Bucolic complexes

B. Brešar[1], J. Chalopin[2], V. Chepoi[2], T. Gologranc[3], and D. Osajda[4,5]

[1]Faculty of Natural Sciences and Mathematics, University of Maribor,
Koroška cesta 160, SI-2000 Maribor, Slovenia
bostjan.bresar@uni-mb.si

[2]Laboratoire d'Informatique Fondamentale, Aix-Marseille Université and CNRS,
Faculté des Sciences de Luminy, F-13288 Marseille Cedex 9, France
{jeremie.chalopin, victor.chepoi}@lif.univ-mrs.fr

[3]Institute of Mathematics, Physics and Mechanics,
Jadranska 19, SI-1000 Ljubljana, Slovenia
tanja.gologranc@imfm.si

[4]Universität Wien, Fakultät für Mathematik
Nordbergstraße 15, 1090 Wien, Austria

[5]Instytut Matematyczny, Uniwersytet Wrocławski (on leave),
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
dosaj@math.uni.wroc.pl

Abstract. We introduce and investigate bucolic complexes, a common generalization of systolic complexes and of CAT(0) cubical complexes. They are defined as simply connected prism complexes satisfying some local combinatorial conditions. We study various approaches to bucolic complexes: from graph-theoretic and topological perspective, as well as from the point of view of geometric group theory. In particular, we characterize bucolic complexes by some properties of their 2–skeleta and 1–skeleta (that we call bucolic graphs), by which several known results are generalized. We also show that locally-finite bucolic complexes are contractible, and satisfy some nonpositive-curvature-like properties.

1. Introduction

CAT(0) cubical complexes and systolic (simplicial) complexes constitute two classes of polyhedral complexes that have been intensively explored over last decades. Both CAT(0) cubical and systolic complexes exhibit various properties typical for spaces with different types of nonpositive curvature. Hence groups of isomorphisms of such complexes provide numerous examples of groups with interesting properties. Both CAT(0) cubical complexes and systolic complexes can be nicely characterized via their 1- and 2-skeleta. It turns out that their 1-skeleta — median graphs and bridged graphs — are intensively studied in various areas of discrete mathematics (see Section 3 for related results and references).

In this article we introduce a notion of *bucolic complexes* — polyhedral complexes being a common generalization of CAT(0) cubical [25, 38], systolic [20, 26, 30], and weakly systolic [33] complexes, and we initiate a regular study of them. Analogously to CAT(0) cubical

and systolic complexes, bucolic complexes are defined as simply connected prism complexes satisfying some local combinatorial conditions (see Subsection 2.4 for details). Our main result on bucolic complexes is the following characterization via their 1- and 2-skeleta (see Section 2 for explanations of all the notions involved).

Theorem 1. *For a prism complex* $\mathbf{X}$*, the following conditions are equivalent:*

   (i) $\mathbf{X}$ *is a bucolic complex;*

  (ii) *the 2-skeleton* $\mathbf{X}^{(2)}$ *of* $\mathbf{X}$ *is a connected and simply connected triangle-square flag complex satisfying the wheel, the 3-cube, and the 3-prism conditions;*

 (iii) *the 1-skeleton* $G(\mathbf{X})$ *of* $\mathbf{X}$ *is a connected weakly modular graph that does not contain induced subgraphs of the form* $K_{2,3}$*,* W_4*, and* W_4^-*, i.e.,* $G(\mathbf{X})$ *is a bucolic graph not containing infinite hypercubes.*

Moreover, if $\mathbf{X}$ *is a connected flag prism complex satisfying the wheel, the cube, and the prism conditions, then the universal cover* $\widetilde{\mathbf{X}}$ *of* $\mathbf{X}$ *is bucolic.*

As an immediate corollary we obtain an analogous characterization (Corollary 3 in Section 5) of *strongly bucolic complexes* — the subclass of bucolic complexes containing products of systolic complexes but not all weakly systolic complexes (see Subsection 2.4 for details). The condition (iii) in the above characterization is a global condition — weak modularity concerns balls of arbitrary radius; cf. Section 2. Thus the theorem — and in particular the last assertion — may be seen as a local-to-global result concerning polyhedral complexes. It is an analogue of the Cartan-Hadamard theorem appearing in various contexts of non-positive-curvature: CAT(0) spaces [7], Gromov hyperbolic spaces [25], systolic and weakly systolic complexes [30, 33].

The 1–skeleta of CAT(0) cubical complexes are exactly the median graphs [20, 37] which constitute a central graph class in metric graph theory (see [6] and the references therein). In the literature there are numerous structural and other characterizations of median graphs. In particular, median graphs are the retracts of hypercubes [2], and can be obtained via so-called iterated gated amalgamations from cubes [29, 40]. The general framework of fiber-complemented graphs was introduced in [14, 15] and allows to prove such decomposition and retraction results. From this perspective, *bucolic graphs* are exactly the fiber-complemented graphs whose elementary gated subgraphs are weakly-bridged; more precisely, the 1–skeleta of bucolic complexes admit the following characterization.

Theorem 2. *For a graph* $G = (V, E)$ *not containing infinite cliques, the following conditions are equivalent:*

   (i) G *is a retract of the (weak) Cartesian product of weakly bridged (respectively, bridged) graphs;*

  (ii) G *is a weakly modular graph not containing induced* $K_{2,3}$*,* W_4*, and* W_4^- *(respectively,* $K_{2,3}$*,* W_4^-*,* W_4*, and* W_5*), i.e.,* G *is a bucolic (respectively, strongly bucolic) graph;*

 (iii) G *is a weakly modular graph not containing* $K_{2,3}$ *and* W_4^- *in which all elementary (or prime) gated subgraphs are edges or 2-connected weakly bridged (respectively, bridged) graphs.*

Moreover, if G is finite, then the conditions (i)-(iii) are equivalent to the following condition:

(iv) *G can be obtained by successive applications of gated amalgamations from Cartesian products of 2-connected weakly bridged (respectively, bridged) graphs.*

Theorem 2 allows us to show further non-positive-curvature-like properties of bucolic complexes. The first one completes the analogy with the Cartan-Hadamard theorem.

Theorem 3. *Locally-finite bucolic complexes are contractible.*

In particular, the above theorem provides local conditions (the ones appearing in the definition of bucolic complexes) for a prism complex implying asphericity. Note that this is one of not so many local asphericity criterions known — most of them appears in the case of some nonpositive curvature.

Similarly to the case of CAT(0) cubical groups and weakly systolic groups, we think that groups acting on bucolic complexes form an important class and deserve further studies. We believe that they have similar properties as groups acting on nonpositively curved spaces, and that they may provide many interesting examples. In the current paper we indicate two basic results on such groups.

Theorem 4. *If* **X** *is a locally-finite bucolic complex and F is a finite group acting by cell automorphisms on* **X***, then there exists a prism π of* **X** *which is invariant under the action of F. The center of the prism π is a point fixed by F.*

A standard argument gives the following immediate consequence of Theorem 4.

Corollary 1. *Let F be a group acting geometrically by automorphisms on a locally-finite bucolic complex* **X***. Then F contains only finitely many conjugacy classes of finite subgroups.*

We prove Theorems 3 and 4 only for locally-finite bucolic complexes. We do not know whether these theorems hold for non-locally-finite bucolic complexes and we leave this as an open question.

Article's structure. In the following Section 2 we introduce all the notions used later on. In Section 3, we review the related work on which our paper is based or which is generalized in our paper. In Section 4, we provide the characterization of bucolic graphs (Theorem 2). A proof of the main characterization of bucolic complexes (Theorem 1) is presented in Section 5. In Section 6, we prove the contractibility and the fixed point result for locally-finite bucolic complexes (Theorems 3 and 4). In Section 7, we complete the proof of Theorem 2 in the non-locally-finite case.
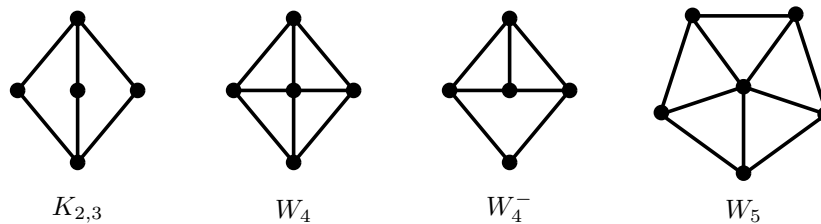
FIGURE 1. $K_{2,3}$, the wheel W_4, the almost-wheel W_4^-, and the wheel W_5.

2. Preliminaries

2.1. **Graphs.** All graphs $G = (V, E)$ occurring in this paper are undirected, connected, without loops or multiple edges, but not necessarily finite or locally-finite. For two vertices u and v of a graph G, we will write $u \sim v$ if u and v are adjacent and $u \nsim v$, otherwise. We will use the notation $v \sim A$ to note that a vertex v is adjacent to all vertices of a set A and the notation $v \nsim A$ if v is not adjacent to any of the vertices of A. For a subset $A \subseteq V$, the subgraph of $G = (V, E)$ *induced by* A is the graph $G(A) = (A, E')$ such that $uv \in E'$ if and only if $uv \in E$. $G(A)$ is also called a *full subgraph* of G. We will say that a graph H is *not an induced subgraph* of G if H is not isomorphic to any induced subgraph $G(A)$ of G.

By an (a, b)-*path* in a graph G we mean a sequence of vertices $P = (x_0 = a, x_1, \ldots, x_{k-1}, x_k = b)$ such that any two consecutive vertices x_i and x_{i+1} of P are different and adjacent (notice that in general we may have $x_i = x_j$ if $|i - j| \geq 2$). If $k = 2$, then we call P a *2-path* of G. If $x_i \neq x_j$ for $|i - j| \geq 2$, then P is called a *simple* (a, b)-*path*. A graph $G = (V, E)$ is *2-connected* if any two vertices a, b of G can be connected by two vertex-disjoint (a, b)-paths. Equivalently, a graph G is 2-connected if G has at least 3 vertices and $G(V \smallsetminus \{v\})$ is connected for any vertex $v \in V$, i.e., G remains connected after removing from G any vertex v and the edges incident to v.

The *wheel* W_k is a graph obtained by connecting a single vertex – the *central vertex* c – to all vertices of the k-cycle $(x_1, x_2, \ldots, x_k, x_1)$; the *almost wheel* W_k^- is the graph obtained from W_k by deleting a spoke (i.e., an edge between the central vertex c and a vertex x_i of the k-cycle), see Figure 1. The *extended* 5-*wheel* $\widehat{W}_5$ is a 5-wheel W_5 plus a 3-cycle (a, x_1, x_2, a) such that $a \nsim c, x_3, x_4, x_5$.

The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v of a graph G is the length of a shortest (u, v)-path. For a vertex v of G and an integer $r \geq 1$, we will denote by $B_r(v, G)$ the *ball* in G (and the subgraph induced by this ball) of radius r centered at v, i.e., $B_r(v, G) = \{x \in V : d(v, x) \leq r\}$. More generally, the *r-ball around a set* $A \subseteq V$ is the set (or the

subgraph induced by) $B_r(A, G) = \{v \in V : d(v, A) \leq r\}$, where $d(v, A) = \min\{d(v, x) : x \in A\}$. As usual, $N(v) = B_1(v, G) \smallsetminus \{v\}$ denotes the set of neighbors of a vertex v in G. The *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)-paths, that is, of all vertices (metrically) *between* u and v: $I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}$. An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any pair of its vertices. The smallest convex subgraph containing a given subgraph S is called the *convex hull* of S and is denoted by $\mathrm{conv}(S)$. An induced subgraph of G (or the corresponding vertex set A) is called *locally convex* if it includes the interval of G between any pair of its vertices at distance two having a common neighbor in A. A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$.

Definition 2.1 (Retraction). A *retraction* φ of a graph G is an idempotent nonexpansive mapping of G into itself, that is, $\varphi^2 = \varphi : V(G) \to V(G)$ with $d(\varphi(x), \varphi(y)) \leq d(x, y)$ for all $x, y \in W$ (equivalently, a retraction is a simplicial idempotent map $\varphi : G \to G$). The subgraph of G induced by the image of G under φ is referred to as a *retract* of G.

Definition 2.2 (Mooring). A map $f : V(G) \to V(G)$ is a *mooring* of a graph G onto u if the following holds:

(1) $f(u) = u$ and for every $v \neq u$, $f(v) \sim v$ and $d(f(v), u) = d(v, u) - 1$.
(2) for every edge vw of G, $f(v)$ and $f(w)$ coincide or are adjacent.

A graph G is *moorable* if, for every vertex u of G, there exists a mooring of G onto u.

Mooring can be viewed as a combing property of graphs — the notion coming from geometric group theory [23]. Let u be a distinguished vertex ("base point") of a graph G. Two shortest paths $P(x, u), P(y, u)$ in G connecting two adjacent vertices x, y to u are called 1-*fellow travelers* if $d(x', y') \leq 1$ holds for each pair of vertices $x' \in P(x, u), y' \in P(y, u)$ with $d(x, x') = d(y, y')$. A *geodesic* 1-*combing* of G with respect to the base point u comprises shortest paths $P(x, u)$ between u and all vertices x such that $P(x, u)$ and $P(y, u)$ are 1-fellow travelers for any edge xy of G. One can select the combing paths so that their union is a spanning tree T_u of G that is rooted at u and preserves the distances from u to all vertices. The neighbor $f(x)$ of $x \neq u$ in the unique path in T_u connecting x with the root u will be called the *father* of x (set also $f(u) = u$). Then f is a mooring of G onto u (vice-versa, any mooring of G onto u can be viewed as a geodesic 1-combing with respect to u). A geodesic 1-combing of G with respect to u thus amounts to a tree T_u preserving the distances to the root u such that if x and y are adjacent in G then $f(x)$ and $f(y)$ either coincide or are adjacent in G.

Definition 2.3 (Gated amalgam). An induced subgraph H of a graph G is *gated* [22] if for every vertex x outside H there exists a vertex x' in H (the *gate* of x) such that $x' \in I(x, y)$ for any y of H. A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G.

Gated sets are convex and the intersection of two gated sets is gated. By Zorn lemma there exists a smallest gated subgraph containing a given subgraph S, called the *gated hull* of S. A graph G is said to be *elementary* [14] if the only proper gated subgraphs of G are singletons.

Let G_i, $i \in I$ be an arbitrary family of graphs. The *Cartesian product* $\square_{i \in I} G_i$ is a graph whose vertices are all functions $x : i \mapsto x_i$, $x_i \in V(G_i)$. Two vertices x, y are adjacent if there exists an index $j \in I$ such that $x_j y_j \in E(G_j)$ and $x_i = y_i$ for all $i \neq j$. Note that Cartesian product of infinitely many nontrivial graphs is disconnected. Therefore, in this case the connected components of the Cartesian product are called *weak Cartesian products*. Since in our paper all graphs are connected, for us a Cartesian product graph will always mean a weak Cartesian product graph. A graph with at least two vertices is said to be *prime* [4, 14] if it is neither a Cartesian product nor a gated amalgam of smaller graphs. A *strong product* $\boxtimes_{i \in I} G_i$ is a graph whose vertices are all functions $x : i \mapsto x_i$, $x_i \in V(G_i)$. Two vertices x, y are adjacent if for all indices $i \in I$ either $x_i = y_i$ or $x_i y_i \in E(G_i)$.

Definition 2.4 (Weakly modular graphs)**.** A graph G is *weakly modular with respect to a vertex u* if its distance function d satisfies the following triangle and quadrangle conditions (see Figure 2):

- *Triangle condition* $\mathrm{TC}(u)$: for any two vertices v, w with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.
- *Quadrangle condition* $\mathrm{QC}(u)$: for any three vertices v, w, z with $d(v, z) = d(w, z) = 1$ and $2 = d(v, w) \leq d(u, v) = d(u, w) = d(u, z) - 1$, there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.

A graph G is *weakly modular* [3] if G is weakly modular with respect to any vertex u.

Median, bridged, and weakly bridged graphs constitute three important subclasses of weakly modular graphs.

Definition 2.5 (Median graphs)**.** A graph G is *median* if it is a bipartite weakly modular graph not containing $K_{2,3}$ as induced subgraphs.

Median graphs can be also defined in many other equivalent ways [27, 31, 41]. For example, median graphs are exactly the graphs in which every triplet of vertices u, v, w has a unique *median*, i.e., a vertex lying simultaneously in $I(u, v)$, $I(v, w)$, and $I(w, u)$.

Definition 2.6 (Bridged and weakly bridged graphs)**.** A graph G is *bridged* if it is weakly modular and does not contain induced 4- and 5-cycles. A graph G is *weakly bridged* if G is a weakly modular graph and does not contain 4-cycles.

There exist other equivalent definitions of bridged graphs [24, 39]. Bridged graphs are exactly the graphs that do not contain isometric cycles of length greater than 3. Alternatively, a graph G is bridged if and only if the balls $B_r(A, G)$ around convex sets A of G are convex. Analogously, a graph G is weakly bridged if and only if G has convex balls $B_r(x, G)$ and does not contain induced C_4 [21].
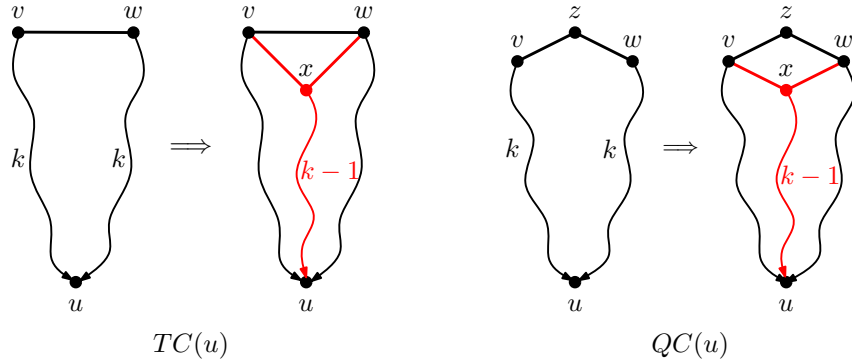
FIGURE 2. Triangle and quadrangle conditions

Median, bridged, and weakly bridged graphs are pre-median graphs: a graph G is *pre-median* [14,15] if G is a weakly modular graph without induced $K_{2,3}$ and W_4^-. An important property of pre-median graphs is that within this class, a graph is elementary if and only if it is prime [14]. Chastand [14,15] proved that pre-median graphs are fiber-complemented graphs (a definition follows). Any gated subset S of a graph G gives rise to a partition F_a ($a \in S$) of the vertex-set of G; viz., the *fiber* F_a of a relative to S consists of all vertices x (including a itself) having a as their gate in S. According to Chastand [14,15], a graph G is called *fiber-complemented* if for any gated set S all fibers F_a ($a \in S$) are gated sets of G.

2.2. **Prism complexes.** In this paper, we consider a particular class of cell complexes (compare e.g. [13, p. 111-115]), called prism complexes, in which all cells are prisms of finite dimension. Cubical and simplicial cell complexes are particular instances of prism complexes. Although most of the notions presented below can be defined for all cell complexes and some of them for topological spaces, we will introduce them only for prism complexes.

An *abstract simplicial complex* is a family $\mathbf{X}$ of finite subsets (of a given set) called *simplices* which is closed for intersections and inclusions, i.e., $\sigma, \sigma' \in \mathbf{X}$ and $\sigma'' \subset \sigma$ implies that $\sigma \cap \sigma', \sigma'' \in \mathbf{X}$. For an abstract simplicial complex $\mathbf{X}$, denote by $V(\mathbf{X})$ and $E(\mathbf{X})$ the set of all 0-dimensional and 1-dimensional simplices of $\mathbf{X}$ and call the pair $G(\mathbf{X}) = (V(\mathbf{X}), E(\mathbf{X}))$ the *1-skeleton* of $\mathbf{X}$. Conversely, for a graph G not containing infinite cliques, one can derive an abstract simplicial complex $\mathbf{X}_{\text{simpl}}(G)$ (the *clique complex* of G) by taking the vertex sets of all complete subgraphs (cliques) as simplices of the complex. By a *simplicial complex* we will mean the geometric realization of an abstract simplicial complex. It is a cell complex with cells corresponding to abstract simplices, being (topologically) solid simplices.

A *prism* is a convex polytope which is a Cartesian product of a finite number of finite-dimensional simplices. Faces of a prism are prisms of smaller dimensions. Particular instances of prisms are simplices and cubes (products of intervals). A *prism complex* is a cell complex $\mathbf{X}$ in which all cells are prisms so that the intersection of two prisms is empty or a common face of each of them. *Cubical complexes* are prism complexes in which all cells are cubes and simplicial complexes are prism complexes in which all cells are simplices. The *1-skeleton*

$G(\mathbf{X}) = \mathbf{X}^{(1)}$ of a prism complex $\mathbf{X}$ has the 0-dimensional cells of $\mathbf{X}$ as vertices and the 1-dimensional cells of $\mathbf{X}$ as edges. The 1-skeleton of a prism of $\mathbf{X}$ is a *Hamming graph*, i.e., the Cartesian product of complete subgraphs of $G(\mathbf{X})$. For vertices v, w or a set of vertices A of a prism complex $\mathbf{X}$ we will write $v \sim w$, $v \sim A$ (or $v \nsim w$, $v \nsim A$) if and only if a similar relation holds in the graph $G(\mathbf{X})$. Note that a prism complex $\mathbf{X}$ is connected if and only $G(\mathbf{X}) = \mathbf{X}^{(1)}$ is a connected graph. In this paper, all prism complexes we consider are connected. The 2-skeleton $\mathbf{X}^{(2)}$ of $\mathbf{X}$ is a *triangle-square complex* obtained by taking the 0-dimensional, 1-dimensional, and 2-dimensional cells of $\mathbf{X}$. A prism complex $\mathbf{X}$ is *simply connected* if every continuous map $S^1 \to X$ is null-homotopic. Note that $\mathbf{X}$ is simply connected if and only if $\mathbf{X}^{(2)}$ is simply connected. The *star* $\operatorname{St}(v, \mathbf{X})$ of a vertex v in a prism complex $\mathbf{X}$ is the subcomplex consisting of the union of all cells of $\mathbf{X}$ containing v.

For every graph G that does not contain infinite cliques or infinite hypercubes as induced subgraphs, G gives rise to four cell complexes: to a prism complex $\mathbf{X}_{\text{prism}}(G)$, to a simplicial complex $\mathbf{X}_{\text{simpl}}(G)$, to a cubical complex $\mathbf{X}_{\text{cube}}(G)$, and to a triangle-square complex $\mathbf{X}_{\text{tr-sq}}(G)$. The *prism complex* $\mathbf{X}_{\text{prism}}(G)$ spanned by G has P as a prism if and only if the 1-skeleton of P is an induced subgraph of G which is a Hamming graph. Analogously, one can define the *simplicial complex* $\mathbf{X}_{\text{simpl}}(G)$ and the *cubical complex* $\mathbf{X}_{\text{cube}}(G)$ of G as the complexes consisting of all complete subgraphs and all induced cubes of G as cells, respectively. In the same way, in the *triangle-square complex* $\mathbf{X}_{\text{tr-sq}}(G)$ of G, the triangular and square cells are spanned by the 3-cycles and induced 4-cycles of G. The triangle-square complex $\mathbf{X}_{\text{tr-sq}}(G)$ of G coincides with the 2-skeleton of the prism complex $\mathbf{X}_{\text{simpl}}(G)$. Notice also that $\mathbf{X}_{\text{simpl}}(G) \cup \mathbf{X}_{\text{cube}}(G) \subseteq \mathbf{X}_{\text{prism}}(G)$. In general, these four complexes can be pairwise distinct, but the graph G is the 1-skeleton of all these four complexes.

An abstract simplicial complex $\mathbf{X}$ is a *flag complex* (or a *clique complex*) if any set of vertices is included in a simplex of $\mathbf{X}$ whenever each pair of its vertices is contained in a simplex of $\mathbf{X}$ (in the theory of hypergraphs this condition is called *conformality*; see for example [6]). A flag simplicial complex can therefore be recovered from its underlying graph $G(\mathbf{X})$ by the formula $\mathbf{X} = \mathbf{X}_{\text{simpl}}(G(\mathbf{X}))$: the complete subgraphs of $G(\mathbf{X})$ are exactly the simplices of $\mathbf{X}$. Analogously, a prism (respectively, a cubical, or a triangle-square) complex $\mathbf{X}$ is a *flag complex* if $\mathbf{X} = \mathbf{X}_{\text{prism}}(G(\mathbf{X}))$ (respectively, $\mathbf{X} = \mathbf{X}_{\text{cube}}(G(\mathbf{X}))$ or $\mathbf{X} = \mathbf{X}_{\text{tr-sq}}(G(\mathbf{X}))$). All complexes occurring in this paper are flag complexes.

Let $\mathbf{X}(W_k) := \mathbf{X}_{\text{tr-sq}}(W_k)$ and $\mathbf{X}(W_k^-) := \mathbf{X}_{\text{tr-sq}}(W_k^-)$ be the triangle-square (or the prism) complexes whose underlying graphs are the graphs W_k and W_k^-, respectively (the first consists of k triangles and the second consists of $k - 2$ triangles and one square). Analogously, let $\mathbf{X}(\widehat{W_5}) = \mathbf{X}_{\text{simpl}}(\widehat{W_5})$ be the 2-dimensional simplicial complex made of 6 triangles and whose underlying graph is the extended 5-wheel $\widehat{W_5}$.

As morphisms between cell complexes we consider all *cellular maps*, i.e., maps sending (linearly) cells to cells. An *isomorphism* is a bijective cellular map being a linear isomorphism on each cell. A *covering (map)* of a cell complex $\mathbf{X}$ is a cellular surjection $p : \widetilde{\mathbf{X}} \to \mathbf{X}$ such that $p|_{\operatorname{St}(\widetilde{v}, \widetilde{\mathbf{X}})} : \operatorname{St}(\widetilde{v}, \widetilde{\mathbf{X}}) \to \operatorname{St}(v, \mathbf{X})$ is an isomorphism for every vertex v in $\mathbf{X}$, and every vertex $\widetilde{v} \in \widetilde{\mathbf{X}}$ with $p(\widetilde{v}) = v$; compare [28, Section 1.3]. The space $\widetilde{\mathbf{X}}$ is then called a *covering space*. A

universal cover of **X** is a simply connected covering space $\widetilde{\mathbf{X}}$. It is unique up to isomorphism; cf. [28, page 67]. In particular, if **X** is simply connected, then its universal cover is **X** itself. A group F *acts by automorphisms* on a cell complex **X** if there is a homomorphism $F \to \mathrm{Aut}(\mathbf{X})$ called an *action of F*. The action is *geometric* (or F *acts geometrically*) if it is proper (i.e. cells stabilizers are finite) and cocompact (i.e. the quotient $\mathbf{X}/F$ is compact).

2.3. **CAT(0) cubical complexes and systolic complexes.** A *geodesic triangle* $\Delta = \Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic between each pair of vertices (the sides of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x_1', x_2', x_3')$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x_i', x_j') = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

Definition 2.7 (CAT(0) spaces)**.** A geodesic metric space (X, d) is a *CAT(0) (or a nonpositively curved) space* if the geodesic triangles in **X** are thinner than their comparison triangles in the Euclidean plane [25], i.e, if $\Delta(x_1, x_2, x_3)$ is a geodesic triangle of X and y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x_1', x_2']$ of the comparison triangle $\Delta(x_1', x_2', x_3')$ such that $d_{\mathbb{E}^2}(x_i', y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x_3', y')$.

CAT(0) spaces satisfy many nice metric and convexity properties and can be characterized in various ways (for a full account of this theory consult the book [13]). For example, any two points of a CAT(0) space can be joined by a unique geodesic. CAT(0) property is also equivalent to convexity of the function $f : [0, 1] \to X$ given by $f(t) = d(\alpha(t), \beta(t))$, for any geodesics α and β (which is further equivalent to convexity of the neighborhoods of convex sets). This implies that CAT(0) spaces are contractible.

Definition 2.8 (CAT(0) cubical complexes)**.** A cubical complex **X** is a *CAT(0) cubical complex* if **X** endowed with intrinsic ℓ_2-metric is a CAT(0) space.

Gromov [25] gave a beautiful characterization of CAT(0) cubical complexes as simply connected cubical complexes satisfying the following combinatorial condition: if three k-cubes pairwise intersect in a $(k-1)$-cube and all three intersect in a $(k-2)$-cube, then are included in a $(k+1)$-dimensional cube. This condition can be equivalently formulated as the requirement that the links of 0-cubes are simplicial flag complexes.

Now we briefly recall the definitions of systolic and weakly systolic simplicial complexes, which are both considered as simplicial complexes with combinatorial nonpositive curvature. For an integer $k \geq 4$, a flag simplicial complex **X** is *locally k-large* if every cycle consisting of less than k edges in any of its links of simplices has some two consecutive edges contained in a 2-simplex of this link, i.e., the links do not contain induced cycles of length $< k$.

Definition 2.9 (Systolic and weakly systolic complexes)**.** A simplicial complex is *k-systolic* if it is locally k-large, connected and simply connected. A flag simplicial complex is *systolic* if it is 6-systolic [20, 26, 30]. A simplicial complex **X** is *weakly systolic* [21, 33] if **X** is flag, connected and simply connected, locally 5-large, and satisfies the following local condition:
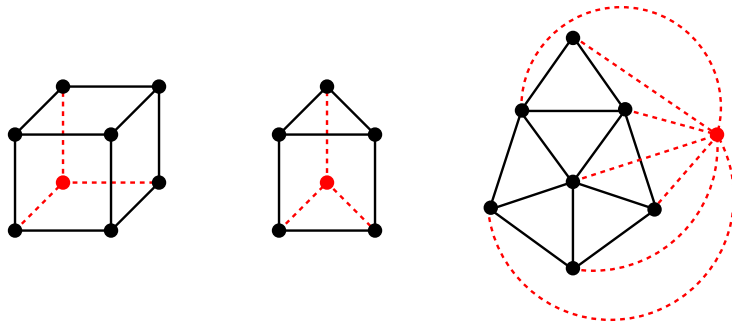
FIGURE 3. The 3-cube condition (left), the 3-prism condition (middle), and the $\widehat{W_5}$-wheel condition (right).

$\widehat{W_5}$-*wheel condition*: for each extended 5-wheel $\mathbf{X}(\widehat{W_5})$ of $\mathbf{X}$, there exists a vertex v adjacent to all vertices of this extended 5-wheel (see Fig. 3, right).

2.4. **Bucolic complexes and bucolic graphs.** In this subsection we define central objects of the article: bucolic complexes and bucolic graphs[1].

Definition 2.10 (Bucolic complexes). A prism complex $\mathbf{X}$ is *bucolic* if it is flag, connected and simply connected, and satisfies the following three local conditions:

Wheel condition: the 1-skeleton $\mathbf{X}^{(1)}$ of $\mathbf{X}$ does not contain induced W_4 and satisfies the $\widehat{W_5}$-wheel condition;

Cube condition: if $k \geq 2$ and three k-cubes of $\mathbf{X}$ pairwise intersect in a $(k-1)$-cube and all three intersect in a $(k-2)$-cube, then they are included in a $(k+1)$-dimensional cube of $\mathbf{X}$;

Prism condition: if a cube and a simplex of $\mathbf{X}$ intersect in a 1-simplex, then they are included in a prism of $\mathbf{X}$.

A bucolic complex $\mathbf{X}$ is *strongly bucolic* if $G(\mathbf{X})$ does not contain induced W_5, i.e., a prism complex $\mathbf{X}$ is strongly bucolic if it is flag, connected, simply connected, and satisfies the cube and prism conditions, as well as the following local condition:

Strong-wheel condition: the 1-skeleton $\mathbf{X}^{(1)}$ of $\mathbf{X}$ does not contain induced W_4 and W_5.

As we already noticed, subject to simple connectivity, the wheel condition characterizes the weakly systolic complexes in the class of flag simplicial complexes. On the other hand, the cube condition for cubical complexes is equivalent to Gromov's condition of flagness of links. Finally, the prism condition for prism complexes shows how simplices and cubes of $\mathbf{X}$ give rise to prisms.

Now, we consider the 2-dimensional versions of the last two conditions. We introduce two combinatorial conditions for a triangle-square complex $\mathbf{X}$:

3-Cube condition: any three squares of **X**, pairwise intersecting in an edge, and all three intersecting in a vertex of **X**, are included in the 2-skeleton of a 3-dimensional cube (see Fig. 3, left);

3-Prism condition: any house (i.e., a triangle and a square of **X** sharing an edge) is included in the 2-skeleton of a 3-dimensional prism (see Fig. 3, middle).

We conclude this section with the definition of bucolic and strongly bucolic graphs.

Definition 2.11 (Bucolic graphs). A graph G is *bucolic* if it is weakly modular, does not contain infinite cliques and does not contain induced subgraphs of the form $K_{2,3}$, W_4, and W_4^-. A bucolic graph is *strongly bucolic* if it does not contain induced W_5.

3. Related work

In his seminal paper [25], among many other results, Gromov characterized CAT(0) cubical complexes as simply connected cubical complexes in which the links of 0-cubes are simplicial flag complexes. Subsequently, Sageev [38] introduced and investigated the concept of (combinatorial) hyperplanes of CAT(0) cubical complexes, showing in particular that each hyperplane is itself a CAT(0) cubical complex and partitions the complex into two CAT(0) cubical complexes. These two results identify CAT(0) cubical complexes as the basic objects in a "high-dimensional Bass-Serre theory", and CAT(0) and nonpositively–curved cubical complexes have thus been studied extensively in geometric group theory. For instance, many well-known classes of groups are known to act nicely on CAT(0) cubical complexes.

It was shown in [20, 37] that the 1-skeleta of CAT(0) cubical complexes are exactly the median graphs. This result establishes a bridge between two seemingly different mathematical structures and two different areas of mathematics. Median graphs and related structures (median algebras, event structures, copair Helly hypergraphs) occur in different areas of discrete mathematics, universal algebra, and theoretical computer science. Median graphs, median algebras, and CAT(0) cubical complexes have many nice properties and admit numerous characterizations. All median structures are intimately related to hypercubes: median graphs are isometric subgraphs of hypercubes; in fact, by a classical result of Bandelt [2] they are the retracts of hypercubes into which they embed isometrically. This isometric embedding of each median graph into a hypercube canonically defines on the associated CAT(0) cubical complex a space with walls "à la Haglund-Paulin" and Sageev's hyperplanes. It was also shown by Isbell [29] and van de Vel [40] that every finite median graph G can be obtained by gated amalgams from hypercubes, thus showing that K_2 is the only prime median graph. Median graphs also have a remarkable algebraic structure, which is induced by the ternary operation on the vertex set that assigns to each triplet of vertices the unique median vertex, and their algebra can be characterized using four natural axioms [7, 29] among all discrete ternary algebras. For more detailed information about median structures, the interested reader can consult the survey [6] and the books [27, 31, 41].

Bridged graphs are the graphs in which all isometric cycles have length 3. It was shown in [24, 39] that the bridged graphs are exactly the graphs in which the metric convexity

satisfies one of the basic properties of the CAT(0) geometry: neighborhoods of convex sets are convex. Combinatorial and structural aspects of bridged graphs have been investigated in [1, 16, 35]. In particular, it was shown in [1] that finite bridged graphs are dismantlable. Similarly to the local-to-global characterization of CAT(0) cubical complexes of [25], it was shown in [20] that the clique complexes of bridged graphs are exactly the simply connected simplicial flag complexes in which the links of vertices do not contain induced 4- and 5-cycles. These complexes have been rediscovered and investigated in depth by Januszkiewicz and Swiatkowski [30], and, independently by Haglund [26], who called them "systolic complexes" and considered them as simplicial complexes satisfying combinatorial nonpositive curvature property. In general, these complexes are not CAT(0). More recently, Osajda [33] proposed a generalization of systolic complexes still preserving most of the structural properties of systolic complexes: the resulting weakly systolic complexes and their 1-skeleta – the weakly bridged graphs – have been investigated and characterized in [21].

The structure theory of graphs based on Cartesian multiplication and gated amalgamation was further elaborated for more general classes of graphs. Some of the results for median graphs have been extended to quasi-median graphs introduced by Mulder [31] and further studied in [8]: quasi-median graphs are the weakly modular graphs not containing induced $K_{2,3}$ and $K_4 - e$ and they can be characterized as the retracts of Hamming graphs (finite quasi-median graphs can be obtained from complete graphs by Cartesian products and gated amalgamations). More recently, Bandelt and Chepoi [4] presented a similar decomposition scheme of weakly median graphs (the weakly modular graphs in which the vertex x in the triangle and quadrangle conditions is unique) and characterized the prime graphs with respect to this decomposition: the hyperoctahedra and their subgraphs, the 5-wheel W_5, and the 2-connected plane bridged graphs. Generalizing the proof of the decomposition theorem of [4], Chastand [14, 15] presented a general framework of fiber-complemented graphs allowing to establish many general properties, previously proved only for particular classes of graphs. An important subclass of fiber-complemented graphs is the class of pre-median graphs [14, 15]. It is an open problem to characterize all prime (elementary) fiber-complemented or pre-median graphs (see [14, p. 121]).

Since CAT(0) cubical complexes and systolic simplicial complexes can be both characterized via their 1-skeleta and via simple connectivity and local conditions, a natural question is to find a common generalization of such complexes which still obey the combinatorial non-positive curvature properties. The prism complexes derived from fiber-complemented graphs is a potential source of such cell complexes. In [11], answering a question from [12], the first step in this direction was taken, and the 1-skeleta of prism complexes resulting from clique complexes of chordal graphs by applying Cartesian products and gated amalgams have been characterized. It was also shown that, endowed with the l_2-metric, such prism complexes are CAT(0) spaces.

In this paper, we continue this line of research and characterize the graphs G which are retracts of Cartesian products of weakly bridged and bridged graphs. We show (cf. Theorem 2) that these graphs are exactly the bucolic and strongly bucolic graphs. We also establish that

the finite bucolic (respectively, strongly bucolic) graphs are exactly the graphs obtained by gated amalgamations of Cartesian products of weakly bridged (respectively, bridged) graphs, thus answering Question 1 from [11]. This also provides a partial answer to Chastand's problem mentioned above (by showing that the weakly bridged graphs are exactly the prime graphs of pre-median graphs without W_4 and that the bridged graphs are the prime graphs of pre-median graphs without W_4 and W_5) and extends the analogous results on finite median, quasi-median, and weakly median graphs. Our previous result can be viewed as a characterization of 1-skeleta of bucolic complexes. We also characterize (Theorem 1) bucolic complexes via their 2-skeleta by showing that they are exactly the simply connected triangle-square complexes satisfying the 3-cube and 3-prism conditions and not containing W_4, W_5, and W_5^- (this answers Question 2 from [11]) and, together with the first result, generalizes the characterizations of CAT(0) cubical complexes, systolic and weakly systolic complexes via theirs 1- and 2-skeleta.

Then we prove that the locally-finite bucolic complexes are contractible (Theorem 3). Thus the three results constitute a version of the Cartan-Hadamard theorem, saying that under some local conditions the complex is aspherical, i.e. its universal covering space is contractible. Only limited number of such local characterizations of asphericity is known, and most of them refer to the notion of nonpositive curvature; cf. e.g. [13, 23, 25, 30, 33]. In fact bucolic complexes exhibit many nonpositive-curvature-like properties. Besides the Cartan-Hadamard theorem we prove the fixed point theorem for finite groups acting on locally-finite bucolic complexes (Theorem 4), and we conclude that groups acting geometrically on such complexes have finitely many conjugacy classes of finite subgroups (Corollary 1). Counterparts of such results are known for other nonpositively curved spaces; cf. e.g. [13, 21, 30, 33]. Thus our classes of complexes and groups acting on them geometrically form new classes of combinatorially nonpositively curved complexes and groups (see e.g. [21, 25, 30, 33] for more background) containing the CAT(0) cubical and systolic classes of objects. A question of studying such unification theories was raised often by various researchers, e.g. by Januszkiewicz and Świątkowski (personal communication). Due to our knowledge, bucolism is the first generalization of the CAT(0) cubical and systolic worlds studied up to now.

The class of bucolic complexes is closed under taking finite Cartesian products and gated amalgamations. Thus the class of groups acting geometrically on them is also closed under similar operations. It should be noticed that both systolic and CAT(0) cubical groups satisfy some strong (various for different classes) restrictions; cf. e.g. [33] and references therein. It implies that there are groups that are neither systolic nor CAT(0) cubical but which act geometrically on our complexes. In particular, in view of Theorem 4 and the fixed point theorems for systolic and CAT(0) complexes (compare [13, 21]), the free product of a systolic group with a CAT(0) cubical group amalgamated over a finite subgroup always acts geometrically on a complex from our class. Note however that such a product is often not systolic neither CAT(0) cubical. Another example with these properties is the Cartesian product of two systolic but not CAT(0) cubical groups.

4. Bucolic graphs

In this section, we prove the following characterization of bucolic graphs:

Theorem 2. *For a graph $G = (V, E)$ not containing infinite cliques, the following conditions are equivalent:*

(i) *G is a retract of the (weak) Cartesian product of weakly bridged (respectively, bridged) graphs;*

(ii) *G is a weakly modular graph not containing induced $K_{2,3}$, W_4, and W_4^- (respectively, $K_{2,3}$, W_4^-, W_4, and W_5), i.e., G is a bucolic (respectively, strongly bucolic) graph;*

(iii) *G is a weakly modular graph not containing $K_{2,3}$ and W_4^- in which all elementary (or prime) gated subgraphs are edges or 2-connected weakly bridged (respectively, bridged) graphs.*

Moreover, if G is finite, then the conditions (i)-(iii) are equivalent to the following condition:

(iv) *G can be obtained by successive applications of gated amalgamations from Cartesian products of 2-connected weakly bridged (respectively, bridged) graphs.*

The most difficult part of the proof of the theorem are the implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i). The main step in the proof of (ii)$\Rightarrow$(iii) is showing that all primes of weakly modular graphs not containing induced W_4 and W_4^- are 2-connected weakly bridged graphs or K_2. To prove (iii)$\Rightarrow$(i), we need to show that weakly bridged graphs are moorable. For locally-finite graphs this was proven in [21]. The mooring of non-locally-finite weakly bridged graphs is established in Section 7. Then, we deduce the theorem from the results of [5,14,15].

4.1. Gated closures of triangles.
In this section, we prove that if G is a weakly modular graph not containing induced 4-wheels W_4 and almost 4-wheels W_4^-, then the gated hull of any triangle of G is a weakly bridged graph. Additionally, if G does not contain induced 5-wheels W_5, then the gated hull of a triangle is a bridged graph.

Lemma 4.1. *Let G be a graph without induced W_4, W_4^- and satisfying the triangle condition. Then G does not contain an induced W_n^- for $n > 4$.*

Proof. Suppose by way of contradiction that W_n^- is an induced subgraph of G and suppose that G does not contain induced W_k^- for any $3 < k < n$. Let $(x_1, x_2, \ldots, x_n, x_1)$ be the outer cycle C of W_n^- and consider a vertex c adjacent to all vertices of C except x_1. We apply the triangle condition to the triple x_1, x_2, x_{n-1} and find a vertex $a \in N(x_1) \cap N(x_2) \cap N(x_{n-1})$. Note that if $a \sim c$, then x_1, x_2, c, x_n, a induce W_4 if a is adjacent to x_n or W_4^- otherwise. Thus $a \not\sim c$. If $n = 5$, then the vertices x_4, a, x_2, c, x_3 induce either a W_4 if x_3 is adjacent to a, or a W_4^- otherwise. Now, if $n \geq 6$ and if a is not adjacent to $x_3, x_4, \ldots, x_{n-3}$ or x_{n-2}, the subgraph induced by the vertices $a, x_2, x_3, \ldots, x_{n-1}, c$ has an induced subgraph isomorphic to one of the forbidden induced subgraphs W_k^-, where $k < n$. Thus a is adjacent to all vertices of C except maybe x_n. The vertices a, x_3, c, x_{n-1}, x_4 induce W_4, if $n = 6$, or W_4^- otherwise, a contradiction. $\square$

Let H be an induced subgraph of a graph G. A 2-path $P = (a, v, b)$ of G is H-*fanned* if $a, v, b \in V(H)$ and if there exists an (a, b)-path P' in H not passing via v and such that v is adjacent to all vertices of P', i.e., $v \sim P'$. Notice that P' can be chosen to be an induced path of G. A path $P = (x_0, x_1, \ldots, x_{k-1}, x_k)$ of G with $k > 2$ is H-*fanned* if every three consecutive vertices (x_i, x_{i+1}, x_{i+2}) of P form an H-fanned 2-path. When H is clear from the context (typically when $H = G$), we say that P is *fanned*. If the endvertices of a 2-path $P = (a, v, b)$ coincide or are adjacent, then P is fanned. Here is a simple generalization of this remark (whose immediate proof is left to the reader).

Lemma 4.2. *If $P = (x_0, x_1, \ldots, x_k)$ is a fanned path and the vertices x_{i-1} and x_{i+1} coincide or are adjacent, then the paths $P' = (x_0, \ldots, x_{i-2}, x_{i+1}, x_{i+2}, \ldots, x_k)$ in the first case and $P'' = (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ in the second case are also fanned.*

In the remaining auxiliary results of this section, G is a weakly modular graph without induced W_4 and W_4^-. By Lemma 4.1, G does not contain W_k^- with $k > 3$.

Lemma 4.3. *If $C = (x, u, y, v, x)$ is an induced 4-cycle of G, then no simple 2-path of C is fanned.*

Proof. Suppose that the simple 2-path $P = (u, y, v)$ is fanned. Let $R = (u, t_1, \ldots, t_m, t_{m+1} = v)$ be a shortest (u, v)-path such that $y \sim R$ (such a path exists because P is fanned). Necessarily, R is an induced path of G. Since C is induced, $m \geq 1$ and $t_i \neq x$ for all $i \in \{1, \ldots, m\}$. If t_1 is adjacent to x, then the vertices x, u, y, v, t_1 induce W_4 if t_1 is adjacent to v, or W_4^- otherwise. Suppose now that t_1 is not adjacent to x and let $i \geq 2$ be the smallest index such that t_i is adjacent to x. Since R is a shortest path, the cycle $(x, u, t_1, \ldots, t_i, x)$ is induced. Thus the vertices $x, u, t_1, \ldots, t_i, y$ induce a forbidden W_{i+2}^-. $\square$

Let v be a common neighbor of vertices a and b of G. For an (a, b)-path P, we denote by $D_v(P)$ the distance sum $D_v(P) := \sum_{x \in P} d(x, v)$.

Lemma 4.4. *Let $P = (a = x_0, x_1, \ldots, x_m = b)$ be a fanned (a, b)-path not containing v, let $k = \max\{d(x_i, v) : x_i \in P\}$ and j be the smallest index so that $d(x_j, v) = k$. If $k \geq 2$, and $j \notin \{0, m\}$ then*

(1) *either $x_{j-1} = x_{j+1}$ and the path $P' = (x_0, \ldots, x_{j-2}, x_{j+1}, x_{j+2}, \ldots, x_m)$ is fanned,*
(2) *either $x_{j-1} \sim x_{j+1}$ and the path $P'' = (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$ is fanned,*
(3) *or there exists a vertex y such that $d(y, v) = k - 1$ and the path $P''' = (x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m)$ is fanned.*

Proof. If $x_{j-1} = x_{j+1}$ or $x_{j-1} \sim x_{j+1}$, then Lemma 4.2 implies that the paths P' and P'' are fanned. So, suppose that x_{j-1} and x_{j+1} are different and non-adjacent. Note that $d(x_{j-1}, v) = k - 1$ and $k - 1 \leq d(x_{j+1}, v) \leq k$. If $d(x_{j+1}, v) = k - 1$, then we can use the quadrangle condition for vertices v, x_{j-1}, x_j and x_{j+1} and find a vertex $z \in N(x_{j-1}) \cap N(x_{j+1})$ such that $d(v, z) = k - 2$ ($z = v$ if $k = 2$). Since z and x_j are not adjacent, the 4-cycle $(z, x_{j-1}, x_j, x_{j+1}, z)$ is induced. Since P is fanned, the 2-path (x_{j-1}, x_j, x_{j+1}) is fanned as well, contradicting Lemma 4.3.

Thus $d(x_{j+1}, v) = k$. Applying the triangle condition to the triple v, x_j, x_{j+1}, we can find a common neighbor y of x_j and x_{j+1} with $d(v, y) = k - 1$. Note that $y \neq x_{j-1}$ since $x_{j-1} \nsim x_{j+1}$. Assume first $x_{j-1} \nsim y$. Then we can apply the quadrangle condition to the vertices x_{j-1}, x_j, y, v, and find a vertex $z \in N(x_{j-1}) \cap N(y)$ with $d(z, v) = k - 2$ ($z = v$ if $k = 2$). Clearly, z is not adjacent to x_j and x_{j+1}. Hence, the cycle $(x_{j-1}, x_j, y, z, x_{j-1})$ is induced. Since the 2-path (x_{j-1}, x_j, x_{j+1}) is fanned, there exists a (x_{j-1}, x_{j+1})-path Q_0 not containing x_j such that $x_j \sim Q_0$. As a consequence, (Q_0, y) is a (x_{j-1}, y)-path of G not passing via x_j whose all vertices are adjacent to x_j. Therefore the 2-path (x_{j-1}, x_j, y) of the induced 4-cycle $(x_{j-1}, x_j, y, z, x_{j-1})$ is fanned, contradicting Lemma 4.3. This implies that x_{j-1} must be adjacent to y. Then $P''' = (x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_m)$ is a path of G. We claim that P''' is fanned. Indeed, all 2-paths of P''', except the three consecutive 2-paths $(x_{j-2}, x_{j-1}, y), (y, x_{j+1}, x_{j+2}), (x_{j-1}, y, x_{j+1})$, are also 2-paths of P, hence they are fanned. The 2-path (x_{j-1}, y, x_{j+1}) is fanned because y is adjacent to all vertices of the path (x_{j-1}, x_j, x_{j+1}). Since the 2-path (x_j, x_{j+1}, x_{j+2}) is fanned, there is an (x_j, x_{j+2})-path R such that $x_{j+1} \sim R$. Then all vertices of the (y, x_{j+2})-path (y, R) are adjacent to x_{j+1}, whence the 2-path (y, x_{j+1}, x_{j+2}) is fanned. Analogously, one can show that the 2-path (x_{j-2}, x_{j-1}, y) is fanned, showing that P''' is fanned. $\qquad\square$

From the proof of Lemma 4.4, since each of $D_v(P'), D_v(P''), D_v(P''')$ is smaller than $D_v(P)$, we conclude that if $v \sim a, b$ and P is a fanned (a, b)-path not containing v with minimal distance sum $D_v(P)$, then $k = 1$. Therefore, we obtain the following result:

Corollary 2. *If $v \sim a, b$ and if P is a fanned (a, b)-path avoiding v with minimal distance sum $D_v(P)$, then $v \sim P$.*

Let $\lhd$ be a well-order on $V(G)$. Let $T = \{a_0, b_0, c_0\}$ be a triangle in G. We define a subgraph K of G by (possibly transfinite) induction as follows. Let H_0, H_1, H_2 be the subgraphs respectively induced by $\{a_0\}, \{a_0, b_0\}$ and $\{a_0, b_0, c_0\}$. Given an ordinal α, assume that for every $\beta < \alpha$, we have defined H_β, and let $H_{<\alpha}$ be the subgraph induced by $\bigcup_{\beta < \alpha} V(H_\beta)$. Let

$$X = \{v \in V(G) \smallsetminus V(H_{<\alpha}) : \text{ there exist } x, y \in V(H_{<\alpha}) \text{ such that } v \sim x, y\}.$$

If X is nonempty, then let v be the least element of $(X, \lhd)$ and define H_α to be the subgraph of G induced by $V(H_{<\alpha} \cup \{v\})$. Otherwise, if X is empty, then set $K := H_{<\alpha}$.

Lemma 4.5. *For any ordinal α, H_α is 2-connected and any 2-path of H_α is K-fanned. Moreover, K is 2-connected and any 2-path of K is K-fanned.*

Proof. We proceed by induction on α. Clearly, $H_0, H_1, H_2 = T$ fulfil these properties. Assume by induction hypothesis that for every $\beta < \alpha$, H_β is 2-connected and that any 2-path of H_β is K-fanned.

We first show that $H_{<\alpha}$ is 2-connected and that any 2-path of $H_{<\alpha}$ is K-fanned. Consider any three vertices $a, b, u \in V(H_{<\alpha})$. There exists $\beta < \alpha$ such that $a, b, u \in V(H_\beta)$. By the induction hypothesis, there exists a path from a to b in $H_\beta \smallsetminus \{u\}$. Since $H_\beta \smallsetminus \{u\}$ is a subgraph of $H_{<\alpha} \smallsetminus \{u\}$, a is not disconnected from b in $H_{<\alpha} \smallsetminus \{u\}$, and thus $H_{<\alpha}$ is 2-connected. For

every 2-path (a, b, c) in $H_{<\alpha}$, there exists $\beta < \alpha$ such that $a, b, c \in V(H_\beta)$. By the induction hypothesis, the 2-path (a, b, c) is K-fanned.

If $K = H_{<\alpha}$, we are done. Otherwise, let v be the unique vertex of $V(G)$ such that $V(H_\alpha) = V(H_{<\alpha}) \cup \{v\}$. By the definition of H_α, v has at least two neighbors x, y in $H_{<\alpha}$.

Suppose that H_α is not 2-connected. Consider three distinct vertices $a, b, u \in V(H_\alpha)$. If $a, b \in V(H_{<\alpha})$, we know there exists a path from a to b in $H_{<\alpha} \smallsetminus \{u\}$. Without loss of generality, assume now that $b = v$ and $u \neq x$. We know that there exists a path from a to x in $H_{<\alpha} \smallsetminus \{u\}$ and consequently, there exists a path from a to $b = v$ in $H_\alpha \smallsetminus \{u\}$ since $x \sim v$. Consequently, for every $a, b, u \in V(H_\alpha)$, u does not disconnect a from b, i.e., H_α is 2-connected.

We will prove that any 2-path of H_α is K-fanned. It suffices to consider the 2-paths Q of H_α that contain v, since all other 2-paths lie in $H_{<\alpha}$ and are K-fanned.

Case 1. $Q = (a, v, c)$.

Since $H_{<\alpha}$ is connected and $a, c \in V(H_{<\alpha})$, there exists an (a, c)-path R in $H_{<\alpha}$. Since any 2-path of $H_{<\alpha}$ is K-fanned by induction hypothesis, R itself is K-fanned. As $H_{<\alpha}$ is a subgraph of K, R belongs to K. Among all K-fanned (a, c)-paths belonging to K and avoiding v, let $P = (a = x_0, x_1, \ldots, x_m = c)$ be chosen in such a way that the distance sum $D_v(P) = \sum_{x_i \in P} d(v, x_i)$ is minimized (note that P does not necessarily belong to H_α). By Corollary 2, $v \sim P$ and thus the 2-path Q is K-fanned.

Case 2. $Q = (c, b, v)$.

If c and v coincide or are adjacent, then Q is trivially fanned. Thus we may assume that $c \neq v$, and $c \not\sim v$. Since v has at least two neighbors in $H_{<\alpha}$, there exists a vertex $a \in H_{<\alpha}$ adjacent to v and different from b. Since $H_{<\alpha}$ is 2-connected and $a, c \in H_{<\alpha}$, there exists an (a, c)-path P_0 in $H_{<\alpha}$ that avoids b. The paths P_0 and (P_0, b) are K-fanned because all their 2-paths are fanned by the induction hypothesis. Hence, there exists at least one K-fanned (a, b)-path (P_0, b) that passes via c, avoids v, and all vertices of P_0 are different from b. Among all such (a, b)-paths (P_0, b) of K (i.e., that pass c, avoid v, the vertices of P_0 are different from b, and are K-fanned), let $P = (a = x_0, x_1, \ldots, x_m, x_{m+1} = c, b)$ be chosen in such a way that $D_v(P)$ is minimized. Since v and $x_{m+1} = c$ are different and not adjacent, $k = \max\{d_G(x_i, v) : x_i \in P\} \geq 2$. Let j be the smallest index such that $d(x_j, v) = k$.

First suppose that $j \neq m + 1$. By Lemma 4.4, the vertices a and b can be connected by one of the paths P', P'', P''' derived from P. These paths are K-fanned, contain the vertex c, avoid the vertex v, and all three have smaller distance sums than P. In case of P' and P'' we obtain a contradiction with the minimality choice of P. Analogously, in case of P''' we obtain the same contradiction except if the vertex y coincides with b, i.e., b is adjacent to the vertices x_{j-1}, x_j, and x_{j+1}. In this case, $d(x_j, v) = 2$ and $x_{j-1} \sim v$. Consider the 2-path (c, b, x_{j+1}). By construction, the path $R = (x_{m+1} = c, x_m, \ldots, x_{j+2}, x_{j+1})$ is K-fanned and avoids b. Applying Lemma 4.4 and Corollary 2 with b and R, there exists a K-fanned (c, x_{j+1})-path R' avoiding b such that $b \sim R'$. Consequently, there is a path (R', x_j, x_{j-1}, v) in K from c to v in the neighborhood of b and thus (c, b, v) is K-fanned.

Now suppose that $j = m + 1$, i.e., v is adjacent to all vertices of P except $x_{m+1} = c$. From the choice of P we conclude that $b \neq x_m$. If $b \not\sim x_m$, then $C = (v, x_m, c, b, v)$ is an induced 4-cycle.

Since the 2-path (b, c, x_m) is K-fanned and simple, we obtain a contradiction with Lemma 4.3. Finally, if b is adjacent to x_m, then the (simple) 2-path (c, b, v) is K-fanned because c and v are connected in K by the (simple) 2-path (c, x_m, v) and x_m is adjacent to b. $\qquad \square$

Lemma 4.6. *For any ordinal α, $H_{<\alpha}$ and H_α do not contain induced 4-cycles.*

Proof. Again we proceed by induction on α.

Suppose by induction hypothesis that for every $\beta < \alpha$, H_β does not contain induced 4-cycles. If there exists a 4-cycle (a, b, c, d, a) in $H_{<\alpha}$, there exists $\beta < \alpha$ such that $a, b, c, d \in V(H_\beta)$. Since H_β is an induced subgraph of $H_{<\alpha}$, (a, b, c, d, a) is an induced 4-cycle of H_β, contradicting the induction hypothesis.

If $K = H_{<\alpha}$, we are done. Otherwise, let v be the unique vertex of $V(G)$ such that $V(H_\alpha) = V(H_{<\alpha}) \cup \{v\}$. Suppose by way of contradiction that H_α contains an induced 4-cycle C. Then necessarily v belongs to C. Let $C = (v, a, b, c, v)$. Since by Lemma 4.5 the 2-paths of H_α are K-fanned, the simple 2-path (a, b, c) of C is fanned and we obtain a contradiction with Lemma 4.3. Consequently, H_α does not contain induced 4-cycles. $\qquad \square$

Lemma 4.7. *K is the gated hull of T in G.*

Proof. Let A be the gated hull of T. First we prove that all vertices of K belong to A. Suppose by way of contradiction that $K \smallsetminus A \neq \varnothing$. From all vertices in $K \smallsetminus A$ we choose v with smallest α, such that $v \notin H_{<\alpha}$, $v \in H_\alpha$, i.e., all vertices from $H_{<\alpha}$ are contained in A. Since $v \in H_\alpha$, it has at least two neighbors in $H_{<\alpha}$ and thus in A. Therefore there is no gate of v in A, a contradiction.

On the other hand, since G is weakly modular, K is gated if and only if for every $x, y \in K$ at distance at most 2, any common neighbor v of x and y also belongs to K [4, 18]. This is obviously true by the definition of K. $\qquad \square$

Summarizing, we obtain the main result of this subsection.

Proposition 1. *Let G be a locally-finite weakly modular graph not containing induced W_4 and W_4^-. Then the gated hull of any triangle T of G is a 2-connected weakly bridged graph. Additionally, if G does not contain induced W_5, then the gated hull of T is a 2-connected bridged graph.*

Proof. By Lemma 4.7, the gated hull of T is the 2-connected subgraph K of G constructed by our procedure. Since K is a convex subgraph of a weakly modular graph G, K itself is a weakly modular graph. By Lemma 4.6, the graph K does not contain induced 4-cycles, thus K is weakly bridged by [21, Theorem 3.1(iv)]. If, additionally, G does not contain 5-wheels, then G does not contain induced 5-cycles because in a weakly bridged graph any induced 5-cycle is included in a 5-wheel. Then K is a weakly modular graph without induced 4- and 5-cycles, thus K is bridged. $\qquad \square$

4.2. **Proof of Theorem 2.** We first prove the implication (i)$\Rightarrow$(ii). First, bridged and weakly bridged graphs are weakly modular. Weakly bridged graphs do not contain induced

$K_{2,3}, W_4$, and W_4^- because they do not contain induced 4-cycles. Bridged graphs additionally do not contain induced W_5. Weakly modular graphs are closed by taking (weak) Cartesian products (this holds also when there are infinite number of factors in weak Cartesian products, since the distances between vertices in a weak Cartesian product are finite). If a (weak) Cartesian product $\square_{i \in I} H_i$ contains an induced $K_{2,3}, W_4, W_5$ or W_4^-, then necessarily this graph occurs in one of the factors H_i. This follows from the fact that in a product, each triangle comes from one factor, and two opposite edges of a square must come from the same factor. As a consequence, Cartesian products $H = \square_{i \in I} H_i$ of weakly bridged graphs do not contain induced $K_{2,3}, W_4$, and W_4^-. Analogously, Cartesian products $H = \square_{i \in I} H_i$ of bridged graphs do not contain induced $K_{2,3}, W_4, W_4^-$, and W_5. If G is a retract of H, then G is an isometric subgraph of H, and therefore G does not contain induced $K_{2,3}, W_4, W_4^-$ in the first case and induced $K_{2,3}, W_4, W_4^-$ and W_5 in the second case. It remains to notice that the triangle and quadrangle conditions are preserved by retractions, thus G is a weakly modular graph, establishing that (i)$\Rightarrow$(ii).

Now suppose that G is a weakly modular graph satisfying the condition (ii) of Theorem 2. Then G is a pre-median graph. By [14, Theorem 4.13], any pre-median graph is fiber-complemented. By [14, Lemma 4.8], this implies that any gated subgraph H of G is elementary if and only if it is prime. Note that the gated hull of any edge in G is either the edge itself, or it is included in a triangle by weak modularity, and by Proposition 1 we find that the gated hull of this edge is a 2-connected (weakly) bridged graph. Hence every elementary (= prime) gated subgraph is a 2-connected (weakly) bridged graph or an edge. This establishes the implication (ii)$\Rightarrow$(iii).

To prove the implication (iii)$\Rightarrow$(i), we will use [15, Theorem 3.2.1] and [21, Theorem 5.1]. By Chastand [15, Theorem 3.2.1], any fiber-complemented graph G whose primes are moorable graphs is a retract of the Cartesian product of its primes. Note that elementary gated subgraphs of G, enjoying (iii), are edges and 2-connected weakly bridged graphs. In [21, Theorem 5.1], it is shown that locally-finite weakly bridged graphs are moorable. Proposition 6 in Section 7 extends this result to non-locally-finite graphs. Thus, by [15, Theorem 3.2.1] G is a retract of the Cartesian product of its primes, establishing the implication (iii)$\Rightarrow$(i) of Theorem 2.

Now, for finite graphs we show that (iv) $\Longleftrightarrow$ (ii). As noticed above, bridged and weakly bridged graphs are weakly modular and do not contain induced $K_{2,3}, W_4$, and W_4^-. Bridged graphs additionally do not contain induced W_5. Weakly modular graphs are closed by Cartesian products and gated amalgams. Moreover, if G is the Cartesian product or the gated amalgam of two graphs G_1 and G_2, then G contains an induced $K_{2,3}$ (respectively, W_4, W_4^-, W_5) if and only if G_1 or G_2 does. Therefore (iv)$\Rightarrow$(ii). Conversely, suppose that G is a finite bucolic (respectively, strongly bucolic) graph. Then G is a pre-median graph. By [14, Theorem 4.13], any pre-median graph is fiber-complemented. Then according to [14, Theorem 5.4], G can be obtained from Cartesian products of elementary (=prime) graphs by a sequence of gated amalgamations. By Proposition 1, any elementary graph of G is either an edge or

a 2-connected weakly bridged graph (respectively, a 2-connected bridged graph). Thus the implication (ii)$\Rightarrow$(iv) in Theorem 2 holds. This concludes the proof of Theorem 2.

5. Bucolic complexes and their skeleta

In this section, we prove the following local-to-global characterization of bucolic complexes via properties of their 1- and 2-skeleta:

Theorem 1. *For a prism complex* $\mathbf{X}$, *the following conditions are equivalent:*

(i) $\mathbf{X}$ *is a bucolic complex;*

(ii) *the 2-skeleton* $\mathbf{X}^{(2)}$ *of* $\mathbf{X}$ *is a connected and simply connected triangle-square flag complex satisfying the wheel, the 3-cube, and the 3-prism conditions;*

(iii) *the 1-skeleton* $G(\mathbf{X})$ *of* $\mathbf{X}$ *is a connected weakly modular graph that does not contain induced subgraphs of the form* $K_{2,3}$, W_4, *and* W_4^-, *i.e.,* $G(\mathbf{X})$ *is a bucolic graph not containing infinite hypercubes.*

Moreover, if $\mathbf{X}$ *is a connected flag prism complex satisfying the wheel, the cube, and the prism conditions, then the universal cover* $\widetilde{\mathbf{X}}$ *of* $\mathbf{X}$ *is bucolic.*

As an immediate corollary we obtain the following analogous characterization of strongly bucolic complexes:

Corollary 3. *For a prism complex* $\mathbf{X}$, *the following conditions are equivalent:*

(i) $\mathbf{X}$ *is a strongly bucolic complex;*

(ii) *the 2-skeleton* $\mathbf{X}^{(2)}$ *of* $\mathbf{X}$ *is a connected and simply connected triangle-square flag complex satisfying the strong-wheel, the 3-cube, and the 3-prism conditions;*

(iii) *the 1-skeleton* $G(\mathbf{X})$ *of* $\mathbf{X}$ *is a connected weakly modular graph not containing induced subgraphs of the form* $K_{2,3}$, W_4, W_4^-, *and* W_5, *i.e.,* $G(\mathbf{X})$ *is a strongly bucolic graph not containing infinite hypercubes as induced subgraphs;*

Moreover, if $\mathbf{X}$ *is a connected flag prism complex satisfying the strong-wheel, the cube, and the prism conditions, then the universal cover* $\widetilde{\mathbf{X}}$ *of* $\mathbf{X}$ *is strongly bucolic.*

5.1. **Auxiliary results.** We start this section with several auxiliary properties of triangle-square flag complexes occurring in condition (ii) of Theorem 1. Throughout this and the next subsections, we will denote such triangle-square complexes by $\mathbf{X}$, assume that they are connected, and use the shorthand $G := G(\mathbf{X})$ for the 1-skeleton of $\mathbf{X}$. We denote by $\mathbf{X}(C_3)$ and $\mathbf{X}(C_4)$ the triangle-square complex consisting of a single triangle and a single square, respectively. Let $\mathbf{X}(H) = \mathbf{X}(C_3 + C_4)$ be the complex consisting of a triangle and a square sharing one edge; its graph is the house H and with some abuse of notation, we call the complex itself a *house*. The *twin-house* $\mathbf{X}(2H)$ is the complex consisting of two triangles and two squares, which can be viewed as two houses glued along two incident edges or as a domino and a kite glued along two incident edges (for an illustration, see Fig. 4, left). Let also $\mathbf{X}(W_k)$ and $\mathbf{X}(W_k^-)$ be the triangle-square complexes whose underlying graphs are W_k and W_k^-: the first consists of k triangles and the second consists of $k-2$ triangles and one
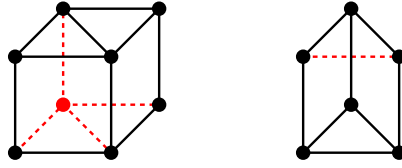
FIGURE 4. On the left, a twin-house (in black) included in a double prism (Lemma 5.2). On the right, a double house (in black) included in a prism (Lemma 5.3).

square. The complex $\mathbf{X}(CW_3)$ consists of three squares sharing a vertex and pairwise sharing edges (its graph is the cogwheel CW_3). The *triangular prism* $\mathbf{X}(Pr) = \mathbf{X}(C_3 \times K_2)$ consists of the surface complex of the 3-dimensional triangular prism (two disjoint triangles and three squares pairwise sharing an edge). The *double prism* $\mathbf{X}(2Pr)$ consists of two prisms $\mathbf{X}(Pr)$ sharing a square (See Fig. 4, left). Finally, the *double-house* $\mathbf{X}(H + C_4) = \mathbf{X}(2C_4 + C_3)$ is the complex consisting of two squares and a triangle, which can be viewed as a house plus a square sharing with the house two incident edges, one from the square and another from the triangle (see Fig. 4, right). In the following results, we use the notation $G = G(\mathbf{X})$.

Lemma 5.1. *If $\mathbf{X}$ is a triangle-square flag complex, then its 1-skeleton G does not contain induced $K_{2,3}$ and W_4^-.*

Proof. If G contains $K_{2,3}$ or W_4^-, then, since $\mathbf{X}$ is a flag complex, we will obtain two squares intersecting in two edges, which is impossible. □

Lemma 5.2. *If $\mathbf{X}$ satisfies the 3-prism condition, then any twin-house $\mathbf{X}(2H)$ of $\mathbf{X}$ is included in $\mathbf{X}$ in a double prism $\mathbf{X}(2Pr)$.*

Proof. Let u, v, w, x_1, x_2 be the vertices of one house and u, v, w, y_1, y_2 be the vertices of another house, where the edge uv is common to the two squares uvx_2x_1 and uvy_2y_1, and where the edge vw is common to the two triangles vwx_2 and vwy_2. By the 3-prism condition, there exists a vertex a adjacent in G to x_1, u, w that is not adjacent to x_2, v. Analogously, there exists a vertex b adjacent to u, y_1, w that is not adjacent to y_2, v. If $a \neq b$, the graph induced by a, b, u, v, w is either $K_{2,3}$ if $a \not\sim b$, or W_4^- otherwise; in both cases, we get a contradiction with Lemma 5.1. Thus $a = b$, and since $a \not\sim v, x_2, y_2$, the vertices $a, u, v, w, x_1, x_2, y_1, y_2$ induce a double prism. □

Lemma 5.3. *If $\mathbf{X}$ satisfies the 3-prism condition, then any double-house $\mathbf{X}(H + C_4)$ in $\mathbf{X}$ is included in a prism $\mathbf{X}(Pr)$, i.e., G does not contain an induced double-house $H + C_4$.*

Proof. Suppose by contradiction that G contains an induced double-house having x, y, u, v, w, z as the set of vertices, where uvw is a triangle and $xyvu$ and $xuwz$ are two squares of this house. By 3-prism condition, there exists a vertex a different from z (since $y \not\sim z$) that is adjacent to x, y, w and that is not adjacent to u, v. Thus, the vertices z, a, w, u, x induce either $K_{2,3}$ if $a \not\sim z$ or W_4^- otherwise. In both cases, we get a contradiction with Lemma 5.1. □

Lemma 5.4. *If* $\mathbf{X}$ *satisfies the 3-prism condition and does not contain* $\mathbf{X}(W_4)$, *then* $\mathbf{X}$ *does not contain* $\mathbf{X}(W_k^-)$ *for any* $k \geq 5$.

Proof. Suppose by way of contradiction that $\mathbf{X}$ contains $\mathbf{X}(W_k^-)$, where k is the smallest value for which this subcomplex exists. Since, by Lemma 5.1, G does not contain W_4^-, necessarily $k \geq 5$. Denote the vertices of $\mathbf{X}(W_k^-)$ by $q, x_1, x_2, \ldots, x_k$ where $x_1, x_2, \ldots, x_k$ induce a cycle and where q is adjacent to $x_1, \ldots, x_{k-1}$ but not to x_k. By the 3-prism condition applied to the house induced by $q, x_{k-1}, x_k, x_1, x_2$, there exists p in G such that $p \sim x_{k-1}, x_k, x_2$ and $p \nsim q, x_1$. If $p \sim x_3$, then the vertices x_3, p, x_2, q, x_{k-1} induce W_4 if $x_3 \sim x_{k-1}$ (i.e., if $k = 5$), or W_4^- otherwise; in both cases, we get a contradiction. Thus $p \nsim x_3$. Let j be the smallest index greater than 3 such that $p \sim x_j$. Since $p \sim x_{k-1}$, j is well-defined. But then, the vertices $q, p, x_2, \ldots, x_j$ induce W_j^- with $j < k$, contradicting the choice of k. $\qquad\square$

Lemma 5.5. *Let* $\mathbf{X}$ *be a triangle-square flag complex such that* $G(\mathbf{X})$ *satisfies the triangle and the quadrangle conditions* $TC(v)$ *and* $QC(v)$, *for some basepoint* v. *Then* $\mathbf{X}$ *is simply connected.*

Proof. A *loop* in $\mathbf{X}$ is a sequence $(w_1, w_2, \ldots, w_k, w_1)$ of vertices of $\mathbf{X}$ consecutively joined by edges. To prove the lemma it is enough to show that every loop in $\mathbf{X}$ can be freely homotoped to a constant loop v. By contradiction, let A be the set of loops in $G(\mathbf{X})$, which are not freely homotopic to v, and assume that A is non-empty. For a loop $C \in A$ let $r(C)$ denote the maximal distance $d(w, v)$ of a vertex $w \in C$ to the basepoint v. Clearly $r(C) \geq 2$ for any loop $C \in A$ (otherwise C would be null-homotopic). Let $B \subseteq A$ be the set of loops C with minimal $r(C)$ among loops in A. Let $r := r(C)$ for some $C \in B$. Let $D \subseteq B$ be the set of loops having minimal number e of edges in the r-sphere around v, i.e. with both endpoints at distance r from v. Further, let $E \subseteq D$ be the set of loops with the minimal number m of vertices at distance r from v.

Consider a loop $C = (w_1, w_2, \ldots, w_k, w_1) \in E$. We can assume without loss of generality that $d(w_2, v) = r$. We distinguish two cases corresponding to the triangle or quadrangle condition that we apply to them.

Case 1: $d(w_1, v) = r$ *or* $d(w_3, v) = r$. Assume without loss of generality that $d(w_1, v) = r$. Then, by the triangle condition $TC(v)$, there exists a vertex $w \sim w_1, w_2$ with $d(w, v) = r - 1$. Observe that the loop $C' = (w_1, w, w_2, \ldots, w_k, w_1)$ belongs to B – in $\mathbf{X}$ it is freely homotopic to C by a homotopy going through the triangle $w w_1 w_2$. The number of edges of C' lying on the r-sphere around v is less than e (we removed the edge $w_1 w_2$). This contradicts the choice of the number e.

Case 2: $d(w_1, v) = d(w_3, v) = r - 1$. By the quadrangle condition $QC(v)$, there exists a vertex $w \sim w_1, w_3$ with $d(w, v) = r - 2$. Again, the loop $C' = (w_1, w, w_3, \ldots, w_k, w_1)$ is freely homotopic to C (via the square $w_1 w_2 w_3 w$). Thus C' belongs to D and the number of its vertices at distance r from v is equal to $m - 1$. This contradicts the choice of the number m.

In both cases above we get contradiction. It follows that the set A is empty and hence the lemma is proved. $\qquad\square$

5.2. Proof of (i)⟹(ii). Since the cube condition implies the 3-cube condition and the prism condition implies the 3-prism condition, if $\mathbf{X}$ is a bucolic complex, then its 2-skeleton $\mathbf{X}^{(2)}$ satisfies the condition (ii), thus (i)⟹(ii).

5.3. Proof of (ii)⟹(iii). To prove the implication (ii)⟹(iii) of Theorem 1, from now on, we suppose that $\mathbf{X}$ is a connected (but not necessarily simply connected) triangle-square flag complex satisfying the wheel, the 3-prism, and the 3-cube conditions. The most difficult part of the proof is to show that the 1-skeleton of $\mathbf{X}$ is weakly modular. To show this, we closely follow the proof method of a local-to-global characterization of weakly systolic complexes provided by Osajda [33] using the level-by-level construction of the universal cover of $\mathbf{X}$.

5.3.1. Structure of the construction. We construct the universal cover $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ as an increasing union $\bigcup_{i\geq 1}\widetilde{\mathbf{X}}_i$ of triangle-square complexes. The complexes $\widetilde{\mathbf{X}}_i$ are in fact spanned by concentric combinatorial balls $\widetilde{B}_i$ in $\widetilde{\mathbf{X}}$. The covering map f is then the union $\bigcup_{i\geq 1}f_i$, where $f_i:\widetilde{\mathbf{X}}_i\to\mathbf{X}$ is a locally injective cellular map such that $f_i|_{\widetilde{\mathbf{X}}_j}=f_j$, for every $j\leq i$. We denote by $\widetilde{G}_i=G(\widetilde{\mathbf{X}}_i)$ the underlying graph of $\widetilde{\mathbf{X}}_i$. We denote by $\widetilde{S}_i$ the set of vertices $\widetilde{B}_i\smallsetminus\widetilde{B}_{i-1}$.

Pick any vertex v of $\mathbf{X}$ as the basepoint. Define $\widetilde{B}_0=\{\widetilde{v}\}:=\{v\},\widetilde{B}_1:=B_1(v,G)$, and $f_1:=\mathrm{Id}_{B_1(v,G)}$. Let $\widetilde{\mathbf{X}}_1$ be the triangle-square complex spanned by $B_1(v,G)$. Assume that, for $i\geq 1$, we have constructed the vertex sets $\widetilde{B}_1,\dots,\widetilde{B}_i$, and we have defined the triangle-square complexes $\widetilde{\mathbf{X}}_1\subseteq\cdots\subseteq\widetilde{\mathbf{X}}_i$ (for any $1\leq j<k\leq i$ we have an identification map $\widetilde{\mathbf{X}}_j\to\widetilde{\mathbf{X}}_k$) and the corresponding cellular maps $f_1,\dots,f_i$ from $\widetilde{\mathbf{X}}_1,\dots,\widetilde{\mathbf{X}}_i$, respectively, to $\mathbf{X}$ so that the graph $\widetilde{G}_i=G(\widetilde{\mathbf{X}}_i)$ and the complex $\widetilde{\mathbf{X}}_i$ satisfy the following conditions:

(P$_i$) $B_j(\widetilde{v},\widetilde{G}_i)=\widetilde{B}_j$ for any $j\leq i$;

(Q$_i$) $\widetilde{G}_i$ is weakly modular with respect to $\widetilde{v}$ (i.e., $\widetilde{G}_i$ satisfies the conditions TC($\widetilde{v}$) and QC($\widetilde{v}$));

(R$_i$) for any $\widetilde{u}\in\widetilde{B}_{i-1}$, f_i defines an isomorphism between the subgraph of $\widetilde{G}_i$ induced by $B_1(\widetilde{u},\widetilde{G}_i)$ and the subgraph of G induced by $B_1(f_i(\widetilde{u}),G)$;

(S$_i$) for any $\widetilde{w},\widetilde{w}'\in\widetilde{B}_{i-1}$ such that the vertices $w=f_i(\widetilde{w}),w'=f_i(\widetilde{w}')$ belong to a square $ww'uu'$ of $\mathbf{X}$, there exist $\widetilde{u},\widetilde{u}'\in\widetilde{B}_i$ such that $f_i(\widetilde{u})=u,f_i(\widetilde{u}')=u'$ and $\widetilde{w}\widetilde{w}'\widetilde{u}\widetilde{u}'$ is a square of $\widetilde{\mathbf{X}}_i$.

(T$_i$) for any $\widetilde{w}\in\widetilde{S}_i:=\widetilde{B}_i\smallsetminus\widetilde{B}_{i-1}$, f_i defines an isomorphism between the subgraphs of $\widetilde{G}_i$ and of G induced by $B_1(\widetilde{w},\widetilde{G}_i)$ and $f_i(B_1(\widetilde{w},\widetilde{G}_i))$.

It can be easily checked that $\widetilde{B}_1,\widetilde{G}_1,\widetilde{\mathbf{X}}_1$ and f_1 satisfy the conditions (P$_1$),(Q$_1$),(R$_1$),(S$_1$), and (T$_1$). Now we construct the set $\widetilde{B}_{i+1}$, the graph $\widetilde{G}_{i+1}$ having $\widetilde{B}_{i+1}$ as the vertex-set, the triangle-square complex $\widetilde{\mathbf{X}}_{i+1}$ having $\widetilde{G}_{i+1}$ as its 1-skeleton, and the map $f_{i+1}:\widetilde{\mathbf{X}}_{i+1}\to\mathbf{X}$. Let

$$Z=\{(\widetilde{w},z):\widetilde{w}\in\widetilde{S}_i\text{ and }z\in B_1(f_i(\widetilde{w}),G)\smallsetminus f_i(B_1(\widetilde{w},\widetilde{G}_i))\}.$$

On Z we define a binary relation $\equiv$ by setting $(\widetilde{w},z)\equiv(\widetilde{w}',z')$ if and only if $z=z'$ and one of the following two conditions is satisfied:

(Z1) $\widetilde{w}$ and $\widetilde{w}'$ are the same or adjacent in $\widetilde{G}_i$ and $z\in B_1(f_i(\widetilde{w}),G)\cap B_1(f_i(\widetilde{w}'),G)$;

(Z2) there exists $\widetilde{u}\in\widetilde{B}_{i-1}$ adjacent in $\widetilde{G}_i$ to $\widetilde{w}$ and $\widetilde{w}'$ and such that $f_i(\widetilde{u})f_i(\widetilde{w})zf_i(\widetilde{w}')$ is a square-cell of $\mathbf{X}$.

In what follows, the above relation will be used in the inductive step to construct $\widetilde{\mathbf{X}}_{i+1}$, f_{i+1} and all related objects.

5.3.2. *Definition of* $\widetilde{G}_{i+1}$. In this subsection, performing the inductive step, we define $\widetilde{G}_{i+1}$ and f_{i+1}. First however we show that the relation $\equiv$ defined in the previous subsection is an equivalence relation. The set of vertices of the graph $\widetilde{G}_{i+1}$ will be then defined as the union of the set of vertices of the previously constructed graph $\widetilde{G}_i$ and the set of equivalence classes of $\equiv$.

Lemma 5.6. *The relation* $\equiv$ *is an equivalence relation on* Z.

Proof. For any vertex $\widetilde{w} \in \widetilde{B}_i$, we will denote by $w = f_i(\widetilde{w})$ its image in X under f_i. Since the binary relation $\equiv$ is reflexive and symmetric, it suffices to show that $\equiv$ is transitive. Let $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ and $(\widetilde{w}', z') \equiv (\widetilde{w}'', z'')$. We will prove that $(\widetilde{w}, z) \equiv (\widetilde{w}'', z'')$. By definition of $\equiv$, we conclude that $z = z' = z''$. By definition of $\equiv$, $z \in B_1(w, G) \cap B_1(w', G) \cap B_1(w'', G)$.

If $\widetilde{w} \sim \widetilde{w}''$ (in $\widetilde{G}_i$), then by definition of $\equiv$, $(\widetilde{w}, z) \equiv (\widetilde{w}'', z)$ and we are done. If $\widetilde{w} \not\sim \widetilde{w}''$ and if there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$, then by (R_i) applied to $\widetilde{u}$, we obtain that $u \sim w, w''$ and $w \not\sim w''$. Since $(\widetilde{w}, z), (\widetilde{w}'', z) \in Z$, we have $z \sim w, w''$. Moreover, if $z \sim u$, then by (R_i) applied to u, there exists $\widetilde{z} \in \widetilde{B}_i$, such that $\widetilde{z} \sim \widetilde{u}, \widetilde{w}, \widetilde{w}''$ and $f_i(\widetilde{z}) = z$. Thus $(\widetilde{w}, z), (\widetilde{w}', z) \notin Z$, which is a contradiction. Consequently, if $\widetilde{w} \not\sim \widetilde{w}''$ and if there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$ and $f_i(\widetilde{u}) = u$, then $uwzw''$ is an induced square in G, and by condition (Z2), we are done. Therefore, in the rest of the proof, we will make the following assumptions and show that they lead to a contradiction.

(A_1) $\widetilde{w} \not\sim \widetilde{w}''$;
(A_2) there is no $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}''$.

Claim 1. For any couple $(\widetilde{w}, z) \in Z$ the following properties hold:

(A_3) there is no neighbor $\widetilde{z} \in \widetilde{B}_{i-1}$ of $\widetilde{w}$ such that $f_i(\widetilde{z}) = z$;
(A_4) there is no neighbor $\widetilde{u} \in \widetilde{B}_{i-1}$ of $\widetilde{w}$ such that $u \sim z$;
(A_5) there are no $\widetilde{x}, \widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $y \sim z$.

Proof. If $\widetilde{w}$ has a neighbor $\widetilde{z} \in \widetilde{B}_{i-1}$ such that $f_i(\widetilde{z}) = z$, then $(\widetilde{w}, z) \notin Z$, a contradiction. This establishes (A_3).

If $\widetilde{w}$ has a neighbor $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $u \sim z$, then by (R_i) applied to $\widetilde{u}$, there exists $\widetilde{z} \in \widetilde{B}_i$ such that $\widetilde{z} \sim \widetilde{u}, \widetilde{w}$. Thus $(\widetilde{w}, z) \notin Z$, a contradiction, establishing (A_4).

If there exist $\widetilde{x}, \widetilde{y} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $y \sim z$, then $yxwz$ is an induced square in G. From (S_i) applied to $\widetilde{y}, \widetilde{x}$, there exists $\widetilde{z} \in \widetilde{B}_i$ such that $\widetilde{z} \sim \widetilde{y}, \widetilde{w}$ and $f_i(\widetilde{z}) = z$. Thus $(\widetilde{w}, z) \notin Z$, a contradiction, and therefore (A_5) holds as well. $\qquad\square$

We distinguish three cases depending on which of the conditions (Z1) or (Z2) are satisfied by the pairs $(\widetilde{w}, z) \equiv (\widetilde{w}', z')$ and $(\widetilde{w}', z') \equiv (\widetilde{w}'', z'')$.

Case 1: $\widetilde{w}'$ is adjacent in $\widetilde{G}_i$ to both $\widetilde{w}$ and $\widetilde{w}''$.

By (Q_i), the graph $\widetilde{G}_i$ satisfies the triangle condition $\mathrm{TC}(\widetilde{v})$, thus there exist two vertices $\widetilde{u}, \widetilde{u}' \in \widetilde{S}_{i-1}$ such that $\widetilde{u}$ is adjacent to $\widetilde{w}, \widetilde{w}'$ and $\widetilde{u}'$ is adjacent to $\widetilde{w}', \widetilde{w}''$. By (A_2), $\widetilde{u} \nsim \widetilde{w}''$, $\widetilde{u}' \nsim \widetilde{w}$, $\widetilde{u} \neq \widetilde{u}'$.

If $\widetilde{u} \sim \widetilde{u}'$, then by (T_i) applied to $\widetilde{w}'$ and by $(A_3)\&(A_4)$, the vertices u, u', w, w', w'', z induce W_5 in G. By $\mathrm{TC}(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. By (R_i) applied to $\widetilde{u}$ and $\widetilde{u}'$, we get $x \notin \{u, u', w, w', w''\}$ and $x \sim u, u'$. From $(A_4)\&(A_5)$, we get $x \neq z$ and $x \nsim z$. Since G satisfies the $\widehat{W_5}$-wheel condition, there exists a vertex y of G adjacent to x, u, u', w, w', w'', z. By (R_i) applied to $\widetilde{u}$, there exists $\widetilde{y} \sim \widetilde{w}, \widetilde{u}, \widetilde{x}$ and thus $\widetilde{y} \in \widetilde{B}_{i-1}$, contradicting the property (A_4).

Suppose now that $\widetilde{u} \nsim \widetilde{u}'$. Then $i \geq 2$ and by $\mathrm{QC}(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. From $(A_4)\&(A_5)$, $x \neq z$ and $x \nsim z$. Consequently, z, w, w', w'', u, u', x induce a W_6^-, contradicting Lemma 5.4.

Case 2: $\widetilde{w}$ and $\widetilde{w}'$ are adjacent in $\widetilde{G}_i$, and there exists $\widetilde{u}' \in \widetilde{B}_{i-1}$ adjacent to $\widetilde{w}'$ and $\widetilde{w}''$ such that $u'w'w''z$ is a square-cell of $\mathbf{X}$.

By $(A_1)\&(A_2)$, $\widetilde{w} \nsim \widetilde{w}''$ and $\widetilde{u}' \nsim \widetilde{w}$. By the triangle condition $\mathrm{TC}(\widetilde{v})$ for $\widetilde{G}_i$, there exists a vertex $\widetilde{u} \in \widetilde{B}_{i-1}$ different from $\widetilde{u}'$ and adjacent to $\widetilde{w}$ and $\widetilde{w}'$. By $(A_3)\&(A_4)$, $u \neq z$ and $u \nsim z$. By (A_2), $\widetilde{u} \nsim \widetilde{w}''$.

If $\widetilde{u} \sim \widetilde{u}'$, by (T_i) applied to w', z, w, w', u, u', w'' induce a W_5^-, contradicting Lemma 5.4. Thus $\widetilde{u} \nsim \widetilde{u}'$. By the quadrangle condition $\mathrm{QC}(\widetilde{v})$ for $\widetilde{G}_i$, there exists a vertex $\widetilde{x} \in \widetilde{S}_{i-2}$ adjacent to $\widetilde{u}$ and $\widetilde{u}'$. From $(A_4)\&(A_5)$, $x \neq z$ and $x \nsim z$. By (T_i) applied to $\widetilde{w}'$ and by (R_i) applied to $\widetilde{u}'$, we get that z, w, w', w'', u, u', x induce a twin-house. By Lemma 5.2 there exists y in G such that $y \sim w, w'', u', x$ and $y \nsim u, z$. By (R_i) applied to u', there exists $\widetilde{y} \in \widetilde{B}_i$ such that $\widetilde{y} \sim \widetilde{u}', \widetilde{w}'', \widetilde{x}$. By (S_i) applied to $\widetilde{u}, \widetilde{x}$ and to the square $uxyw$, we get $\widetilde{y} \sim \widetilde{w}$. Consequently, $\widetilde{y} \in \widetilde{S}_{i-1}$, $\widetilde{y} \sim \widetilde{w}, \widetilde{w}''$, contradicting (A_2).

Case 3: There exist $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i-1}$ such that the vertex $\widetilde{u}$ is adjacent in $\widetilde{G}_i$ to $\widetilde{w}, \widetilde{w}'$, the vertex $\widetilde{u}'$ is adjacent to $\widetilde{w}', \widetilde{w}''$, and $uwzw'$ and $u'w'zw''$ are square-cells of $\mathbf{X}$.

From $(A_1)\&(A_2)$, $\widetilde{w} \nsim \widetilde{w}''$, $\widetilde{u} \neq \widetilde{u}'$, $\widetilde{u} \nsim \widetilde{w}''$, and $\widetilde{u}' \nsim \widetilde{w}$. From (A_3), $u \neq z \neq u'$ and $z \nsim u, u'$. If $\widetilde{u} \sim \widetilde{u}'$, by (T_i) applied to w' and by (R_i) applied to u, u', the vertices z, w, w', w'', u, u' induce a double-house, which is impossible from Lemma 5.3. Thus $\widetilde{u} \nsim \widetilde{u}'$.

By $\mathrm{QC}(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$. By $(A_4)\&(A_5)$, $x \neq z$ and $x \nsim z$. By (T_i) applied to w' and by (R_i) applied to u, u', the vertices z, w, w', w'', u, u', x induce CW_3. Thus, by the 3-cube condition, there exists a vertex y of G such that $y \sim x, w, w''$ and $y \nsim z, w', u, u'$. By (R_i) applied to $\widetilde{x}$, there is $\widetilde{y} \in \widetilde{B}_i$ such that $\widetilde{y} \sim \widetilde{x}$. By (S_i) applied to $\widetilde{u}, \widetilde{x}$ and to the square $uxyw$, we have $\widetilde{y} \sim \widetilde{w}$. By (S_i) applied to $\widetilde{u}', \widetilde{x}$ and to the square $u'xyw''$, we get $\widetilde{y} \sim \widetilde{w}''$. Consequently, $\widetilde{y} \in \widetilde{S}_{i-1}$, $\widetilde{y} \sim \widetilde{w}, \widetilde{w}''$, contradicting (A_2). $\hspace{1em}\square$

Let $\widetilde{S}_{i+1}$ denote the equivalence classes of $\equiv$, i.e., $\widetilde{S}_{i+1} = Z/_{\equiv}$. For a couple $(\widetilde{w}, z) \in Z$, we will denote by $[\widetilde{w}, z]$ the equivalence class of $\equiv$ containing $(\widetilde{w}, z)$. Set $\widetilde{B}_{i+1} := \widetilde{B}_i \cup \widetilde{S}_{i+1}$. Let $\widetilde{G}_{i+1}$ be the graph having $\widetilde{B}_{i+1}$ as the vertex set in which two vertices $\widetilde{a}, \widetilde{b}$ are adjacent if and only if one of the following conditions holds:

(1) $\widetilde{a}, \widetilde{b} \in \widetilde{B}_i$ and $\widetilde{ab}$ is an edge of $\widetilde{G}_i$,

(2) $\widetilde{a} \in \widetilde{B}_i$, $\widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{b} = [\widetilde{a}, z]$,

(3) $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$, $\widetilde{a} = [\widetilde{w}, z]$, $\widetilde{b} = [\widetilde{w}, z']$ for a vertex $\widetilde{w} \in \widetilde{B}_i$, and $z \sim z'$ in the graph G.

Finally, we define the map $f_{i+1} : \widetilde{B}_{i+1} \to V(\mathbf{X})$ in the following way: if $\widetilde{a} \in \widetilde{B}_i$, then $f_{i+1}(\widetilde{a}) = f_i(\widetilde{a})$, otherwise, if $\widetilde{a} \in \widetilde{S}_{i+1}$ and $\widetilde{a} = [\widetilde{w}, z]$, then $f_{i+1}(\widetilde{a}) = z$. Notice that f_{i+1} is well-defined because all couples representing $\widetilde{a}$ have one and the same vertex z in the second argument. In the sequel, all vertices of $\widetilde{B}_{i+1}$ will be denoted with a tilde and their images in G under f_{i+1} will be denoted without tilde, e.g. if $\widetilde{w} \in \widetilde{B}_{i+1}$, then $w = f_{i+1}(\widetilde{w})$.

5.3.3. *Properties of $\widetilde{G}_{i+1}$ and f_{i+1}.* In this subsection we check our inductive assumptions, verifying the properties (P_{i+1}) through (T_{i+1}) for $\widetilde{G}_{i+1}$ and f_{i+1} defined above. In particular it allows us to define the corresponding complex $\widetilde{\mathbf{X}}_{i+1}$.

Lemma 5.7. *$\widetilde{G}_{i+1}$ satisfies the property (P_{i+1}), i.e., $B_j(v, \widetilde{G}_{i+1}) = \widetilde{B}_j$ for any $j \leq i + 1$.*

Proof. By definition of edges of $\widetilde{G}_{i+1}$, any vertex $\widetilde{b}$ of $\widetilde{S}_{i+1}$ is adjacent to at least one vertex of $\widetilde{B}_i$ and all such neighbors of $\widetilde{b}$ are vertices of the form $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, z]$ for a couple $(\widetilde{w}, z)$ of Z. By definition of Z, $\widetilde{w} \in \widetilde{S}_i$, whence any vertex of $\widetilde{S}_{i+1}$ is adjacent only to vertices of $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$. Therefore, the distance between the basepoint $\widetilde{v}$ and any vertex $\widetilde{a} \in \widetilde{B}_i$ is the same in the graphs $\widetilde{G}_i$ and $\widetilde{G}_{i+1}$. On the other hand, the distance in $\widetilde{G}_{i+1}$ between $\widetilde{v}$ and any vertex $\widetilde{b}$ of $\widetilde{S}_{i+1}$ is $i + 1$. This shows that indeed $B_j(v, \widetilde{G}_{i+1}) = \widetilde{B}_j$ for any $j \leq i + 1$. □

Lemma 5.8. *$\widetilde{G}_{i+1}$ satisfies the property (Q_{i+1}), i.e., the graph $\widetilde{G}_{i+1}$ is weakly modular with respect to the basepoint $\widetilde{v}$.*

Proof. First we show that $\widetilde{G}_{i+1}$ satisfies the triangle condition $\mathrm{TC}(\widetilde{v})$. Pick two adjacent vertices $\widetilde{x}, \widetilde{y}$ having in $\widetilde{G}_{i+1}$ the same distance to $\widetilde{v}$. Since by Lemma 5.7, $\widetilde{G}_{i+1}$ satisfies the property (P_{i+1}) and the graph $\widetilde{G}_i$ is weakly modular with respect to $\widetilde{v}$, we can suppose that $\widetilde{x}, \widetilde{y} \in \widetilde{S}_{i+1}$. From the definition of the edges of $\widetilde{G}_{i+1}$, there exist two couples $(\widetilde{w}, z), (\widetilde{w}, z') \in Z$ such that $\widetilde{w} \in \widetilde{B}_i$, z is adjacent to z' in G, and $\widetilde{x} = [\widetilde{w}, z], \widetilde{y} = [\widetilde{w}, z']$. Since $\widetilde{w}$ is adjacent in $\widetilde{G}_{i+1}$ to both $\widetilde{x}$ and $\widetilde{y}$, the triangle condition $\mathrm{TC}(\widetilde{v})$ is established.

Now we show that $\widetilde{G}_{i+1}$ satisfies the quadrangle condition $\mathrm{QC}(\widetilde{v})$. Since the properties (P_{i+1}) and (Q_i) hold, it suffices to consider a vertex $\widetilde{x} \in \widetilde{S}_{i+1}$ having two nonadjacent neighbors $\widetilde{w}, \widetilde{w}'$ in $\widetilde{S}_i$. By definition of $\widetilde{G}_{i+1}$, there exists a vertex z of $\mathbf{X}$ and couples $(\widetilde{w}, z), (\widetilde{w}', z) \in Z$ such that $\widetilde{x} = [\widetilde{w}, z]$ and $\widetilde{x} = [\widetilde{w}', z]$. Hence $(\widetilde{w}, z) \equiv (\widetilde{w}', z)$. Since $\widetilde{w}$ and $\widetilde{w}'$ are not adjacent, by condition (Z2) in the definition of $\equiv$ there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ adjacent to $\widetilde{w}$ and $\widetilde{w}'$, whence $\widetilde{x}, \widetilde{w}, \widetilde{w}'$ satisfy $\mathrm{QC}(\widetilde{v})$. □

We first prove that the mapping f_{i+1} is a graph homomorphism (preserving edges) from $\widetilde{G}_{i+1}$ to G. In particular, this implies that two adjacent vertices of $\widetilde{G}_{i+1}$ are mapped in G to different vertices.

Lemma 5.9. *f_{i+1} is a graph homomorphism from $\widetilde{G}_{i+1}$ to G, i.e., for any edge $\widetilde{ab}$ of $\widetilde{G}_{i+1}$, ab is an edge of G.*

Proof. Consider an edge $\widetilde{a}\widetilde{b}$ of $\widetilde{G}_{i+1}$. If $\widetilde{a}, \widetilde{b} \in \widetilde{B}_i$, the lemma holds by (R_i) or (T_i) applied to $\widetilde{a}$. Suppose that $\widetilde{a} \in \widetilde{S}_{i+1}$. If $\widetilde{b} \in \widetilde{B}_i$, then $\widetilde{a} = [\widetilde{b}, a]$, and ab is an edge of G. If $\widetilde{b} \in \widetilde{B}_{i+1}$, then the fact that $\widetilde{a}$ and $\widetilde{b}$ are adjacent implies that there exists a vertex $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{a} = [\widetilde{w}, a], \widetilde{b} = [\widetilde{w}, b]$ and such that $a \sim b$ in G. $\square$

We now prove that f_{i+1} is locally surjective at any vertex in $\widetilde{B}_i$.

Lemma 5.10. *If $\widetilde{a} \in \widetilde{B}_i$ and if $b \sim a$ in G, then there exists a vertex $\widetilde{b}$ of $\widetilde{G}_{i+1}$ adjacent to $\widetilde{a}$ such that $f_{i+1}(\widetilde{b}) = b$.*

Proof. If $\widetilde{a} \in \widetilde{B}_{i-1}$, the lemma holds by (R_i). Suppose that $\widetilde{a} \in \widetilde{S}_i$ and consider $b \sim a$ in G. If $\widetilde{a}$ has a neighbor $\widetilde{b} \in \widetilde{B}_i$ mapped to b by f_i, we are done. Otherwise $(\widetilde{a}, b) \in Z$, $[\widetilde{a}, b] \sim \widetilde{a}$ in $\widetilde{G}_{i+1}$ and $[\widetilde{a}, b]$ is mapped to b by f_{i+1}. $\square$

Before proving the local injectivity of f_{i+1}, we formulate a technical lemma.

Lemma 5.11. *Let $(\widetilde{w}, a), (\widetilde{w}', a) \in Z$ be such that $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$. If $(\widetilde{w}, b) \in Z$ and $b \sim w'$ in G, then $\widetilde{w} \sim \widetilde{w}'$, $(\widetilde{w}', b) \in Z$ and $(\widetilde{w}, b) \equiv (\widetilde{w}', b)$.*

Proof. First suppose that $\widetilde{w} \not\sim \widetilde{w}'$. Since $(\widetilde{w}, a) \equiv (\widetilde{w}', a)$, there exists $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}'$ and $wuw'a$ is an induced square in G. In G, $b \sim w, w'$, thus b, w, u, a, w' induce $K_{2,3}$ if $b \not\sim a, u$, W_4 if $b \sim a, u$, or W_4^- otherwise. In any case, we get a contradiction.

Thus $\widetilde{w} \sim \widetilde{w}'$. If $(\widetilde{w}', b) \notin Z$, then there exists $\widetilde{b} \in \widetilde{B}_i$ such that $\widetilde{b} \sim \widetilde{w}'$ and $f_i(\widetilde{b}) = b$. In G, wbw' is a triangle, thus $\widetilde{b} \sim \widetilde{w}$ by condition (R_i) applied to $\widetilde{b}$. This implies that $(\widetilde{w}, b) \notin Z$. Consequently, $(\widetilde{w}, b), (\widetilde{w}', b) \in Z$ and $(\widetilde{w}, b) \equiv (\widetilde{w}', b)$ since $\widetilde{w} \sim \widetilde{w}'$. $\square$

We can now prove that f_{i+1} is locally injective.

Lemma 5.12. *If $\widetilde{a} \in \widetilde{B}_{i+1}$ and $\widetilde{b}, \widetilde{c}$ are distinct neighbors of $\widetilde{a}$ in $\widetilde{G}_{i+1}$, then $b \neq c$.*

Proof. First note that if $\widetilde{b} \sim \widetilde{c}$, the assertion holds by Lemma 5.9; in the following we assume that $\widetilde{b} \not\sim \widetilde{c}$. If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, the lemma holds by (R_i) or (T_i) applied to $\widetilde{a}$. Suppose first that $\widetilde{a} \in \widetilde{B}_i$. If $\widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, then $\widetilde{b} = [\widetilde{a}, b]$ and $\widetilde{c} = [\widetilde{a}, c]$, and thus $b \neq c$. If $\widetilde{b} \in \widetilde{B}_i$ and $\widetilde{c} = [\widetilde{a}, c] \in \widetilde{S}_{i+1}$, then $(\widetilde{a}, b) \notin Z$, and thus $c \neq b$. Thus, let $\widetilde{a} \in \widetilde{S}_{i+1}$.

If $\widetilde{b}, \widetilde{c} \in \widetilde{B}_i$ and $\widetilde{a} \in \widetilde{S}_{i+1}$, then $\widetilde{a} = [\widetilde{b}, a] = [\widetilde{c}, a]$. Since $(\widetilde{b}, a) \equiv (\widetilde{c}, a)$ and since $\widetilde{b} \not\sim \widetilde{c}$, there exists $\widetilde{u} \in \widetilde{B}_{i-1}$ such that $\widetilde{u} \sim \widetilde{b}, \widetilde{c}$ and $abuc$ is an induced square of G. This implies that $b \neq c$.

If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, then there exist $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$, $\widetilde{c} = [\widetilde{w}', c]$, and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{w}', a]$. If $b = c$, then $[\widetilde{w}, b] = [\widetilde{w}', b] = [\widetilde{w}', c]$ by Lemma 5.11, and thus $\widetilde{b} = \widetilde{c}$, which is impossible.

If $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{S}_i$, then there exists $\widetilde{w} \in \widetilde{S}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$ and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{c}, a]$. If $\widetilde{w} \sim \widetilde{c}$, then $(\widetilde{w}, c) \notin Z$ and thus, $(\widetilde{w}, c) \neq (\widetilde{w}, b)$, i.e., $b \neq c$. If $\widetilde{w} \not\sim \widetilde{c}$, since $[\widetilde{w}, a] = [\widetilde{c}, a]$, there exists $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{w}, \widetilde{c}$ and such that $acxu$ is an induced square of G. Since $\widetilde{w}$ and $\widetilde{c}$ are not adjacent, by (R_i) applied to $\widetilde{u}$, w and c are not adjacent as well. Since $w \sim b$, this implies that $b \neq c$. $\square$

Lemma 5.13. *If $\widetilde{a} \sim \widetilde{b}, \widetilde{c}$ in $\widetilde{G}_{i+1}$, then $\widetilde{b} \sim \widetilde{c}$ if and only if $b \sim c$.*

27

Proof. If $\widetilde{b} \sim \widetilde{c}$, then $b \sim c$ by Lemma 5.9. Conversely, suppose that $b \sim c$ in G. If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{B}_i$, then $\widetilde{b} \sim \widetilde{c}$ by condition (R_i) applied to $\widetilde{a}$. Therefore, further we will assume that at least one of the vertices $\widetilde{a}, \widetilde{b}, \widetilde{c}$ does not belong to $\widetilde{B}_i$.

First suppose that $\widetilde{a} \in \widetilde{B}_i$. If $\widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, $\widetilde{b} = [\widetilde{a}, b]$ and $\widetilde{c} = [\widetilde{a}, c]$. Since $b \sim c$, by construction, we have $\widetilde{b} \sim \widetilde{c}$ in $\widetilde{G}_{i+1}$. Suppose now that $\widetilde{b} = [\widetilde{a}, b] \in S_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$. If there exists $\widetilde{b}' \sim \widetilde{c}$ in $\widetilde{G}_i$ such that $f_i(\widetilde{b}') = b$, then by (R_i) applied to $\widetilde{c}$, $\widetilde{a} \sim \widetilde{b}'$ and $(\widetilde{a}, b) \notin Z$, which is a contradiction. Thus $(\widetilde{c}, b) \in Z$ and since $\widetilde{c} \sim \widetilde{a}$, $[\widetilde{c}, b] = [\widetilde{a}, b] = \widetilde{b}$, and consequently, $\widetilde{c} \sim \widetilde{b}$. Therefore, let $\widetilde{a} \in \widetilde{S}_{i+1}$.

If $\widetilde{b}, \widetilde{c} \in \widetilde{B}_i$ and $\widetilde{a} \in \widetilde{S}_{i+1}$, then $\widetilde{a} = [\widetilde{b}, a] = [\widetilde{c}, a]$ and either $\widetilde{b} \sim \widetilde{c}$, or there exists $\widetilde{u} \in \widetilde{S}_{i-1}$ such that $\widetilde{u} \sim \widetilde{b}, \widetilde{c}$ and $ubac$ is an induced square in G, which is impossible because $b \sim c$.

If $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$ and $\widetilde{c} \in \widetilde{B}_i$, then there exists $\widetilde{w} \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$ and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{c}, a]$. By Lemma 5.11, $(\widetilde{c}, b) \in Z$ and $\widetilde{b} = [\widetilde{w}, b] = [\widetilde{c}, b]$. Consequently, $\widetilde{c} \sim \widetilde{b}$.

If $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{S}_{i+1}$, there exist $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ such that $\widetilde{b} = [\widetilde{w}, b]$, $\widetilde{c} = [\widetilde{w}', c]$ and $\widetilde{a} = [\widetilde{w}, a] = [\widetilde{w}', a]$. If $\widetilde{w} \sim \widetilde{c}$ or $\widetilde{w}' \sim \widetilde{b}$, then $\widetilde{b} \sim \widetilde{c}$ because $b \sim c$. Suppose now that $\widetilde{w} \nsim \widetilde{c}$, $\widetilde{w}' \nsim \widetilde{b}$. From previous case applied to $\widetilde{a}, \widetilde{b} \in \widetilde{S}_{i+1}$ (respectively, $\widetilde{a}, \widetilde{c} \in \widetilde{S}_{i+1}$) and $\widetilde{w}' \in \widetilde{B}_i$ (respectively, $\widetilde{w} \in \widetilde{B}_i$), it follows that $w \nsim c$ and $w' \nsim b$. If $\widetilde{w} \sim \widetilde{w}'$, then a, b, w, w', c induce W_4 in G, which is impossible. Since $[\widetilde{w}, a] = [\widetilde{w}', a]$, there exists $\widetilde{u} \in \widetilde{S}_{i-1}$, such that $\widetilde{u} \sim \widetilde{w}, \widetilde{w}'$ and such that $awuw'$ is an induced square in G. If $u \sim b$, then by (R_i) applied to u, there exists $\widetilde{b}' \in \widetilde{B}_i$ mapped to b by f_i such that $\widetilde{b}' \sim \widetilde{u}, \widetilde{w}$ and thus $(\widetilde{w}, b) \notin Z$, which is a contradiction. Using the same arguments, we have that $u \nsim c$ and thus, a, b, c, w', u, w induce W_5^- in G, which is impossible. $\qquad\square$

We can now prove that the image under f_{i+1} of an induced triangle or square is an induced triangle or square.

Lemma 5.14. *If $\widetilde{a}\widetilde{b}\widetilde{c}$ is a triangle in $\widetilde{G}_{i+1}$, then abc is a triangle in G. If $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ is an induced square of $\widetilde{G}_{i+1}$, then $abcd$ is an induced square in G. In particular, the graph $\widetilde{G}_{i+1}$ does not contain induced $K_{2,3}$ and W_4^-.*

Proof. For triangles, the assertion follows directly from Lemma 5.9. Consider now a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$. From Lemmas 5.9 and 5.12, the vertices a, b, c, and d are pairwise distinct and $a \sim b$, $b \sim c$, $c \sim d$, $d \sim a$. From Lemma 5.13, $a \nsim c$ and $b \nsim d$. Consequently, $abcd$ is an induced square in G.

Now, if $\widetilde{G}_{i+1}$ contains an induced $K_{2,3}$ or W_4^-, from the first assertion and Lemma 5.13 we conclude that the image under f_{i+1} of this subgraph will be an induced $K_{2,3}$ or W_4^- in the graph G, a contradiction. $\qquad\square$

The second assertion of Lemma 5.14 implies that replacing all 3-cycles and all induced 4-cycles of $\widetilde{G}_{i+1}$ by triangle- and square-cells, we will obtain a triangle-square flag complex, which we denote by $\widetilde{\mathbf{X}}_{i+1}$. Then obviously $\widetilde{G}_{i+1} = G(\widetilde{\mathbf{X}}_{i+1})$. The first assertion of Lemma 5.14 and the flagness of $\mathbf{X}$ imply that f_{i+1} can be extended to a cellular map from $\widetilde{\mathbf{X}}_{i+1}$ to $\mathbf{X}$: f_{i+1} maps a triangle $\widetilde{a}\widetilde{b}\widetilde{c}$ to the triangle abc of $\mathbf{X}$ and a square $\widetilde{a}\widetilde{b}\widetilde{c}\widetilde{d}$ to the square $abcd$ of $\mathbf{X}$.

Lemma 5.15. *f_{i+1} satisfies the conditions (R_{i+1}) and (T_{i+1}).*

Proof. From Lemmas 5.12 and 5.13, we know that for any $\widetilde{w} \in \widetilde{B}_{i+1}$, f_{i+1} induces an isomorphism between the subgraph of $\widetilde{G}_{i+1}$ induced by $B_1(\widetilde{w}, \widetilde{G}_{i+1})$ and the subgraph of G induced by $f_{i+1}(B_1(\widetilde{w}, \widetilde{G}_{i+1}))$. Consequently, the condition (T_{i+1}) holds. From Lemma 5.10, we know that $f_{i+1}(B_1(\widetilde{w}, \widetilde{G}_{i+1})) = B_1(w, G)$ and consequently (R_{i+1}) holds as well. □

Lemma 5.16. *For any* $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_i$ *such that the vertices* $w = f_{i+1}(\widetilde{w}), w' = f_{i+1}(\widetilde{w}')$ *belong to a square* $ww'u'u$ *of* $\mathbf{X}$*, there exist* $\widetilde{u}, \widetilde{u}' \in \widetilde{B}_{i+1}$ *such that* $f_{i+1}(\widetilde{u}) = u, f_{i+1}(\widetilde{u}') = u'$ *and* $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ *is a square of* $\widetilde{\mathbf{X}}_{i+1}$*, i.e.,* $\widetilde{\mathbf{X}}_{i+1}$ *satisfies the property* (S_{i+1})*.*

Proof. By Lemma 5.15 applied to $\widetilde{w}$ and $\widetilde{w}'$, we know that in $\widetilde{G}_{i+1}$ there exist a unique $\widetilde{u}$ (respectively, a unique $\widetilde{u}'$) such that $\widetilde{u} \sim \widetilde{w}$ (respectively, $\widetilde{u}' \sim \widetilde{w}'$) and $f_{i+1}(\widetilde{u}) = u$ (respectively, $f_{i+1}(\widetilde{u}) = u'$). By Lemma 5.15, $\widetilde{w}$ (respectively, $\widetilde{w}'$) is the unique neighbor of $\widetilde{u}$ (resp. $\widetilde{u}$) mapped to w (respectively, w') by f_{i+1}.

Note that if $\widetilde{w}, \widetilde{w}' \in \widetilde{B}_{i-1}$, the lemma holds by condition (S_i). Let us assume further that $\widetilde{w} \in \widetilde{S}_i$.

Case 1. $\widetilde{w}' \in \widetilde{S}_{i-1}$.

If $\widetilde{u}' \in \widetilde{B}_{i-1}$, by (S_i) applied to $\widetilde{w}'$ and $\widetilde{u}'$, we conclude that $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$.

If $\widetilde{u}' \in \widetilde{S}_i$ and $\widetilde{u} \in \widetilde{S}_{i-1}$, then Lemma 5.15 applied to $\widetilde{w}$, implies that $\widetilde{u}$ is not adjacent to $\widetilde{w}'$. Thus, by the quadrangle condition $QC(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-2}$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{w}'$. Hence, w, w', u, u', x induce in G a forbidden $K_{2,3}, W_4^-$, or W_4, which is impossible.

Suppose now that $\widetilde{u}', \widetilde{u} \in \widetilde{S}_i$. By $TC(\widetilde{v})$, there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ different from $\widetilde{w}'$ such that $\widetilde{x} \sim \widetilde{u}, \widetilde{w}$. Since G does not contain W_4^- or W_4, $x \nsim u', w'$ and the vertices u, w, w', u', x induce a house. By the 3-prism condition there exists y in G such that $y \sim x, u', w'$ and $y \nsim u, w$. Since $x \nsim w'$, by R_i applied to $\widetilde{w}$, $\widetilde{x} \nsim \widetilde{w}'$. Applying $QC(\widetilde{v})$, there exists $\widetilde{z} \in \widetilde{S}_{i-2}$ such that $\widetilde{z} \sim \widetilde{x}, \widetilde{w}'$ and $\widetilde{z} \nsim \widetilde{w}$. Since $\widetilde{z} \in \widetilde{S}_{i-2}$, $\widetilde{z} \nsim \widetilde{u}'$ and thus by R_{i+1} applied to $\widetilde{w}'$, $z \nsim u'$. Consequently, $z \neq y$. Thus, from Lemma 5.14, $xzw'w$ is an induced square of G and y, x, z, w', w induce a $K_{2,3}$ if $z \nsim y$ and W_4^- otherwise, which is impossible. Note that if $\widetilde{u}'$ has a neighbor $\widetilde{u}_2$ in $\widetilde{B}_i$ mapped to u , then, exchanging the roles of $\widetilde{u}'$ and $\widetilde{w}$, we also get a contradiction. Suppose now that neither $\widetilde{w}$ nor $\widetilde{u}'$ has a neighbor in $\widetilde{B}_i$ mapped to u. Thus, $(\widetilde{w}, u), (\widetilde{u}', u) \in Z$ and since $\widetilde{w}' \in \widetilde{S}_{i-1}$ is adjacent to $\widetilde{w}$ and $\widetilde{u}'$, $(\widetilde{w}, u) \equiv (\widetilde{u}', u)$. Consequently, $\widetilde{w}\widetilde{w}'\widetilde{u}'[\widetilde{w}, u]$ is a square of $\widetilde{G}_{i+1}$ which is mapped by f_{i+1} to the square $ww'u'u$.

Case 2. $\widetilde{w}' \in \widetilde{S}_i$.

If $\widetilde{u} \in \widetilde{S}_{i-1}$, exchanging the role of $\widetilde{w}'$ and $\widetilde{u}$, we are in the previous case and thus there exists $\widetilde{u}'' \sim \widetilde{w}', \widetilde{u}$ such that $f_{i+1}(\widetilde{u}'') = u'$. By Lemma 5.15, we get that $\widetilde{u}' = \widetilde{u}''$ and we are done. For the same reasons, if $\widetilde{u}' \in \widetilde{S}_{i-1}$, applying Case 1 with $\widetilde{w}'$ in the role of $\widetilde{w}$ and $\widetilde{u}'$ in the role of $\widetilde{w}'$, we are done.

If $\widetilde{u} \in \widetilde{S}_i$, by $TC(\widetilde{v})$ there exists $\widetilde{x} \in \widetilde{B}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{u}$. Thus, in G there exists $x \sim u, w$ and, since G does not contain W_4 or W_4^-, $x \nsim u', w'$. Applying the 3-prism condition, we get y in G such that $y \sim u', w', x$ and $y \nsim u, w$. Applying the previous case to $\widetilde{w}, \widetilde{x}$ and the square $wxyw'$ of G, we know that there exists $\widetilde{y} \in \widetilde{B}_i$ such that $\widetilde{w}\widetilde{x}\widetilde{y}\widetilde{w}'$ is an induced square in $\widetilde{G}_{i+1}$. From Lemma 5.15 applied to $\widetilde{w}'$, we deduce that $\widetilde{y} \sim \widetilde{u}'$. Applying (S_i) to $\widetilde{x}, \widetilde{y}$ and to the

square $xyu'u$, we get that $\widetilde{u} \sim \widetilde{u}'$, thus $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$. If $\widetilde{u}' \in \widetilde{S}_i$, then exchanging the roles of $\widetilde{w}, \widetilde{w}', \widetilde{u}, \widetilde{u}'$ we also get that $\widetilde{w}\widetilde{w}'\widetilde{u}'\widetilde{u}$ is a square in $\widetilde{G}_{i+1}$.

Suppose now that $\widetilde{w}$ has no neighbor in $\widetilde{B}_i$ mapped to u and that $\widetilde{w}'$ has no neighbor in $\widetilde{B}_i$ mapped to u'. Thus, there exist $[\widetilde{w}, u]$ and $[\widetilde{w}', u']$ in $\widetilde{S}_{i+1}$. By TC($\widetilde{v}$), there exists $\widetilde{x} \in \widetilde{S}_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{w}'$. In G, $x \sim w, w'$ and $x \nsim u, u'$ since G does not contain W_4^- or W_4. Applying the 3-prism condition, there is a vertex y in G such that $y \sim u, u'$ and $y \nsim w, w'$. By (R_i) applied to $\widetilde{x}$, there exists $\widetilde{y}$ in $\widetilde{B}_i$ such that $\widetilde{y} \sim \widetilde{x}$ and $\widetilde{y} \nsim \widetilde{w}, \widetilde{w}'$. If $\widetilde{y}$ has a neighbor in $\widetilde{B}_i$ mapped to u, then applying the previous case to $\widetilde{w}, \widetilde{x}$ and the square $wxyu$, we conclude that $\widetilde{w}$ has a neighbor in $\widetilde{B}_i$ mapped to u, which is impossible. Consequently, $(\widetilde{y}, u) \in Z$, and since there is $\widetilde{x} \in S_{i-1}$ such that $\widetilde{x} \sim \widetilde{w}, \widetilde{y}$ and $wxyu$ is an induced square in G, $(\widetilde{y}, u) \equiv (\widetilde{w}, u)$. Using the same arguments, one can show that there exists $(\widetilde{y}, u') \in [\widetilde{w}', u']$. Since yuu' is a triangle in G, and since $[\widetilde{w}, u] = [\widetilde{y}, u]$ and $[\widetilde{w}', u'] = [\widetilde{y}, u']$, there is an edge in $\widetilde{G}_{i+1}$ between $[\widetilde{w}, u]$ and $[\widetilde{w}', u']$. Consequently, $\widetilde{w}\widetilde{w}'[\widetilde{w}', u'][\widetilde{w}, u]$ is a square of $\widetilde{G}_{i+1}$ satisfying the lemma. $\qquad\square$

5.3.4. *The universal cover* $\widetilde{\mathbf{X}}$. Let $\widetilde{\mathbf{X}}_v$ denote the triangle-square complex obtained as the directed union $\bigcup_{i\geq 0} \widetilde{\mathbf{X}}_i$ with a vertex v of $\mathbf{X}$ as the basepoint. Denote by $\widetilde{G}_v$ the 1-skeleton of $\widetilde{\mathbf{X}}_v$. Since each $\widetilde{G}_i$ is weakly modular with respect to $\widetilde{v}$, the graph $\widetilde{G}_v$ is also weakly modular with respect to $\widetilde{v}$. Thus the complex $\widetilde{\mathbf{X}}_v$ is simply connected by virtue of Lemma 5.5. Let also $f = \bigcup_{i\geq 0} f_i$ be the map from $\widetilde{\mathbf{X}}_v$ to $\mathbf{X}$.

Lemma 5.17. *For any* $\widetilde{w} \in \widetilde{\mathbf{X}}_v$, $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ *is isomorphic to* $St(w, \mathbf{X})$ *where* $w = f(\widetilde{w})$. *Consequently,* $f : \widetilde{\mathbf{X}}_v \to \mathbf{X}$ *is a covering map.*

Proof. Note that, since $\widetilde{\mathbf{X}}_v$ is a flag complex, a vertex $\widetilde{x}$ of $\widetilde{\mathbf{X}}_v$ belongs to $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ if and only if either $\widetilde{x} \in B_1(\widetilde{w}, \widetilde{G}_v)$ or $\widetilde{x}$ has two non-adjacent neighbors in $B_1(\widetilde{w}, \widetilde{G}_v)$.

Consider a vertex $\widetilde{w}$ of $\mathbf{X}_v$. Let i be the distance between $\widetilde{v}$ and $\widetilde{w}$ in $\widetilde{G}_v$ and consider the set $\widetilde{B}_{i+2}$. Then the vertex-set of $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ is included in $\widetilde{B}_{i+2}$. From (R_{i+2}) we know that f is an isomorphism between the graphs induced by $B_1(\widetilde{w}, \widetilde{G}_v)$ and $B_1(w, G)$.

For any vertex x in $St(w, \mathbf{X}) \smallsetminus B_1(w, G)$ there exists an induced square $wuxu'$ in G. From (R_{i+2}), there exist $\widetilde{u}, \widetilde{u}' \sim \widetilde{w}$ in $\widetilde{G}_v$ such that $\widetilde{u} \nsim \widetilde{u}'$. From (S_{i+2}) applied to $\widetilde{w}, \widetilde{u}$ and since $\widetilde{w}$ has a unique neighbor $\widetilde{u}'$ mapped to u', there exists a vertex $\widetilde{x}$ in $\widetilde{G}_v$ such that $f(\widetilde{x}) = x$, $\widetilde{x} \sim \widetilde{u}, \widetilde{u}'$ and $\widetilde{x} \nsim \widetilde{w}$. Consequently, f is a surjection from $V(St(\widetilde{w}, \widetilde{\mathbf{X}}_v))$ to $V(St(w, \mathbf{X}))$.

Suppose by way of contradiction that there exist two distinct vertices $\widetilde{u}, \widetilde{u}'$ of $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ such that $f(\widetilde{u}) = f(\widetilde{u}') = u$. If $\widetilde{u}, \widetilde{u}' \sim \widetilde{w}$, by condition (R_{i+1}) applied to $\widetilde{w}$ we get a contradiction. Suppose now that $\widetilde{u} \sim \widetilde{w}$ and $\widetilde{u}' \nsim \widetilde{w}$ and let $\widetilde{z} \sim \widetilde{w}, \widetilde{u}'$. This implies that w, u, z are pairwise adjacent in G. Since f is an isomorphism between the graphs induced by $B_1(\widetilde{w}, \widetilde{G}_v)$ and $B_1(w, G)$, we conclude that $\widetilde{z} \sim \widetilde{u}$. But then f is not locally injective around $\widetilde{z}$, contradicting the condition (R_{i+2}). Suppose now that $\widetilde{u}, \widetilde{u}' \nsim \widetilde{w}$. Let $\widetilde{a}, \widetilde{b} \sim \widetilde{u}, \widetilde{w}$ and $\widetilde{a}', \widetilde{b}' \sim \widetilde{u}', \widetilde{w}'$. If $\widetilde{a}' = \widetilde{a}$ or $\widetilde{a}' = \widetilde{b}$, then applying (R_{i+2}) to $\widetilde{a}'$, we get that $f(\widetilde{u}) \neq f(\widetilde{u}')$. Suppose now that $\widetilde{a}' \notin \{\widetilde{a}, \widetilde{b}\}$. Then the subgraph of G induced by a', w, a, b, u is either $K_{2,3}$ if $a' \nsim a, b$, or W_4 if $a' \sim a, b$, or W_4^- otherwise. In all cases, we get a contradiction.

Hence f is a bijection between the vertex-sets of $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ and $St(w, \mathbf{X})$. Since $\mathbf{X}_v$ is a flag complex, by (R_{i+2}), $\widetilde{a} \sim \widetilde{b}$ in $St(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ if and only if $a \sim b$ in $St(w, \mathbf{X})$. By (R_{i+2}) applied

to w and since $\mathbf{X}$ and $\widetilde{\mathbf{X}}_v$ are flag complexes, $\widehat{ab}\widetilde{w}$ is a triangle in $\mathrm{St}(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ if and only if abw is a triangle in $\mathrm{St}(w, \mathbf{X})$. By (R_{i+2}) and since $\mathbf{X}$ is a flag complex, if $\widehat{ab}\widetilde{c}\widetilde{w}$ is a square in $\mathrm{St}(\widetilde{w}, \widetilde{\mathbf{X}})$, then $abcw$ is a square in $\mathrm{St}(w, \mathbf{X})$. Conversely, by the conditions (R_{i+2}) and (S_{i+2}) and flagness of $\widetilde{\mathbf{X}}_v$, we conclude that if $abcw$ is a square in $\mathrm{St}(w, \mathbf{X})$, then $\widehat{ab}\widetilde{c}\widetilde{w}$ is a square in $\mathrm{St}(\widetilde{w}, \widetilde{\mathbf{X}}_v)$. Consequently, for any $\widetilde{w} \in \widetilde{\mathbf{X}}_v$, f defines an isomorphism between $\mathrm{St}(\widetilde{w}, \widetilde{\mathbf{X}}_v)$ and $\mathrm{St}(w, \mathbf{X})$, and thus f is a covering map. $\qquad\square$

Lemma 5.18. $\widetilde{\mathbf{X}}_v$ *satisfies the 3–prism, the 3–cube, and the* $\widehat{W_5}$*–wheel conditions, and the graph* $\widetilde{G}_v$ *does not contain induced* $K_{2,3}, W_4^-,$ *and* W_4*. Moreover, if G is W_5–free, then $\widetilde{G}_v$ is also W_5–free.*

Proof. If $\widetilde{G}_v$ contains an induced $K_{2,3}$ or W_4^-, there exists i such that $\widetilde{G}_i$ contains an induced $K_{2,3}$ or W_4^-, contradicting Lemma 5.14. Let $C \in \{W_4, W_5\}$ be an induced subgraph of $\widetilde{G}_v$. By Lemma 5.17 applied to the center of the wheel, the subgraph induced by $f(V(C))$ is isomorphic to C. Since G does not contain any induced W_4, the graph $\widetilde{G}_v$ also does not contain any induced W_4 and if G is W_5–free, $\widetilde{G}_v$ is also W_5–free.

3–prism condition: Let $\widetilde{u}\widetilde{u}'\widetilde{w}'\widetilde{w}\widetilde{x}$ be a house in $\widetilde{\mathbf{X}}_v$ where, $\widetilde{u}\widetilde{u}'\widetilde{w}'\widetilde{w}$ is a square and $\widetilde{u}\widetilde{w}\widetilde{x}$ is a triangle. Consider the image of this house by f, i.e. $uu'w'wx$ in $\mathbf{X}$. If the image is not an induced subgraph of $\mathbf{X}$ then, by Lemma 5.17 (applied to $\widetilde{u}$), we have $x \sim w'$ and the vertices x, u, u', w, w' induce a W_4^-, a contradiction. Thus $uu'w'wx$ is an induced house in $\mathbf{X}$. By the 3–prism condition in $\mathbf{X}$, there exists a vertex $y \in G$ such that $y \sim u', w', x$ and $y \nsim u, w$. Since f is locally bijective, there exists $\widetilde{y} \sim \widetilde{x}$ such that $f(\widetilde{y}) = y$. Since f is an isomorphism from $\mathrm{St}(\widetilde{x}, \widetilde{\mathbf{X}}_v)$ to $\mathrm{St}(x, \mathbf{X})$, considering the squares $xyu'u$ and $xyw'w$, we get that $\widetilde{y} \sim \widetilde{u}', \widetilde{w}'$ and $\widetilde{y} \nsim \widetilde{u}, \widetilde{w}$. Thus, $\widetilde{\mathbf{X}}_v$ satisfies the 3–prism condition.

3–cube condition: Consider three squares $\widetilde{x}\widetilde{a}_1\widetilde{b}_3\widetilde{a}_2$, $\widetilde{x}\widetilde{a}_2\widetilde{b}_1\widetilde{a}_3$, $\widetilde{x}\widetilde{a}_3\widetilde{b}_2\widetilde{a}_1$ in $\widetilde{\mathbf{X}}_v$. By Lemma 5.17 applied to $\widetilde{x}$, the images $xa_1b_3a_2$ of $\widetilde{x}\widetilde{a}_1\widetilde{b}_3\widetilde{a}_2$, $xa_2b_1a_3$ of $\widetilde{x}\widetilde{a}_2\widetilde{b}_1\widetilde{a}_3$ and $xa_3b_2a_1$ of $\widetilde{x}\widetilde{a}_3\widetilde{b}_2\widetilde{a}_1$ are squares of $\mathbf{X}$. By the 3–cube condition, in G there exists a vertex y such that $y \sim b_i$ and $y \nsim x, a_i$, for all i. Moreover, for all distinct i, j, $b_i \nsim a_i$ and $b_i \nsim b_j$. By Lemma 5.17 applied to $\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3$, for all distinct i, j, $\widetilde{b}_i \nsim \widetilde{a}_i$ and $\widetilde{b}_i \nsim \widetilde{b}_j$. Since f is locally bijective, there exists $\widetilde{y} \sim \widetilde{b}_1$ such that $f(\widetilde{y}) = y$. Since f is an isomorphism from $\mathrm{St}(\widetilde{b}_1, \widetilde{\mathbf{X}}_v)$ to $\mathrm{St}(b_1, \mathbf{X})$, we get that $\widetilde{y} \sim \widetilde{b}_2, \widetilde{b}_3$ and $\widetilde{y} \nsim \widetilde{a}_2, \widetilde{a}_3$. When considering $\mathrm{St}(\widetilde{b}_2, \widetilde{\mathbf{X}}_v)$, we get that $\widetilde{y} \nsim \widetilde{a}_1$. If $\widetilde{y} \sim \widetilde{x}$, $\widetilde{G}_v$ contains an induced $K_{2,3}$, a contradiction. Thus, $\widetilde{\mathbf{X}}$ also satisfies the 3–cube condition.

$\widehat{W_5}$*–wheel condition:* Consider W_5 in $\widetilde{G}_v$ made of a 5–cycle $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{x}_1)$ and a vertex $\widetilde{c}$ adjacent to all vertices of this cycle. Suppose that there exists a vertex $\widetilde{z}$ such that $\widetilde{z} \sim \widetilde{x}_1, \widetilde{x}_2$ and $\widetilde{z} \nsim \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{c}$. By Lemma 5.17, the vertices $c, x_1, x_2, x_3, x_4, x_5$ are all distinct and they induce W_5 in G. Moreover, $z \notin \{c, x_1, x_2, x_3, x_4, x_5\}$, and $z \nsim c, x_3, x_5$. Similarly, if $z \sim x_4$ then $zx_2x_3x_4$ is a square and, by Lemma 5.17 (applied to $\widetilde{x}_2$), we have $\widetilde{z} \sim \widetilde{x}_4$, a contradiction. By the $\widehat{W_5}$–wheel condition for $\mathbf{X}$, there exists $y \sim c, z, x_1, x_2, x_3, x_4, x_5$ in G. Consider the neighbor $\widetilde{y}$ of $\widetilde{c}$ such that $f(\widetilde{y}) = y$. Since $\mathrm{St}(\widetilde{c}, \widetilde{\mathbf{X}}_v)$ is isomorphic to $\mathrm{St}(c, \mathbf{X})$, $\widetilde{y} \sim \widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5$. Considering the star $\mathrm{St}(\widetilde{x}_1, \widetilde{\mathbf{X}}_v)$, we conclude that $\widetilde{y} \sim \widetilde{z}$. Consequently, $\widetilde{\mathbf{X}}_v$ satisfies the $\widehat{W_5}$–wheel condition. $\qquad\square$

Now, we are ready to complete the proof of the implication (ii)⟺(iii). Let $\mathbf{X}$ be a connected triangle-square flag complex satisfying the local conditions of (ii). By Lemma 5.17, $f : \widetilde{\mathbf{X}}_v \to \mathbf{X}$ is a covering map. By Lemma 5.5, $\widehat{\mathbf{X}}_v$ is simply connected, thus $\widehat{\mathbf{X}}_v$ is the universal cover $\widetilde{\mathbf{X}}$ of $\mathbf{X}$. Therefore the triangle-square complexes $\widetilde{\mathbf{X}}_v, v \in V(\mathbf{X})$, are all universal covers of $\mathbf{X}$, whence they are all isomorphic. Since for each vertex v of $\mathbf{X}$, the graph $\widetilde{G}_v = G(\widetilde{\mathbf{X}}_v)$ is weakly modular with respect to the basepoint v, we conclude that the 1-skeleton $G(\widetilde{\mathbf{X}})$ of $\widetilde{\mathbf{X}}$ is weakly modular with respect to each vertex, thus $G(\widetilde{\mathbf{X}})$ is a weakly modular graph. Since $\widetilde{\mathbf{X}}$ is isomorphic to any $\widetilde{\mathbf{X}}_v$, by Lemma 5.18, $\widetilde{\mathbf{X}}$ satisfies the same local conditions as $\mathbf{X}$. Thus $\widetilde{\mathbf{X}}$ satisfies the wheel, the 3-prism, and the 3-cube conditions. If, additionally, $\mathbf{X}$ is simply connected, then the universal cover $\widetilde{\mathbf{X}}$ is $\mathbf{X}$ itself. Therefore, $\mathbf{X}$ coincides with $\widetilde{\mathbf{X}}_v$ for any choice of the basepoint $v \in V(\mathbf{X})$. Therefore, by what has been proven above, $G(\mathbf{X})$ is a bucolic graph. This establishes the implication (ii)⟹(iii) of Theorem 1.

5.4. **Proof of (iii)⟹(ii).** Let $\mathbf{X}$ be a prism flag complex such that $G := G(\mathbf{X})$ is a weakly modular graph not containing induced W_4. Then G does not contain induced $K_{2,3}$ and W_4^- because G is the 1-skeleton of a flag triangle-square cell complex $\mathbf{X}^{(2)}$ and both $K_{2,3}$ and W_4^- contain squares intersecting on two edges. From Lemma 5.5 we conclude that $\mathbf{X}^{(2)}$ (and therefore $\mathbf{X}$) is simply connected. Thus, it remains to show that $\mathbf{X}$ satisfies the 3-prism, the 3-cube, and the $\widehat{W_5}$-wheel conditions. First suppose that the triangle uvw and the square $uvxy$ define in $\mathbf{X}$ a house. Then w is at distance 2 to the adjacent vertices x and y. By the triangle condition, there exists a vertex w' adjacent to w, x, and y and different from u and v. If w' is adjacent to one or both of the vertices u, v, then we will get a forbidden W_4^- or W_4 induced by u, v, x, y, w'. This establishes the 3-prism condition.

To prove the 3-cube condition, let $xyuv$, $uvwz$, and $uytz$ be three squares of $\mathbf{X}$ pairwise intersecting in edges and all three intersecting in u. If x and w are adjacent, then the vertices v, x, w, u, y, z induce in $\mathbf{X}$ a double house, which is impossible by Lemma 5.3 because $\mathbf{X}$ satisfies the 3-prism condition. Hence $x \nsim w$ and analogously $x \nsim t$ and $t \nsim w$. If x is adjacent to z, then x, y, u, t, z induce in G a forbidden $K_{2,3}$. Thus $x \nsim z$ and analogously $y \nsim w$ and $v \nsim t$. First suppose that $d(x, z) = 2$ in G. Since $d(y, z) = 2$, by the triangle condition there exists a vertex s adjacent to x, y, and z. From what has been shown before, $s \neq u, t$, hence y, u, z, t, s induce $K_{2,3}$, W_4^-, or W_4 depending of whether s is adjacent to none, one or two of the vertices u, t. Thus, $d(x, z) = 3$ and for the same reasons, $d(y, w) = d(v, t) = 3$. By the quadrangle condition there exists a vertex s adjacent to x, w, t and distinct from previous vertices. Since $d(x, z) = d(w, y) = d(t, v) = 3$, $s \nsim z, y, v$. If s is adjacent to u, then s, u, v, w, z induce a forbidden $K_{2,3}$. This shows that in this case the vertices s, t, u, v, w, x, y, z define a 3-cube, establishing the 3-cube condition.

Finally, we establish the $\widehat{W_5}$-wheel condition. Note that $\mathbf{X}$ satisfies the 3-cube and the 3-prism conditions and does not contain a $\mathbf{X}(W_5^-)$ by Lemma 5.4. Pick a 5-wheel defined by a 5-cycle $(x_1, x_2, x_3, x_4, x_5, x_1)$ and a vertex c adjacent to all vertices of this cycle, and suppose that x_0 is a vertex adjacent to x_1 and x_5 and not adjacent to remaining vertices of this 5-wheel. If $d(x_0, x_3) = 3$, then by the quadrangle condition QC(x_0), there exists a vertex

y adjacent to x_0, x_2, x_4 and not adjacent to x_3. Then the vertices c, y, x_2, x_3, x_4 induce a W_4 if y is adjacent to c, and a W_4^- otherwise. So, suppose that $d(x_0, x_3) = 2$. By the triangle condition $\text{TC}(x_0)$, there exists a vertex z adjacent to x_0, x_2, x_3. Suppose that $z \not\sim c$. If $z \sim x_1$, then x_2, x_1, z, x_3, c induce a forbidden W_4. If $z \sim x_5$, the vertices x_1, x_2, c, x_5, z induce a forbidden W_4 if $z \sim x_1$ or a W_4^- otherwise. If $z \not\sim x_1, x_5$, the vertices z, x_2, c, x_5, x_0, x_1 induce a forbidden W_5^-. Thus, $z \sim c$. To avoid a forbidden W_4^- or W_4 induced by z, c, x_1, x_0, x_5, the vertex z must be adjacent to x_1 and x_5. Finally, to avoid W_4 induced by z, c, x_3, x_4, x_5, the vertex z must be adjacent to x_4 as well. As a result, we conclude that z is adjacent to x_0 and to all vertices of the 5-wheel, establishing the $\widehat{W_5}$-wheel condition. This concludes the proof of the implication (iii)$\Rightarrow$(ii).

Before proving the implication (ii)&(iii)$\Rightarrow$(i), we will establish the last assertion of Theorem 1. Let $\mathbf{X}$ be a flag prism complex satisfying the wheel, the cube, and the prism conditions. Then its 2-skeleton $\mathbf{Y} := \mathbf{X}^{(2)}$ is a triangle-square flag complex satisfying the wheel, the 3-cube, and the 3-prism conditions. Let $\widetilde{\mathbf{X}}$ be the universal cover of $\mathbf{X}$. Then the 2-skeleton $\widetilde{\mathbf{X}}^{(2)}$ of $\widetilde{\mathbf{X}}$ is a covering space of $\mathbf{Y}$. But at the same time $\widetilde{\mathbf{X}}^{(2)}$ is simply connected (because the 2-skeleton carries all the information about the fundamental group), so $\widetilde{\mathbf{X}}^{(2)}$ is the universal cover of $\mathbf{Y}$. Since $\widetilde{\mathbf{X}}$ is the prism complex of $\widetilde{\mathbf{X}}^{(2)}$ and $\widetilde{\mathbf{X}}^{(2)} = \widetilde{\mathbf{Y}}$ satisfies the condition (ii) of Theorem 1, we conclude that $\widetilde{\mathbf{X}}$ is a bucolic complex.

5.5. **Proof of (ii)&(iii)$\Rightarrow$(i).** Now, we will show that a flag prism complex $\mathbf{X}$ satisfying the conditions (ii) and (iii) of Theorem 1 also satisfies the cube and the prism conditions. We start with an auxiliary result and some conventions.

Lemma 5.19. *Any prism H (and in particular, any cube) of $\mathbf{X}$ is convex, i.e., $G(H)$ induces a convex subgraph of $G(\mathbf{X})$.*

Proof. If the 1-skeleton $G(H)$ of a prism H is not convex in $G(\mathbf{X})$, then $G(H)$ is not locally convex since $G(\mathbf{X})$ is weakly modular; cf. [18, Theorem 7]. Thus we can find two vertices x, y of H at distance 2 in $G(H)$ having a common neighbor outside H. Since x and y already have two common (non-adjacent) neighbors in H, we will obtain in $G(\mathbf{X})$ a forbidden $K_{2,3}, W_4^-$, or W_4. $\qquad\square$

Further we will use recurrently this result without referring to Lemma 5.19, simply saying that the prisms and the cubes of $\mathbf{X}$ are convex. Furthermore, when we say that some vertex set or some subgraph π of $G(\mathbf{X})$ is a prism or a cube, we mean that π is the 0-skeleton or the 1-skeleton of a prism or a cube of $\mathbf{X}$. In that case, if the meaning is clear from the context, we will denote the resulting prism or cube also by π. Finally, the notation Q_n stands for the n-cube and K_n for the n-clique or n-simplex.

Cube condition. Let q_1, q_2, q_3 be three k-cubes of $\mathbf{X}$ that share a common $(k-2)$-cube q and pairwise share common $(k-1)$-cubes q_{ij}. Note that the vertices of $q_{ij} \smallsetminus q$ span a $(k-2)$-cube and those of $q_i \smallsetminus q_{ij}$ span a $(k-1)$-cube. For a vertex x of q, let x_{ij} be the unique neighbor of x in $q_{ij} \smallsetminus q$. Let x_i be the second common neighbor in q_i of the vertices x_{ij} and x_{ik}; x_i is in $q_i \smallsetminus (q_{ij} \cup q_{ik})$. By the 3-cube condition, there exists a vertex x^* such that $x^* \sim x_1, x_2, x_3$

and $x^* \nsim x_{12}, x_{13}, x_{23}$, and the vertices $x^*, x, x_{12}, x_{13}, x_{23}, x_1, x_2, x_3$ constitute a 3-cube q_x of $\mathbf{X}$. Since $x_2 \in I(x^*, x_{12})$, $x_2 \notin q_1$, and the cube q_1 is convex, $x^* \notin q_1$. For similar reasons, $x^* \notin q_2, q_3$. Now, for another vertex y of q we denote by $y_{12}, y_{13}, y_{23}, y_1, y_2, y_3, y^*$ the vertices defined in the same way as for x and we denote by q_y the 3-cube spanned by these vertices and y. From the definition of these vertices we conclude that all x_i, x_{ij}, y_i, y_{ij} are distinct and for all distinct $i, j \in \{1, 2, 3\}$, $x_{ij} \sim y_{ij}$ as well as $x_i \sim y_i$ hold if and only if $x \sim y$.

Lemma 5.20. *For any $x, y \in q$, for any distinct i, j, k, $x \nsim y_i, y_{ij}$, $x_{ik} \nsim y_i, y_{ij}, y_j$ and $x_i \nsim y_j$.*

Proof. If x (respectively, x_{ik}) is adjacent to y_i or y_{ij}, then since $x \sim x_{ij}$ (respectively, $x_{ik} \sim x_i$), either the cube q_i will contain a triangle or the cube $q_i \setminus q_{ik}$ is not convex, which are impossible. Since $x_{ik} \sim x$ and $x \nsim y_j$, the convexity of q_j ensures that $x_{ik} \nsim y_j$. Finally, the convexity of q_i ensures that $x_i \nsim y_j$, since $y_j \sim y_{ij}$ and $x_i \nsim y_{ij}$. $\qquad\square$

Lemma 5.21. *For any $x, y \in q$, for any distinct i, j, $x^* \nsim y, y_i, y_{ij}$.*

Proof. First suppose by way of contradiction that x^* is adjacent to y or y_{ij}. Since $x^* \notin q_i$, $x^* \sim x_i$, and since $x_i \nsim y, y_{ij}$ by Lemma 5.20, we get a contradiction with the convexity of q_i. Suppose now by way of contradiction that $x^* \sim y_i$. If $x \nsim y$, then $x_i \nsim y_i$ and since both $x_i, y_i \in q_i$ are adjacent to $x^* \notin q_i$, we obtain a contradiction with the convexity of q_i. Now, suppose that $x \sim y$. Then $x_i \sim y_i, x_{ij} \sim y_{ij}$ and the vertices $x_j, x_{ij}, y_{ij}, y_i, x_i, x^*$ define a double-house; by Lemma 5.3, it implies that $x_j \sim y_{ij}$, contradicting Lemma 5.20. Thus, $x^* \nsim y_i$. $\qquad\square$

Lemma 5.22. *The set $\{x^* : x \in q\}$ spans a $(k-2)$-cube q' of $\mathbf{X}$ and the vertices of $q_1 \cup q_2 \cup q_3 \cup q'$ span a $(k+1)$-cube of $\mathbf{X}$.*

Proof. First note that since $y_1 \sim y^*$ and $y_1 \nsim x^*$ by Lemma 5.21, we have that $x^* \neq y^*$. To prove the first assertion of the lemma, since q is a $(k-2)$-cube of $\mathbf{X}$, it suffices to show that $x^* \sim y^*$ if and only if $x \sim y$.

First suppose that x is adjacent to y. Consider the three 2-cubes induced by the 4-cycles $(x_1, x^*, x_2, x_{12}, x_1)$, $(x_1, y_1, y_{12}, x_{12}, x_1)$, and $(x_2, y_2, y_{12}, x_{12}, x_2)$ of $G(\mathbf{X})$. By the 3-cube condition, they are included in a 3-cube of $\mathbf{X}$, i.e., there exists a vertex s adjacent to x^*, y_1, and y_2. Since $(y_1, y_{12}, y_2, y^*, y_1)$ is an induced 4-cycle in the 1-skeleton of the 3-cube q_y, it is also an induced 4-cycle of $G(\mathbf{X})$. Since $G(\mathbf{X})$ does not contain induced $K_{2,3}$, W_4^- or W_4, we conclude that $s = y^*$ or $s = y_{12}$. Since $x^* \sim s$ and $x^* \nsim y_{12}$ from Lemma 5.21, $s = y^*$ and $x^* \sim y^*$. Conversely, suppose that $x^* \sim y^*$ and assume that $x \nsim y$. Then $x_i \nsim y_i$ and $x_{ij} \nsim y_{ij}$. Since $x_i, y_i \in q_i$ and since q_i is convex, we conclude that $d(x_i, y_i) = 2$, (otherwise, (x_i, x^*, y^*, y_i) would be a shortest path from x_i to y_i). Since q_i is a cube, it implies that $d(x, y) = 2$. Let z be a common neighbor of x and y in the cube q and let q_z be the 3-cube spanned by the vertices $z, z_{12}, z_{13}, z_{23}, z_1, z_2, z_3, z^*$. Since $z \sim x, y$, $z_1 \sim x_1, y_1$ and $z^* \sim x^*, y^*$. Consequently, the vertices $x_1, z_1, y_1, y^*, z^*, x^*$ define a double-house, and from Lemma 5.3, it implies that $x_1 \sim y_1$, a contradiction. Therefore, $x^* \sim y^*$ if and only if $x \sim y$, whence q' is a $(k-2)$-cube.

From Lemmas 5.20 and 5.21, and since q' is a $(k-2)$-cube, the vertices of $q_1 \cup q_2 \cup q_3 \cup q'$ span a $(k+1)$-cube of $\mathbf{X}$. $\qquad\square$

Prism condition: Let q be a k-cube intersecting a simplex σ in an edge xy. We will prove that $q \cup \sigma$ is included in a prism of $\mathbf{X}$. Let uv be the unique edge of the k-cube q that is the farthest from xy such that $d(x,u) = d(y,v) = k-1$ and $d(x,v) = d(y,u) = k$. Let $\sigma = \{x, y, w_1, \ldots, w_m\}$. Since q is convex, $d(w_i, u) = d(w_i, v) = k$ for any $i = 1, \ldots, m$.

Lemma 5.23. *If q is a 2-cube (i.e., $k = 1$), then q and σ satisfy the prism condition.*

Proof. By the 3-prism condition, the square $q = xyvu$ together with each triangle xyw_i of σ is included in a 3-prism H_i. Let a_i be the common neighbor in H_i of u, v, and w_i. Then $a_i \neq a_j$ for $i \neq j$, otherwise $w_j \in I(a_i, y)$, contrary to the convexity of H_i. On the other hand, if $a_i \nsim a_j$, by the triangle condition there exists a vertex b adjacent to a_i, a_j, w_j. To avoid a forbidden W_4^- or W_4 induced by a_i, u, v, a_j, b necessarily $b \sim u, v$. But then v, b, a_j, y, w_j induce a forbidden W_4^- or W_4 because $a_j \nsim y$. This shows that $a_i \sim a_j$, i.e., the vertices of σ together with $u, v, a_1, \ldots a_m$ span a prism $K_2 \square K_{m+2}$. $\square$

Now let $k \geq 2$ and proceed by induction on k. Denote by $q' = I(x, u)$ and $q'' = I(y, v)$ the two disjoint $(k-1)$-cubes obtained from q by removing all edges parallel to xy (and uv). Let x' be an arbitrary neighbor of x in q' and let y' be the neighbor of y in q'' such that $xyy'x'$ is a square of q. Finally, let u' be the neighbor of u in q' and v' be the neighbor of v in q'' such that uu' and vv' are parallel in q to xx' and yy'. Then $u' \sim v'$ and $uvv'u'$ is a square of q. Consider the decomposition of the k-cube q with respect to the squares $xyy'x'$ and $uvv'u'$ into four $(k-2)$-cubes $q_x = I(x, u'), q_y = I(y, v'), q_{y'} = I(y', v)$, and $q_{x'} = I(x', u)$. Note that $q_x \cup q_y$ and $q_{x'} \cup q_{y'}$ are two $(k-1)$-cubes constituting q.

By the triangle condition, for each vertex w_i of σ there exists a vertex w_i' adjacent to w_i, x', y'. Since $w_i' \in I(w_i, x') \subset I(w_i, u)$ and $w_i' \in I(w_i, y') \subset I(w_i, v)$, we conclude that $d(w_i', u) = d(w_i', v) = k-1$. By the triangle condition, there exists a vertex a_i adjacent to u and v at distance $k-2$ from w_i' (and at distance $k-1$ from w_i).

Lemma 5.24. *The interval $I(w_i, a_i)$ spans a $(k-1)$-cube q_i such that $q_i \cup q' \cup q''$ is a prism.*

Proof. Since $a_i, u' \in I(u, w_i)$, by the quadrangle condition there exists a vertex $b_i \sim u', a_i$ at distance $k-2$ from w_i. By the 3-prism condition, the square $uu'b_i a_i$ and the triangle $ua_i v$ are included into a 3-prism H. Since H is convex and $v' \in I(u', v)$, necessarily v' belongs to H, whence $b_i \sim v'$. Applying the induction hypothesis to σ and the $(k-1)$-cube $q_x \cup q_y$, we conclude that $I(w_i, b_i)$ spans a $(k-2)$-cube q_i' such that $q_i' \cup q_x \cup q_y$ is a prism $Q_{k-2} \times K_3$. Analogously, applying the induction hypothesis to the 3-simplex $w_i' x' y'$ and the $(k-1)$-cube $q_{x'} \cup q_{y'}$, we conclude that $I(w_i', a_i)$ spans a $(k-2)$-cube q_i'' such that $q_i'' \cup q_{x'} \cup q_{y'}$ is a prism $Q_{k-2} \times K_3$.

We show now that $q_i' \cup q_i''$ is a $(k-1)$-cube. If $q_i' \cap q_i'' \neq \varnothing$, then a simple distance comparison shows that $b_i \in I(w_i', a_i)$. Since $a_i \in I(w_i', u)$ and $u' \in I(b_i, u)$, we obtain that $u' \in I(w_i', u)$, contrary to the assumption that $q_i'' \cup q_{x'}$ is a $(k-1)$-cube. Thus q_i' and q_i'' are disjoint. Now, pick any vertex z of q_i'. Let x_z be the unique neighbor of z in q_x, x_z' be the unique neighbor of x_z in $q_{x'}$, and w_z be the unique neighbor of x_z' in q_i''. We will prove by induction on $r = d(z, b_i)$ that z and w_z are adjacent. If $r = 0$, then we are done because $z = b_i, x_z = u', x_z' = u$, and

$w_z = a_i$. Now, let $r > 0$. Let s be a neighbor of z in the interval $I(z, b_i)$. Since $d(s, b_i) = r - 1$, by the induction assumption the vertex s together with the vertices $x_s \in q_x$, $x'_s \in q_{x'}$, and $w_s \in q''_i$ span a square. Applying the 3-cube condition to this square and the squares $zx_zx_ss, x_zx'_zx'_sx_s$, we conclude that the vertices $z, x_z, x'_z, s, x_s, x'_s, w_s$ are included in a 3-cube. Since this cube is convex and $w_z \in I(x'_z, w_s)$, necessarily w_z belongs to this cube, whence $z \sim w_z$. Finally, we show that w_z is the unique neighbor of z in q''_i. Suppose by way of contradiction that z is adjacent to yet another vertex $t \in q''_i$. Since the cube q''_i is convex, $w_z \sim t$. Let t' be the neighbor of t in $q_{x'}$. Since $t \sim w_z$, necessarily $t' \sim x'_z$. By the 3-prism condition, the square $w_ztt'x'_z$ and the triangle ztw_z are included in a convex 3-prism. Since $x_z \in I(z, x'_z)$, necessarily x_z belongs to this prism, i.e., $x_z \sim t'$. But then the vertex x_z of q_x has two neighbors in the cube $q_{x'}$, contrary to the fact that $q_x \cup q_{x'}$ is a $(k-1)$-cube. This establishes that $q'_i \cup q''_i$ is a $(k-1)$-cube and that $q_i \cup q' \cup q''$ is a prism $Q_{k-1} \square K_3$. □

Lemma 5.25. *If $i \neq j$, then $q_i \cup q_j$ is a k-cube and $q' \cup q'' \cup q_i \cup q_j$ is a prism $Q_{k-1} \square K_4$.*

Proof. First we show that the cubes q_i and q_j are disjoint. If this is not the case and $z \in q_i \cap q_j$, then $d(z, w_i) = d(z, w_j)$. By the triangle condition, there exists a vertex $z_0 \in I(z, w_i) \cap I(z, w_j) \subseteq q_i \cap q_j$ adjacent to w_i and w_j. Since z_0 belongs to q_i and q_j, z_0 has a neighbor $x_0 \in q'$ and a neighbor $y_0 \in q''$ such that $x_0 \sim y_0$ and $x_0 \sim x, y_0 \sim y$. But then the vertices x, y, z_0, x_0, y_0, w_i define a 3-prism, which is not convex because $w_j \in I(x, z_0)$, a contradiction. Hence, the $(k-1)$-cubes q_i and q_j are disjoint.

Now, we show that $a_i \sim a_j$. Suppose by way of contradiction that $a_i \nsim a_j$. Consider the vertices $u' \in q'$ and $v' \in q''$ defined above. Recall that $u' \sim v'$ and $u' \sim u, v' \sim v$. From Lemma 5.24 we know that u' has a unique neighbor b_i in q_i and a unique neighbor b_j in q_j; moreover, $b_i \sim a_i, v'$ and $b_j \sim a_j, v'$. By induction assumption applied to the simplex σ and to the $(k-1)$-cube $q_x \cup q_y$ spanned by the parallel edges xy and $u'v'$, we conclude that $b_i \sim b_j$. Now, applying the case $k = 1$ (Lemma 5.23) to the 4-simplex spanned by b_i, b_j, u', v' and to the 2-cube spanned by u', u, v, v', we will obtain a contradiction. Thus $a_i \sim a_j$.

Finally, we establish that $q_i \cup q_j$ is a k-cube. Pick two adjacent vertices $z' \in q'$ and $z'' \in q''$, and let $x' \in q_i$ and $y' \in q_j$ be their common neighbors. If $z' = u$ and $z'' = v$, then $x' = a_i, y' = a_j$ and $x' \sim y'$ because $a_i \sim a_j$. Otherwise, if $z' \neq u, z'' \neq v$, then $x' \sim y'$ follows from the induction hypothesis applied to σ and the cube spanned by the parallel edges xy and $z'z''$. This shows that indeed $q_i \cup q_j$ is a cube.

Since by Lemma 5.24 $q \cup q_i$ and $q \cup q_j$ are prisms of the form $Q_{k-1} \square K_3$ and $q_i \cup q_j$ is a k-cube, we obtain that $q \cup q_i \cup q_j$ is a prism $Q_{k-1} \square K_4$. □

From Lemma 5.25 we immediately conclude that the vertex set of the union of q with $\cup_{i=1}^m q_i$ spans a prism $Q_k \square K_{m+2}$, thus establishing the prism condition. This also concludes the proof of the implication (ii)&(iii)$\Rightarrow$(i) of Theorem 1 and finishes the proof of Theorem 1.

6. CONTRACTIBILITY AND THE FIXED POINT PROPERTY

In this section, we prove contractibility and the fixed point theorem for finite group actions for locally-finite bucolic complexes. The proofs of both results are based on the fact that in

a locally-finite graph the convex hull of any finite set is finite (this property is no longer true for non-locally-finite bucolic graphs).

6.1. Convex hulls of finite sets.

Proposition 2. *If $G = (V, E)$ is a locally-finite bucolic graph, then the convex hull $\mathrm{conv}(S)$ in G of any finite set $S \subset V$ is finite.*

Proof. By Theorem 2, G is a retract of the (weak) Cartesian product $H = \square_{i \in I} H_i$ of weakly bridged graphs H_i. Each H_i is locally-finite since it is isomorphic to a gated subgraph of G. Note that G is an isometric subgraph of H. For each index $i \in I$, let S_i denote the projection of S in H_i. Since the set S is finite and the distance between any two vertices of S is finite, for all but a finite set I' of indices i the set S_i is a single vertex. Since each set S_i is finite, it is included in a ball of H_i, which is necessarily finite. Since the balls in weakly bridged graphs are convex, we conclude that for each S_i, the convex hull $\mathrm{conv}_{H_i}(S_i)$ of S_i in H_i is finite. The convex hull $\mathrm{conv}_H(S)$ of S in H is the Cartesian product of the convex hulls of the sets $\mathrm{conv}_{H_i}(S_i)$: $\mathrm{conv}_H(S) = \square_{i \in I} \mathrm{conv}_{H_i}(S_i)$. All $\mathrm{conv}_{H_i}(S_i)$ for $i \in I \smallsetminus I'$ are singletons, thus the size of $\mathrm{conv}_H(S)$ equals the size of $\square_{i \in I'} \mathrm{conv}_{H_i}(S_i)$, and thus is finite because I' is finite and each factor $\mathrm{conv}_{H_i}(S_i)$ in this product is finite by what has been shown above.

Since $A := V \cap \mathrm{conv}_H(S)$ is convex in G and it contains the set S, the convex hull of S in G is necessarily included in A. Thus this convex hull is finite, concluding the proof of the proposition. $\square$

Now, we show that Proposition 2 is false for non-locally-finite bucolic graphs. Namely, we present an infinite bridged graph G in which all maximal cliques have size 3 (i.e., the systolic complex whose 1-skeleton is G has dimension 3) and the convex hull of five of its vertices is infinite.

Example. The graph G consists of a graph H of girth 6 (recall that the *girth* of a graph is the length of its smallest cycle) and a vertex c not belonging to H and adjacent to all vertices of H. Obviously G is bridged, has diameter 2, and clique-number 3. The graph H is defined in the following way: it has a set of four pairwise nonadjacent vertices $A = \{a_0, a_1, a_2, a_3\}$ and a one-way infinite path $P = \{b_0, b_1, b_2, \ldots, b_j, \ldots\}$ disjoint from A. In H, b_j is adjacent to a_i if and only if $j = i \mod 4$. For any distinct vertices a_i, a_j, $d_H(a_i, a_j) \geq 3$ and thus any cycle containing a_i and a_j has length at least 6. Any shortest cycle containing only one vertex a_i, has the form $(b_j, b_{j+1}, b_{j+2}, b_{j+3}, b_{j+4}, a_{j \bmod 4}, b_j)$ and has also length at least 6. Thus the girth of H is 6. Now, take the convex hull in G of the 5-point set $A \cup \{b_0\}$. For each j, note that b_j is in the interval $I(b_{j-1}, a_{j \bmod 4})$. Consequently, one can easily show by induction on j that $\mathrm{conv}(A)$ is the whole graph G.

6.2. Contractibility.

Theorem 3. *Locally-finite bucolic complexes are contractible.*

Proof. Let $\mathbf{X}$ be a bucolic complex and let $G = (V, E)$ be its 1-skeleton. Pick any vertex v_0 of G and let $B_k(v_0, G)$ be the ball of radius k centered at v_0. Since G is locally-finite, each

ball $B_k(v_0, G)$ is finite. By Proposition 2 the convex hulls $\mathrm{conv}(B_k(v_0, G)), k \geq 1$, are finite. Hence V is an increasing union of the finite convex sets $\mathrm{conv}(B_k(v_0, G)), k \geq 1$. A subgraph G' of G induced by a convex set of G satisfies the condition (ii) of Theorem 2, thus G' satisfies all other conditions of this theorem, whence G' is bucolic. Hence each subgraph G_k induced by $\mathrm{conv}(B_k(v_0, G))$ is bucolic.

The prism complex $\mathbf{X}$ is an increasing union of the finite bucolic complexes $\mathbf{X}(G_k)$ of the graphs $G_k, k \geq 1$. Thus, to show that $\mathbf{X}$ is contractible, by Whitehead theorem, it suffices to show that each complex $\mathbf{X}(G_k)$ is contractible. By condition (iv) of Theorem 2, the graph G_k can be obtained via Cartesian products of finite weakly bridged graphs using successive gated amalgams. The clique complexes of weakly bridged graphs are exactly the weakly systolic complexes, therefore they are contractible by the results [33]. Cartesian products of contractible topological spaces are contractible, thus the prism complexes resulting from the Cartesian products of prime graphs are contractible. Now, if a graph G' is a gated amalgam of two finite bucolic graphs G_1, G_2 with contractible prism complexes $\mathbf{X}(G_1), \mathbf{X}(G_2)$ along a gated subgraph $G_0 = G_1 \cap G_2$ which also has a contractible prism complex $\mathbf{X}(G_0)$, then by the gluing lemma [9, Lemma 10.3], the prism complex $\mathbf{X}(G')$ of the bucolic graph G' is also contractible. Therefore, for each k, the prism complex $\mathbf{X}(G_k)$ is contractible. This concludes the proof of the contractibility theorem. $\qquad\square$

6.3. **Fixed prism property.**

Theorem 4. *If $\mathbf{X}$ is a locally-finite bucolic complex and F is a finite group acting by cell automorphisms on $\mathbf{X}$, then there exists a prism π of $\mathbf{X}$ which is invariant under the action of F. The center of the prism π is a point fixed by F.*

Proof. Let $\mathbf{X}$ be a bucolic complex and let G denote the 1-skeleton of $\mathbf{X}$. Let F be a finite group acting by cell automorphisms on $\mathbf{X}$ (i.e., any $f \in F$ is a bijection and maps isometrically prisms onto prisms). Then for an arbitrary vertex v of $\mathbf{X}$, its orbit $Fv = \{fv : \ f \in F\}$ is finite. Let G_v be the subgraph of G induced by the convex hull in G of the orbit Fv. Since Fv is finite, the graph G_v is finite by Proposition 2. Moreover, as a convex subgraph of G, G_v satisfies the conditions of Theorem 2(ii), hence G_v is bucolic. Clearly, the prism complex $\mathbf{X}(G_v)$ of G_v is F-invariant. Thus there exists a minimal by inclusion finite non-empty bucolic subgraph $\overline{G}$ of G whose prism complex is F–invariant. We assert that $\mathbf{X}(\overline{G})$ is a single prism, i.e., $\overline{G}$ is the Cartesian product of complete graphs. We prove this assertion in two steps: first we show that $\overline{G}$ is a *box*, (i.e., a Cartesian product of prime graphs), and then we show that each prime graph must be a complete graph. By minimality choice of $\overline{G}$ as an F-invariant bucolic subgraph, we conclude that each proper bucolic subgraph of $\overline{G}$ is not F-invariant. Therefore, the first step of our proof is a direct consequence of the following result.

Proposition 3. *If $\overline{G}$ is a finite bucolic graph, then there exists a box that is invariant under every automorphism of G.*

Proof. If $\overline{G}$ is a box, then the assertion is trivially true. Suppose now that $\overline{G}$ is not a box and assume without loss of generality that each proper bucolic subgraph of $\overline{G}$ is not $Aut(\overline{G})$-invariant. By Theorem 2(iv), $\overline{G}$ is a gated amalgam of two proper nonempty gated subgraphs G' and G'' along a common gated subgraph H_0. Then we say that H_0 is a *gated separator* of $\overline{G}$. Following [10], we will call $U' := G' \smallsetminus H_0$ a *peripheral subgraph* of $\overline{G}$ if U' does not contain any gated separator of $\overline{G}$.

Since $\overline{G}$ is not a box, it contains at least one gated separator, and therefore $\overline{G}$ contains at least one peripheral subgraph (indeed, among all gated separators of $\overline{G}$ it suffices to consider a gated separator H_0 so that $\overline{G}$ is the gated amalgam of G' and G'' along H_0 and G' has minimum size; then $G' \smallsetminus H_0$ is a peripheral subgraph). Let $\mathcal{U} = \{U_i : i \in I\}$ be the family of all peripheral subgraphs of $\overline{G}$, such that G is the gated amalgam of G_i' and G_i'' along the gated separator H_i, where $U_i = G_i' - H_i$ and $G_i'' \neq H_i$. Note that any automorphism f of $\overline{G}$ maps peripheral subgraphs to peripheral subgraphs, thus the subgraph $\bigcup_{i \in I} U_i$ and the subgraph $H = \bigcap_{i \in I} G_i''$ induced by the complement of this union are both $Autov G$)-invariant subgraphs of $\overline{G}$. As an intersection of gated subgraphs of $\overline{G}$, the graph H is either empty or a proper gated subgraph of $\overline{G}$. In the second case, since gated subgraphs of $\overline{G}$ are bucolic, we conclude that H is a proper bucolic $Aut(\overline{G})$-invariant subgraph of $\overline{G}$, contrary to minimality of $\overline{G}$. So, H is empty. By the Helly property for gated sets of a metric space [22], we can find two indices $i, j \in I$ such that the gated subgraphs G_i'' and G_j'' are disjoint. Since $H_i \cap H_j \subseteq G_i'' \cap G_j''$, the gated separators H_i and H_j are disjoint. But in this case, since $U_i = G_i' \smallsetminus H_i$ is peripheral, we conclude that H_j is contained in G_i'' (analogously, H_i is contained in G_j''). Thus $H_i \cup H_j \subseteq G_i'' \cap G_j''$, contrary to the choice of G_i'' and G_j''. This finishes the proof of the proposition. $\square$

Thus $\overline{G}$ is a box, and to finish the proof of Theorem 4 it is enough to show the following.

Proposition 4. *The graph $\overline{G}$ is the Cartesian product of complete graphs, i.e., $\mathbf{X}(\overline{G})$ is a prism.*

Proof. Let $\overline{G} = G_1 \square \cdots \square G_k$, where each factor $G_i, i = 1, \ldots, k$, is a 2-connected finite weakly bridged graph. By [21, Theorem B] every factor G_i is dismantlable. Since dismantlable graphs form a variety (cf. e.g. [32, Theorem 1]), it follows that the strong product $G' = G_1 \boxtimes \cdots \boxtimes G_k$ is dismantlable. Observe that the finite group F acts by automorphisms on G'. By the definition of the strong product, any clique of G' is included in a prism of $\mathbf{X}(G)$. By [34, Theorem A], there exists a clique σ in G' invariant under the action of F. Since F acts by cellular automorphisms on $\mathbf{X}(G)$, it follows that F fixes the minimal prism containing all vertices of σ (treated as vertices of G, and hence of $\mathbf{X}(G)$). By the minimality choice of $\overline{G}$ it follows that $\mathbf{X}(\overline{G})$ is itself a prism. $\square$

This concludes the proof of the fixed prism theorem. $\square$

7. Moorability of weakly bridged graphs

In this section, we extend Theorem 5.1 of [21] and prove that non-locally-finite weakly bridged graphs without infinite cliques are moorable. This result is established in [21] via a LexBFS ordering of vertices, which heavily uses local-finiteness of G. Simple examples show that not every non-locally-finite graph admits a LexBFS ordering. On the other hand, Polat [36] showed that all graphs admit a BFS (breadth-first-search) ordering and, extending the result of [19], he showed that this BFS order provides a mooring of non-locally-finite bridged graphs. In order to circumvent the bottleneck of LexBFS, we refine Polat's definition of BFS and define a well-ordering of the vertices of a graph, which is intermediate between BFS and LexBFS, that we call SimpLexBFS. We show that any (non-locally-finite) graph without infinite cliques admits a SimpLexBFS and that for weakly bridged graphs SimpLexBFS provides a mooring. This will complete the proof of the implication (iii)$\Rightarrow$(i) of Theorem 2.

Definition 7.1. A well-order $\leq$ on the vertex-set $V(G)$ of a graph G is a *SimpLexBFS order* if for every vertex $x \in V(G)$, there exists a mapping $L_x : \{y : y > x\} \to 2^{\{t:t \leq x\}}$ satisfying the following conditions (in what follows we set $L_{(x)}(y) = \bigcup_{t<x} L_t(y)$, for $y \geq x$):

(S1) If $x < y < z$, then $L_x(z) \subseteq L_y(z)$.

(S2) If $x < y$ and $L_{(x)}(x) \neq L_{(x)}(y)$, then $\min_{\leq}\{L_{(x)}(x)\Delta L_{(x)}(y)\} \in L_{(x)}(x)$.

(S3) For $x < y$, we have $L_x(y) = L_{(x)}(y) \cup \{x\}$ if $x \sim y$ and $x \sim t$ for all $t \in L_{(x)}(y)$, and $L_x(y) = L_{(x)}(y)$ otherwise.

Consider a graph G and a SimpLexBFS order $\leq$ on $V(G)$. We now explain how to build a spanning tree using $\leq$. Let u_0 be the least element of $(V(G), \leq)$ and for every vertex $v \neq u_0$, let $f(v) = \min_{\leq}\{u : u \in L_{(v)}(v)\}$; we say that $f(v)$ is the *father* of v and $f : V(G) \to V(G)$ is the *father map* of $\leq$. Note that for every $v \neq u_0$, $f(v) = \min_{\leq}\{u : u \sim v\}$, and thus, $f(v) \leq v$. Since $\leq$ is a well-order, the set of edges $\{vf(v) : v \neq u_0\}$ constitutes a spanning tree of G.

The following lemma provides some basic properties of SimpLexBFS orders and can be easily proved by (transfinite) induction.

Lemma 7.2. *Let $\leq$ be a SimpLexBFS order on $V(G)$, let u_0 be the least element of $(V(G), \leq)$ and let $(L_x)_{x \in V(G)}$ be the corresponding family of mappings. Then the following properties hold:*

(1) $\leq$ is a BFS order, i.e., if $v \leq w$, then $d(v, u_0) \leq d(w, u_0)$;

(2) if $v \neq u_0$, then $d(f(v), u_0) = d(v, u_0) - 1$;

(3) if $f(v) \neq f(w)$, then $v < w$ if and only if $f(v) < f(w)$;

(4) if $w \sim v$, then $f(v) \leq w$.

(5) if $v \leq w$, then $L_{(v)}(w) \cup \{w\}$ is a clique of $V(G)$;

Properties (2)-(4) also hold for all BFS orderings. On the other hand, (5) is the property which distinguishes SimpLexBFS from BFS.

Proposition 5. *If a graph G does not contain infinite cliques and u_0 is an arbitrary vertex of G, then there exists a SimpLexBFS order $\leq$ on $V(G)$ such that u_0 is the least element of $(V(G), \leq)$.*

Proof. We proceed as in the proof of Lemma 3.6 of [36]. Consider an arbitrary well-order $\lhd$ on $V(G)$. We inductively construct a well-order $\le$ on $V(G)$ and a family of mappings $(L_x)_{x \in V(G)}$ satisfying the conditions (S1), (S2), (S3). For every $y \in V(G)$, let $L_{u_0}(y) = \{u_0\}$ if $u_0 \sim y$ and $L_{u_0}(y) = \varnothing$ if $u_0 \nsim y$. Assume that for a set $I \subseteq V(G)$ (including u_0), we have constructed a well-order $\le$ and a family of mappings $(L_x)_{x \in I}$ such that:

(P1) If $x \in I$ and $y \in (V(G) \smallsetminus I) \cup I_{>x}$, then $L_x(y) \subseteq I_{\le x}$, where $I_{>x} := \{t \in I : t > x\}$ and $I_{\le x} := \{t \in I : t \le x\}$.

(P2) If $x, y \in I$ such that $x < y$ and $z \in (V(G) \smallsetminus I) \cup I_>(y)$, then $L_x(z) \subseteq L_y(z)$.

(P3) If $x \in I$, $y \in (V(G) \smallsetminus I) \cup I_{>x}$, and $L_{(x)}(x) \ne L_{(x)}(y)$, then $\min_{\le}\{L_{(x)}(x) \Delta L_{(x)}(y)\} \in L_{(x)}(x)$.

(P4) If $x \in I$ and $y \in (V(G) \smallsetminus I) \cup I_{>x}$, then $L_x(y) = L_{(x)}(y) \cup \{x\}$ if $x \sim y$ and $x \sim t$ for all $t \in L_{(x)}(y)$; and $L_x(y) = L_{(x)}(y)$ otherwise.

(P5) If $y \notin I$, then $L_I(y) \cup \{y\}$ induces a clique of G, where $L_I(y) := \bigcup_{t \in I} L_t(y)$.

If $I = V(G)$, then $\le$ is a SimpLexBFS order on $V(G)$ and we are done. Otherwise, if $I \ne V(G)$, we iteratively define a set L' as follows. Initially, let $L' = \varnothing$ and while there exists $y \in V(G) \smallsetminus I$ such that $L' \subsetneqq L_I(y)$, we add

$$\min_{\le}\{x \in I \smallsetminus L' : \exists y \in V(G) \smallsetminus I \text{ such that } L' \subsetneqq L_I(y)\}$$

to L'. Since by (P5), for each $y \in V(G) \smallsetminus I$, $L_I(y)$ induces a clique of G, and since G does not contain infinite cliques, after a finite number of steps the iteration stops and that there exists $y \in V(G) \smallsetminus I$ such that $L_I(y) = L'$.

Let w be the least element of $(\{y \in V(G) \smallsetminus I : L_I(y) = L'\}, \lhd)$. We extend $\le$ by setting $x < w$ for any $x \in I$. We define L_w as follows: for every $y \notin I \cup \{w\}$, we set $L_w(y) := L_I(y) \cup \{w\}$ if $w \sim y$ and $w \sim t$ for all $t \in L_I(y)$; otherwise, we set $L_w(y) := L_I(y)$. Let $I' := I \cup \{w\}$. To complete the proof of the proposition, it remains to show that I' satisfies the induction properties (P1)-(P5).

For (P1), if $x < w$, then the property holds by the induction hypothesis. If $x = w$, then for every $y \notin I \cup \{w\}$ we have $L_w(y) \subseteq L_I(y) \cup \{w\} \subseteq I'$.

For (P2), if $x < y < w$, then the property holds by the induction hypothesis. If $x < y = w$, then for every $z \notin I'$ we have $L_x(z) \subseteq L_I(z) \subseteq L_w(z)$.

For (P3), if $x < w$, then the property holds by the induction hypothesis. If $x = w$, then for every $y \notin I'$, we have $L_{(w)}(y) = L_I(y)$. By the definition of $L' = L_I(w) = L_{(w)}(w)$, either $L_{(w)}(y) = L'$ or $\min_{\le}\{L' \Delta L_{(w)}(y)\} \in L'$.

For (P4), if $x < w$, then the property holds by the induction hypothesis. If $x = w$, the property holds by the definition of L_w.

For (P5), if $x < w$, then the property holds by the induction hypothesis. If $x = w$, then, by induction hypothesis, $L_I(y) \cup \{y\}$ is a clique of G for every y. If $w \notin L_w(y)$, then $L_w(y) = L_I(y)$ and we are done. If $w \in L_w(y)$, from the definition of $L_w(y)$ it follows that for every $t \in L_I(y) \cup \{y\}$ we have $t \sim w$; consequently, $L_w(y) \cup \{y\} = L_I(y) \cup \{y\} \cup \{w\}$ is a clique of G, and we are done. □

We can now prove the main result of this section.

Proposition 6. *Any weakly bridged graph G without infinite cliques is moorable.*

Proof. We proceed as in the proof of Theorem 5.1 of [21]. Let u_0 be any vertex of G. By Proposition 5, $V(G)$ admits a SimpLexBFS order $\leq$, where u_0 is the least element of $(V(G), \leq)$. Let $(L_x)_{x \in V(G)}$ be the corresponding family of mappings. For every vertex $v \neq u_0$, let $f(v) = \min_{\leq}\{u : u \in L_{(v)}(v)\}$ be the father map of $\leq$.

The following property of weakly bridged graphs immediately follows from the convexity of balls.

Lemma 7.3. *If u, v, v', w are four vertices of a weakly bridged graph G such that $u \sim v, v'$ and $v, v' \in I(u, w)$, then $v \sim v'$.*

We now prove that G satisfies the fellow-traveler property and that f is a mooring of G.

Lemma 7.4. *If $v \sim w$, then either $f(v) = f(w)$ or $f(v) \sim f(w)$; additionally, if $v \leq w$, then either $f(w) = v$ or $f(w) \sim v$. In particular, the father map f is a mooring of G onto u_0.*

Proof. Let $w' = f(w)$ and $v' = f(v)$. To prove the first assertion of the lemma, we proceed by induction on $i + 1 = \max\{d(u_0, v), d(u_0, w)\}$.

Case 1. $d(u_0, v) < d(u_0, w)$.

Since $\leq$ is a BFS order (Lemma 7.2(1)), we have $v \leq w$. By Lemma 7.3, v and w' either coincide or are adjacent. In the first case we are done because v and therefore w' are adjacent to $f(v)$. If v and w' are adjacent, since $i = d(u, v) = d(u, w')$, the vertices v' and $f(w')$ coincide or are adjacent by the induction assumption. Again, if $v' = f(w')$, we are done. Now suppose that v' and $f(w')$ are adjacent. Since $w' = f(w)$, we have $w' \leq v$ (by Lemma 7.2(4)), and by the induction hypothesis, $v' \sim w'$. This concludes the analysis of Case 1.

Case 2. $d(u_0, v) = d(u_0, w) = i + 1$.

Suppose, without loss of generality that $v \leq w$. If the vertices v' and w' coincide, then we are done. If $v' \neq w'$, then $v' \leq w'$ because $v \leq w$, and thus $v' \not\sim w$. If v' and w' are adjacent, then the vertices v, w, w', v' define a 4-cycle. Since G is weakly bridged, this cycle cannot be induced and since $v' \not\sim w$, we have $w' \sim v$. So, assume by way of contradiction that the vertices v' and w' are not adjacent in G. If $v \sim w'$, then $v', w' \in I(v, u_0)$ by Lemma 7.3, and we get $v' \sim w'$, contrary to our assumptions. Consequently, $v' \not\sim w$ and $w' \not\sim v$.

Since G is weakly modular, by $TC(u_0)$, there exists $s \sim v, w$ such that $d(u_0, s) = i$. Denote by S the set of all such vertices s. For every $s \in S$, since $s, v' \in I(v, u_0)$ (respectively, $s, v' \in I(w, u_0)$) and since G is weakly bridged, $s \sim v'$ (respectively, $s \sim w'$). For every $s \in S$, since $f(v) = v'$, $v' \leq s$ and thus $f(v') \leq f(s)$ and $f(s) \sim v'$ by the induction hypothesis. For the same reasons, for every $s \in S$, we have $f(s) \sim w'$. For every $p \sim v', w'$ and any vertex $s \in S$, the cycle (p, v', s, w', p) cannot be induced and thus $p \sim s$.

Claim 2. For every $s \in S$, $L_{(v')}(v') \neq L_{(v')}(s)$.

Proof of Claim 2. Suppose $L_{(v')}(v') = L_{(v')}(s)$. If $L_{(v')}(w') = L_{(v')}(v')$, then we obtain $L_{v'}(s) = L_{(v')}(s) \cup \{v'\} = L_{(v')}(v') \cup \{v'\}$ (since $L_{(v')}(v') \cup \{v'\}$ is a clique and $v' \sim s$) and $L_{v'}(w') = L_{(v')}(v')$ (since $v' \nsim w'$). Consequently, $v' = \min_{\le}\{L_{v'}(w')\Delta L_{v'}(s)\} = \min_{\le}\{L_{(w')}(w')\Delta L_{(w')}(s)\}$ and thus $s < w'$, a contradiction. Otherwise, if $L_{(v')}(w') \ne L_{(v')}(v')$, let $p = \min_{\le}\{L_{(v')}(w')\Delta L_{(v')}(s)\}$. Since $L_{(v')}(v') = L_{(v')}(s)$, we conclude that $p \in L_{(v')}(s)$. Consequently, $p = \min_{\le}\{L_{(w')}(w')\Delta L_{(w')}(s)\}$ and thus $s < w'$, a contradiction. $\square$

Claim 3. Let s_0 be the least vertex of $(S, \le)$ and let $p = \min_{\le}\{L_{(v')}(v')\Delta L_{(v')}(s_0)\}$. Then for every $s \in S$ we have $L_{(p)}(v') = L_{(p)}(w') = L_{(p)}(s)$ and $p \nsim s$.

Proof of Claim 3. By the definition of p, $L_{(p)}(v') = L_{(p)}(s_0)$. If $L_{(p)}(w') \ne L_{(p)}(v')$, then $q = \min_{\le}\{L_{(p)}(v')\Delta L_{(p)}(w')\} \in L_{(p)}(v')$ since $v' \le w'$. Consequently, $\min_{\le}\{L_{(p)}(s_0)\Delta L_{(p)}(w')\} \in L_{(p)}(s_0)$, and hence $s_0 \le w'$, a contradiction.

Thus $L_{(p)}(w') = L_{(p)}(v') = L_{(p)}(s_0)$. For every $s \in S$ and $p' \in L_{(p)}(v') = L_{(p)}(w')$ we have $p' \sim s$. Since $L_{(p)}(v')$ is a clique, we get $L_{(p)}(v') \subseteq L_{(p)}(s)$. Moreover, since $v' < s$, we have $L_{(p)}(v') = L_{(p)}(s)$. Since $p \notin L_p(s_0)$ and $L_p(v) = L_{(p)}(s_0) \cup \{p\}$ is a clique, we conclude that $p \nsim s_0$. If there exists $s_1 \in S$ such that $p \sim s_1$, then $L_p(s_1) = L_{(p)}(s_1) \cup \{p\} = L_p(v)$. In this case, $p = \min_{\le}(L_p(s_1)\Delta L_p(s_0)) \in L_p(s_1)$, and thus $s_1 \le s_0$, contrary to the choice of s_0. $\square$

Let s_0 be the least vertex of $(S, \le)$ and let $s' = f(s_0)$. By the induction assumption, we know that $s' \sim v', w'$ because $v' < w' < s_0$. Moreover, since $d(s', u_0) = i - 1$, we have $s' \nsim v, w$.

Let $p = \min_{\le}\{L_{(v')}(v')\Delta L_{(v')}(s_0)\}$. From Claim 3, $s_0 \nsim p$. Since $p \le s_0$, $d(u_0, p) \le i$, and thus, if $p \sim v$ (respectively, $p \sim w$), then $s_0, p \in I(u_0, v)$ (respectively, $s_0, p \in I(u_0, w)$). By Lemma 7.3, $s_0 \sim p$, a contradiction. If $p \sim w'$, then $p \sim v', w'$, and thus $p \sim s_0$, a contradiction.

Since $s_0 \nsim p$, we conclude that $s' = f(s_0) \ne p$. If $s' < p$, then $f(v') \le s' < p$ and thus $f(v') = \min_{\le} L_p(v') = \min_{\le} L_p(s_0) = f(s_0) = s'$; consequently, $s', p \in L_p(v')$ and thus $p \sim s'$. If $p < s'$, then $p = f(v')$ and by the induction assumption, $s' \sim p$.

Consequently, v, w, v', w', s_0, s' and p induce in G a $\widehat{W_5}$. Thus, by the $\widehat{W_5}$-condition, there exists a vertex $t \sim v, w, v', w', s_0, s', p$. Hence $t \in S$, and $t \sim p$, contradicting Claim 3. This finishes the analysis of Case 2 and concludes the proof of the first assertion of the lemma.

Finally, we claim that the mapping f is a mooring of G onto u_0. Indeed, for every $v \ne u_0$, we have $v \sim f(v)$ and $d(f(v), u_0) = d(v, u_0) - 1$. Moreover, for any edge vw of G, from the first assertion it follows that either $f(v) = f(w)$ or $f(v) \sim f(w)$, i.e., f is indeed a mooring. $\square$

This concludes the proof of Proposition 6. $\square$

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