Packing Bipartite Graphs with Covers of Complete Bipartite Graphs

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Abstract. For a set $\mathcal{S}$ of graphs, a perfect $\mathcal{S}$-packing ($\mathcal{S}$-factor) of a graph G is a set of mutually vertex-disjoint subgraphs of G that each are isomorphic to a member of $\mathcal{S}$ and that together contain all vertices of G. If G allows a covering (locally bijective homomorphism) to a graph H, then G is an H-cover. For some fixed H let $\mathcal{S}(H)$ consist of all H-covers. Let $K_{k,\ell}$ be the complete bipartite graph with partition classes of size k and ℓ, respectively. For all fixed $k, \ell \geq 1$, we determine the computational complexity of the problem that tests if a given bipartite graph has a perfect $\mathcal{S}(K_{k,\ell})$-packing. Our technique is partially based on exploring a close relationship to pseudo-coverings. A pseudo-covering from a graph G to a graph H is a homomorphism from G to H that becomes a covering to H when restricted to a spanning subgraph of G. We settle the computational complexity of the problem that asks if a graph allows a pseudo-covering to $K_{k,\ell}$ for all fixed $k, \ell \geq 1$.

1 Introduction

Throughout the paper we consider undirected graphs with no loops and no multiple edges. Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be some fixed set of mutually vertex-disjoint graphs. A set of (not necessarily vertex-induced) mutually vertex-disjoint subgraphs of G, each isomorphic to a member of $\mathcal{S}$, is called an *$\mathcal{S}$-packing*. Packings naturally generalize matchings (the case in which $\mathcal{S}$ only contains edges). They arise in many applications, both practical ones such as exam scheduling [12], and theoretical ones such as the study of degree constraint graphs (cf. the survey [11]). If $\mathcal{S}$ consists of a single subgraph S, we write *S-packing* instead of $\mathcal{S}$-packing. The problem of finding an S-packing of a graph G that packs the maximum number of vertices of G is NP-hard for all fixed connected S on at least three vertices, as shown by Hell and Kirkpatrick [13].

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A packing of a graph is *perfect* if every vertex of the graph belongs to one of the subgraphs of the packing. Perfect packings are also called *factors* and from now on we call a perfect $\mathcal{S}$-packing an $\mathcal{S}$*-factor*. We call the corresponding decision problem the $\mathcal{S}$-FACTOR problem. For a survey on graph factors we refer to [18].

Our Focus. We study a relaxation of $K_{k,\ell}$-factors, where $K_{k,\ell}$ denotes the *biclique* (complete connected bipartite graph) with partition classes of size k and ℓ, respectively. In order to explain this relaxation we must introduce some new terminology. A *homomorphism* from a graph G to a graph H is a vertex mapping $f : V_G \to V_H$ satisfying the property that $f(u)f(v)$ belongs to E_H whenever the edge uv belongs to E_G. If for every $u \in V_G$ the restriction of f to the neighborhood of u, i.e. the mapping $f_u : N_G(u) \to N_H(f(u))$, is bijective then we say that f is a *locally bijective* homomorphism or a *covering* [2,16]. The graph G is then called an H*-cover* and we write $G \xrightarrow{B} H$. Locally bijective homomorphisms have applications in distributed computing [1] and in constructing highly transitive regular graphs [3]. For a specified graph H, we let $\mathcal{S}(H)$ consist of all H-covers. This paper studies $\mathcal{S}(K_{k,\ell})$**-factors of bipartite graphs**.

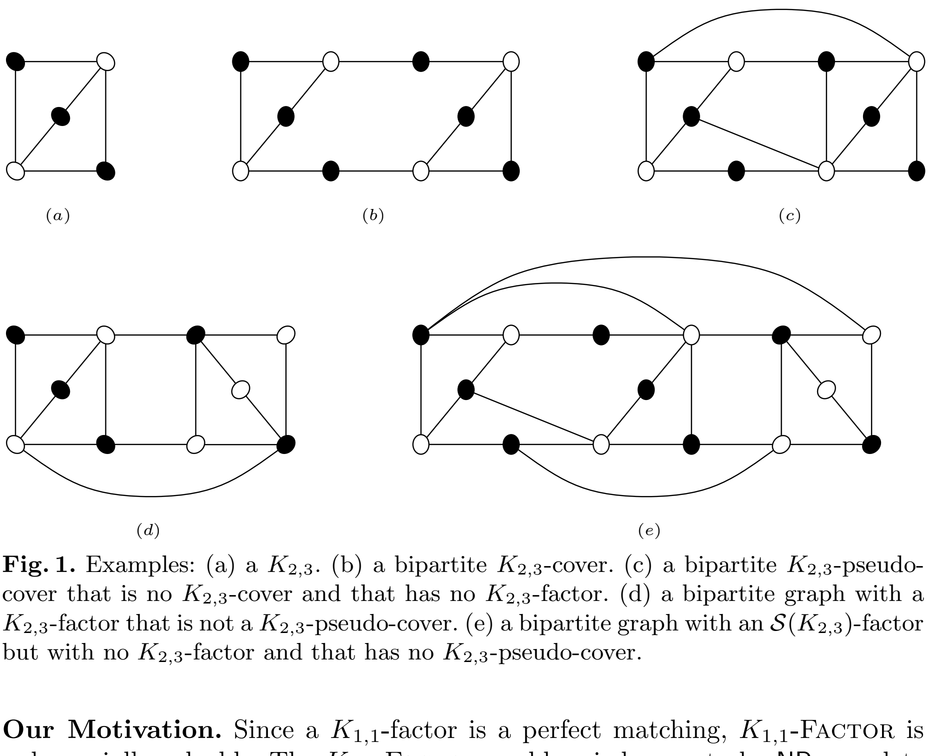

Fig. 1. Examples: (a) a $K_{2,3}$. (b) a bipartite $K_{2,3}$-cover. (c) a bipartite $K_{2,3}$-pseudo-cover that is no $K_{2,3}$-cover and that has no $K_{2,3}$-factor. (d) a bipartite graph with a $K_{2,3}$-factor that is not a $K_{2,3}$-pseudo-cover. (e) a bipartite graph with an $\mathcal{S}(K_{2,3})$-factor but with no $K_{2,3}$-factor and that has no $K_{2,3}$-pseudo-cover.

Our Motivation. Since a $K_{1,1}$-factor is a perfect matching, $K_{1,1}$-FACTOR is polynomially solvable. The $K_{k,\ell}$-FACTOR problem is known to be NP-complete for all other $k, \ell \geq 1$, due to the earlier mentioned result of [13]. These results have some consequences for our relaxation. In order to explain this, we make the following observation, which holds because only a tree has a unique cover (namely the tree itself).

Observation 1. $\mathcal{S}(K_{k,\ell}) = \{K_{k,\ell}\}$ *if and only if* $\min\{k,\ell\} = 1$.

Hence, the above results immediately imply that $\mathcal{S}(K_{1,\ell})$-Factor is only polynomially solvable if $\ell = 1$; it is NP-complete otherwise. What about our relaxation for $k, \ell \geq 2$? Note that, for these values of k, ℓ, the size of the set $\mathcal{S}(K_{k,\ell})$ is unbounded. The only result known so far is for $k = \ell = 2$; Hell, Kirkpatrick, Kratochvíl and Kříž [14] showed that $\mathcal{S}(K_{2,2})$-Factor is NP-complete for general graphs, as part of their computational complexity classification of restricted 2-factors.

For bipartite graphs, the following is known. Firstly, Monnot and Toulouse [17] researched path factors in bipartite graphs and showed that the $K_{2,1}$-Factor problem stays NP-complete when restricted to the class of bipartite graphs. Secondly, we observed that as a matter of fact the proof of the NP-completeness result for $\mathcal{S}(K_{2,2})$-Factor in [14] is even a proof for bipartite graphs.

Our interest in bipartite graphs stems from a close relationship of $\mathcal{S}(K_{k,\ell})$-factors of bipartite graphs and so-called $K_{k,\ell}$-pseudo-covers, which originate from topological graph theory and have applications in the area of distributed computing [4,5]. A homomorphism f from a graph G to a graph H is a *pseudo-covering* from G to H if there exists a spanning subgraph G' of G such that f is a covering from G' to H. In that case G is called an *H-pseudo-cover* and we write $G \xrightarrow{P} H$. The computational complexity classification of the H-Pseudo-Cover problem that tests if $G \xrightarrow{P} H$ for input G is still open, and our paper can also be seen as a first investigation into this question. We explain the exact relationship between factors and pseudo-coverings in detail, later on.

Our Results and Paper Organization. Section 2 contains additional terminology, notations and some basic observations. In Section 3 we pinpoint the relationship between factors and pseudo-coverings. In Section 4 we completely classify the computational complexity of the $\mathcal{S}(K_{k,\ell})$-Factor problem for bipartite graphs. Recall that $\mathcal{S}(K_{1,1})$-Factor is polynomially solvable on general graphs. We first prove that $\mathcal{S}(K_{1,\ell})$-Factor is NP-complete on bipartite graphs for all fixed $\ell \geq 2$. By applying our result in Section 3, we then show that NP-completeness of every remaining case can be shown by proving NP-completeness of the corresponding $K_{k,\ell}$-Pseudo-Cover problem.

We classify the complexity of $K_{k,\ell}$-Pseudo-Cover in Section 5. We show that it is indeed NP-complete on bipartite graphs for all fixed pairs $k, \ell \geq 2$ by adapting the hardness construction of [14] for restricted 2-factors. In contrast to $\mathcal{S}(K_{k,\ell})$-Factor, we show that $K_{k,\ell}$-Pseudo-Cover is polynomially solvable for all $k, \ell \geq 1$ with $\min\{k,\ell\} = 1$.

In Section 6 we further discuss the relationships between pseudo-coverings and locally constrained homomorphisms, such as the earlier mentioned coverings. We shall see that as a matter of fact the NP-completeness result for $K_{k,\ell}$-Pseudo-Cover for fixed $k, \ell \geq 3$ also follows from a result of Kratochvíl, Proskurowski and Telle [15] who prove that $K_{k,\ell}$-Cover is NP-complete for $k, \ell \geq 3$. This problem tests whether $G \xrightarrow{B} K_{k,\ell}$ for a given graph G. However, the same authors [15] show that $K_{k,\ell}$-Cover is polynomially solvable when $k = 2$ or $\ell = 2$. Hence, for those cases we need to rely on our proof in Section 5.

2 Preliminaries

From now on let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_\ell\}$ denote the partition classes of $K_{k,\ell}$, where we assume $k \leq \ell$. If $k = 1$ then we say that x_1 is the *center* of $K_{1,\ell}$. We denote the degree of a vertex u in a graph G by $\deg_G(u)$.

Recall that a homomorphism f from a graph G to a graph H is a pseudo-covering from G to H if there exists a spanning subgraph G' of G such that f is a covering from G' to H. We would like to stress that this is *not* the same as saying that f is a vertex mapping from V_G to V_H such that f restricted to some spanning subgraph G' of G becomes a covering. The reason is that in the latter setting it may well happen that f is not a homomorphism from G to H. For instance, f might map two adjacent vertices of G to the same vertex of H. However, there is an alternative definition which turns out to be very useful for us. In order to present it we need the following notations.

We let $f^{-1}(x)$ denote the set $\{u \in V_G \mid f(u) = x\}$. For a subset $S \subseteq V_G$, $G[S]$ denotes the *induced subgraph* of G by S, i.e., the graph with vertex set S and edges uv whenever $uv \in E_G$. For $xy \in E_H$ with $x \neq y$, we write $G[x, y] = G[f^{-1}(x) \cup f^{-1}(y)]$. Because f is a homomorphism, $G[x, y]$ is a bipartite graph with partition classes $f^{-1}(x)$ and $f^{-1}(y)$.

Proposition 1 ([4]). *A homomorphism f from a graph G to a graph H is a pseudo-covering if and only if $G[x, y]$ contains a perfect matching for all $x, y \in V_H$. Consequently, $|f^{-1}(x)| = |f^{-1}(y)|$ for all $x, y \in V_H$.*

Let f be a pseudo-covering from a graph G to a graph H. We then sometimes call the vertices of H *colors* of vertices of G. Due to Proposition 1, $G[x, y]$ must contain a perfect matching M_{xy}. Let $uv \in M_{xy}$ for $xy \in E_H$. Then we say that v is a *matched neighbor* of u, and we call the set of matched neighbors of u the *matched neighborhood* of u.

3 How Factors Relate to Pseudo-covers

Theorem 1. *Let G be a graph on n vertices. Then G is a $K_{k,\ell}$-pseudo-cover if and only if G has an $\mathcal{S}(K_{k,\ell})$-factor and G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$.*

Proof. First suppose $G = (V, E)$ is a $K_{k,\ell}$-pseudo-cover. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then f is a homomorphism from G to $K_{k,\ell}$, which is a bipartite graph. Consequently, G must be bipartite as well. Let A and B denote the partition classes of G. Then we may without loss of generality assume that $f(A) = X$ and $f(B) = Y$. Due to Proposition 1 we then find that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. By the same proposition we find that each $G[x_i, y_j]$ contains a perfect matching M_{ij}. We define the spanning subgraph $G' = (V, \bigcup_{ij} M_{ij})$ of G and observe that every component in G' is a $K_{k,\ell}$-cover. Hence G has an $\mathcal{S}(K_{k,\ell})$-factor.

Now suppose G has an $\mathcal{S}(K_{k,\ell})$-factor. Also suppose G is bipartite with partition classes A and B such that $|A| = \frac{kn}{k+\ell}$ and $|B| = \frac{\ell n}{k+\ell}$. Since $\{F_1, \ldots, F_p\}$ is an $\mathcal{S}(K_{k,\ell})$-factor, there exists a covering f_i from F_i to $K_{k,\ell}$ for $i = 1, \ldots, p$. Let f be the mapping defined on V such that $f(u) = f_i(u)$ for all $u \in V$, where f_i is the (unique) covering that maps u to some vertex in $K_{k,\ell}$. Let A_X be the set of vertices of A that are mapped to a vertex in X and let A_Y be the set of vertices of A that are mapped to a vertex in Y. We define subsets B_X and B_Y of B in the same way. This leads to the following equalities:

$$\begin{aligned} |A_X| + |A_Y| &= \tfrac{kn}{k+\ell} \\ |B_X| + |B_Y| &= \tfrac{\ell n}{k+\ell} \\ |A_Y| &= \tfrac{\ell}{k}|B_X| \\ |B_Y| &= \tfrac{\ell}{k}|A_X|. \end{aligned}$$

Suppose $\ell \neq k$. Then this set of equalities has a unique solution, namely, $|A_X| = \frac{kn}{k+\ell} = |A|, |A_Y| = |B_X| = 0$, and $|B_Y| = \frac{\ell n}{k+\ell} = |B|$. Hence, we find that f maps all vertices of A to vertices of X and all vertices of B to Y. This means that f is a homomorphism from G to $K_{k,\ell}$ that becomes a covering when restricted to the spanning subgraph obtained by taken the disjoint union of the subgraphs $\{F_1, \ldots, F_p\}$. In other words, f is a pseudo-covering from G to $K_{k,\ell}$, as desired.

Suppose $\ell = k$. Then solving this set of equalities yields solutions of the form $|A_X| = |B_Y| = \alpha$ and $|A_Y| = |B_X| = \frac{1}{2}n - \alpha$ with arbitrary α. However, in this case we have that $|V_{F_i} \cap A| = |V_{F_i} \cap B|$ for $i = 1, \ldots, p$. Then we can without loss of generality assume that each f_i maps $V_{F_i} \cap A$ to X and $V_{F_i} \cap B$ to Y; so, $|A_X| = |A| = |B_Y| = |B|$ and $|A_Y| = |B_X| = 0$. This completes the proof of Theorem 1. □

4 Classifying the $\mathcal{S}(K_{k,\ell})$-Factor Problem

Here is the main theorem of this section.

Theorem 2. *The* $\mathcal{S}(K_{k,\ell})$-Factor *problem is solvable in polynomial time for* $k = \ell = 1$. *Otherwise it is* NP*-complete, even for the class of bipartite graphs.*

We prove Theorem 2 as follows. First we consider the case when $\min\{k, \ell\} = 1$ Due to Observation 1 the $\mathcal{S}(K_{1,1})$-Factor problem is equivalent to the problem of finding a perfect matching, which can be solved in polynomial time. We continue with the case when $k = 1$ and $\ell \geq 2$. Recall that for general graphs NP-completeness of this case immediately follows from Observation 1 and the earlier mentioned result of Hell and Kirkpatrick [13]. However, we consider bipartite graphs. For this purpose, a result by Monnot and Toulouse [17] is of importance for us. Here, P_k denotes a path on k vertices.

Theorem 3 ([17]). *For any fixed* $k \geq 3$*, the* P_k-Factor *problem is* NP*-complete for the class of bipartite graphs.*

We use Theorem 3 to show the result below, the proof of which is omitted.

Proposition 2. *For any fixed* $\ell \geq 2$, $\mathcal{S}(K_{1,\ell})$-FACTOR *and* $K_{1,\ell}$-FACTOR *are* NP*-complete, even for the class of bipartite graphs.*

The following statement allows us to consider the $K_{k,\ell}$-PSEUDO-COVER problem in order to finish the proof of Theorem 2. We omit its proof.

Proposition 3. *Fix arbitrary integers* $k, \ell \geq 2$*. If the* $K_{k,\ell}$-PSEUDO-COVER *problem is* NP*-complete, then so is the* $\mathcal{S}(K_{k,\ell})$-FACTOR *problem for the class of bipartite graphs.*

5 Classifying the $K_{k,\ell}$-PSEUDO-COVER Problem

Here is the main theorem of this section.

Theorem 4. *The* $K_{k,\ell}$-PSEUDO-COVER *problem can be solved in polynomial time for any fixed* k, ℓ *with* $\min\{k, \ell\} = 1$*. Otherwise it is* NP*-complete.*

In order to prove Theorem 4 we first analyse the polynomial cases.

Proposition 4. *The* $K_{k,\ell}$-PSEUDO-COVER *problem can be solved in polynomial time for any fixed* k, ℓ *with* $\min\{k, \ell\} = 1$*.*

Proof. For $k = \ell = 1$, the problem is equivalent to finding a perfect matching in a graph. Let $k = 1$ and $\ell \geq 2$. Let G be an input graph of $K_{1,\ell}$-PSEUDO-COVER. By Theorem 1, we may without loss of generality assume that G is bipartite with partition classes A and B such that $|A| = \frac{n}{1+\ell}$ and $|B| = \frac{\ell n}{1+\ell}$. Because $\ell \geq 2$, we can distinguish between A and B. Then we replace each vertex $a \in A$ by ℓ copies $a^1, \ldots, a^\ell$, and we make each a^i adjacent to all neighbors of a. This leads to a bipartite graph G', the partition classes of which have the same size. Then G has a $K_{1,\ell}$-PSEUDO-COVER if and only if G' has a perfect matching. □

We now prove that $K_{k,\ell}$-PSEUDO-COVER is NP-complete for all $k, \ell \geq 2$. Our proof is inspired by the proof of Hell, Kirkpatrick, Kratochvíl, and Kříž in [14]. They consider the problem of testing if a graph has an $\mathcal{S}_L$-factor for any set $\mathcal{S}_L$ of cycles, the length of which belongs to some specified set L. This is useful for our purposes because of the following. If $L = \{4, 8, 12, \ldots, \}$ then an $\mathcal{S}_L$-factor of a bipartite graph G with partition classes A and B of size $\frac{n}{2}$ is an $\mathcal{S}(K_{2,2})$-factor of G that is also a $K_{2,2}$-pseudo-cover of G by Theorem 1. However, for $k = \ell \geq 3$, this is not longer true, and when $k \neq \ell$ the problem is not even "symmetric" anymore. Below we show how to deal with these issues. We refer to Section 6 for an alternative proof for the case $k, \ell \geq 3$. However, our construction for $k, \ell \geq 2$ does not become simpler when we restrict ourselves to $k, \ell \geq 2$ with $k = 2$ or $\ell = 2$. Therefore, we decided to present our NP-completeness result for *all* k, ℓ with $k, \ell \geq 2$.

Recall that we denote the partition classes of $K_{k,\ell}$ by $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_\ell\}$ with $k \leq \ell$. We first state a number of useful lemmas (proofs are

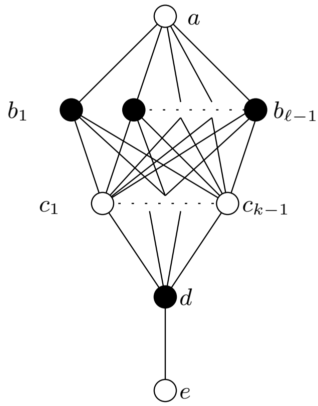

Fig. 2. The graph $G_1(k,\ell)$

omitted). Hereby, we use the alternative definition in terms of perfect matchings, as provided by Proposition 1, when we argue on pseudo-coverings.

Let $G_1(k,\ell)$ be the graph in Figure 2. It contains a vertex a with $\ell-1$ neighbors $b_1,\ldots,b_{\ell-1}$ and a vertex d with $k-1$ neighbors $c_1,\ldots,c_{k-1}$. For any $i\in[1,\ell-1]$, $j\in[1,k-1]$, it contains an edge b_ic_j. Finally, it contains a vertex e which is only adjacent to d.

Lemma 2. *Let $G_1(k,\ell)$ be an induced subgraph of a bipartite graph G such that only a and e have neighbors outside $G_1(k,\ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then $f(a)=f(e)$. Moreover, a has only one matched neighbor outside $G_1(k,\ell)$ and this matched neighbor has color $f(d)$, where d is the only matched neighbor of e inside $G_1(k,\ell)$.*

Lemma 3. *Let G be a bipartite graph that contains $G_1(k,\ell)$ as an induced subgraph such that only a and e have neighbors outside $G_1(k,\ell)$ and such that a and e have no common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_1(k,\ell)$ and by adding a new vertex u adjacent to all neighbors of a and e outside $G_1(k,\ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$ such that u has exactly one neighbor v of a in its matched neighborhood. Then G is a $K_{k,\ell}$-pseudo-cover.*

Let $G_2(k,\ell)$ be the graph in Figure 3. It contains k vertices $u_1,\ldots,u_k$. It also contains $(k-1)k$ vertices $v_{h,i}$ for $h=1,\ldots,k-1$, $i=1,\ldots,k$, and $(k-1)(\ell-1)$ vertices $w_{i,j}$ for $i=1,\ldots,k-1$, $j=1,\ldots,\ell-1$. For $h=1,\ldots,k-1$, $i=1,\ldots,k$, $j=1,\ldots,\ell-1$, $G_2(k,\ell)$ contains an edge $u_iv_{h,i}$ and an edge $v_{h,i}w_{h,j}$.

Lemma 4. *Let G be a bipartite graph that has $G_2(k,\ell)$ as an induced subgraph such that only u-vertices have neighbors outside $G_2(k,\ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then each u_i has exactly one matched neighbor t_i outside $G_2(k,\ell)$. Moreover, $|f(\{u_1,\ldots,u_k\})|=1$ and $|f(\{t_1,\ldots,t_k\})|=k$.*

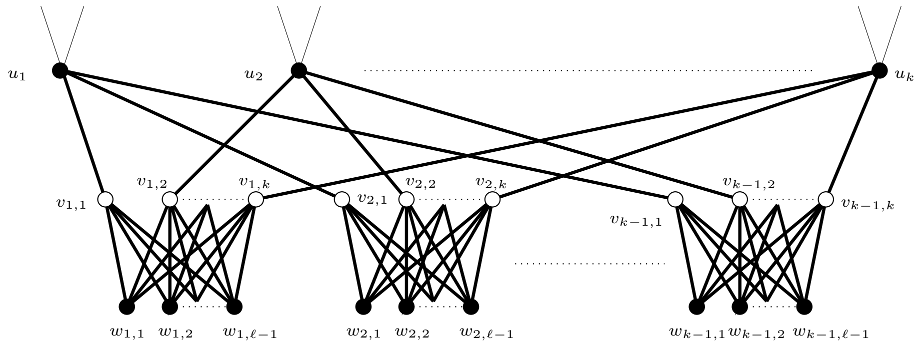

Fig. 3. The graph $G_2(k, \ell)$

Lemma 5. *Let G be a bipartite graph that has $G_2(k, \ell)$ as an induced subgraph such that only u-vertices have neighbors outside $G_2(k, \ell)$ and such that no two u-vertices have a common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_2(k, \ell)$ and by adding a new vertex s that is adjacent to the neighbors of all u-vertices outside $G_2(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$ such that s has exactly one neighbor t_i of every u_i in its matched neighborhood. Then G is a $K_{k,\ell}$-pseudo-cover.*

Let $G_3(k, \ell)$ be the graph defined in Figure 4. It contains k copies of $G_1(k, \ell)$, where we denote the a-vertex and e-vertex of the i^{th} copy by a_i and e_i, respectively. It also contains a copy of $G_2(k, \ell)$ with edges $e_i u_i$ and $a_i u_{i+1}$ for

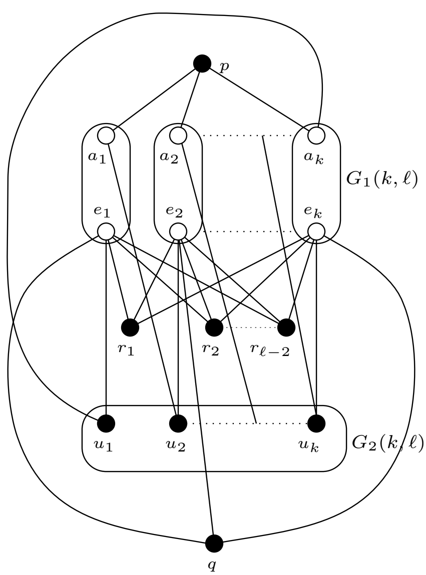

Fig. 4. The graph $G_3(k, \ell)$

$i = 1, \ldots, k$ (where $u_{k+1} = u_1$). The construction is completed by adding a vertex p adjacent to all a-vertices and by adding vertices $q, r_1, \ldots, r_{\ell-2}$ that are adjacent to all e-vertices. Here we assume that there is no r-vertex in case $\ell = 2$.

Lemma 6. *Let G be a bipartite graph that has $G_3(k, \ell)$ as an induced subgraph, such that only p and q have neighbors outside $G_3(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then either every a_i is a matched neighbor of p and no e_i is a matched neighbor of q, or else every e_i is a matched neighbor of q and no a_i is a matched neighbor of p.*

Lemma 7. *Let G be a graph that has $G_3(k, \ell)$ as an induced subgraph such that only p and q have neighbors outside $G_3(k, \ell)$ and such that p and q do not have a common neighbor. Let G' be the graph obtained from G by removing all vertices of $G_3(k, \ell)$ and by adding a new vertex r^* that is adjacent to all neighbors of p and q outside $G_3(k, \ell)$. Let f be a pseudo-covering from G' to $K_{k,\ell}$ such that either all vertices in the matched neighborhood of r^* in G' are neighbors of p in G, or else are all neighbors of q in G. Then G is a $K_{k,\ell}$-pseudo-cover.*

Let $G_4(k, \ell)$ be the graph in Figure 5. It is constructed as follows. We take k copies of $G_3(\ell, k)$. We denote the p-vertex and the q-vertex of the i^{th} copy by $p_{1,i}$ and $q_{1,i}$, respectively. We take ℓ copies of $G_3(k, \ell)$. We denote the p-vertex and the q-vertex of the j^{th} copy by $p_{2,j}$ and $q_{2,j}$, respectively. We add an edge between any $p_{1,i}$ and $p_{2,j}$.

Lemma 8. *Let G be a bipartite graph that has $G_4(k, \ell)$ as an induced subgraph such that only the q-vertices have neighbors outside $G_4(k, \ell)$. Let f be a pseudo-covering from G to $K_{k,\ell}$. Then either every $p_{1,i}p_{2,j}$ is in a perfect matching and all matched neighbors of every q-vertex are in $G_4(k, \ell)$, or else no edge $p_{1,i}p_{2,j}$ is in a perfect matching and all matched neighbors of every q-vertex are outside $G_4(k, \ell)$.*

We are now ready to present our NP-completeness reduction. This finishes the proof of Theorem 4.

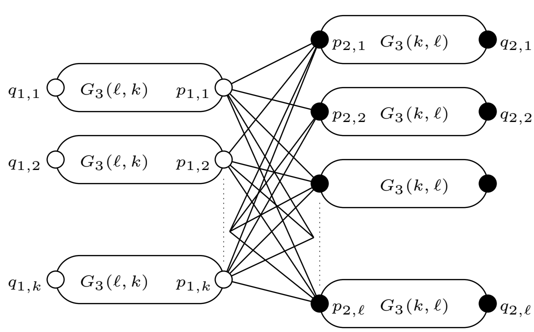

Fig. 5. The graph $G_4(k, \ell)$

Proposition 5. *The* $K_{k,\ell}$-Pseudo-Cover *problem is* NP*-complete for any fixed* k, ℓ *with* $k, \ell \geq 2$.

Proof. We reduce from the problem $k+\ell$-Dimensional Matching, which is NP-complete as $k+\ell \geq 3$ (see [10]). In this problem, we are given $k+\ell$ mutually disjoint sets $Q_{1,1}, \ldots, Q_{1,k}, Q_{2,1}, \ldots, Q_{2,\ell}$, all of equal size m, and a set H of hyperedges $h \in \Pi_{i=1}^{k} Q_{1,i} \times \Pi_{j=1}^{\ell} Q_{2,j}$. The question is whether H contains a *$k+\ell$-dimensional matching*, i.e., a subset $M \subseteq H$ of size $|M| = m$ such that for any distinct pairs $(q_{1,1}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,\ell})$ and $(q'_{1,1}, \ldots, q'_{1,k}, q'_{2,1}, \ldots, q'_{2,\ell})$ in M we have $q_{1,i} \neq q'_{1,i}$ for $i = 1, \ldots, k$ and $q_{1,j} \neq q_{1,j'}$ for $j = 1, \ldots, \ell$.

Given such an instance, we construct a bipartite graph G with partition classes V_1 and V_2. First we put all elements in $Q_{1,1} \cup \ldots \cup Q_{1,k}$ in V_1, and all elements in $Q_{2,1} \cup \ldots \cup Q_{2,\ell}$ in V_2. Then we introduce an extra copy of $G_4(k,\ell)$ for each hyperedge $h = (q_{1,1}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,\ell})$ by adding the missing vertices and edges of this copy to G. We observe that indeed G is bipartite. We also observe that G has polynomial size.

We claim that $((Q_{1,1}, \ldots, Q_{1,k}, Q_{2,1}, \ldots, Q_{2,\ell}), H)$ admits a $k+\ell$-dimensional matching M if and only if G is a $K_{k,\ell}$-pseudo-cover.

Suppose $((Q_{1,1}, \ldots, Q_{1,k}, Q_{2,1}, \ldots, Q_{2,\ell}), H)$ admits a $k+\ell$-dimensional matching M. We define a homomorphism f from G to $K_{k,\ell}$ as follows. For $1 \leq i \leq k$, we define $f(q) = x_i$ if $q \in Q_{1,i}$, and for $1 \leq j \leq \ell$ we define $f(q) = y_j$ if $q \in Q_{2,j}$. For each hyperedge $h = (q_{1,1}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,\ell})$, we let $f(p_{1,i}) = x_i$ for $i = 1, \ldots, k$ and $f(p_{2,j}) = y_j$ for $j = 1, \ldots, \ell$.

For all $h \in M$, we let every q-vertex of h has all its matched neighbors in the copy of $G_4(k,\ell)$ that corresponds to h. Since M is a $k+\ell$-dimensional matching, the matched neighbors of every q-vertex are now defined. We then apply Lemma 7 and find that G is a $K_{k,\ell}$-pseudo-cover.

Conversely, suppose that f is a pseudo-covering from G to $K_{k,\ell}$. By Lemma 8, every q-vertex has all its matched neighbors in exactly one copy of $G_4(k,\ell)$ that corresponds to a hyperedge h such that the matched neighbor of *every* q-vertex in h is as a matter of fact in that copy $G_4(k,\ell)$. We now define M to be the set of all such hyperedges. Then M is a $k+\ell$-dimensional matching: any q-vertex appears in exactly one hyperedge of M. □

6 Further Research on Pseudo-coverings

Pseudo-coverings are closely related to the so-called locally constrained homomorphisms, which are homomorphisms with some extra restrictions on the neighborhood of each vertex. In Section 1 we already defined a covering which is also called a locally bijective homomorphism. There are two other types of such homomorphisms. First, a homomorphism from a graph G to a graph H is called *locally injective* or a *partial covering* if for every $u \in V_G$ the restriction of f to the neighborhood of u, i.e. the mapping $f_u : N_G(u) \to N_H(f(u))$, is injective. Second, a homomorphism from a graph G to a graph H is called *locally surjective* or a *role assignment* if the mapping $f_u : N_G(u) \to N_H(f(u))$ is surjective for every $u \in V_G$. See [7] for a survey.

The following observation is insightful. Recall that $G[x, y]$ denotes the induced bipartite subgraph of a graph G with partition classes $f^{-1}(x)$ and $f^{-1}(y)$ for some homomorphism f from G to a graph H.

Observation 9 ([9]). *Let f be a homomorphism from a graph G to a graph H. For every edge xy of H,*

- *f is locally bijective if and only if $G[x, y]$ is 1-regular (i.e., a perfect matching) for all $xy \in E_H$;*
- *f is locally injective if and only if $G[x, y]$ has maximum degree at most one (i.e., a matching) for all $xy \in E_H$;*
- *f is locally surjective if and only if $G[x, y]$ has minimum degree at least one for all $xy \in E_H$.*

By definition, every covering is a pseudo-covering. We observe that this is in line with Proposition 1 and Observation 9. Furthermore, by these results, we find that every pseudo-covering is a locally surjective homomorphism. This leads to the following result, the proof of which is omitted.

Proposition 6. *For any fixed graph H, if H-Cover is NP-complete, then so is H-Pseudo-Cover.*

Due to Proposition 6, the NP-completeness of $K_{k,\ell}$-Pseudo-Cover for $k, \ell \geq 3$ also follows from the NP-completeness of $K_{k,\ell}$-Cover for these values of k, ℓ. The latter is shown by Kratochvíl, Proskurowski and Telle [15]. However, these authors show in the same paper [15] that $K_{k,\ell}$-Cover is solvable in polynomial time for the cases k, ℓ with $\min\{k, \ell\} \leq 2$. Hence for these cases we have to rely on our proof in Section 5.

Another consequence of Proposition 6 is that H-Pseudo-Cover is NP-complete for all k-regular graphs H for any $k \geq 3$ due to a hardness result for the corresponding H-Cover [6]. However, a complete complexity classification of H-Pseudo-Cover is still open, just as dichotomy results for H-Partial Cover and H-Cover are not known, whereas for the locally surjective case a complete complexity classification is known [8]. Hence, for future research we will try to classify the computational complexity of the H-Pseudo-Cover problem. So far we have obtained some partial results but a complete classification seems already problematic for trees (we found many polynomially solvable and NP-complete cases).

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