \xrightarrow{B} M$. Hence $G \xrightarrow{B} M$ is a necessary condition for $G \xrightarrow{B} K$, and we therefore call G a K-candidate.

Now let G be a K-candidate. Then, by Equation (1) of Section 2.3, the vertex set of G can be partitioned into two blocks B_1, B_2 such that

- for all $u \in B_1$, $|N_G(u) \cap B_1| = 0$ and $|N_G(u) \cap B_2| = 3$
- for all $u \in B_2$, $|N_G(u) \cap B_1| = 2$ and $|N_G(u) \cap B_2| = 2$.

We use Equation (2) of Section 2.3 to deduce that $3|B_1| = 2|B_2|$. Since $|B_1| + |B_2| = |V_G|$, this means that there exists an integer $k \geq 1$ such that $|B_1| = 2k$ and $|B_2| = 3k$. This implies that G has $5k$ vertices: $2k$ vertices of degree 3 that are adjacent only to vertices of degree 4, and $3k$ vertices of degree 4 that are adjacent to two vertices of degree 3 and two vertices of degree 4.

For our NP-completeness proof, we modify G as follows. Because G is a K-candidate, we can take two adjacent vertices u and v of $\deg_G(u) = 3$ and $\deg_G(v) = 4$. We replace the edge (u, v) by a chain of $q \geq 1$ diamonds $D_1, \ldots, D_q$ as described in Figure 8. Each diamond D_i has vertices a_i, b_i, c_i, d_i, e_i and edges $(a_i, b_i), (a_i, c_i), (a_i, d_i), (b_i, c_i), (b_i, d_i), (c_i, d_i), (c_i, e_i), (d_i, e_i)$. For $i = 1, \ldots, q-1$, two diamonds D_i and D_{i+1} are connected via edge (e_i, a_{i+1}). The chain of diamonds is connected to G by diamond D_1 via edge $(e_0, a_1) = (u, a_1)$ and by diamond D_q via edge $(e_q, a_{q+1}) = (e_q, v)$. We call the resulting graph G^q a diamond graph of G for diamond pair (u, v). We observe that G^q is a K-candidate as well. Also note that the first vertex in a diamond pair has degree 3 and the second vertex has degree 4. For $i = 1, \ldots, q$, we say that D_i is a diamond of G^q.

The next lemma is exactly what we need for our NP-completeness proof.

Lemma 19. *The K-cover problem is* NP-*complete even for the class of diamond graphs of connected K-candidates.*

Proof. Recall that K-cover is NP-complete for connected graphs [22]. Also recall that only K-candidates allow a locally bijective homomorphism to K. Then, because we can check in polynomial time whether a graph G is a K-candidate,

we may assume without loss of generality that an instance graph of the K-cover problem is a connected K-candidate. Recall that we denote the vertices of K by $1, 2, 3, 4, 5$ and its edges by $(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (3, 4),$ $(3, 5), (4, 5)$; see Figure 7.

Let G^q be a diamond graph of a connected K-candidate G for diamond pair (u, v). We claim the following:

$$G \xrightarrow{B} K \text{ if and only if } G^q \xrightarrow{B} K.$$

Suppose $G \xrightarrow{B} K$. Without loss of generality we assume that u has color 5 and v has color 1. Then we assign color 1 to all a_i, color 2 to all b_i, color 3 to all c_i, color 4 to all d_i and color 5 to all e_i. Hence $G^q \xrightarrow{B} K$.

Suppose $G^q \xrightarrow{B} K$. Let f' be a locally bijective homomorphism from G^q to K. We may assume without loss of generality that $f'(a_1) = 1$ and $f'(b_1) = 2$. Then $f'(u) = 5$ and $f'(\{c_1, d_1\}) = \{3, 4\}$. Consequently, $f'(e_1) = 5$. Then $f'(a_2) = 1$ and so on. Continuing this way we find that $f'(e_q) = 5$ and $f'(v) = 1$. Hence the restriction f of $f' : V_{G^q} \to V_K$ to V_G is a witness for $G \xrightarrow{B} K$. ∎

6.1.2. Properties of the Gadget. We present two lemmas that give a number of useful properties of proper pseudo-regular labelings of diamond graphs of K-candidates when we carefully choose the value for q. Lemma 20 shows amongst others that such a labeling is injective on the neighborhood of any vertex in a diamond, and Lemma 21 shows that such a labeling uses exactly 5 labels.

Lemma 20. *Let G be a connected K-candidate on $5k$ vertices with diamond graph G^q for diamond pair (u, v) for some $q \geq k + 3$ such that $q + k$ is a prime number. If ℓ is a proper pseudo-regular labeling of G^q, then $|\ell(V_{D_i})| = 5$ and $\ell(e_{i-1}) \notin \ell(V_{D_i} \setminus \{e_i\})$ for all $1 \leq i \leq q$.*

Proof. Note that G^q is connected, because G is connected. We write $p = q + k$. Then $|V_{G^q}| = 5k + 5q = 5p$ and p is a prime number. Since G^q is not regular, $|\ell(V_{G^q})| > 1$ and by Observation 18, we then find $|\ell(V_{G^q})| = 5$ or $|\ell(V_{G^q})| = p = q + k \geq 2k + 3 \geq 5$. Let D_i be a diamond for some $1 \leq i \leq n$. Recall that we defined $u = e_0$ and $v = a_{q+1}$. We prove the lemma by a sequence of claims. Let $\ell(a_i) = 1$.

Claim 1. *We may assume $\ell(b_i) = 2$.*

We prove this claim as follows. Suppose $\ell(b_i) = 1$.

Suppose $\ell(c_i) = 1$. Suppose $\ell(d_i) = 1$ as well. Then b_i only has neighbors with color 1. Hence, each vertex with the same color as b_i, namely color 1, has only neighbors with color 1. Because G^q is connected, this would mean that $\ell(V_{G^q}) = \{1\}$, so $|\ell(V_{G^q})| = 1 < 5$. This is not possible. So $\ell(d_i) \neq 1$. We assume $\ell(d_i) = 2$. Since $G^q[1, 2]$ contains a perfect matching, we then obtain $\ell(e_{i-1}) = \ell(e_i) = 2$. Then b_i and d_i only have neighbors with colors 1 and 2. Hence, each vertex with the same color as b_i, namely color 1, and each vertex with the same color as d_i, namely color 2, has only neighbors with colors 1 and 2. Since G^q is connected, this would mean that $\ell(V_{G^q}) = \{1, 2\}$, so $|\ell(V_{G^q})| = 2 < 5$. This is not possible. Hence $\ell(c_i) \neq 1$, say $\ell(c_i) = 2$.

If $\ell(d_i) = 1$ then by symmetry we can return to the previous case. Suppose $\ell(d_i) = 2$. Then $\ell(e_i) = 1$ or $\ell(e_i) = 2$, as otherwise $G^q[2, \ell(e_i)]$ does not contain a perfect matching. In both cases, however, $|\ell(V_{G^q})| = 2 < 5$. This not possible. So $\ell(d_i) \notin \{1, 2\}$, say $\ell(d_i) = 3$. If $\ell(e_{i-1}) = 1$ then $G^q[1]$ is not regular. If $\ell(e_{i-1}) = 2$, then $G^q[1, 3]$ does not contain a perfect matching. If $\ell(e_{i-1}) = 3$, then $G^q[1, 2]$ does not contain a perfect matching. So $\ell(e_{i-1}) \notin \{1, 2, 3\}$, say $\ell(e_{i-1}) = 4$. Then $G^q[1, 4]$ does not contain a perfect matching. Hence $\ell(b_i) \neq 1$. From now on we assume $\ell(a_i) = 1$ and $\ell(b_i) = 2$.

Claim 2. *We may assume $\ell(c_i) = 3$.*

We prove this claim as follows. Suppose $\ell(c_i) \in \{1, 2\}$.

First suppose $\ell(c_i) = 1$. Suppose $\ell(d_i) = 1$. Since $G^q[1, 2]$ has a perfect matching, $\ell(e_{i-1}) = \ell(e_i) = 2$. Then $|\ell(V_{G^q})| = 2 < 5$. This is not possible. So $\ell(d_i) \neq 1$. Suppose $\ell(d_i) = 2$. Then, since b_i only has one neighbor with color 2, $G^q[2]$ is 1-regular. This implies that $\ell(e_i) \neq 2$. If $\ell(e_i) > 2$, then $G[2, \ell(e_i)]$ does not contain a perfect matching, because b_i with color 2 does not have a neighbor with color $\ell(e_i)$. Hence $\ell(e_i) = 1$. Then $|\ell(V_{G^q})| = 2 < 5$. This is not possible. So $\ell(d_i) \notin \{1, 2\}$, say $\ell(d_i) = 3$. If $\ell(e_i) \in \{1, 2, 3\}$, then $|\ell(V_{G^q})| = 3 < 5$. This is not possible. So $\ell(e_i) \notin \{1, 2, 3\}$, say $\ell(e_i) = 4$. Since $G^q[1, 4]$ contains a perfect matching, $\ell(e_{i-1}) = 4$. Then $G^q[1, 2]$ does not have a perfect matching. Hence $\ell(c_i) \neq 1$.

Suppose $\ell(c_i) = 2$. If $\ell(d_i) = 1$ then by symmetry we can return to a previous case. If $\ell(d_i) = 2$, then $G^q[1, 2]$ does not contain a perfect matching. So $\ell(d_i) \notin \{1, 2\}$, say $\ell(d_i) = 3$. Since $G^q[2, 3]$ has a perfect matching, $\ell(e_i) = 3$. Then $G^q[1, 2]$ does not allow a perfect matching. Hence $\ell(c_i) \neq 2$. From now on we assume $\ell(a_i) = 1$, $\ell(b_i) = 2$, and $\ell(c_i) = 3$.

Claim 3. *We may assume $\ell(d_i) = 4$.*

We prove this claim as follows. Suppose $\ell(d_i) \in \{1, 2, 3\}$. If $\ell(d_i) = 1$ or $\ell(d_i) = 2$ then by symmetry we can return to a previous case. Suppose $\ell(d_i) = 3$. Since $G^q[2, 3]$ has a perfect matching, $\ell(e_i) = 2$. Then $G^q[1, 3]$ does not contain a perfect matching. Hence $\ell(d_i) \neq 3$. From now on we assume $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, and $\ell(d_i) = 4$.

Claim 4. *We may assume $\ell(e_i) = 5$.*

We prove this claim as follows. Suppose $\ell(e_i) \in \{1, 2, 3, 4\}$. First suppose $\ell(e_i) = 1$. Since $G^q[1, 2]$ has a perfect matching, $\ell(a_{i+1}) = 2$. Then $|\ell(V_{G^q})| = 4 < 5$. This is not possible. So $\ell(e_i) \neq 1$.

Suppose $\ell(e_i) = 2$. Since $G^q[1, 2]$ has a perfect matching, $\ell(a_{i+1}) = 1$. Then $G^q[2, 3]$ does not have a perfect matching. Hence $\ell(e_i) \neq 2$.

Suppose $\ell(e_i) = 3$. Since $G^q[3, 4]$ has a perfect matching, $\ell(a_{i+1}) = 4$. Then $G^q[2, 3]$ does not have a perfect matching. Hence $\ell(e_i) \neq 3$. By symmetry $\ell(e_i) \neq 4$. From now on we assume $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, $\ell(d_i) = 4$, and $\ell(e_i) = 5$.

Note that we have deduced above that $|\ell(V_{D_i})| = 5$. Hence we are left to prove $\ell(e_{i-1}) \notin \{1, 2, 3, 4\}$. Suppose $\ell(e_{i-1}) = 1$. Then e_{i-1} has neighbors colored $1, 2, 3, 4$. This is not possible, since $\deg_{G^q}(e_{i-1}) = 3$. Suppose $\ell(e_{i-1}) = 2$. Then the two neighbors of e_{i-1} outside D_i have colors 3 and 4. Then $G^q[1, 2]$ does not have a perfect matching. Suppose $\ell(e_{i-1}) = 3$. Then e_{i-1} must have neighbors colored $1, 2, 4, 5$. This is not possible, since $\deg_{G^q}(e_{i-1}) = 3$. By symmetry, $\ell(e_{i-1}) \neq 4$. Hence e_{i-1} does not have a color in $\{1, 2, 3, 4\}$. This completes the proof of Lemma 20. ∎

Lemma 21. *Let G be a connected K-candidate on $5k$ vertices with diamond graph G^q for diamond pair (u, v) for some $q \geq k + 3$ such that $q + k$ is a prime number. If ℓ is a proper pseudo-regular labeling of G^q then $|\ell(V_{G^q})| = 5$.*

Proof. Note that G^q is connected, because G is connected. We write $p = q + k$. Then $|V_{G^q}| = 5k + 5q = 5p$ and p is a prime number. By Observation 18, we then find $|\ell(V_{G^q})| = 5$ or $|\ell(V_{G^q})| = p$. We note that $p = q + k \geq 2k + 3 \geq 5$.

Suppose $|\ell(V_{G^q})| = p > 5$. By our choice of q, we have $\sum_{i=1}^{q} |V_{D_i}| = 5q \geq 2q \geq q + k + 3 = p + 3 > p$. Hence there exists a vertex u in a diamond D_i with the same color as a vertex v in a diamond D_j. By Lemma 20, we find $i \neq j$, say $i < j$. We choose u and v such that there do not exist two vertices in $G[D_i \cup \cdots \cup D_{j-1}]$ that have the same color. By Lemma 20, we can write $\ell(a_i) = 1$, $\ell(b_i) = 2$, $\ell(c_i) = 3$, $\ell(d_i) = 4$ and $\ell(e_i) = 5$. By the same lemma, we then obtain $\ell(e_{i-1}) \notin \{1, 2, 3, 4\}$. If $\ell(e_{i-1}) = 5$ then $\ell(a_{i+1}) = 1$, and consequently, $|\ell(V_{G^q})| = 5 < p$. So we can write $\ell(e_{i-1}) = 6$.

By construction of G^q, every vertex of G^q has either degree 3 or 4. The following two claims are helpful.

Claim 1. *Each vertex x with $\ell(x) \in \{1, 3, 4\}$ has $\deg_{G^q}(x) = 4$.*

We prove this claim as follows. Let x be a vertex in G^q. Suppose $\ell(x) = 1$. Then $\{2, 3, 4, 6\} \subseteq \ell(N_{G^q}(x))$. Hence $\deg_{G^q}(x) = 4$. Suppose $\ell(x) = 3$. Then $\{1, 2, 4, 5\} \subseteq \ell(N_{G^q}(x))$. Hence $\deg_{G^q}(x) = 4$. Suppose $\ell(x) = 4$. Then $\{1, 2, 3, 5\} \subseteq \ell(N_{G^q}(x))$. Hence $\deg_{G^q}(x) = 4$.

Claim 2. *Each vertex y with $\ell(y) \in \{2, 5\}$ has $\deg_{G^q}(y) = 3$.*

We prove this claim by contradiction. Suppose there exists a vertex y in G^q with $\deg(y) = 4$ and $\ell(y) \in \{2, 5\}$. Suppose $\ell(y) = 2$. Then $\ell(N_{G^q}(y)) = \{1, 3, 4\}$. By Claim 1, y has three neighbors of degree four. This is not possible by construction of G^q.

Suppose $\ell(y) = 5$. Then $\ell(N_{G^q}(y)) = \{3, 4, \ell(a_{i+1})\}$. Since $i < j$, vertex a_{i+1} belongs to a diamond D_{i+1}. By Lemma 20, we know that $|\ell(N_{G^q}(a_{i+1}))| = 4$. Then each vertex x with $\ell(x) = \ell(a_{i+1})$ has $\deg_{G^q}(x) = 4$. Then the neighbor of y that has color $\ell(a_{i+1})$ has degree four. By Claim 1, the neighbors of y with colors 3 and 4 have degree four as well. Hence y has three neighbors of degree four. This is not possible by construction of G^q. This finishes the proof of Claim 2.

We will use Claim 1 and 2 to show a contradiction, namely that none of the colors 1, 2, 3, 4, 5 can occur on D_j.

First we show $1 \notin \ell(D_j)$. By Claim 1, only vertices a_j, c_j, d_j can have color 1. Suppose $\ell(a_j) = 1$. From our choice of D_i and D_j, all vertices in $D_i \cup D_{i+1} \cup \cdots \cup D_{j-1}$ have a different color, i.e., $|\ell(D_i \cup D_{i+1} \cup \cdots \cup D_{j-1})| = 5(j-i)$. Then $\ell(e_{j-1}) \notin \{2, 3, 4\}$. This implies $\ell(\{b_j, c_j, d_j\}) = \{2, 3, 4\}$ and $\ell(e_{j-1}) = 6$. We then obtain $\ell(V_G) = \ell(D_i \cup \cdots \cup D_{j-1})$, so $p = |\ell(V_G)| = 5(j-i)$. Since p is a prime number not equal to 5, this is not possible. Hence $\ell(a_j) \neq 1$. Suppose $\ell(c_j) = 1$ (respectively $\ell(d_j) = 1$). Then $\ell(d_j) \in \{3, 4\}$ (respectively $\ell(c_j) \in \{3, 4\}$) and $\ell(\{b_j, e_j\}) = \{2, 6\}$. Then a vertex with color in $\{3, 4\}$ is adjacent to a vertex with color 6. This is not possible. Hence $1 \notin \ell(D_j)$.

We show $2 \notin \ell(D_j)$. By Claim 2, only b_j and e_j can have color 2. If $\ell(b_j) = 2$, then $1 \in \ell(\{a_j, c_j, d_j\})$. This is not possible as proved above. If $\ell(e_j) = 2$, then either $1 \in \ell(\{c_j, d_j\})$, or else $\ell(\{c_j, d_j\}) = \{3, 4\}$ which implies $\ell(a_j) = 1$. So, also in this case we find $1 \in \ell(D_j)$, which is not possible as we saw before. Hence $2 \notin \ell(D_j)$.

We show $3 \notin \ell(D_j)$. By Claim 1, only vertices a_j, c_j, d_j can have color 3. Suppose $\ell(a_j) = 3$. Since $1 \notin \ell(D_j)$, the neighbor of a_j with color 1 is e_{j-1}. This is not possible, since we chose D_i and D_j such that all vertices in $D_i \cup D_{i+1} \cup \cdots \cup D_{j-1}$ have different colors. If $\ell(c_j) = 3$ or $\ell(d_j) = 3$ we would have $1 \in \ell(D_j)$. Hence $3 \notin \ell(D_j)$. By symmetry, we obtain $4 \notin \ell(D_j)$.

Finally, we show $5 \notin \ell(D_j)$. By Claim 2, only vertices b_j and e_j can have color 5. In both cases, at least one of the colors 3, 4 is a color of a vertex in D_j. This is not possible as shown above. This finishes the proof of Lemma 21. ∎

6.1.3. The Reduction.

We are now ready to show that the following problems are NP-complete:

1. Proper symmetric regular labeling
2. Proper perfect-regular labeling
3. Proper regular labeling
4. Proper pseudo-regular labeling
5. Proper perfect-regular coloring
6. Proper regular coloring
7. Proper pseudo-regular coloring

Proof. We start with proving NP-membership. First of all, we can efficiently check if a given labeling ℓ for an n-vertex graph G is proper, and if it is one of the seven labelings above. Indeed, by their definitions, we only have to check $O(n)$ subgraphs $G[i]$ for

- being edgeless (all the colorings)
- regularity (all the labelings)
- containing a perfect matching (the symmetric regular labelings)

and only $O(n^2)$ subgraphs $G[i, j]$ of G for

- regular bipartiteness (the (symmetric) regular labelings, and the (perfect-)regular colorings)
- semiregular bipartiteness (the semiregular labelings and colorings)
- being a perfect matching (the perfect-regular labelings and colorings)
- containing a perfect matching (the pseudo-regular labelings and colorings)

To check if a graph contains a perfect matching we can use the polynomial-time algorithm for finding a maximum-size matching in [12]. The other conditions are also easy to check. Hence, all the seven graph problems are in NP.

We now prove NP-completeness by a reduction from K-cover [22]. Let G be a connected K-candidate, so $|V_G| = 5k$ for some $k \geq 1$. We construct a diamond graph G^q of G for some diamond pair (u, v), where we choose q such that $q \geq k + 3$ and $p = q + k$ is a prime number. We can find such a q in polynomial time due to Bertrand's postulate that states that for each integer $n \geq 4$ there exists a prime number p in the interval $[n, 2n - 2]$. By taking $n = 2k + 3 \geq 4$, we then find that there exists a prime number p in the interval $[2k + 3, 4k + 4]$. Hence we can find an appropriate value for q in the interval $[k + 3, 3k + 4]$. By Lemma 19, we may consider G^q as our instance graph for the K-cover problem (note $G \overset{B}{\to} K$ if and only if $G^q \overset{B}{\to} K$ as shown in the proof of Lemma 19).

We claim that the following statements are equivalent.

(i) $G^q \overset{B}{\to} K$.
(ii) G^q has a proper perfect-regular coloring.
(iii) G^q has a proper regular coloring.
(iv) G^q has a proper pseudo-regular coloring.
(v) G^q has a proper perfect-regular labeling.
(vi) G^q has a proper symmetric regular labeling.
(vii) G^q has a proper regular labeling.
(viii) G^q has a proper pseudo-regular labeling.

The proof of this claim is as follows.

(i) $\Rightarrow$ (ii)

Suppose $G^q \overset{B}{\to} K$. Let f' be a locally bijective homomorphism from G^q to K (so $|f'(V_G)| = 5$). Recall that $V_K = \{1, 2, 3, 4, 5\}$. So f' is a proper perfect-regular coloring by Observation 2.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (viii)

This follows directly from the definitions, as noted in Observation 1.
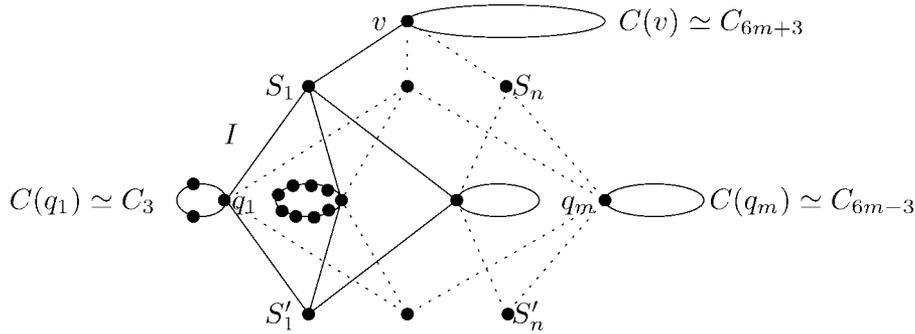
FIG. 9. An example of a C_3-minimizer I^* of a hypergraph $(Q, \mathcal{S})$.

(ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii)

This follows directly from the definitions, as noted in Observation 1.

(viii) $\Rightarrow$ (i)

Suppose G^q allows a proper pseudo-regular labeling ℓ. By Lemma 20, $|\ell(D_1)| = 5$. Let $\ell(a_1) = 1$, $\ell(b_1) = 2$, $\ell(c_1) = 3$, $\ell(d_1) = 4$ and $\ell(e_1) = 5$. By Lemma 20, $\ell(e_0) \notin \{1, 2, 3, 4\}$. Since $|\ell(V_G)| = 5$ due to Lemma 21, we then find that $\ell(e_0) = 5$. This means that ℓ defines a locally bijective homomorphism from G to K. ∎

6.2. Connected Colorings

We start with a lemma that we need later on.

Lemma 22. *Let ℓ be a proper connected coloring of a graph G. Let $x_1, \ldots, x_k$ be a sequence of k different colors from $\ell(V_G)$ such that $G[x_i, x_{i+1}]$ is not edgeless for $1 \le i \le k - 1$. Then, for each vertex r in G with color x_1, there exists a path $P = r_1 r_2 \ldots r_k$ from $r_1 = r$ to some vertex r_k such that $\ell(r_h) = x_h$ for $h = 1, \ldots, k$.*

Proof. We prove the statement by induction on k. Let $k = 1$. Since $G[x_1, x_2]$ contains an edge, r has a neighbor r_2 with color x_2. Let $k \ge 2$. By the induction hypothesis, there exists a path $P' = r_1 r_2 \ldots r_{k-1}$ from $r_1 = r$ to r_{k-1} such that $\ell(r_i) = x_i$ for $i = 1, \ldots, k - 1$. Since $G[x_{k-1}, x_k]$ is not edgeless, every vertex in $G[x_{k-1}, x_k]$ has degree at least one, by definition of a connected coloring. This means that r_{k-1} needs a neighbor with color x_k. Because all colors in $\{x_1, \ldots, x_k\}$ are different, there exists a neighbor $r_k \notin \{r_1, \ldots, r_{k-2}\}$ of r_{k-1} that has color x_k. Hence, we have proven Lemma 22. ∎

6.2.1. The Gadget.
A hypergraph $(Q, \mathcal{S})$ is a set $Q = \{q_1, \ldots, q_m\}$ together with a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of Q. A *2-coloring* of a hypergraph $(Q, \mathcal{S})$ is a partition of Q into $Q_1 \cup Q_2$ such that $Q_1 \cap S_j \ne \emptyset$ and $Q_2 \cap S_j \ne \emptyset$ for $1 \le j \le n$. In our proof, we use reduction from the following, well-known, NP-complete problem (cf. [18]).

Hypergraph 2-colorability
Instance: A hypergraph $(Q, \mathcal{S})$.
Question: Does $(Q, \mathcal{S})$ have a 2-coloring?

We call a hypergraph $(Q, \mathcal{S})$ with $\emptyset \notin \mathcal{S}$, $\bigcup_j S_j = Q$ and $S_j \ne S_k$ for all $j \ne k$ nontrivial. It is easy to see that the Hypergraph 2-colorability restricted to nontrivial hypergraphs remains NP-complete. With a hypergraph $(Q, \mathcal{S})$ we associate its incidence graph I, which is a bipartite graph on $Q \cup \mathcal{S}$, where (q, S) forms an edge if and only if $q \in S$.

Let C_i denote the cycle on i vertices. Given the incidence graph I of a non-trivial hypergraph $(Q, \mathcal{S})$ we construct the following graph. First we make a copy S' for each $S \in \mathcal{S}$. We add edges (S', q) if and only if $q \in S$. Let $\mathcal{S}' = \{S'_1, \ldots, S'_n\}$. Then we glue a cycle $C(q_i) \simeq C_{6i-3}$ to q_i for $1 \le i \le m$. We add a new vertex v and edges from v to all vertices in $\mathcal{S}$. Finally, we glue a cycle $C(v) \simeq C_{6m+3}$ to v. We call the resulting graph I^* the C_3-minimizer of $(Q, \mathcal{S})$. See Figure 9 for an example. Note that I^* is connected because $(Q, \mathcal{S})$ is non-trivial.

6.2.2. Properties of the Gadget.
In Lemma 23, we show that proper connected colorings of C_3-minimizers use exactly three colors. This is crucial information for our NP-completeness proof.

Lemma 23. *Let I^* be the C_3-minimizer of a non-trivial hypergraph $(Q, \mathcal{S})$. If ℓ is a proper connected coloring of I^* then $|\ell(V_{I^*})| = 3$.*

Proof. Suppose ℓ is a proper connected coloring of I^*. We note that, by definition, two neighbors must be mapped to different colors. We write $\ell(q_1) = 1$. Let the other two vertices of $C(q_1)$ be s, t with $\ell(s) = 2$ and $\ell(t) = 3$. If q_1 only has neighbors with color 2 or 3 then $\ell(V_{I^*}) = \{2, 3\}$, and we are done.

Suppose q_1 has a neighbor in $\mathcal{S} \cup \mathcal{S}'$ with a color not in $\{2, 3\}$. Then all vertices of I^* with color 1 have at least degree three. By a sequence of claims, we show that $|\ell(I_{I^*})| = |V_{I^*}|$. This gives us a contradiction to our assumption that ℓ is proper.

Claim 1. *Colors $2, 3$ are not in $\ell(V_{I^*} \setminus \{s, t\})$.*

We prove Claim 1 by contradiction. Suppose $\ell(w) \in \{2, 3\}$ for some $w \in V_{I^*} \setminus \{s, t\}$. By symmetry, we may assume $\ell(w) = 2$. We consider three cases.

First suppose $w \in V_{C(p)} \setminus \{p\}$ for some $p \in (Q \setminus \{q_1\}) \cup \{v\}$. Then w needs a neighbor with color 1. Recall that such a neighbor must have degree at least three. The only candidate is p. However, w also needs a neighbor with color 3. This neighbor of color 3 must be adjacent to a vertex with color 1. Since $|C(p)|$ contains at least nine vertices, the latter vertex (the one with color 1) is on $C(p)$ and has degree two. Since every vertex of color 1 has degree at least three, this is not possible.

Next suppose $w = p$ for some $p \in Q \cup \{v\}$. Let x be a neighbor of w on $C(p)$. Then x must have color 1 or 3. Color 1 is not possible since x has degree $2 < 3$. Color 3 is not possible, since then x has a (degree-two) neighbor y on $C(p)$ with color 1.

Finally suppose $w = S$ for some $S \in \mathcal{S} \cup \mathcal{S}'$. Then w must have a neighbor p', which is in $Q \cup \{v\}$, with color 3. By symmetry (use color 3 instead of color 2), we can return to the previous case. This finishes the proof of Claim 1.

Claim 2. *For all $p \in Q \cup \{v\}$, $|\ell(V_{C(p)})| = |V_{C(p)}|$.*

For $p = q_1$, the condition of Claim 2 is satisfied. For all other p we prove Claim 2 by contradiction. Let $|\ell(V_{C(p)})| < |V_{C(p)}|$ for some $p \in (Q \setminus \{q_1\}) \cup \{v\}$. This means that at least two vertices in $C(p)$ share the same color.

First suppose $z \in V_{C(p)} \setminus \{p\}$ has color $\ell(p)$. By Claim 1, color 2 is not a color of any vertex in $C(p)$. Since p is a cutvertex of I^* and since I^* is a connected graph, any path from z (with color $\ell(p)$) to a vertex with color 2 contains p (with color $\ell(p)$). This is not possible due to Lemma 22. Hence p is the only vertex in $C(p)$ with color $\ell(p)$.

Since two vertices in $C(p)$ have the same color, we obtain $\ell(u_1) = \ell(u_2)$ for some $u_1, u_2 \in V_{C(p)} \setminus \{p\}$. Let P be the path from u_1 to p that does not use u_2. Let P^* be the path from u_2 to p that does not use u_1. We apply Lemma 22 with $x_1 = \ell(u_1)$ and $x_k = 2$. Then we find that P and P^* must use exactly the same $|V_P| = |V_{P^*}|$ colors (in exactly the same order). We choose u_1 and u_2 such that P and P^* are maximal. Let v_1 be the neighbor of u_1 not on P, and let v_2 be the neighbor of u_2 not on P^*. If v_1 is not equal to v_2, then v_1 and v_2 must have the same color by definition of a connected coloring. This contradicts the maximality of P and P^*. Hence $v_1 = v_2$.

So we have found that two colors, namely $\ell(p)$ and $\ell(v_1)$, appear once on $C(p)$ while all other colors appear exactly twice on $C(p)$. This means that $|C(p)|$ is even. This is not possible, since $|C(p)|$ is odd by construction. Hence, we have proven Claim 2.

Claim 3. *On any two cycles $C(p)$ and $C(q)$ with $p, q \in Q \cup \{v\}$, only p and q can share the same color, i.e., if $\ell(V_{C(p)}) \cap \ell(V_{C(q)}) \neq \emptyset$ then $\ell(p) = \ell(q)$ and $\ell(V_{C(p)} \setminus \{p\}) \cap \ell(V_{C(q)} \setminus \{q\}) = \emptyset$.*

We prove Claim 3 as follows. For some $p, q \in Q \cup \{v\}$ with $p \neq q$, let x be a common color of $C(p)$ and $C(q)$, i.e., $x \in \ell(V_{C(p)}) \cap \ell(V_{C(q)})$. We may assume without loss of generality that $|V_{C(p)}| < |V_{C(q)}|$.

First suppose $x = \ell(p)$. Below we prove that q is the only vertex of $C(q)$ that can have color x by showing that any vertex with color x must have degree at least three. Due to Claim 2, both neighbors of p on $C(p)$ have a different color. Suppose these colors are the only colors that the neighbors of p have, i.e., $|\ell(N_{I^*}(p))| = 2$. Then $|\ell(V_{I^*})| = |V_{C(p)}|$. This not possible: by Claim 2 and our choice of p and q, the number of different colors on I^* is at least $|V_{C(v)}| \geq |V_{C(q)}| > |V_{C(p)}|$. So on the neighborhood of p at least three different colors are used. This means that any vertex with color $\ell(p)$ must have degree at least three. Hence, if $x \in \ell(V_{C(q)})$ then q is the only vertex of $C(q)$ that can have color $\ell(p)$.

Now suppose $x \neq \ell(p)$, say $x = \ell(u) = \ell(u')$ for some $u \in C(p) \setminus \{p\}$ and $u' \in C(q)$. Below we show that this case is not possible. Then we are done.

Suppose $u' = q$. By Claim 2, both neighbors of u have a different color, so $|\ell(N_{I^*}(q))| = 2$. By Claim 2, the two neighbors of q in $C(q)$ do not have the same color. This means that the color x^* of a vertex $u^* \in N_{I^*}(u) \setminus \{p\}$ is a color of a neighbor of q in $C(q)$. Then we can replace x by x^* and u by u^*. Hence, we may without loss of generality assume that $u' \neq q$.

Let P_{up} and P_{up}^* be the two (vertex-disjoint) paths from u to p in $C(p)$. By Claim 2, all vertices on P_{up} and P_{up}^* have different colors. By our choice of p and q, $C(q)$ contains at least six more vertices than $C(p)$. Then, by Lemma 22, $C(q)$ contains a path $P_{u'p'}$ from u' to a vertex $p' \neq q$ with the same colors as P_{up} or with the same colors as P_{up}^*, so p' has color $\ell(p)$. Since $p' \neq q$, p' has degree two. This is not possible, since we already showed that q is the only vertex of $C(q)$ that can have the same color as p. This finishes the proof of Claim 3.

To prove the lemma we need a claim that is stronger than Claim 3.

Claim 4. *$\ell(V_{C(p)}) \cap \ell(V_{C(q)}) = \emptyset$ for all $p, q \in Q \cup \{v\}$ with $p \neq q$.*

We prove Claim 4 as follows. Suppose $\ell(V_{C(p)}) \cap \ell(V_{C(q)}) \neq \emptyset$ for some $p, q \in Q \cup \{v\}$ with $p \neq q$. By Claim 3, $\ell(p) = \ell(q)$. Let r_1 be a neighbor of p on $C(p)$. Then r_1 has degree two in I^*. Let $r_2 \neq p$ be the other neighbor of r_1. By definition of a connected coloring, q must have a neighbor with color $\ell(r_1)$. By Claim 3, $\ell(r_1)$ is the color of a vertex $S \in \mathcal{S} \cup \mathcal{S}'$. Again by definition, S must have a neighbor with color $\ell(r_2)$. By construction, this neighbor is a vertex $q' \in Q \cup \{v\}$. By Claim 2, $\ell(r_2) \neq \ell(p)$. Hence $q' \neq p$. Then $C(p)$ and $C(q')$ have common color $\ell(r_2)$. This violates Claim 3 and completes the proof of Claim 4.

By Claim 2 and Claim 4, all vertices in the union of all cycles $C(p)$ over $p \in Q \cup \{v\}$ are mapped to different colors. As $(Q, \mathcal{S})$ is non-trivial, any two $S_j, S_k \in \mathcal{S}$ with $j \neq k$ represent two different subsets of Q. Hence they cannot have the same color. The same holds for any two $S_j', S_k' \in \mathcal{S}'$ because they are copies of sets $S_j, S_k \in \mathcal{S}$. Furthermore, all S_j' are not adjacent to v. This means that $\mathcal{S}$ and $\mathcal{S}'$ do not share
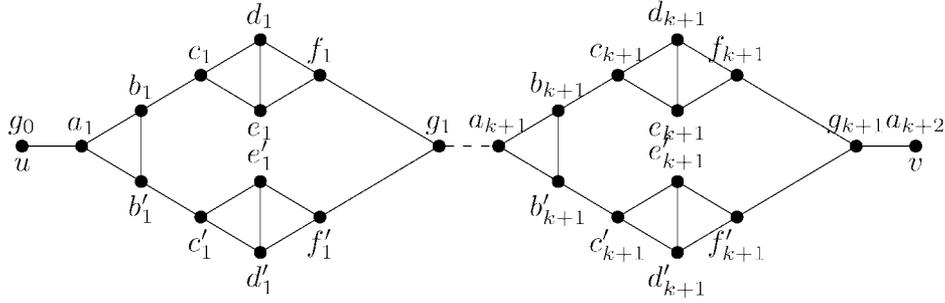
FIG. 10. The chain of $k + 1$ multi-diamonds that replace edge $e_k = (u, v)$.

any colors, i.e., $\ell(\mathcal{S}) \cap \ell(\mathcal{S}') = \emptyset$. Hence we have obtained $|\ell(\mathcal{S} \cup \mathcal{S}')| = |\mathcal{S} \cup \mathcal{S}'|$.

Suppose some $S \in \mathcal{S} \cup \mathcal{S}'$ has the same color as a vertex u of some $C(p)$. Then the colors of the neighbors of u on $C(p)$ must appear on the neighbors of S, which all belong to $Q \cup \{v\}$. This violates Claim 4. Hence $|\ell(V_{I^*})| = |V_{I^*}|$, so ℓ is not proper. This finishes the proof of Lemma 23. ∎

6.2.3. The Reduction. We are now ready to show that the PROPER CONNECTED COLORING problem is NP-complete.

Proof. To show membership in NP we must verify the following for a given labeling ℓ of an n-vertex graph G. First we must check $O(n)$ subgraphs $G[i]$ for being edgeless. Second, we must check $O(n^2)$ subgraphs $G[i, j]$ of G for minimum degree. Both checks can be performed in polynomial time. Hence, the problem is a member of NP.

We prove NP-completeness by a reduction from the Hypergraph 2-colorability problem restricted to non-trivial hypergraphs. Let $(Q, \mathcal{S})$ be a non-trivial hypergraph. We construct its C_3-minimizer I^* and claim that the following three statements are equivalent. Note that we prove a bit more than required by including statement (ii). The reason we do this is made clear in Remark 24.

(i) $(Q, \mathcal{S})$ has a 2-coloring.
(ii) I^* admits a proper connected coloring ℓ with $|\ell^{-1}(i)| \geq 2$ for all $1 \leq i \leq |\ell(V_G)|$.
(iii) I^* admits a proper connected coloring.

The proof is as follows.

$(i) \Rightarrow (ii)$

Suppose $(Q, \mathcal{S})$ has a 2-coloring $Q_1 \cup Q_2$. Define $\ell(v) = 1$, $\ell(S) = 2$ for all $S \in \mathcal{S} \cup \mathcal{S}'$, $\ell(q) = 1$ for all $q \in Q_1$ and $\ell(q) = 3$ for all $q \in Q_2$. Finish the coloring in the obvious way. Note that we use exactly three colors. Since $C(v)$ contains at least six vertices, $|\ell^{-1}(i)| \geq 2$ for all $1 \leq i \leq 3$.

$(ii) \Rightarrow (iii)$

This is trivial.

$(iii) \Rightarrow (i)$

Suppose I^* has a proper connected coloring ℓ. By Lemma 23 we find $|\ell(V_{I^*})| = 3$. Since $C(q_1)$ is isomorphic to C_3, we find that the three vertices of $C(q_1)$ have three

different colors, say 1,2,3. Then, by definition, any vertex colored by 1 (resp. 2,3) has a neighbor colored by 2 (resp. 3, 1) and a neighbor colored by 3 (resp. 1, 2). We assume without loss of generality that $\ell(v) = 1$. Then $\ell(S_j) \in \{2, 3\}$ for all j. If $\ell(S'_j) = 1$ for some j, then S'_j needs a neighbor of color 2 and a neighbor of color 3. Both neighbors are adjacent to S_j that has color 2 or 3. This is not possible. Hence we find $\ell(S'_j) \in \{2, 3\}$ for all j. We define $Q_1 = \{q \in Q \mid \ell(q) = 1\}$ and $Q_2 = Q \backslash Q_1$. Since each S'_j needs at least two neighbors with different colors and at least one neighbor with color 1, the partition $Q_1 \cup Q_2$ is a 2-coloring of $(Q, \mathcal{S})$. ∎

Remark 24. *The equivalence of statements (i)–(iii) above, combined with Theorem 16, shows that Theorem 17 holds for the leader election problem in model (1).*

6.3. Semiregular Colorings

Our proof combines new arguments with the ingredients of the proofs in both Sections 6.1 and 6.2.

6.3.1. The Gadget. Recall that the H-cover problem asks whether there exists a locally bijective homomorphism from an instance graph G to a fixed graph H. In our NP-completeness proof we use a reduction from the K_4-cover problem, where K_4 is the complete graph on four vertices denoted by 1,2,3, and 4. The K_4-cover problem is NP-complete for connected graphs [20].

Let G be a connected graph with $G \xrightarrow{B} K_4$. Because $G \xrightarrow{B} K_4$ and K_4 is 3-regular, we find that G is 3-regular. By definition, any locally bijective homomorphism f from G to K_4 is a pseudo-regular labeling. Then we can use Observation 18 to deduce that $|f^{-1}(i)| = \frac{|V_G|}{4}$ for $i = 1, \ldots, 4$. This means that $|V_G| = 4q$ for some $q \geq 1$. Since $2|E_G| = 3|V_G|$, we then obtain $|E_G| = \frac{3|V_G|}{2} = 6q$. We therefore call a graph G that is 3-regular and that has $|V_G| = 4q$ and $|E_G| = 6q$ for some $q \geq 1$ a K_4-candidate.

For our NP-completeness proof we modify a K_4-candidate as follows. Let $E_G = \{e_1, e_2, \ldots, e_m\}$. For each edge e_k, we do as follows. First we choose an orientation of e_k, say $e_k = (u, v)$ is oriented from u to v. Then we replace e_k by a chain of $k + 1$ multi-diamonds $D_1(k), \ldots, D_{k+1}(k)$, as described in Figure 10. Each multi-diamond $D_{k+1}(k)$ has vertices

$$a_i(k), b_i(k), b'_i(k), c_i(k), c'_i(k), d_i(k), d'_i(k), e_i(k), e'_i(k),$$
$$f_i(k), f'_i(k), g_i(k).$$

We call the vertices in a multi-diamond $D_i(k)$ that are not equal to $a_i(k)$ or $g_i(k)$ the inner vertices of $D_i(k)$. If no confusion is possible, we will write a_i for $a_i(k)$, b_i for $b_i(k)$ etc. A chain of multi-diamonds is connected to G by multi-diamond $D_1(k)$ via edge $(g_0(k), a_1(k)) = (u, a_1(k))$ and by multi-diamond $D_{k+1}(k)$ via edge $(g_{k+1}(k), a_{k+2}(k)) = (g_{k+1}(k), v)$. Note that, in contrast to the diamonds in Section 6.1.1, this operation is not symmetric: we would have obtained a different graph if we had chosen to orient e_k from v to u. After performing the operation above for each edge in G, we obtain a graph G' that we call a multi-diamond graph of G. We observe that G' is a K_4-candidate as well.

The next lemma is exactly what we need for our NP-completeness proof.

Lemma 25. *The K_4-cover problem is NP-complete even for the class of multi-diamond graphs of connected K_4-candidates.*

Proof. Recall that K_4-cover is NP-complete for connected graphs [20]. Also recall that only K_4-candidates allow a locally bijective homomorphism to K_4. Then, because we can check in polynomial time whether a graph G is a K_4-candidate, we may assume without loss of generality that an instance graph of the K_4-cover problem is a connected K_4-candidate. Recall that we denote the vertices of K_4 by $1, \ldots, 4$.

Let G' be a multi-diamond graph of a K_4-candidate G with $|E_G| = m$. We claim that $G \xrightarrow{B} K_4$ if and only if $G' \xrightarrow{B} K_4$.

Suppose $G \xrightarrow{B} K_4$. For all $1 \le k \le m$, we proceed as follows. Let $(u, v) = e_k$ be an edge in G. We may assume without loss of generality that u has color 1 and v has color 2. For $1 \le i \le k+1$, we assign color 1 to all c_i, c'_i, g_i, color 2 to all a_i, d_i, d'_i, color 3 to all b_i, e'_i, f_i and color 4 to all b'_i, e_i, f'_i. This way we obtain a witness for $G' \xrightarrow{B} K_4$.

Suppose $G' \xrightarrow{B} K_4$. Let ℓ' be a locally bijective homomorphism from G' to K_4. Let $(u, v) = e_k$ be an edge in G. We may assume without loss of generality that $\ell'(g_0) = \ell'(u) = 1$ and $\ell'(a_1) = 2$. Then $\ell'(\{b_1, b'_1\}) = \{3, 4\}$, and consequently, $\ell'(\{c_1, c'_1\}) = \{1\}$. We may assume without loss of generality that $\ell'(b_1) = 3$ and $\ell'(b'_1) = 4$. Then $\ell'(\{d_1, e_1\}) = \{2, 4\}$ and $\ell'(\{d'_1, e'_1\}) = \{2, 3\}$. Then $\ell'(f_1) = 3$ and $\ell'(f'_1) = 4$. Consequently $\ell'(g_1) = 1$ and $\ell'(a_2) = 2$. Continuing this way, we find that $\ell'(g_{k+1}) = 1$ and $\ell'(v) = \ell'(a_{k+2}) = 2$. Hence, the restriction ℓ of $\ell' : V_{G'} \to V_{K_4}$ to V_G is a witness for $G \xrightarrow{B} K_4$. ∎

6.3.2. Properties of the Gadget. In Lemmas 26–29, we state a number of useful properties of semiregular colorings of multi-diamond graphs. The first lemma gives us a relation between the colors of different a-vertices and g-vertices.

Lemma 26. *Let G' be a multi-diamond graph of a connected K_4-candidate G with $|E_G| = m$. If ℓ is a semiregular*

coloring of G' then for all $1 \le i \le k+1$, for all $1 \le k \le m$ and for all $1 \le j \le k'+1$, for all $1 \le k' \le m$,

$$\ell(g_i(k)) = \ell(g_j(k')) \quad and \quad \ell(a_{i+1}(k)) = \ell(a_{j+1}(k'))$$
$$\Longleftrightarrow$$
$$\ell(g_{i-1}(k)) = \ell(g_{j-1}(k')) \quad and \quad \ell(a_i(k)) = \ell(a_j(k')).$$

Proof. Let $1 \le i \le k+1$ for some $1 \le k \le m$, and let $1 \le j \le k'+1$ for some $1 \le k' \le m'$.

Suppose $\ell(g_i(k)) = \ell(g_j(k'))$ and $\ell(a_{i+1}(k)) = \ell(a_{j+1}(k'))$. By definition of a semiregular coloring, $\ell(\{f_i(k), f'_i(k)\}) = \ell(\{f_j(k'), f'_j(k')\})$. Without loss of generality, we may assume $\ell(f_i(k)) = \ell(f_j(k'))$ and $\ell(f'_i(k)) = \ell(f'_j(k'))$. Consequently, $\ell(\{d_i(k), e_i(k)\}) = \ell(\{d_j(k'), e_j(k')\})$. Then $\ell(c_i(k)) = \ell(c_j(k'))$. This means we have $\ell(b_i(k)) = \ell(b_j(k'))$. By symmetry, we obtain $\ell(b'_i(k)) = \ell(b'_j(k'))$. Hence, we find $\ell(a_i(k)) = \ell(a_j(k'))$ and consequently $\ell(g_{i-1}(k)) = \ell(g_{j-1}(k'))$.

We show the reverse implication by the same arguments.
∎

The next lemma shows that if a multi-diamond graph G' does not cover K_4, then the vertices in any multi-diamond of G' must all have different colors. So, the neighbors of the inner vertices in a multi-diamond are all different.

Lemma 27. *Let G' be a multi-diamond graph of a connected K_4-candidate G with $|E_G| = m$. Let ℓ be a semiregular coloring of G'. If G' does not cover K_4, then $|\ell(D_i(k))| = 12$ for all $1 \le i \le k+1$, for all $1 \le k \le m$.*

Proof. Let $D_i = D_i(k)$ be a multi-diamond in G'. Note that c_i, d_i, e_i have different colors. We write $\ell(d_i) = 1$, $\ell(e_i) = 2$ and $\ell(c_i) = 3$.

Claim 1. *We may assume $\ell(b_i) = 4$ and $\ell(b'_i) \notin \{\ell(d'_i), \ell(e'_i)\}$.*

Note that $\ell(b_i) \ne 3$. We will also show $\ell(b_i) \notin \{1, 2\}$. We write $x = \ell(f_i)$ and $y = \ell(g_i)$. Suppose $\ell(b_i) \in \{1, 2\}$. By symmetry, we may assume $\ell(b_i) = 1$. Then either $\ell(a_i) = 2$ and $\ell(b'_i) = x$, or $\ell(a_i) = x$ and $\ell(b'_i) = 2$.

First suppose $\ell(a_i) = 2$ and $\ell(b'_i) = x$. Then c'_i and g_i have the same color. Since f'_i is a neighbor of g_i, this implies that $\ell(f'_i)$ must be the color of a neighbor of c'_i. Suppose $\ell(f'_i) = x$. By definition of a semiregular coloring, c'_i also has at least two neighbors with color x. Then either $\ell(d'_i) = \ell(f'_i)$ or $\ell(e'_i) = \ell(f'_i)$. Both cases are not possible because adjacent vertices may not share the same color. If $\ell(f'_i) \ne x$, then $\ell(f'_i)$ must still be a color of a neighbor of c'_i. This neighbor can not be b'_i since $\ell(b'_i) = x$. So it must be d'_i or e'_i, and we obtain the same contradiction.

Now suppose $\ell(a_i) = x$ and $\ell(b'_i) = 2$. Then $\ell(c'_i) = 3$. Since $\ell(c'_i) = \ell(c_i)$ and c_i has two neighbors with color 1, we then find that c'_i has two neighbors with color 1. Since b'_i already has color 2, this implies that $\ell(e'_i) = \ell(d'_i) = 1$. This is not possible by definition of a semiregular coloring.

From the above, we conclude that $\ell(b_i) \notin \{1,2,3\}$. So we may write $\ell(b_i) = 4$. By symmetry, we find $\ell(b_i') \notin \{\ell(d_i'), \ell(e_i')\}$.

Claim 2. *We may assume $\ell(f_i) = 5$ and $\ell(c_i') \neq \ell(f_i')$.*

Note that $\ell(f_i) \notin \{1,2\}$. We will also show $\ell(f_i) \notin \{3,4\}$.

First suppose $\ell(f_i) = 3$. We write $x = \ell(a_i)$ and $y = \ell(b_i')$. Since f_i must have a neighbor with color 4, we obtain $\ell(g_i) = 4$. Consequently, either $\ell(f_i') = x$ or $\ell(f_i') = y$. In the first case, if $\ell(f_i') = x$, then $y = \ell(b_i')$ must belong to $\{\ell(d_i'), \ell(e_i')\}$. By Claim 1, this is not possible. In the second case, if $\ell(f_i') = y$, we use $4 = \ell(b_1) = \ell(g_i)$ to find that $\ell(c_i') = \ell(d_i')$ or $\ell(c_i') = \ell(e_i')$. Both options are not possible.

Now suppose $\ell(f_i) = 4$. Then $\{\ell(a_i), \ell(b_i')\} = \{1,2\}$ and $\ell(g_i) = 3$. Then, for $i = 1,2,3,4$, each vertex u in G' with color i has $N(u) = \{1,2,3,4\}\setminus\{i\}$. Then $G' \xrightarrow{B} K_4$. This is a contradiction.

From the above, we conclude that $\ell(f_i) \notin \{1,2,3,4\}$. So we may write $\ell(f_i) = 5$. By symmetry, we obtain $\ell(c_i') \neq \ell(f_i')$.

Claim 3. *We may assume $\ell(a_i) = 6$ and $6 \notin \{\ell(b_i'), \ell(c_i'), \ell(d_i'), \ell(e_i'), \ell(f_i')\}$.*

Note that $\ell(a_i) \neq 4$. We will also show that $\ell(a_i) \notin \{1,2,3,5\}$. Since a_i has a neighbor, namely b_i, with color 4 whereas d_i and e_i do not have such a neighbor, we obtain $\ell(a_i) \notin \{1,2\}$. Suppose $\ell(a_i) \in \{3,5\}$. Then $\ell(b_i') \in \{1,2\}$. We may assume without loss of generality that $\ell(b_i') = 1$. However, then colors 1 and 4 belong to adjacent vertices, namely b_i' and b_i. Since $\ell(d_i) = 1$, and $4 \notin \ell(N_{G'}(d_i)) = \{2,3,5\}$, this is not possible. Hence $\ell(a_i) \notin \{1,2,3,4,5\}$. So we may write $\ell(a_i) = 6$. By symmetry, we obtain $6 \notin \{\ell(b_i'), \ell(c_i'), \ell(d_i'), \ell(e_i'), \ell(f_i')\}$.

Claim 4. *We may assume $\ell(b_i') = 7$.*

Note that $\ell(b_i') \notin \{1,2,3,4,6\}$. Suppose $\ell(b_i') = 5$. Then $\ell(N_{G'}(b_i'))$ must contain $\{1,2,4,6\}$. This is not possible, since $\deg_{G'}(b_i') = 3$. Hence $\ell(b_i') > 6$, so we may write $\ell(b_i') = 7$.

Claim 5. *We may assume $\ell(c_i') = 8$.*

Note that $\ell(c_i') \notin \{1,2,3,7\}$. The color of c_i' cannot be 4 because then 3 and 6 are colors of adjacent vertices, namely the neighbors d_i' and e_i' of c_i', and this is not possible. By Claim 3, $\ell(c_i') \neq 6$.

Suppose $\ell(c_i') = 5$. Then $\ell(g_i) = 7$ and $\ell(\{d_i', e_i'\}) = \{1,2\}$. Consequently, $\ell(f_i') = 3$. Then colors 3 and 7 belong to adjacent vertices (namely f_i' and g_i). However, this is not possible, since $\ell(c_i) = 3$ and $7 \notin \ell(N_{G'}(c_i)) = \{1,2,4\}$. Hence $\ell(c_i') > 7$, so we may write $\ell(c_i') = 8$.

Claim 6. *We may assume $\ell(d_i') = 9$ and $\ell(e_i') = 10$.*

We will first show that $\ell(d_i') > 8$. First note that $\ell(d_i') \notin \{1,2,3,4,8\}$. Suppose $\ell(d_i') = 5$. Then $\ell(e_i') \in \{1,2\}$. This is

not possible since $8 \notin \ell(N_{G'}(d_i)) \cup \ell(N_{G'}(e_i)) = \{1,2,3,5\}$. By Claim 3, $\ell(d_i') \neq 6$. Suppose $\ell(d_i') = 7$. Then 6 is the color of e_i' or f_i'. By Claim 3, this is not possible. Hence, indeed $\ell(d_i') > 8$ and we we may write $\ell(d_i') = 9$.

By symmetry, $\ell(e_i') > 8$. Since d_i' and e_i' are adjacent, $\ell(d_i') \neq \ell(e_i')$. Hence $\ell(e_i') > 9$, and we may write $\ell(e_i') = 10$.

Claim 7. *We may assume $\ell(f_i') = 11$.*

Note that $\ell(f_i') \notin \{1,2,3,4,7,9,10\}$. Suppose $\ell(f_i') = 5$. Then $\ell(N_{G'}(f_i'))$ must contain $\{1,2,9,10\}$. This is not possible since $\deg_{G'}(f_i') = 3$. By Claim 3, $\ell(f_i') \neq 6$. By Claim 2, $\ell(f_i') \neq 8$. Hence $\ell(f_i') > 10$, and we may write $\ell(f_i') = 11$.

Claim 8. *We may assume $\ell(g_i') = 12$.*

Note that $\ell(g_i') \notin \{1,2,3,4,5,7,8,9,10,11\}$. Suppose $\ell(g_i') = 6$. Then $\ell(N_{G'}(g_i'))$ contain colors $4,5,7,11$. This is not possible, since $\deg_{G'}(g_i') = 3$. Hence $\ell(g_i') > 11$, and we may write $\ell(g_i') = 12$. This completes the proof of Lemma 27. ∎

The next lemma shows the following. If G' does not cover K_4, then the color of each $g_i(k)$ cannot be assigned to the vertices of any multi-diamond $D_j(k')$ excluding $g_j(k')$, and the same holds for the color of each $a_i(k)$. We need this lemma to prove in Lemma 29 that also the neighbors of vertices that are not inner vertices of multi-diamonds have different colors.

Lemma 28. *Let G' be a multi-diamond of a connected K_4-candidate G with $|E_G| = m$. Let ℓ be a semiregular coloring of G'. If G' does not cover K_4, then*

(i) $\ell(g_i(k)) \notin \ell(D_j(k') \setminus \{g_j(k')\})$ for all $1 \leq i \leq k+1$, for all $1 \leq k \leq m$ and for all $1 \leq j \leq k'+1$, for all $1 \leq k' \leq m$.

(ii) $\ell(a_i(k)) \notin \ell(D_j(k') \setminus \{a_j(k')\})$ for all $1 \leq i \leq k+1$, for all $1 \leq k \leq m$ and for all $1 \leq j \leq k'+1$, for all $1 \leq k' \leq m$.

(iii) $\{\ell(g_0(k)), \ell(a_{k+2}(k))\} \cap \ell(D_j(k')) = \emptyset$ for all $1 \leq j \leq k'+1$, for all $1 \leq k, k' \leq m$.

Proof. Note that for $1 \leq k \leq m$, we have defined vertices $a_{k+2}(k)$ and $g_0(k)$, while we did not define $a_0(k)$ and $g_{k+2}(k)$. We first prove (i), then (ii) and then (iii).

(i) We use a proof by contradiction. Suppose there exist indices i,j,k,k' such that there is a vertex $u \in D_j(k') \setminus \{g_j(k')\}$ with $\ell(u) = \ell(g_i(k))$. Then $(i,k) \neq (j,k')$ due to Lemma 27. We note that there exist two vertices $v, w \in N_{G'}(u) \cap D_j(k')$ with $(v,w) \in E_{G'}$. Because $\ell(u) = \ell(g_i(k))$, we find that at least one of the vertices v, w, has a color in $\{\ell(f_i(k)), \ell(f_i'(k))\}$. We may assume without loss of generality that $\ell(v) = \ell(f_i(k))$. Then w has a color in $\{\ell(d_i(k)), \ell(e_i(k))\}$. We may assume without loss of generality that $\ell(w) = \ell(d_i(k))$. Then we have found that $\ell(d_i(k))$ and $\ell(g_i(k))$ are colors of adjacent vertices (namely of u and w). We consider $D_i(k)$ again. By definition of a semiregular coloring, $\ell(g_i(k))$ must be a color of one of the neighbors of $d_i(k)$. Since these neighbors are $c_i(k), e_i(k), f_i(k)$, we find that $\ell(g_i(k))$ appears twice on
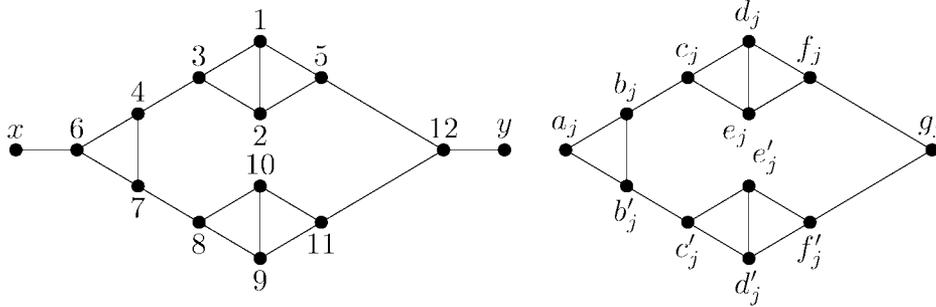
FIG. 11.   The two multi-diamonds in the proof of Lemma 28 (ii).

$D_i(k)$. This is a contradiction to Lemma 27. Hence we have proven (i).

(ii) Also here, we use a proof by contradiction. Note that we cannot use the same proof as for statement (i) because the two cases are not symmetric. Suppose there exist indices i, j, k, k' such that there is a vertex $u \in D_j(k') \setminus \{a_j(k')\}$ with $\ell(u) = \ell(a_i(k))$. Then $(i, k) \neq (j, k')$ due to Lemma 27.

By Lemma 27, we may assume $\ell(d_i(k)) = 1, \ell(e_i) = 2, \ell(c_i(k)) = 3, \ell(b_i(k)) = 4, \ell(f_i(k)) = 5, \ell(a_i(k)) = 6, \ell(b'_i(k)) = 7, \ell(c'_i(k)) = 8, \ell(d'_i(k)) = 9, \ell(e'_i(k)) = 10, \ell(f'_i(k)) = 11, \ell(g_i(k)) = 12$; see the left hand side of Figure 11. We write x and y for $\ell(g_{i-1}(k))$ and $\ell(a_{i+1}(k))$, respectively.

We write a_j for $a_j(k')$, and so on; see the right hand side of Figure 11. Due to (i), g_j does not have color $\ell(a_i(k)) = 6$. By symmetry, we are done (i.e., we get our desired contradiction) if we can show that the color $\ell(v)$ of each $v \in \{b_j, c_j, d_j, f_j\}$ is not equal to 6. We consider these four cases separately.

Suppose $\ell(c_j) = 6$. Then $\{\ell(d_j), \ell(e_j)\} \cap \{4, 7\} \neq \emptyset$. We may assume without loss of generality that $\ell(d_j) = 4$. Then c_j and d_j both have a neighbor with color 7. By Lemma 27, we then obtain $\ell(e_j) = 7$. Since e_j has a neighbor with color 8, we then find $\ell(f_j) = 8$. Then colors 4 and 8 belong to adjacent vertices (namely d_j and f_j) but this is impossible, since it is not the case in $D_i(k)$. Hence $\ell(c_j) \neq 6$.

Suppose $\ell(f_j) = 6$. Then $\{\ell(d_j), \ell(e_j)\} \cap \{4, 7\} \neq \emptyset$. We may assume without loss of generality that $\ell(d_j) = 4$. Then f_j and d_j both have a neighbor with color 7. By Lemma 27, we then obtain $\ell(e_j) = 7$. Since e_j has a neighbor with color 8, we then find $\ell(c_j) = 8$. Then colors 4 and 8 belong to adjacent vertices (namely d_j and c_j) but this is impossible, since it is not the case in $D_i(k)$. Hence $\ell(f_j) \neq 6$.

Suppose $\ell(d_j) = 6$. Then $\{\ell(c_j), \ell(f_j)\} \cap \{4, 7\} \neq \emptyset$. We assume without loss of generality that $4 \in \{\ell(c_j), \ell(f_j)\}$. Suppose $\ell(c_j) = 4$. Then c_j and d_j both have a neighbor with color 7. By Lemma 27, we then obtain $\ell(e_j) = 7$. Since e_j must have a neighbor with color 8, we then find $\ell(f_j) = 8$. Then a vertex with color 8, namely f_j, has a neighbor with color 6, namely d_j. This is not possible, since $\ell(b'_i(k)) = 8$ and $4 \notin \ell(N_G(b'_i(k)))$. Hence $\ell(c_j) \neq 4$. Suppose $\ell(f_j) = 4$. Then f_j and d_j both have a neighbor with color 7. Again, by Lemma 27, we obtain $\ell(e_j) = 7$. Since e_j must have a neighbor with color 8, we then obtain $\ell(c_j) = 8$. Then we find the

same contradiction, namely two adjacent vertices with colors 6 and 8. Hence $\ell(d_j) \neq 6$.

Suppose $\ell(b_j) = 6$. If $i \geq 2$, then b_j must have a neighbor with color $x = \ell(g_{i-1}(k))$. This is not possible, since $\ell(g_{i-1}(k))$ is not a color of $D_j(k) \setminus \{g_j(k)\}$, as proved in part (i) of this lemma. Hence, $i = 1$ and $y = \ell(a_2(k))$. Note that $a_2(k)$ is a vertex of a multi-diamond, by construction of G'.

Since b_j has color 6, colors 4 and 7 must be colors of neighbors of b. If c_j has color 4 or 7, then 6 is the color of two vertices of $D_j(k)$, namely of b_j and f_j. This is not possible due to Lemma 27. Hence, $\ell(\{a_j, b'_j\}) = \{4, 7\}$. We may assume without loss of generality that $\ell(a_j) = 7$ and $\ell(b'_j) = 4$. Consequently, we obtain $\ell(c'_j) = 3, \ell(\{d'_j, e'_j\}) = \{1, 2\}, \ell(f'_j) = 5$ and $\ell(g_j) = 12$. Then either $\ell(f_j) = \ell(a_2(k)) = y$, or $\ell(f_j) = 11$. We consider each case. Suppose $\ell(f_j) = \ell(a_2(k)) = y$. In the same way as we have proven that f_j does not have color $\ell(a_1(k))$ we can show that f_j does not have color $\ell(a_2(k))$ either. Suppose $\ell(f_j) = 11$. Then $\ell(\{d_j, e_j\}) = \{9, 10\}$ and consequently $\ell(c_j) = 8$. This is not possible, since a vertex with color 8, namely c_j, does not have a neighbor with color 6. This way we have proven statement (ii).

(iii) We use a proof by contradiction. Let $u = a_{k+2}(k)$ for some $1 \leq k \leq m$. Let $v \in D_j(k')$ for some $1 \leq j \leq k' + 1$ and some $1 \leq k' \leq m$ such that v and u have the same color $\ell(v) = \ell(u)$. Then v has a neighbor v' in $D_j(k') \setminus \{a_i(k), g_i(k')\}$. Since $\ell(u) = \ell(v)$, there exists a vertex $u' \in N_{G'}(u)$ such that $\ell(v') = \ell(u')$. We observe that either $u' = a_1(k^*)$ or $u' = g_{k'+1}(k^*)$ for some $1 \leq k^* \leq m$. By (i) and (ii), the color of $a_1(k^*)$ and the color of $g_{k+1}(k^*)$ cannot be assigned to some inner vertex of a multi-diamond $D_j(k')$. So we obtain a contradiction. The case $u = g_0(k)$ uses exactly the same arguments. This completes the proof of Lemma 28.   ∎

The proof of the next lemma explains why we replaced each edge in a K_4-candidate by a different number of multi-diamonds. It shows that the neighbors of all vertices have different colors if G' does not cover K_4. Hence our semiregular coloring is a perfect-regular coloring.

Lemma 29.   *Let G' be a multi-diamond graph of a connected K_4-candidate G. Let ℓ be a semiregular coloring of G'. If G' does not cover K_4, then ℓ is a perfect-regular coloring of G'.*

Proof. Let $|E_G| = m$. By definition of a perfect-regular coloring, we prove this lemma by showing that the neighbors of any vertex $u \in V_{G'}$ all have different colors. This statement is true if u is an inner vertex of a multi-diamond, due to Lemma 27.

Suppose $u = g_i(k)$ for some $1 \leq i \leq k+1$ and some $1 \leq k \leq m$. By Lemma 27, we obtain $\ell(f_i(k)) \neq \ell(f_i'(k))$. By Lemma 28, we obtain $\ell(a_{i+1}(k)) \notin \{\ell(f_i(k)), \ell(f_i'(k))\}$. Hence all neighbors of u have a different color. The case $u = a_i(k)$ for $1 \leq i \leq k+1$ uses the same arguments.

In the two remaining cases, we have $u = g_0(k)$ or $u = a_{k+2}(k)$, respectively, for some $1 \leq k \leq m$. First suppose $u = g_0(k)$ for some $1 \leq k \leq m$. Suppose u has neighbors v, w in G' with $\ell(v) = \ell(w)$. We distinguish three possibilities. First, if $\{v, w\} = \{a_1(k'), g_{k^*+1}(k^*)\}$ for some $1 \leq k', k^* \leq m$, then we obtain a contradiction due to Lemma 28. Second, suppose $\{v, w\} = \{a_1(k'), a_1(k^*)\}$, say $v = a_1(k')$ and $w = a_1(k^*)$, for some $1 \leq k' \leq k^* - 1$. Note $u = g_0(k') = g_0(k^*)$. So we have $\ell(g_0(k')) = \ell(g_0(k^*))$ and $\ell(a_1(k')) = \ell(a_1(k^*))$. We apply Lemma 26 for $k'+1$ times in order to obtain $\ell(a_{k'+2}(k')) = \ell(a_{k'+2}(k^*))$. This contradicts Lemma 28. Third, suppose $\{v, w\} = \{g_{k'+1}(k'), g_{k^*+1}(k^*)\}$ for some $1 \leq k' \leq k^* - 1$. This third possible case uses the same arguments as the second one. Hence we obtain that all neighbors of u have a different color.

Suppose $u = a_{k+2}(k)$ for some $1 \leq k \leq m$. We can prove that all neighbors of u have a different color in the same way as we did for $u = g_0(k)$. This completes the proof of Lemma 29. ∎

6.3.3. The Reduction.

We are now ready to show that the Proper semi-regular coloring problem is NP-complete.

Proof. To show membership in NP, we must verify the following for a given labeling ℓ of a graph G. First we must check $O(n)$ subgraphs $G[i]$ for being edgeless. Second, we must check $O(n^2)$ subgraphs $G[i, j]$ of G for semiregular bipartiteness. Both checks can be performed in polynomial time. Hence, the problem is a member of NP.

We prove NP-completeness by a reduction from the K_4-cover problem. By Lemma 25, we may consider a multi-diamond graph G' of a connected K_4-candidate G as our instance. We claim that the following three statements are equivalent. Note that we prove a bit more than required by including statement (ii). The reason we do this is made clear in Remark 30.

  (i) G' covers K_4.
 (ii) G' has a proper semiregular coloring ℓ with $|\ell^{-1}(i)| \geq 2$ for all $1 \leq i \leq |\ell(V_G)|$.
(iii) G' has a proper semiregular coloring.

The proof is as follows.

(i) $\Rightarrow$ (ii)

Suppose G' covers K_4. Let f be a locally bijective homomorphism from G' to K_4. By Observation 2, f is a proper perfect-regular coloring, and consequently, a proper semiregular coloring of G' that uses four colors. On any multi-diamond of G' these four colors appear at least twice. Hence $|f^{-1}(i)| \geq 2$ for all $1 \leq i \leq |f(V_G)|$.

(ii) $\Rightarrow$ (iii)

This is trivial.

(iii) $\Rightarrow$ (i)

Suppose G' does not cover K_4. We show that G' does not have a proper semiregular coloring.

Since we can map each vertex of G' to a unique color, G has at least one semiregular coloring. So, let ℓ be a semiregular coloring of G'. By Lemma 29, ℓ is a perfect-regular coloring, and consequently, a pseudo-regular labeling of G'. We show that any vertex u that does not belong to some multi-diamond of G' has a unique color $\ell(u) = x$, in other words $|\ell^{-1}(x)| = 1$. Since ℓ is a pseudo-regular labeling, we then apply Observation 18 to conclude that all vertices in G' have a unique color. Hence, ℓ is not proper and we are done.

So, let $\ell(u) = x$ be a color of a vertex $u \in V_{G'}$ that is not in any multi-diamond, i.e., $u = g_0(k)$ or $u = a_{k+2}(k)$ for some $1 \leq k \leq m$. First suppose $u = g_0(k)$. We will use a proof by contradiction to show that $x \notin \ell(V_{G'} \backslash \{u\})$.

Suppose $x = \ell(v)$ for some vertex $v \in V_{G'} \backslash \{u\}$. By Lemma 28, v does not belong to some multi-diamond. Hence $v = a_{k'+2}(k')$ or $v = g_0(k')$ for some $1 \leq k' \leq m$. We write $y = \ell(a_1(k))$. Since ℓ is a perfect-regular coloring of G', v has a neighbor v' with $\ell(v') = y$. By Lemma 28, $v' \neq g_{k^*+1}(k^*)$ for some $1 \leq k^* \leq m$. Hence $v' = a_1(k^*)$ for some $1 \leq k^* \leq m$. By construction of G' (also see Fig. 11), we then have $v = g_0(k^*)$. So we have $x = \ell(g_0(k)) = \ell(g_0(k^*))$ and $y = \ell(a_1(k)) = \ell(a_1(k^*))$. We may assume without loss of generality that $k \leq k^* - 1$. We apply Lemma 25 $k + 1$ times to obtain $\ell(a_{k+2}(k)) = \ell(a_{k+2}(k^*))$. This contradicts Lemma 28. The case $u = a_{k+2}(k)$ can be proven using the same arguments. Hence u is the only vertex in G' with color x. As explained earlier on, we may now conclude that G' does not allow a proper semiregular coloring. ∎

Remark 30. *The equivalence of statements (i)–(iii) above, combined with Theorem 12, shows that Theorem 17 holds for the leader election problem in model (5).*

Remark 31. *As perfect-regular and regular colorings are semiregular colorings, NP-completeness of Proper perfect-regular coloring and Proper regular coloring also follows from the proof for Proper semi-regular coloring; we use the same arguments after replacing each occurrence of the adjective "semiregular" by "regular" or "perfect-regular," respectively.*

7. THE PROOF OF THEOREM 5

The proofs for Semi-regular k-labeling and Connected k-labeling are given in Section 3. Here is the proof for the other nine statements. We first analyze the proofs in Section 6.

This immediately gives us NP-completeness results for many values of k. Then we discuss the remaining (small) values of k for each of the nine labelings separately.

7.1. Using the Proofs for Proper Labelings

We first discuss the case $k = 1$. In Section 6, we showed membership of NP for every proper labeling problem by explaining how to check in polynomial time if a given labeling of a graph G is a required labeling. This implies that the case $k = 1$ is polynomial-time solvable (because for $k = 1$ we have $\ell \equiv 1$). We now consider every reduction in Section 6 to obtain a number of NP-completeness results.

Recall the reduction in Section 6.1.3. For any $k \geq 5$, we let G be a K-candidate on at least k vertices (such that a labeling of diamond graph G^q that uses at most k labels is proper). Then we can replace statements (ii)–(viii) in the reduction in Section 6.1.3 by

(ii) G^q has a perfect-regular coloring that uses at most k colors;
(iii) G^q has a regular coloring that uses at most k colors;
(iv) G^q has a pseudo-regular coloring that uses at most k colors;
(v) G^q has a symmetric regular labeling that uses at most k labels;
(vi) G^q has a perfect-regular labeling that uses at most k labels;
(vii) G^q has a regular labeling that uses at most k labels;
(viii) G^q has a pseudo-regular labeling that uses at most k labels,

and repeat all arguments made in this reduction. We then find that the following problems are NP-complete for any fixed $k \geq 5$:

- Perfect regular k-coloring
- Regular k-coloring
- Pseudo-regular k-coloring
- Symmetric regular k-labeling
- Perfect-regular k-labeling
- Regular k-labeling
- Pseudo-regular k-labeling

Note that for $k \leq 4$ we cannot use the argument above, because $|V_K| = 5$.

Recall the reduction in Section 6.2.3. For any $k \geq 3$ we let (Q, S) be a non-trivial hypergraph with $|Q| \geq k$ (such that a connected coloring of C_3-minimizer I^* that uses at most k labels is proper). Then we can replace statement (iii) in the reduction in Section 6.2.3 by

(iii) I^* admits a connected coloring that uses at most k colors,

and repeat all arguments made in this reduction. We then find that the connected k-coloring problem is NP-complete for any fixed $k \geq 3$. Note that for $k \leq 2$ we cannot use the argument above, because any connected coloring of I^* uses at least three colors due to the presence of a triangle.

Recall the reduction in Section 6.3.3. For any $k \geq 4$, we let G be a K_4-candidate on at least k vertices (such that a labeling of a multi-diamond graph G' that uses at most k labels is proper). Then we replace statement (iii) in the reduction

in Section 6.3.3 by the following three statements (see also Remark 31):

(iii) G' has a perfect-regular coloring using at most k colors;
(iv) G' has a regular coloring using at most k colors;
(v) G' has a semiregular coloring using at most k colors,

and repeat all arguments made in this reduction. We then find that Perfect-regular k-coloring, Regular k-coloring and Semi-regular k-coloring are NP-complete for any fixed $k \geq 4$; note that we already proved above that the first two problems are NP-complete for any fixed $k \geq 5$. New arguments are required for $k \leq 3$, because $|V_{K_4}| = 4$.

7.2. The Remaining Values of k

To complete our dichotomy results, we use reductions from several NP-complete problems. We explain them below.

7.2.1. Known NP-Complete Problems used in the Proofs. For several cases, we use a variant on the Hypergraph 2-colorability problem for some specific classes of hypergraphs, which we explain below.

A hypergraph (Q, S) with incidence graph I is called p-regular if every $q \in Q$ belongs to exactly p sets in S, or equivalently, if $\deg_I(q) = p$ for all $q \in Q$. Hypergraph (Q, S) is called m-uniform if every set $S \in S$ contains exactly m elements of Q, or equivalently, if $\deg_I(S) = m$ for all $S \in S$.

For any $1 \leq k \leq m - 1$, an m-uniform hypergraph (Q, S) is said to have a (k-in-m)-coloring if there exists a partition of Q into $Q_1 \cup Q_2$ such that $|Q_1 \cap S_j| = k$ (and consequently $|Q_2 \cap S_j| = m - k$) for each $S_j \in S$. Then, for fixed integers k, m with $1 \leq k \leq m - 1$, one can define the following decision problem.

> Hypergraph (k-in-m)-colorability
> *Instance:* An m-uniform hypergraph (Q, S).
> *Question:* Does (Q, S) have a (k-in-m)-coloring?

Kratochvíl [19] proved the following result, which is very useful for us; in our proofs we will use reductions from Hypergraph (1-in-3)-colorability for 3-regular 3-uniform hypergraphs and Hypergraph (3-in-6)-colorability for 3-regular 6-uniform hypergraphs.

Theorem 32 ([19]). *For every $p \geq 3, m \geq 3$ and $1 \leq k \leq m - 1$, the Hypergraph (k-in-m)-colorability problem restricted to the class of p-regular m-uniform hypergraphs is* NP-*complete.*

We also use the Graph k-colorability problem. This problem asks if a given graph G is k-colorable for some fixed integer k. The Graph k-colorability problem is well known to be NP-complete for all fixed $k \geq 3$ (cf. [18]).

Finally, we make again use of the H-cover problem. We take $H = K_{3,3}$ and $H = K_{4,4}$. Here, $K_{p,p}$ denotes the complete bipartite graph, in which each partition class consists of p vertices. Both problems are NP-complete [13].

7.2.2. Perfect-Regular Labelings.

We are left to prove that the perfect regular k-labeling problem is polynomial-time solvable for $k = 2$ and NP-complete for $3 \leq k \leq 4$.

Proof. Let $k = 2$. Then G has a perfect-regular labeling with at most two colors if and only if one of the following conditions holds:

- G has no edge (then one color can be used).
- Each connected component of G has two vertices (then one color can be used).
- Each connected component of G is a path $u_1u_2u_3u_4$ (then u_1 and u_4 get color 1 and u_2, u_3 get color 2).
- Each connected component of G is a cycle on $4p$ vertices (then the vertices can be colored in order 1,1,2,2,1,1 and so on, when going along the cycle).

We can check all three conditions in polynomial time.

Let $k = 3$. We prove that perfect-regular 3-labeling is NP-complete by a reduction from $K_{3,3}$-cover. We observe that any graph that allows a locally bijective homomorphism to $K_{3,3}$ is (3,3)-regular bipartite. Since we can check this condition in polynomial time, we may assume without loss of generality that our instance graph G of $K_{3,3}$-cover is (3,3)-regular bipartite. We denote the vertices of $K_{3,3}$ by $1, 2, 3, 1', 2', 3'$ such that the edges of $K_{3,3}$ are of the form (i, j) for any $i \in \{1, 2, 3\}$ and $j \in \{1', 2', 3'\}$.

We claim that $G \overset{B}{\to} K_{3,3}$ holds if and only if G has a perfect-regular labeling of at most three colors.

Suppose $G \overset{B}{\to} K_{3,3}$ holds. Let f be a locally bijective homomorphism from G to $K_{3,3}$. We replace each occurrence of color $1'$ by 1, each occurrence of color $2'$ by 2, and each occurrence of color $3'$ by 3. The resulting labeling is perfect-regular.

Suppose G has a perfect-regular labeling ℓ using at most 3 labels. Since G is (3,3)-regular bipartite, every vertex in G has three neighbors. By definition of a perfect-regular labeling, each such neighbor must have a different label. Hence, $\ell(V_G) = \{1, 2, 3\}$. Let the partition classes of G be A and B. Then we replace each occurrence of label 1 on B by $1'$, each occurrence of label 2 on B by $2'$, and each occurrence of label 3 on B by $3'$. The resulting mapping is a locally bijective homomorphism from G to $K_{3,3}$.

Let $k = 4$. We prove NP-completeness by a reduction from $K_{4,4}$-cover. The proof uses exactly the same arguments as in the previous case. ∎

7.2.3. Perfect-Regular Colorings.

We are left to prove that the perfect regular k-coloring problem is polynomial-time solvable for $2 \leq k \leq 3$.

Proof. Let $k = 2$. Let G be a graph. Then G has a perfect-regular coloring with at most two colors if and only if G has no edge (then one color can be used), or each connected component of G has exactly one edge. We can check both conditions in polynomial time.

Let $k = 3$. Let G be a graph. Recall that the cycle on three vertices is denoted C_3, and let P_3 denote the path on three vertices. We first check (in polynomial time) if G has a perfect-regular coloring with at most two colors. Suppose not. Then we check if G has a perfect-regular coloring with exactly three colors. This is the case if and only if each connected component of G is a cycle of length divisible by three (then $G \overset{B}{\to} C_3$), or if each connected component of G is a path on three vertices (then $G \overset{B}{\to} P_3$). Clearly, we can check these two conditions in polynomial time. ∎
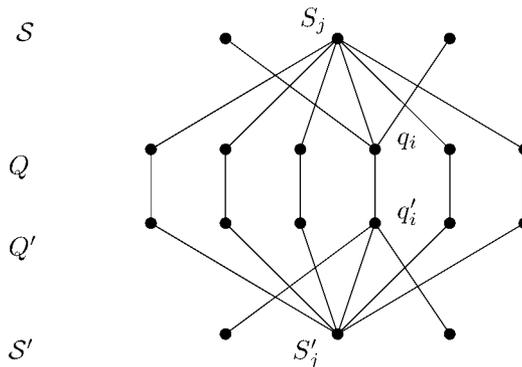
7.2.4. Connected Colorings.

We are left to prove that the connected k-coloring problem is polynomial-time solvable for $k = 2$.

Proof. Let $k = 2$. Let G be a graph. Then G has a connected coloring with at most two colors if and only if G has no edge (then one color can be used), or each connected component of G is bipartite and contains at least one edge. We can check both conditions in polynomial time. ∎

7.2.5. Semiregular Colorings.

We are left to prove that the semi-regular k-coloring problem is polynomial-time solvable for $k = 2$ and NP-complete for $k = 3$.

Proof. Let $k = 2$. Let G be a graph. Then G has a semiregular coloring with at most two colors if and only if G has no edge (then one color can be used), or G is semiregular bipartite. We can check both conditions in polynomial time.

Let $k = 3$. We show NP-completeness by reducing from the Hypergraph (3-in-6)-colorability problem for the class of 3-regular 6-uniform hypergraphs. By Theorem 32, this problem is NP-complete. Let $(Q, \mathcal{S})$ be a 3-regular 6-uniform hypergraph $(Q, \mathcal{S})$ with $Q = \{q_1, \ldots, q_m\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$. We modify its incidence graph I as follows.

First, we make a copy S_j' of each $S_j \in \mathcal{S}$ and a copy q_i' of each $q_i \in Q$. For all i, j, we add an edge (q_i', S_j') if and only if $q_i \in S_j$. For each $1 \leq i \leq m$, we add an edge (q_i, q_i'). We denote the resulting graph by I'. Note that, in I', all q_i and all copies q_i' have degree four, while all S_j and all copies S_j' have degree six. See Figure 12 for an example of I'. For

FIG. 12. An example of a graph I' obtained from a 3-regular 6-uniform hypergraph $(Q, \mathcal{S})$.

clarity reasons, we only display the adjacencies of one set $S_j \in \mathcal{S}$ and one element $q_i \in Q$, together with their copies in $\mathcal{S}'$ and Q', respectively. We prove that I' admits a semiregular coloring with at most 3 colors if and only if $(Q, \mathcal{S})$ admits a 3-in-6 coloring.

Suppose $(Q, \mathcal{S})$ admits a 3-in-6 coloring $Q_1 \cup Q_2$. For all S_j, we define $\ell(S_j) = \ell(S_j') = 3$. For all $q_i \in Q_1$, we let $\ell(q_i) = 1$ and $\ell(q_i') = 2$. For all $q_i \in Q_2$, we let $\ell(q_i) = 2$ and $\ell(q_i') = 1$. Since $(Q, \mathcal{S})$ is 6-uniform and 3-regular, ℓ is a semiregular coloring of I' using exactly 3 colors.

Suppose I' admits a semiregular coloring ℓ using at most 3 colors. Since ℓ is a semiregular coloring, all vertices in I' with the same color must have the same degree. Since vertices in Q have degree four and vertices in $\mathcal{S}$ have degree six, ℓ uses at least two colors. By construction, there exists an edge in I' between any q_i and its copy q_i', which are both of degree four. Since adjacent vertices have different colors, we then find that there are at least two colors, say $1, 2$, that appear only on vertices of degree four. Consequently, ℓ uses exactly three colors such that all vertices of degree four have color 1 or 2, and all vertices of degree six have the same color, say color 3. Since each q_i and its copy q_i' are adjacent, we then find that either q_i, q_i' are colored by $1, 2$, respectively, or else by $2, 1$, respectively. Consequently, if a vertex S_j has p neighbors labeled by 1, then its copy S_j' has $6 - p$ neighbors labeled by 1. Since S_j and S_j' have the same color, we obtain $p = 3$. Then each S_j has 3 neighbors labeled by 1 and 3 neighbors labeled by 2. Thus, $Q_1 \cup Q_2$ with $Q_1 = \{q \in Q | \ell(q) = 1\}$ and $Q_2 = Q \setminus Q_1$ is a 3-in-6-coloring of $(Q, \mathcal{S})$. ∎

7.2.6. Regular Colorings and (Symmetric) Regular Labelings. We are left to prove that problem regular k-coloring is polynomial-time solvable for $k = 2$ and NP-complete for $k = 3$, and that the problems symmetric regular k-labeling and regular k-labeling are polynomial-time solvable for $k = 2$ and NP-complete for $3 \leq k \leq 4$.

Proof. Let $k = 2$. Let G be a graph.

We find that G has a regular coloring with at most two colors if and only if G has no edge (then one color can be used), or G is regular bipartite. We can check both conditions in polynomial time.

We now show how to check in polynomial time if G admits a regular labeling with at most two colors. First we check (in polynomial time) if G has a regular labeling with one color. This is the case if and only if G is regular. So, if G is regular then we are done.

Suppose G is not regular. We (efficiently) check if the vertices of G can be split into two different classes according to their degrees. If not, then G does not have a regular labeling with at most two colors. This is because in that case the number of different degrees in G is at least three. However, in any regular labeling of G, two vertices with the same color have the same degree.

Suppose V_G can indeed be partitioned into two sets V_1 and V_2 such that all vertices in V_1 (resp. V_2) have the same degree. We color all vertices in V_1 by 1 and all vertices in V_2

by 2. Then we are left to check if this labeling is a regular labeling, which we can do in polynomial time (i.e., we check if both $G[1]$ and $G[2]$ are regular, and if $G[1, 2]$ is regular bipartite).

We now show how to check in polynomial time whether G admits a symmetric regular labeling with at most two colors. We first check (in polynomial time) if G has a symmetric regular labeling with one color. This is the case if and only if G is regular and admits a perfect matching in case its vertices have odd degree. If so, we are done. Suppose this is not the case.

First, we (efficiently) check if G is regular. If so, then G does not have a symmetric regular labeling with at most two colors. This can be seen as follows. Suppose G is d-regular for some $d \geq 1$. Since G does not have a symmetric regular labeling with at most one color, d is odd, and G does not admit a perfect matching. If G admits a symmetric regular labeling with two colors, say $1, 2$, then $G[1, 2]$ is regular bipartite. Then $G[1, 2]$, and consequently, G would admit a perfect matching due to König's Theorem (cf. [5]). This is not possible.

Suppose G is not regular. Also, in any symmetric regular labeling two vertices with the same color have the same degree. Furthermore, as noted at the start of this proof, we can efficiently check if a given labeling is a symmetric regular labeling. Hence, we can perform similar steps as in the previous algorithm (for regular labelings).

Let $3 \leq k \leq 4$. We prove NP-completeness by reduction from Hypergraph (3-in-6)-colorability for 3-regular 6-uniform hypergraphs. Due to Theorem 32 this problem is NP-complete. Given a hypergraph $(Q, \mathcal{S})$ with incidence graph I, we prove that the following statements are equivalent.

(i) $(Q, \mathcal{S})$ has a (3-in-6)-coloring.
(ii) I admits a regular coloring with at most three colors.
(iii) I admits a symmetric regular labeling with at most three colors.
(iv) I admits a symmetric regular labeling with at most four colors.
(v) I admits a regular labeling with at most three colors.
(vi) I admits a regular labeling with at most four colors.

(i) $\Rightarrow$ (ii)

Consider a (3-in-6)-coloring $Q_1 \cup Q_2$ of $(Q, \mathcal{S})$. For each $q \in Q_i$, $i = 1, 2$, let $\ell(q) = i$. For each $S \in \mathcal{S}$, let $\ell(S) = 3$. Then, each vertex labeled by 1 or 2 has three neighbors labeled by 3, and each vertex labeled by 3 has three neighbors labeled by 1 and three neighbors labeled by 2. Then ℓ is a regular coloring of I that uses at most three colors.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi)

This is trivial, or follows directly from the definitions.

(ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi)

This is trivial, or follows directly from the definitions.

(vi) $\Rightarrow$ (i)

Consider a regular labeling $\ell : V_I \rightarrow \{1, 2, 3, 4\}$ of I. Since $(Q, \mathcal{S})$ is 3-regular 6-uniform, there exists an integer p such that $|\mathcal{S}| = p$ and $|Q| = 2p$. Since ℓ is a regular labeling of I, vertices with the same color have the same
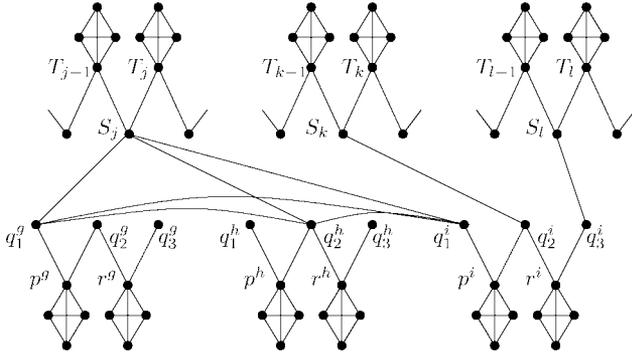
FIG. 13. An example of a graph I' obtained from a 3-regular 3-uniform hypergraph (Q, S).

degree in I. Because ℓ is also a pseudo-regular labeling, we find by Observation 18 that all color classes in a connected component of I must have the same size. Hence $|\ell(S)| = 1$ and $|\ell(Q)| = 2$. We assume without loss of generality that $\ell(Q) = \{1, 2\}$ and $\ell(S) = \{3\}$. Since $|\ell^{-1}(1)| = |\ell^{-1}(2)|$ and since all the vertices of S must have the same number of neighbors labeled by 1 (resp. 2), each vertex of S has three neighbors in $\ell^{-1}(1)$ and three neighbors in $\ell^{-1}(2)$. Thus $\ell^{-1}(1) \cup \ell^{-1}(2)$ is a (3-in-6)-coloring of (Q, S). ∎

7.2.7. Pseudo-Regular Colorings and Labelings.
We are left to prove that pseudo-regular k-coloring is polynomial-time solvable for $k = 2$ and NP-complete for $3 \leq k \leq 4$, and that pseudo-regular k-labeling is NP-complete for $2 \leq k \leq 4$.

Proof. Let $k = 2$. Let G be a graph. Then G has a pseudo-regular coloring with at most two colors if and only if G has no edge (then one color can be used), or else G is bipartite and admits a perfect matching. Both conditions can be checked in polynomial time.

Below we prove that the problem of deciding if a graph G has a pseudo-regular labeling with at most two colors is NP-complete. We use reduction from the hypergraph (1-in-3)-colorability problem for 3-regular 3-uniform hypergraphs. This problem is NP-complete due to Theorem 32. Consider a 3-regular 3-uniform hypergraph (Q, S) where $Q = \{q_1, \ldots, q_m\}$ and $S = \{S_1, \ldots, S_n\}$. We observe that its incidence graph I is 3-regular. From I we construct a graph I' as follows.

Since I is 3-regular and bipartite, each $q_i \in Q$ has three neighbors S_1^i, S_2^i, and S_3^i in S (with possibly $S_a^h = S_b^i$ for some $h \neq i$). We remove all edges between Q and S and replace each $q_i \in Q$ by three copies q_1^i, q_2^i and q_3^i with new edges $(S_1^i, q_1^i), (S_2^i, q_2^i)$, and (S_3^i, q_3^i). For each S_j, we add edges between all its three neighbors in the graph constructed so far. So, if $S_j = S_a^g = S_b^h = S_c^i$ then its three neighbors are q_a^g, q_b^h, q_c^i with $|\{g, h, i\}| = 3$, and the subgraph induced by S_j and q_a^g, q_b^h, q_c^i is isomorphic to K_4. Furthermore, each vertex q_a^i belongs to exactly one K_4 and each vertex S_j belongs to exactly one K_4.

We now add a set of $n - 1$ new vertices $\mathcal{T} = T_1, \ldots, T_{n-1}$, together with edges (S_j, T_j) and (T_j, S_{j+1}) for $j = 1, \ldots, n-1$. To each T_j we glue a graph $K_4(T_j)$ isomorphic to K_4. For each $1 \leq i \leq m$, we add two new vertices p^i, r^i with edges $(q_1^i, p^i), (p^i, q_2^i), (q_2^i, r^i)$ and (r^i, q_3^i). Finally, we glue a K_4 denoted by $K_4(p^i)$ to each p^i, and a K_4 denoted by $K_4(r^i)$ to each r^i. See Figure 13 for an example of a graph I', where we assume that S contains a set $S_j = \{q_g, q_h, q_i\}$, and Q contains an element q_i that belongs to set S_j, S_k, S_l. In this figure, only the relevant edges and vertices of I' are depicted (all other vertices and edges have been omitted for clarity). Note that I' is connected, even in case I was not. We now prove that (Q, S) admits a (1-in-3)-coloring if and only if I' admits a pseudo-regular labeling with at most two colors.

Suppose that (Q, S) admits a (1-in-3)-coloring $Q_1 \cup Q_2$. We let $\ell(S_j) = 1$ for all $1 \leq j \leq n$ and $\ell(T_j) = 2$ for all $1 \leq j \leq n-1$. We label two of the three remaining vertices in $K_4(T_j)$ by 1 and the other remaining one by 2. For all $q_i \in Q_1$ we let $\ell(q_1^i) = \ell(q_2^i) = \ell(q_3^i) = 1$ and $\ell(p^i) = \ell(r^i) = 2$, and in $K_4(p^i)$ and $K^4(r^i)$ we label two of their three remaining vertices by 1 and the other remaining one by 2. For all $q_i \in Q_2$ we let $\ell(q_1^i) = \ell(q_2^i) = \ell(q_3^i) = 2$ and $\ell(p^i) = \ell(r^i) = 1$. We label, both in $K_4(p^i)$ and in $K^4(r^i)$, two of their three remaining vertices by 2 and the other remaining one by 1. This way we have defined a labeling ℓ of I'.

Below we show that ℓ is a pseudo-regular labeling of I'. We first observe that both $I'[1]$ and $I'[2]$ are 1-regular graphs. To see this, recall that $Q_1 \cup Q_2$ is a 1-in-3 coloring of (Q, S). Hence each S_j (which has $\ell(S_j) = 1$) contains exactly one element in Q_1, i.e., has only one neighbor labeled by 1. Furthermore, all vertices u in each $K_4(T_j)$, in each $K_4(p^i)$ and in each $K_4(r^i)$ have exactly one neighbor labeled by $\ell(u)$, by definition of ℓ. The same is true for any vertex q_a^i that is a copy of a vertex q_i.

We are left to show that $I'[1, 2]$ contains a perfect matching M. All $K_4(T_j), K_4(p^i), K_4(r^i)$ contain two vertices labeled 1 and two vertices label 2. In each of them we chose two matching edges whose end vertices are labeled by 1,2, respectively, to be in M. We now consider the remaining vertices, which are all vertices in S_j and all copies q_a^i. By construction of I', each vertex q_a^i belongs to exactly one K_4 and each vertex S_j belongs to exactly one K_4. Consider such a K_4, say $I'[S_j, q_a^g, q_b^h, q_c^i]$. By definition of ℓ, such a K_4 consists of exactly two vertices labeled by 2 and two vertices labeled by 1. Hence, also here we can pick two matching edges for M. We conclude that ℓ is a pseudo-regular labeling of I'.

To prove the reverse implication, suppose I' admits a pseudo-regular labeling ℓ using at most two colors. Since I' is not regular, we obtain $\ell(V_{I'}) = \{1, 2\}$ as otherwise, i.e., in case $\ell(V_{I'}) = \{1\}$, $I'[1] = I'$ would not be regular.

Consider any glued $K_4(T_j)$ or $K_4(p^i)$ or $K_4(q^i)$. If all vertices of such a K_4 are labeled with the same color, we would have $|\ell(V_G| = 1$. As we already noticed, this is not possible. Hence ℓ uses both the two colors 1 and 2 on such a K_4. Since $I'[1, 2]$ admits a perfect matching, we even find that

FIG. 14. A graph G with glued copies of C_3. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

two vertices of such a K_4 are labeled by 1 and the other two are labeled by 2. This means that both $I'[1]$ and $I'[2]$ are 1-regular. Consequently, by construction of I', we obtain for all $1 \le i \le n$, $\ell(q_1^i) = \ell(q_2^i) = \ell(q_3^i)$ and $\ell(p^i) = \ell(r^i)$ with $\ell(q_1^i) \ne \ell(p_i)$. Assume without loss of generality that $\ell(T_1) = 2$. Then, again by construction of I', we obtain that each S_j has label 1 and each T_j has label 2. We define $Q_1 = \{q_i \mid \ell(q_1^i) = 1\}$ and $Q_2 = Q \setminus Q_1$.

We claim that $Q_1 \cup Q_2$ is a (1-in-3)-coloring of $(Q, \mathcal{S})$. In order to see this consider an arbitrary $S_j \in \mathcal{S}$. By construction of I', S_j belongs to exactly one K_4. Let q_a^g, q_b^h, q_c^i be the neighbors of S_j in this K_4. Since $I'[1]$ is 1-regular, $\ell(S_j) = 1$, and its neighbors in $\mathcal{T}$ have label 2, there is exactly one vertex, say q_a^g, in $\{q_a^g, q_b^h, q_c^i\}$ with label 1, while the other two vertices have label 2. Consequently, there is exactly one $q \in S_j$ that belongs to Q_1, namely q_g. Hence $Q_1 \cup Q_2$ is a (1-in-3)-coloring of $(Q, \mathcal{S})$.

Let $k = 3$. For our NP-completeness proofs, we use reduction from the graph 3-colorability problem. This problem is NP-complete (cf. [18]), even for the class of graphs that are not regular (as otherwise we could just add one vertex adjacent to an arbitrary vertex of G to make G non-regular).

Let G be a non-regular instance graph of graph 3-colorability. To each vertex v of G, we glue a C_3. We denote this C_3 by $C_3(v)$ and its three vertices by v, v', v'' (where v' and v'' are the new vertices). We denote the resulting graph by G'; see Figure 14 for an example. We show that the following statements are equivalent.

(i) G is 3-colorable,
(ii) G' admits a pseudo-regular coloring using at most three colors,
(iii) G' admits a pseudo-regular labeling using at most three colors,

(i) $\Rightarrow$ (ii)

Given a coloring ℓ of G using three colors 1, 2, 3, we extend it to G' as follows. For each $v \in V_G$, we give two different colors to v' and v'' that are also distinct from $\ell(v)$. Thus, for each $i \in [1, 3]$, $G'[i]$ is empty. Moreover, for all $1 \le i, j \le 3$ with $i \ne j$, $G'[i, j]$ admits a perfect matching. This can be seen as follows. For each $v \in V_G$, all three vertices v, v', v'' have different colors. So exactly one of the edges in each $C_3(v)$ has its end vertices colored by i, j, respectively, and that edge will be in the perfect matching.

(ii) $\Rightarrow$ (iii)
This follows directly from the definitions.
(iii) $\Rightarrow$ (i)
Consider a pseudo-regular labeling ℓ of G' using at most three colors. Note that G contains an edge, because G is not regular. Hence there exists a vertex $v \in V_G$ that has at least one neighbor distinct from v', v'' in G'. If $\ell(v) = \ell(v') = \ell(v'')$, then $G' = G'[1]$. However, since G is not regular, G' is not regular. So this is not possible, and we may assume without loss of generality that v and v' have different colors, say $\ell(v) = 1$ and $\ell(v') = 2$. Since $G'[1, 2]$ admits a perfect matching, $\ell(v'') \ne 2$. Suppose $\ell(v'') = 1$. Then, $\ell(V_{G'}) = \{1, 2\}$, $G'[2]$ is edgeless and $G'[1]$ is 1-regular. Since $G'[1]$ is 1-regular, $\ell(u) = 2$ for some neighbor u of v. Since $G'[2]$ is edgeless, $\ell(u') = \ell(u'') = 1$, but then $G'[1, 2]$ does not admit a perfect matching. Consequently, we may assume $\ell(v'') = 3$. Then, $G'[2]$ and $G'[3]$ are edgeless.

Suppose all vertices in V_G have color 1. Then $G'[1] = G$. This is not possible since G is not regular. Hence there exists a vertex $w \in V_G$ with $\ell(w) \ne 1$. Then we may assume without loss of generality that $\ell(w) = 2$. Since $G'[2]$ and $G'[3]$ are edgeless, we may again assume without loss of generality that $\ell(w') = 1$ and $\ell(w'') = 3$. Consequently $G'[1]$ is also edgeless. Hence, ℓ is a 3-coloring of G' and its restriction to G is a 3-coloring of G.

Let $k = 4$. For our NP-completeness proofs we use reduction from graph 4-colorability. This problem is NP-complete (cf. [18]), even for the class of graphs that are not regular (as otherwise we could just add one vertex adjacent to an arbitrary vertex of G) and that are not bipartite (as those graphs can be efficiently checked and colored with at most two colors).

Let G be a non-bipartite non-regular instance graph of graph 4-colorability. To each vertex v of G, we glue a K_4. We denote this K_4 by $K_4(v)$ and its four vertices by v, v', v'', v^* (where v', v'' and v^* are the new vertices). We denote the resulting connected graph by G'. We show that the following statements are equivalent.

(i) G is 4-colorable,
(ii) G' admits a pseudo-regular coloring using at most four colors,
(iii) G' admits a pseudo-regular labeling using at most four colors,

(i) $\Rightarrow$ (ii)

Given a coloring ℓ of G using four colors 1,2,3,4, we extend it to G' as follows. For each $v \in V_G$, we give three different colors to v', v'' and v^* that are also distinct from $\ell(v)$. Thus, for each $1 \le i \le 4$, $G'[i]$ is empty. Moreover, for all $1 \le i, j \le 4$ with $i \ne j$, $G'[i, j]$ admits a perfect matching. This can be seen as follows. For each $v \in V_G$, all four vertices v, v', v'', v^* have different colors. So exactly one of the edges in each $K_4(v)$ has its end vertices colored by i, j, respectively, and that edge will be in the perfect matching.
(ii) $\Rightarrow$ (iii)
This follows directly from the definitions.

(iii) $\Rightarrow$ (i)

Consider a pseudo-regular labeling ℓ of G' using at most four colors. Note that G contains an edge, because G is not regular. Hence there exists a vertex $v \in V_G$ that has at least one neighbor distinct from v', v'', v^* in G'.

If $\ell(v) = \ell(v') = \ell(v'') = \ell(v^*) = 1$, then $G' = G'[1]$. However, since G is not regular, G' is not regular. So this is not possible, and we may assume without loss of generality that v and v' have different colors, say $\ell(v) = 1$ and $\ell(v') = 2$.

Suppose $\ell(v'') = 1$. Since $G[1, 2]$ admits a perfect matching, $\ell(v^*) \neq 1$. Suppose $\ell(v^*) = 2$. This means that $\ell(V_{G'}) = \{1, 2\}$ and that $G[1]$ and $G[2]$ are 1-regular. Consequently, for each $u \in N_G(v)$, $\ell(u) = 2$, and thus G is 2-colorable. This is not possible since we assume that G is not bipartite. Suppose $\ell(v^*) = 3$. This means that $\ell(V_{G'}) = \{1, 2, 3\}$, that $G[1]$ is 1-regular and that $G[2]$ and $G[3]$ are edgeless. Let u be a neighbor of v in G. Since $G[1]$ is 1-regular, we may assume without loss of generality that $\ell(u) = 2$. Since $G[2]$ is edgeless, $\ell(\{u', u'', u^*\}) \subseteq \{1, 3\}$. Since $G[3]$ is edgeless, there are at least two vertices among u', u'', u^* that are labeled by 1. However, this is not possible, since $G[1, 2]$ would not admit a perfect matching in that case. Consequently, $\ell(v'') \neq 1$.

Suppose $\ell(v'') = 2$. Since $G[1, 2]$ admits a perfect matching, this implies that $\ell(v^*) = 1$. Then, by symmetry, we return to the previous case. Thus, we may assume without loss of generality that $\ell(v'') = 3$ and $\ell(v^*) = 4$. Then, $G'[2], G'[3], G'[4]$ are edgeless.

Suppose all vertices in V_G have color 1. Then $G'[1] = G$. This is not possible, since G is not regular. Hence there exists a vertex $w \in V_G$ with $\ell(w) \neq 1$. Then we may assume without loss of generality that $\ell(w) = 2$. Since $G'[2], G'[3]$ and $G'[4]$ are edgeless, we may then assume without loss of generality that $\ell(w') = 1$, $\ell(w'') = 3$ and $\ell(w^*) = 4$. Consequently $G'[1]$ is also edgeless. Hence, ℓ is a 4-coloring of G' and its restriction to G is a 4-coloring of G. ∎

Acknowledgments

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