Asymptotic ensemble stabilizability of the Bloch equation

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Abstract

In this paper we are concerned with the stabilizability to an equilibrium point of an ensemble of non interacting half-spins. We assume that the spins are immersed in a static magnetic field, with dispersion in the Larmor frequency, and are controlled by a time varying transverse field. Our goal is to steer the whole ensemble to the uniform "down" position.

Two cases are addressed: for a finite ensemble of spins, we provide a control function (in feedback form) that asymptotically stabilizes the ensemble in the "down" position, generically with respect to the initial condition. For an ensemble containing a countable number of spins, we construct a sequence of control functions such that the sequence of the corresponding solutions pointwise converges, asymptotically in time, to the target state, generically with respect to the initial conditions.

The control functions proposed are uniformly bounded and continuous.

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1. Introduction

Ensemble controllability (also called *simultaneous controllability*) is a notion introduced in [1, 2, 3] for quantum systems described by a family of parameter-dependent ordinary differential equations; it concerns the possibility of finding control functions that compensate the dispersion in the parameters and drive the whole family (ensemble) from some initial state to some prescribed target state.

Such an issue is motivated by recent engineering applications, such as, for instance, quantum control (see for instance [3, 4, 5, 6] and references therein), distributed parameters systems and PDEs [7, 8, 9, 10, 11], and flocks of identical systems [12].

General results for the ensemble controllability of linear and nonlinear systems, in continuous and discrete time, can be found in the recent papers [13, 14, 15, 16, 17].

This paper deals with the simultaneous control of an ensemble of half-spins immersed on a magnetic field, where each spin is described by a magnetization vector $\boldsymbol{M} \in \mathbb{R}^3$, subject to the dynamics $\frac{d\boldsymbol{M}}{dt} = -\gamma \boldsymbol{M} \times \boldsymbol{B}(\boldsymbol{r}, t)$, where $\boldsymbol{B}(\boldsymbol{r}, t)$ is a magnetic field composed by a static component directed along the z-axis, and a time varying component on the xy-plane, called *radio-frequency (rf) field*, and γ denotes the gyromagnetic ratios of the spins. In this system, since all spins are controlled by the same magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$, spatial dispersion in the amplitude of the magnetic field gives rise to the following inhomogeneities in the dynamics: *rf inhomogeneity*, caused by dispersion in the radio-frequency field, and a spread in the Larmor frequency, given by dispersion of the static component of the field. This problem arises, for instance, in NMR spectroscopy (see [18] and references in [19, 3, 4]).

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The task of controlling such system is wide, multi-faceted and very rich, depending on the cardinality of the set of the spin to be controlled (and the topology of this set), on the particular notion of controllability addressed, and on the functional space where control functions live.

The above-cited articles [1, 2, 3] are concerned with both rf inhomogeneity and Larmor dispersion, with dispersion parameters that belong to some compact domain $\mathcal{D}$. The magnetization vector of the system is thus a function on $\mathcal{D}$, taking values in the unit sphere of $\mathbb{R}^3$, and ensemble controllability has to be intended as convergence in the $L^\infty(\mathcal{D}, \mathbb{R}^3)$-norm. The controllability result is achieved by means of Lie algebraic techniques coupled with adiabatic evolution, and holds for both bounded and unbounded controls.

In [4], the authors focus on systems subject to Larmor dispersions, and provide a complete analysis of controllability properties of the ensemble in different scenarios, such as: bounded/unbounded controls; finite time/asymptotic controllability; approximate/exact controllability in the $L^2(\mathcal{D}, \mathbb{R}^3)$ norm; boundedness/unboundedness of the set $\mathcal{D}$. In particular, results on exact local controllability with unbounded controls are provided.

In this paper we consider an ensemble of Bloch equations presenting Larmor dispersion, with frequencies belonging to some possibly unbounded subset $\mathcal{E} \subset \mathbb{R}$. Coupling a Lyapunov function approach with some tools of dynamical systems theory, we exhibit a control function (in feedback form) that approximately drives, asymptotically in time and generically with respect to the initial conditions, all spins to the "down" position. Two cases are addressed: if the set $\mathcal{E}$ is finite, our strategy provides exact exponential stabilizability in infinite time, while in the case where $\mathcal{E}$ is a countable collection of energies, our approach implies asymptotic pointwise convergence towards the target state.

Feedback control is a widely used tool for stabilization of control-affine systems (see for instance [20, 21] and references therein).

Concerning the stabilization of ensembles, we mention two papers using this approach: in [19], the author aims at stabilizing an ensemble of interacting spins along a reference trajectory; the result is achieved by showing, by means of Lie-algebraic methods, that the distance between the state of the system and the target trajectory is a Lyapunov function. In [22], Jurdjevic-Quinn conditions are applied to stabilize an ensemble of harmonic oscillator.

The feedback form of the control guarantees more robustness with respect to open-loop controls, and gives rise to a continuous bounded control, more easy to implement in practical situations. We stress that, in the finite dimensional case, the implementation of the control requires the knowledge of the *bulk magnetization* of all spin, which is accessible through classical measurements (see for instance [23, 19]). We finally remark that the control proposed in this paper is very similar to the *radiation damping effect* arising in NMR (see [24, 25]); we comment this fact in the conclusion.

The structure of the paper is the following: in Section 2 we state the problem in general form; in Section 3 we tackle the finite dimensional case, while in Section 4 we analyze the case of a countable family of systems. Section 5 is devoted to some numerical results.

2. Statement of the problem

We consider an ensemble of non-interacting spins immersed in a static magnetic field of strength $B_0(\boldsymbol{r})$, directed along the z-axis, and a time varying transverse field $(B_x(t), B_y(t), 0)$ (rf field), that we can control. Bloch equation for this system takes then the form

$$\frac{\partial \boldsymbol{M}}{\partial t}(\boldsymbol{r}, t) = \begin{pmatrix} 0 & -B_0(\boldsymbol{r}) & B_y(t) \\ B_0(\boldsymbol{r}) & 0 & -B_x(t) \\ -B_y(t) & B_x(t) & 0 \end{pmatrix} \boldsymbol{M}((\boldsymbol{r}), t) \tag{1}$$

(here for simplicity we set $\gamma = 1$). For more details, we mention the monograph [26].

Since the dependence on the spatial coordinate $\boldsymbol{r}$ appears only in $B_0(\boldsymbol{r})$, we can represent $M(\boldsymbol{r}, t)$ as a collection of time-dependent vectors $X_e(t) = (x_e(t), y_e(t), z_e(t))$, where $e = B_0(\boldsymbol{r})$, each one belonging to the unit sphere $S^2 \subset \mathbb{R}^3$ and subject to the law

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{z}_e \end{pmatrix} = \begin{pmatrix} 0 & -e & u_2 \\ e & 0 & u_1 \\ -u_2 & -u_1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \end{pmatrix}, \tag{2}$$

with $u_1(t) = -B_x(t)$ and $u_2(t) = B_y(t)$. The Larmor frequencies e of the spins in the ensemble take value in some subset $\mathcal{E} \subset I$ of an interval I [1]. Depending on the spatial distribution of the spins, $\mathcal{E}$ could be a finite set, an infinite countable set, or an interval.

We are concerned with the following control problem:

(P) *Design a control function* $\mathbf{u} : [0, +\infty) \to \mathbb{R}^2$ *such that for every* $e \in \mathcal{E}$ *the solution of equation* (2) *is driven to* $X_e = (0, 0, -1)$.

To face this problem, we consider the Cartesian product $\boldsymbol{S} = \prod_{e \in \mathcal{E}} S^2$, whose elements are the collections $\boldsymbol{X} = \{X_e\}_{e \in \mathcal{E}}$ such that $X_e \in S^2$ for every $e \in \mathcal{E}$. Depending on the structure of $\mathcal{E}$, $\boldsymbol{X}$ can be a finite or an infinite countable collection of states $X_e \in S^2$, or a function $\boldsymbol{X} : \mathcal{E} \to S^2$ belonging to some functional space. The collection $\boldsymbol{X}$ of magnetic moments evolves according to the equation

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}, \mathbf{u}), \tag{3}$$

where $\boldsymbol{F}$ denotes the collection $\boldsymbol{F} = \{F_e\}_{e \in \mathcal{E}}$ of tangent vectors to S^2, with $F_e(\boldsymbol{X}, \mathbf{u}) = \begin{pmatrix} 0 & -e & u_2 \\ e & 0 & u_1 \\ -u_2 & -u_1 & 0 \end{pmatrix} X_e$, and $\mathbf{u} = (u_1, u_2)$.

Some remarks on the existence of solutions for equation (3) are in order, and will be provided case by case. Assuming that these issues are already fixed, we define the two states $\boldsymbol{X}^+ = \{X_e : X_e = (0, 0, 1)\ \forall e \in \mathcal{E}\}$ and $\boldsymbol{X}^- = \{X_e : X_e = (0, 0, -1)\ \forall e \in \mathcal{E}\}$, and rewrite the problem **(P)** as

(P') *Design a control function* $\mathbf{u} : [0, +\infty) \to \mathbb{R}^2$ *such that the solution of equation* (3) *is driven to* $\boldsymbol{X} = \boldsymbol{X}^-$.

We remark that the notion of convergence of $\boldsymbol{X}(\cdot)$ towards $\boldsymbol{X}^-$ in problem **(P')** has to be specified case by case, depending on the structure of the set $\mathcal{E}$ and on the topology of $\boldsymbol{S}$.

3. Finite dimensional case

First of all, we consider the case in which the set $\mathcal{E}$ is a finite collection of pairwise distinct energies, that is $\mathcal{E} = (e_1, \ldots, e_p)$ such that $e_k \in I$, $k = 1, \ldots, p$ and $e_k \neq e_j$ if $i \neq j$. We recall that the state space $\boldsymbol{S}$ of the system is the finite product of p copies of S^2.

Lemma 1. *Assume that all energy levels e_i are pairwise distinct. Let $\mathcal{I} = \{\boldsymbol{X} \in \boldsymbol{S} : x_{e_i} = y_{e_i} = 0\ \forall i = 1, \ldots, p\}$. Then every solution of the the control system* (2) *with control*

$$\begin{cases} u_1 = \sum_{i=1}^p y_{e_i} \\ u_2 = \sum_{i=1}^p x_{e_i} \end{cases} \tag{4}$$

tends to $\mathcal{I}$ as $t \to +\infty$.

Proof. Consider the function $V(\boldsymbol{X}) = \sum_{i=1}^p z_{e_i}$, and let $\Xi(\cdot)$ be a solution of (3) with the control given in (4). We notice that $\dot{V}(\Xi(t)) = -\left(\sum_{i=1}^p x_{e_i}\right)^2 - \left(\sum_{i=1}^p y_{e_i}\right)^2$, therefore it is non-positive on the whole $\boldsymbol{S}$, and it is zero only on the set $\mathcal{M} = \{\boldsymbol{X} \in \boldsymbol{S} : \sum_{i=1}^p x_{e_i} = \sum_{i=1}^p y_{e_i} = 0\}$. We can then apply La Salle invariance principle to conclude that, for every initial condition, $\Xi(t)$ tends to the largest invariant subset of $\mathcal{M}$.

[1] for simplicity in the exposition, here we will consider the interval I bounded. The results holds true also if I is unbounded

Consider a trajectory $\Xi(\cdot)$ entirely contained in $\mathcal{M}$. Since $\mathbf{u}=0$, then for every i we have that

$$\Xi_i(t) = \begin{pmatrix} \cos(e_i t) & -\sin(e_i t) & 0 \\ \sin(e_i t) & \cos(e_i t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{e_i}(0) \\ y_{e_i}(0) \\ z_{e_i}(0) \end{pmatrix}.$$

By definition, for every $t \geq 0$ it holds $\sum_{i=1}^{p} x_{e_i}(t) = \sum_{i=1}^{p} y_{e_i}(t) = 0$. Differentiating these equalities $p-1$ times and evaluating at $t=0$ we obtain the two conditions

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ e_1 & e_2 & \dots & e_p \\ \vdots & \vdots & & \\ e_1^{p-1} & e_2^{p-1} & \dots & e_p^{p-1} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & \dots & 1 \\ e_1 & e_2 & \dots & e_p \\ \vdots & \vdots & & \\ e_1^{p-1} & e_2^{p-1} & \dots & e_p^{p-1} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The determinant of the Vandermonde matrix here above is given by $\prod_{1\leq i<j\leq p}(e_i - e_j)$, which is non-zero under the assumptions. Therefore the two equations are satisfied if and only if $\Xi(0) \in \mathcal{I}$. It is immediate to see that $\mathcal{I}$ is the largest invariant subset of $\mathcal{M}$. $\square$

The set $\mathcal{I}$ is composed by a collection of 2^p isolated points $\boldsymbol{Q}^k = (Q_1^k, \dots, Q_p^k)$, $k = 1, \dots, 2^p$, where $Q_j^k = (0, 0, \alpha_j^k)$ and $|\alpha_j^k| = 1$. These points are equilibria for the closed-loop system (2)-(4). We distinguish three cases:

- if $\alpha_j^k = -1$ for every j, then $\boldsymbol{Q}^k = \boldsymbol{X}^-$, and it is an asymptotically stable equilibrium for the system;
- if $\alpha_j^k = 1$ for every j, then $\boldsymbol{Q}^k = \boldsymbol{X}^+$ is an unstable equilibrium for the system; in particular, $\boldsymbol{X}^+$ is a repeller;
- all other points in $\mathcal{I}$ are neither attractor neither repellers, since each of these points is a saddle-point of V.

These facts will be proved in the next section (see Proposition 1, Lemma 5 and Remark 2). We end this one recalling the following property of the basin of attraction.

Lemma 2. *Let $\mathcal{B}$ be the basin of attraction of $\boldsymbol{X}^-$. Then $\mathcal{B}$ is an open neighborhood of $\boldsymbol{X}^-$ and, in the case $p > 1$, there exists at least one $\boldsymbol{Q} \in \mathcal{I} \setminus \{\boldsymbol{X}^+, \boldsymbol{X}^-\}$ such that $\boldsymbol{Q} \in \partial\mathcal{B}$.*

3.1. Linearized system

In order to study the structure of the basin of attraction $\mathcal{B}$, we linearize the system (3)-(4) around a point $\boldsymbol{Q} \in \mathcal{I}$, and we study the corresponding eigenvalues. We will show below that the linearized system is always hyperbolic (when we consider its restriction to the tangent space to the collection of spheres).

The linearization gives

$$\begin{pmatrix} \dot{\boldsymbol{\delta x}} \\ \dot{\boldsymbol{\delta y}} \\ \dot{\boldsymbol{\delta z}} \end{pmatrix} = \begin{pmatrix} K_{\boldsymbol{Q}} & -E & 0 \\ E & K_{\boldsymbol{Q}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\delta x} \\ \boldsymbol{\delta y} \\ \boldsymbol{\delta z} \end{pmatrix} \tag{5}$$

where

$$K_{\boldsymbol{Q}} = \begin{pmatrix} z_1^{\boldsymbol{Q}} & z_1^{\boldsymbol{Q}} & \dots & z_1^{\boldsymbol{Q}} \\ z_2^{\boldsymbol{Q}} & z_2^{\boldsymbol{Q}} & \dots & z_2^{\boldsymbol{Q}} \\ \vdots & \vdots & & \\ z_p^{\boldsymbol{Q}} & z_p^{\boldsymbol{Q}} & \dots & z_p^{\boldsymbol{Q}} \end{pmatrix} \qquad E = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & e_p \end{pmatrix}$$

and $z_i^{\boldsymbol{Q}}$ is the value of the coordinate z_i at the point $\boldsymbol{Q}$. Set moreover $M_{\boldsymbol{Q}} = \left(\begin{smallmatrix} K_{\boldsymbol{Q}} & -E \\ E & K_{\boldsymbol{Q}} \end{smallmatrix}\right)$. Notice that we can write $K_{\boldsymbol{Q}} = \kappa_{\boldsymbol{Q}} \zeta^T$, where $\kappa_{\boldsymbol{Q}} = (z_1^{\boldsymbol{Q}}, \dots, z_p^{\boldsymbol{Q}})^T$ and $\zeta = (1, 1, \dots, 1)^T$, then $\mathrm{rank} K_{\boldsymbol{Q}} = 1$.

In the following, with a little abuse of notation, we will remove the dependence on $\boldsymbol{Q}$ from K, M, κ and its components, specifying it only when necessary.

In order to compute the eigenvalues of the matrix M, we consider the complexification of system (5), that is we set $\xi = \boldsymbol{\delta x} + i\boldsymbol{\delta y}$, observing that $\dot{\xi} = (K + iE)\xi$. It is easy to see that ℓ is an eigenvalue of M if and only if it is also either an eigenvalue of $(K + iE)$ or an eigenvalue of $(K - iE)$, that is, the spectrum of M is equal to the union of the spectra of $(K + iE)$ and $(K - iE)$.

Properties In the following, we will use the following properties of block matrices

(P1) Let M be the block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. If A is invertible, then $\det M = \det(A)\det(D - CA^{-1}B)$. If D is invertible, then $\det M = \det(D)\det(A - BD^{-1}C)$.

(P2) Let A be an invertible matrix of size n, and x, y two n-dimensional vectors. Then $\det(A + xy^T) = \det(A)(1 + y^T A^{-1}x)$.

Lemma 3. *The matrices $(K + iE)$ and $(K - iE)$ are invertible.*

Proof. Assume that E is invertible. Then $\det(K + iE) = \det(iE)\det(\mathbb{1} - iE^{-1}K)$, where $\mathbb{1}$ denotes the n-dimensional identity matrix; since $-iE^{-1}K = -iE^{-1}\kappa\zeta^T = \widetilde{\kappa}\zeta^T$, by **(P2)** we have that $\det(\mathbb{1} + \widetilde{\kappa}\zeta^T) = (1 + \zeta^T\widetilde{\kappa}) \neq 0$, since $\zeta^T\widetilde{\kappa}$ is purely imaginary.

If E is not invertible, up to permutations and relabeling we assume that $e_1 = 0$. We suitably add or subtract the first row of K to all other ones, in order to get that

$$\det(K + iE) = \det \begin{pmatrix} z_1 & z_1 & \dots & z_1 \\ 0 & ie_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & ie_p \end{pmatrix} = (i)^{p-1} z_1 e_2 \cdots e_p.$$

The same arguments prove that $(K - iE)$ is invertible. □

Lemma 4. *For every $i = 1, \dots, p$ the matrices $(K + iE + ie_i\mathbb{1})$, $(K + iE - ie_i\mathbb{1})$, $(K - iE + ie_i\mathbb{1})$ and $(K - iE - ie_i\mathbb{1})$ are invertible.*

Proof. By contradiction, assume that ie_i is an eigenvalue of $K + iE$, that is there exists a vector $v \in \mathbb{C}^p$ such that $(K + iE)v = ie_i v$, that is $Kv = (i(e_i - e_1)v_1, \dots, i(e_i - e_p)v_p)^T$; since $Kv = \kappa(\zeta^T v)$ and all the components of κ are different from zero, this implies that $\zeta^T v = 0$ and therefore $Kv = 0$. Then $i(E - e_i\mathbb{1})v = 0$, that is $v_j = 0$ for every $j \neq i$, therefore $(Kv)_l = z_l \sum_{k=1}^p v_k = z_l v_i$ for every l. Since $Kv = 0$, then $v = 0$. Then ie_i cannot be an eigenvalue of $K + iE$.

Assume now that $-ie_i$ is an eigenvalue of $K + iE$. Analogously, we have that there exists $v \in \mathbb{C}^p$ such that $(Kv)_j = -i(e_i + e_j)v_j$ for every j. But as above $(Kv)_j = z_j(\sum_{l=1}^p v_l)$. If there exists some j such that $e_i + e_j = 0$, reasoning as above we can prove that $v = 0$. If not, we can write $v_j = \frac{iz_j}{e_i + e_j}\sum_{l=1}^p v_l$ for every j. Then $\sum_{j=1}^p v_j = i\sum_{j=1}^p \frac{z_j}{e_i + e_j}\sum_{l=1}^p v_l$, that is $i\sum_{j=1}^p \frac{z_j}{e_i + e_j} = 1$, which is a contradiction, since all z_{e_i} and e_j are real.

Analogous computations prove that also $(K - iE - ie_i\mathbb{1})$ and $(K - iE + ie_i\mathbb{1})$ are invertible for every $i = 1, \dots, p$. □

Remark 1. *In particular, 0 and $\pm ie_i$, $i = 1, \dots, p$, are not eigenvalues of M.*

We are now ready to prove the hyperbolicity of every fixed point of the linearized system (5).

Proposition 1. *For every $\boldsymbol{Q} \in \mathcal{I}$, all eigenvalues of the matrix $M_{\boldsymbol{Q}}$ have non-zero real part.*

Proof. Assume by contradiction that $i\ell$, $\ell \in \mathbb{R}$, is an eigenvalue of M relative to the eigenvector $(X, Y)^T$ (where $X, Y \in \mathbb{C}^p$), and assume that $X + iY \neq 0$ (if this is the case, then $X - iY \neq 0$ and we can repeat the same argument below with $(K - iE)$). Then $X + iY$ is an eigenvector of

$(K+iE)$ relative to $i\ell$, and we remark that, by Lemma 4, ℓ is different from every of the e_i. Let $\chi, \eta \in \mathbb{R}^p$ be the real vectors such that $X + iY = \chi + i\eta$. Straight computations show that

$$\begin{cases} K\chi - E\eta = -\ell\eta \\ K\eta + E\chi = \ell\chi \end{cases} \Rightarrow \begin{cases} (E-\ell\mathbb{1})\eta = K\chi = (\zeta^T\chi)\kappa \\ (E-\ell\mathbb{1})\chi = -K\eta = -(\zeta^T\eta)\kappa. \end{cases}$$

Since $(E-\ell\mathbb{1})$ is invertible, we have that

$$\chi = -(\zeta^T\eta)(E-\ell\mathbb{1})^{-1}\kappa$$
$$\eta = (\zeta^T\chi)(E-\ell\mathbb{1})^{-1}\kappa,$$

that is, η and χ are parallel and $X+iY = (a+ib)(E-\ell\mathbb{1})^{-1}\kappa$, for some real coefficients a, b. Then $(E-\ell\mathbb{1})^{-1}\kappa$ is a real eigenvector of $(K+iE)$ relative to $i\ell$, which implies that $K(E-\ell\mathbb{1})^{-1}\kappa = 0$ and $E(E-\ell\mathbb{1})^{-1}\kappa = \ell(E-\ell\mathbb{1})^{-1}\kappa$, which is possible only if κ is null. □

Lemma 5. *Let ℓ be an eigenvalue of $K_{\boldsymbol{Q}} + iE$. Then the following equality holds*

$$\sum_{j=1}^{p} \frac{z_j^{\boldsymbol{Q}}(\lambda + i(\mu - e_j))}{\lambda^2 + (e_j-\mu)^2} = 1, \tag{6}$$

where λ and μ denote respectively the real and the imaginary part of ℓ. In particular, all the eigenvalues of $K_{\boldsymbol{X}^+} + iE$ have positive real part and all the eigenvalues of $K_{\boldsymbol{X}^-} + iE$ have negative real part.

Proof. Thank to property **(P2)** and the fact that ℓ is not an eigenvalue of iE, it holds

$$\det(K + iE - \ell\mathbb{1}) = \det(iE - \ell\mathbb{1})(1 + \zeta^T(iE-\ell\mathbb{1})^{-1}\kappa)$$
$$= \det(iE-\ell\mathbb{1})\Big(1 + \sum_{j=1}^{p} \frac{z_j}{ie_j - \ell}\Big).$$

Equation (6) follows from $1 + \sum_{j=1}^{p} \frac{z_j}{ie_j - \ell} = 0$.

In particular, since $z_i^{\boldsymbol{X}^-} = -1$ for every i, for every eigenvalue ℓ of $K_{\boldsymbol{X}^-} + iE$ equation (6) reads

$$\begin{cases} \sum_{j=1}^{p} \frac{\lambda}{\lambda^2 + (e_j-\mu)^2} = -1 \\ \sum_{j=1}^{p} \frac{(\mu - e_j)}{\lambda^2+(e_j-\mu)^2} = 0 \end{cases},$$

which implies $\lambda < 0$. The same argument proves that all eigenvalues of $K_{\boldsymbol{X}^+} + iE$ have positive real part. □

Analogous computations show that every eigenvalue $\ell = \lambda + i\mu$ of $K_{\boldsymbol{Q}} - iE$ satisfies the equation

$$\sum_{j=1}^{p} \frac{z_j^{\boldsymbol{Q}}(\lambda - i(\mu - e_j))}{\lambda^2 + (e_j-\mu)^2} = 1,$$

and that all eigenvalues of $K_{\boldsymbol{X}^-} - iE$ (respectively, $K_{\boldsymbol{X}^+} - iE$) have negative (respectively, positive) real part.

Let us now consider the linearized flow in the tangent space to $\boldsymbol{S}$ at some $\boldsymbol{Q} \in \mathcal{I}$. First of all, we notice that $T_{\boldsymbol{Q}}\boldsymbol{S} = \{(\boldsymbol{\delta x}, \boldsymbol{\delta y}, 0)\}$ for every $\boldsymbol{Q} \in \mathcal{I}$, therefore the linearization of the flow ϕ^t on $T_{\boldsymbol{Q}}\boldsymbol{S}$ can be represented by the matrix M. In particular, Proposition 1 implies that each $\boldsymbol{Q} \in \mathcal{I}$ is a hyperbolic equilibrium for the flow ϕ^t (restricted to $\boldsymbol{S}$).

Remark 2. *For every $\boldsymbol{Q} \neq \boldsymbol{X}^-$, then at least two eigenvalues of $D_{\boldsymbol{Q}}\phi^t$ have positive real part. Indeed, Proposition 1 implies that all eigenvalues have non-zero real part; if all eigenvalue $D_{\boldsymbol{Q}}\phi^t$ had negative real part, there would be a contradiction with the fact none of the $\boldsymbol{Q} \in \mathcal{I} \setminus \{\boldsymbol{X}_-\}$ is a local minimum of the Lyapunov function, that is, none of these equilibria is stable. Since the eigenvalues $D_{\boldsymbol{Q}}\phi^t$ come in conjugate pairs, at least two must have positive real part.*

Theorem 1. *There exists an open dense set $\boldsymbol{G}^p \subset \boldsymbol{S}$ such that for every $\boldsymbol{X}_0 \in \boldsymbol{G}^p$ the solution of the control system* (2)*-*(4) *with initial condition $\boldsymbol{X}_0$ tends asymptotically to $\boldsymbol{X}^-$, with exponential velocity.*

Proof. For $p = 1$, then the basin of attraction of $\boldsymbol{X}^-$ is trivially $S^2 \setminus \boldsymbol{X}^+$. Let us then assume that $p > 1$.

Consider $\boldsymbol{Q} \in \mathcal{I} \setminus (\boldsymbol{X}^+ \cup \boldsymbol{X}^-)$. From Proposition 1, Lemma 5 and Remark 2, we know that the restriction of $D_{\boldsymbol{Q}}\phi^t$ to $T_{\boldsymbol{Q}}\boldsymbol{S}$ satisfies the following properties: there exists a splitting of the tangent space $T_{\boldsymbol{Q}}\boldsymbol{S} = E^-_{\boldsymbol{Q}} \oplus E^+_{\boldsymbol{Q}}$ such that

- there exists $\rho_+ > 1$ such that $\|D\phi^{-t}|_{E^+_{\boldsymbol{Q}}}\| \le \rho_+^{-t}$ and $\dim E^+_{\boldsymbol{Q}} \ge 2$
- there exists $\rho_- < 1$ such that $\|D\phi^{t}|_{E^-_{\boldsymbol{Q}}}\| \le \rho_-^{t}$ and $\dim E^-_{\boldsymbol{Q}} \ge 2$.

Then we can apply Hadamard-Perron Theorem [27] and conclude that there exist two $\mathcal{C}^1$-smooth injectively immersed submanifolds $W^s_{\boldsymbol{Q}}, W^u_{\boldsymbol{Q}} \subset \boldsymbol{S}$ such that

$$W^s_{\boldsymbol{Q}} = \{X \in \boldsymbol{S} : \mathrm{dist}(\phi^t(X), \boldsymbol{Q}) \to 0 \text{ as } t \to +\infty\} \quad \text{and} \quad T_{\boldsymbol{Q}}W^s_{\boldsymbol{Q}} = E^-_{\boldsymbol{Q}} \tag{7}$$

$$W^u_{\boldsymbol{Q}} = \{X \in \boldsymbol{S} : \mathrm{dist}(\phi^{-t}(X), \boldsymbol{Q}) \to 0 \text{ as } t \to +\infty\} \quad \text{and} \quad T_{\boldsymbol{Q}}W^u_{\boldsymbol{Q}} = E^+_{\boldsymbol{Q}}. \tag{8}$$

We recall that every point in $\boldsymbol{S}$ asymptotically reaches $\mathcal{I}$, under the action of the flow ϕ^t. Therefore, the set of all points that do not asymptotically reach $\boldsymbol{X}^-$ is

$$\mathcal{Q} = \bigcup_{\boldsymbol{Q} \in \mathcal{I} \setminus \{\boldsymbol{X}^+ \cup \boldsymbol{X}^-\}} W^s_{\boldsymbol{Q}} \cup \{\boldsymbol{X}^+\}.$$

Set $\boldsymbol{G}^p = \boldsymbol{S} \setminus \mathcal{Q}$, and notice that $\mathcal{Q}$ is a finite union of smooth manifolds of codimension at least 2. This implies that its complement is dense.

Let $\boldsymbol{X}_k$ be a sequence in $\mathcal{Q}$, converging to some $\bar{\boldsymbol{X}} \in \boldsymbol{S}$. By continuity with respect to initial conditions, for every $\epsilon > 0$ and every $T > 0$ there exists $\bar{k}$ such that if $k \ge \bar{k}$, then $|\phi^T(\boldsymbol{X}_k) - \phi^T(\bar{\boldsymbol{X}})| \le \epsilon$, which implies, by smoothness of the Lyapunov function V, that $|V(\phi^T(\boldsymbol{X}_k)) - V(\phi^T(\bar{\boldsymbol{X}}))| \le L\epsilon$, for some $L > 0$.

Since for every t and every k it holds $V(\phi^t(\boldsymbol{X}_k)) \ge \bar{V}$, where $\bar{V} = \min_{\mathcal{I} \setminus \{\boldsymbol{X}^-\}} V$, then we can conclude that for every $\epsilon > 0$ and $T > 0$ we can find $\bar{k}$ such that $V(\phi^T(\bar{\boldsymbol{X}})) \ge V(\phi^T(\boldsymbol{X}_{\bar{k}})) - \epsilon \ge \bar{V} - \epsilon$. Then

$$\phi^t(\bar{\boldsymbol{X}}) \to \mathcal{I} \setminus \{\boldsymbol{X}^-\},$$

that is

$$\overline{\mathcal{Q}} \subset \bigcup_{\boldsymbol{Q} \in \mathcal{I} \setminus \{\boldsymbol{X}^-\}} W^s_{\boldsymbol{Q}} = \mathcal{Q}.$$

□

Remark 3. *Theorem 1 states that the set of "bad" initial conditions - that is, the set of initial condition not converging to the state $\boldsymbol{X}^-$ - is given by the union of the unstable equilibria and of their corresponding stable manifold. In the single spin case, the "bad set" reduces to the stable equilibrium $\boldsymbol{X}^+$, as already pointed out in [19], where a similar feedback control is applied for stabilizing a set of interacting spins.*

4. Countable case

4.1. Existence of solutions

Let us now assume that $\mathcal{E} = \{e_k\}_{k \in \mathbb{N}^+}$ is a sequence of pairwise distinct elements contained in I. The state of the system is represented by the sequence $\boldsymbol{X} = \{X_{e_k}\}_k$, with $X_k \in S^2$, and the state space is the countable Cartesian product $\boldsymbol{S} = \Pi_{k=1}^{\infty} S^2_{e_k}$.

Before trying to solve the problem **(P')**, it is necessary to discuss its well-posedness. To do this, let us consider the function $\boldsymbol{d}$ on the infinite Cartesian product $\boldsymbol{\Pi}^\infty\mathbb{R}^3$:

$$\boldsymbol{d}(\boldsymbol{X},\boldsymbol{X}')=\sum_{k=1}^{\infty} w_k|X_k-X_k'|,$$

where $|X_k-X_k'|^2=|x_{e_k}-x'_{e_k}|^2+|y_{e_k}-y'_{e_k}|^2+|z_{e_k}-z'_{e_k}|^2$ and $\{w_k\}_{k\in\mathbb{N}^+}$ is a positive monotone sequence such that the series $\sum_{k\in\mathbb{N}^+} w_k$ converges. Without loss of generality, here below we put $w_k=2^{-k}$. We now consider the subset $\mathfrak{X}\subset\boldsymbol{\Pi}^\infty\mathbb{R}^3$ of all sequences $\boldsymbol{X}$ such that $\sum_{k=1}^\infty w_k|X_k|$ is finite.

It is immediate to see that $\mathfrak{X}$, endowed with the distance function $\boldsymbol{d}$, is a Banach space. More precisely, it corresponds to a weighted ℓ_1-space. The choice of this space is due to the fact that, in the standard ℓ_1 setting, the norm is not defined on the points under interest - that is, those belonging to the infinite product $\boldsymbol{S}$; moreover, the straightforward extension of the function V as the sum of all the z_{e_k} is not defined as well.

We notice that $\boldsymbol{S}$ is a proper connected subset of the unit sphere in the Banach space $(\mathfrak{X},\boldsymbol{d})$.

Remark 4. *By standard arguments, it is easy to prove that, for every* $-\infty<a<b<+\infty$, $\mathcal{C}([a,b],\boldsymbol{\Pi}^\infty\mathbb{R}^3)$ *is a Banach space with respect to the sup norm*

$$\|\boldsymbol{f}\|_{\mathcal{C}([a,b],\mathfrak{X})}=\sup_{t\in[a,b]}\Big|\sum_{k=1}^{\infty}2^{-k}|f_k(t)|\Big|.$$

We now consider the feedback control $\mathbf{u}=(u_1,u_2)$, defined by

$$\begin{cases} u_1=\sum_{k=1}^\infty 2^{-k}y_{e_k}\\ u_2=\sum_{k=1}^\infty 2^{-k}x_{e_k}\end{cases}\tag{9}$$

and we plug it into the control system (3). The resulting autonomous dynamical system on $\boldsymbol{S}$ is well defined, as the following result states.

Theorem 2. *The Cauchy problem* $\dot{\Xi}=\boldsymbol{F}(\Xi,\mathbf{u})$ *with initial condition in* $\boldsymbol{S}$ *is well-defined.*

Proof. In order to apply the standard existence theorem of solution of ODEs in Banach spaces (see for instance [28, 29]), we need our solution space to be a linear space. Therefore, we consider the Cauchy problem on $\mathfrak{X}$

$$\begin{cases}\dot{\Xi}=\widetilde{\boldsymbol{F}}(\Xi)\\ \Xi(0)=\Xi^0,\end{cases}\tag{10}$$

where $\widetilde{\boldsymbol{F}}=\{\widetilde{F}_k\}_{k=1}^\infty$ is the vector field on $\mathfrak{X}$ defined by

$$\widetilde{F}_k(\boldsymbol{X})=e_kA\psi(X_{e_k})+\varphi\Big(\sum_{j=0}^{\infty}2^{-j}x_{e_j}\Big)B\psi(X_{e_k})+\varphi\Big(\sum_{j=0}^{\infty}2^{-j}y_{e_j}\Big)C\psi(X_{e_k}),$$

where $A=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}$, $C=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$, and $\varphi:\mathbb{R}\to\mathbb{R}$ and $\psi:\mathbb{R}^3\to\mathbb{R}^3$ are the cut-off functions

$$\varphi(x)=\begin{cases}x & \text{if } |x|\le b\\ b & \text{if } |x|\ge b\end{cases}\qquad \psi(\mathbf{w})=\begin{cases}\mathbf{w} & \text{if } |\mathbf{w}|\le a\\ a\frac{\mathbf{w}}{|\mathbf{w}|} & \text{if } |\mathbf{w}|\ge a\end{cases},$$

for some real numbers $a,b>1$. Applying the Picard-Lindelöf Theorem ([28, 29]), we obtain that there exists an interval I_0 containing 0 such that the Cauchy problem (10) admits a unique solution, continuous on I_0.

We notice that, if the initial condition belongs to $\boldsymbol{S}$, then the solution of (10) belongs to $\boldsymbol{S}$ for all $t\in I_0$. Moreover, by computations we can prove that the solution arising from an initial condition in $\boldsymbol{S}$ is well defined for all $t \in \mathbb{R}$. Finally, we observe that $\widetilde{\boldsymbol{F}}|_{\boldsymbol{S}} = \boldsymbol{F}$, therefore the solutions of (10) with initial condition in $\boldsymbol{S}$ coincide with the solutions of the equation $\dot{\boldsymbol{\Xi}} = \boldsymbol{F}(\boldsymbol{\Xi}, \mathbf{u}(\boldsymbol{\Xi}))$ with the same initial condition. □

A direct application of Gronwall inequality yields the following result.

Proposition 2. *The solutions of the Cauchy problem* (10) *depend smoothly on initial conditions.*

4.2. Pointwise convergence to $\boldsymbol{X}^-$

Let us consider the function $V : \boldsymbol{S} \to \mathbb{R}$ defined by $V(\boldsymbol{X}) = \sum_{k=1}^{\infty} 2^{-k} z_{e_k}$. It is easy to see that its time derivative along the integral curves of the vector field $\boldsymbol{F}(\boldsymbol{X}, \mathbf{u}(\boldsymbol{X}))$ satisfies $\dot{V} = -\big(\sum_{k=1} 2^{-k} x_{e_k}\big)^2 - \big(\sum_{k=1} 2^{-k} y_{e_k}\big)^2 \leq 0$. In order to conclude about the stability of these trajectories by means of a La Salle-type argument, we need to prove that $\boldsymbol{S}$ is compact. To do that, let us first recall the following definition (see for instance [30]).

Definition 1. *The product topology $\mathcal{T}$ on $\boldsymbol{S}$ is the coarsest topology that makes continuous all the projections $\pi_k : \boldsymbol{S} \to S^2_{e_k}$.*

By Tychonoff's Theorem, any product of compact topological spaces is compact with respect to the product topology ([30]). This in particular implies that $\boldsymbol{S}$ is compact with respect to $\mathcal{T}$.

As we will see just below, the product topology is equivalent to topology induced by the distance $\boldsymbol{d}$, so $\boldsymbol{S}$ is compact with respect to the latter.

Lemma 6. *Let us denote with $\mathcal{T}_{\boldsymbol{d}}$ the topology on $\boldsymbol{S}$ induced by $\boldsymbol{d}$. We have that $\mathcal{T} = \mathcal{T}_{\boldsymbol{d}}$.*

Proof. By definition, $\mathcal{T} \subset \mathcal{T}_{\boldsymbol{d}}$. If we prove that the open balls (that are a basis for $\mathcal{T}_{\boldsymbol{d}}$) are open with respect to $\mathcal{T}$, then $\mathcal{T}_{\boldsymbol{d}} \subset \mathcal{T}$ and we get the result.

Let $N > 0$ and let us define the function $\boldsymbol{d}^N : \boldsymbol{S} \times \boldsymbol{S} \to [0,1]$ as $\boldsymbol{d}^N(\boldsymbol{X}, \boldsymbol{X}') = \sum_{k=1}^{N} 2^{-k} |\boldsymbol{X}_k - \boldsymbol{X}'_k|$. It is easy to prove that $\boldsymbol{d}^N$ is continuous with respect to $\mathcal{T}$; indeed, the restriction $\boldsymbol{d}|_{\prod_{i=1}^N S^2_i}$ is obviously continuous with respect to the product topology on $\prod_{i=1}^N S^2_{e_i}$, and for every open interval $(a,b) \subset [0,1]$ we have that $(\boldsymbol{d}^N)^{-1}(a,b) = \big(\boldsymbol{d}|_{\prod_{i=1}^N S^2_{e_i}}\big)^{-1}(a,b) \times \prod_{i>N} S^2_{e_i}$, which is open with respect to $\mathcal{T}$.

The sequence $\{\boldsymbol{d}^N\}^N$ converges uniformly to $\boldsymbol{d}$. Indeed, for every $\boldsymbol{X}, \boldsymbol{X}' \in \boldsymbol{S}$ we have that

$$|\boldsymbol{d}^N(\boldsymbol{X}, \boldsymbol{X}') - \boldsymbol{d}(\boldsymbol{X}, \boldsymbol{X}')| = |\sum_{k \geq N+1} 2^{-k} |X_{e_k} - X'_{e_k}|| \leq 2^{-N}.$$

Then $\boldsymbol{d}$ is continuous with respect to $\mathcal{T}$, and this completes the proof. □

Thanks to previous Lemma, we can conclude that $\boldsymbol{S}$ is compact with respect to $\boldsymbol{d}$. In particular, we are able to prove a version of La Salle invariance principle holding for the equation $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}, \mathbf{u}(\boldsymbol{X}))$. We also remark that the Lyapunov function V is continuous with respect to $\mathcal{T}$.

Proposition 3 (Adapted La Salle). *Let us consider the set $\mathcal{M} = \{\boldsymbol{X} \in \boldsymbol{S} : \dot{V}(\boldsymbol{X}) = 0\}$, where we use the notation $\dot{V}(\boldsymbol{X}) = \frac{d}{dt} V(\phi^t(\boldsymbol{X})|_{t=0}$, and ϕ^t denotes the flow associated with the dynamical system $\dot{\boldsymbol{X}} = (\boldsymbol{X}, \mathbf{u}(\boldsymbol{X}))$. Let $\mathcal{I}$ be the largest subset of $\mathcal{M}$ which is invariant for the flow ϕ^t. Then for every $\boldsymbol{X} \in \boldsymbol{S}$ we have that $\phi^t(\boldsymbol{X}) \to \mathcal{I}$ as $t \to +\infty$.*

Proof. The proof of this proposition relies on the compactness of $\boldsymbol{S}$ with respect to the topology $\mathcal{T}_{\boldsymbol{d}}$, and follows standard arguments.

Let $m = \min_{\boldsymbol{X} \in \boldsymbol{S}} V(\boldsymbol{X})$, and fix $\boldsymbol{X}^0 \in \boldsymbol{S}$. By continuity of V, there exists $a = \lim_{t \to +\infty} V(\phi^t(\boldsymbol{X}^0))$, $a \geq m$.

Let $\Omega_{\boldsymbol{X}^0} = \{\boldsymbol{X} \in \boldsymbol{S} : \exists\ (t_n)_n \to +\infty : \phi^{t_n}(\boldsymbol{X}^0) \to \boldsymbol{X}\}$ denote the ω-limit set issued from $\boldsymbol{X}_0$; notice that $\Omega_{\boldsymbol{X}^0}$ is non-empty, since $\boldsymbol{S}$ is compact, therefore for every sequence $(t_n)_n \to +\infty$ there exists a subsequence $(t_{n_k})_k$ such that $\phi^{t_{n_k}}(\boldsymbol{X}^0)$ converges to some point in $\boldsymbol{S}$. It is easy to see that $\Omega_{\boldsymbol{X}^0}$ is invariant for the flow ϕ^t and therefore, since by continuity $V|_{\Omega_{\boldsymbol{X}^0}} = a$, we obtain that $\dot{V}(\boldsymbol{X}) = 0$ for every $\boldsymbol{X} \in \Omega_{\boldsymbol{X}^0}$. This implies that $\Omega_{\boldsymbol{X}^0} \subset \mathcal{M}$.

Let us now prove that $\Omega_{\boldsymbol{X}^0}$ is compact. Consider a sequence $(\boldsymbol{X}_k)_k$ contained in $\Omega_{\boldsymbol{X}^0}$; by compactness of $\boldsymbol{S}$, it converges, up to subsequences, to some $\bar{\boldsymbol{X}} \in \boldsymbol{S}$ (we relabel the indexes). By definition, for every k there exist a sequence $(t_{k_n})_n \to +\infty$ such that $\lim_n \phi^{t_{k_n}}(\boldsymbol{X}^0) = \boldsymbol{X}_k$. Moreover, it is possible to define a divergent sequence $(\tau_k)_k$ such that $\boldsymbol{d}(\phi^{\tau_k}(\boldsymbol{X}_0), \boldsymbol{X}_k) \leq 1/2k$ for every k. Fix $\epsilon > 0$ and choose some $\bar{k} \geq 1/\epsilon$ such that $\boldsymbol{d}(\bar{\boldsymbol{X}}, \boldsymbol{X}_k) \leq \epsilon/2$ for $k \geq \bar{k}$ (possibly taking a suitable subsequence). Then for $k \geq \bar{k}$ we have that $\boldsymbol{d}(\bar{\boldsymbol{X}}, \phi^{\tau_k}(\boldsymbol{X}^0)) \leq \boldsymbol{d}(\bar{\boldsymbol{X}}, \boldsymbol{X}_k) + \boldsymbol{d}(\phi^{\tau_k}(\boldsymbol{X}^0), \boldsymbol{X}_k) \leq \epsilon$. This means that $\bar{\boldsymbol{X}} \in \Omega_{\boldsymbol{X}^0}$, that is $\Omega_{\boldsymbol{X}^0}$ is compact.

Finally, let us assume, by contradiction, that there exist an open neighborhood U of $\Omega_{\boldsymbol{X}^0}$ in $\boldsymbol{S}$ and a sequence $(t_n)_n \to +\infty$ such that $\phi^{t_n}(\boldsymbol{X}^0) \in \boldsymbol{S} \setminus U$ for every n. By compactness of $\boldsymbol{S}$, ϕ^{t_n} converges up to subsequences to some $\bar{\boldsymbol{X}} \in \boldsymbol{S} \setminus U$. But by definition $\bar{\boldsymbol{X}} \in \Omega_{\boldsymbol{X}^0}$, then we have a contradiction.

Let us now set $\mathcal{I} = \cup_{\boldsymbol{X} \in \boldsymbol{S}} \Omega_{\boldsymbol{X}}$. By construction, it is an invariant subset contained in $\mathcal{M}$. □

By definition, $\mathcal{M} = \{\boldsymbol{X} \in \boldsymbol{S} : \sum_{k=1}^{\infty} 2^{-k} x_{e_k} = \sum_{k=1}^{\infty} 2^{-k} y_{e_k} = 0\}$. Now we look for its largest invariant subset. Let $\boldsymbol{X}^0 \in \mathcal{M}$; with the same argument than above, we can see that $\phi^t(\boldsymbol{X}^0) = \{\ (x_{e_k}(t), y_{e_k}(t), z_{e_k}(t))\}_k$ with

$$x_{e_k}(t) = \cos(e_k t) x^0_{e_k} - \sin(e_k t) y^0_{e_k} \qquad y_{e_k}(t) = \sin(e_k t) x^0_{e_k} + \cos(e_k t) y^0_{e_k} \qquad z_{e_k}(t) = z^0_{e_k}.$$

If $\boldsymbol{X}^0$ belongs to an invariant subset of $\mathcal{M}$, then $\sum_{k=1}^{\infty} 2^{-k} x_{e_k}(t) = \sum_{k=1}^{\infty} 2^{-k} y_{e_k}(t) = 0$ for every t. Let us consider the two functions

$$f(t) = \sum_{k=1}^{\infty} 2^{-k} x_{e_k}(t) = \sum_{k=1}^{\infty} 2^{-k} (\cos(e_k t) x^0_{e_k} - \sin(e_k t) y^0_{e_k})$$
$$g(t) = \sum_{k=1}^{\infty} 2^{-k} y_{e_k}(t) = \sum_{k=1}^{\infty} 2^{-k} (\sin(e_k t) x^0_{e_k} + \cos(e_k t) y^0_{e_k}).$$

It is easy to see that both $f(t)$ and $g(t)$ are uniform limits of trigonometric polynomials, that is, they are almost periodic functions (also referred to as *uniform almost periodic functions* or *Bohr almost periodic functions*, see [31, 32]). According to the references [31, 32], the Fourier is series of a (uniform) almost periodic function is computed as follows: for every $\omega \in \mathbb{R}$, we define

$$a(f, \omega) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) e^{-i\omega t}\, dt \qquad a(g, \omega) = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(t) e^{-i\omega t}\, dt.$$

By easy computations, we see that $a(f, \omega)$ and $a(g, \omega)$ are zero for every $\omega \notin \{e_k, -e_k\}_{k \in \mathbb{N}^+}$, and, moreover, that

$$a(f, e_k) = \frac{x^0_{e_k} + i y^0_{e_k}}{2^{k+1}}, \quad a(f, -e_k) = \frac{x^0_{e_k} - i y^0_{e_k}}{2^{k+1}}, \quad a(g, e_k) = \frac{y^0_{e_k} - i x^0_{e_k}}{2^{k+1}}, \quad a(g, -e_k) = \frac{i x^0_{e_k} + y^0_{e_k}}{2^{k+1}}.$$

The Fourier series of f and g are respectively

$$f(t) \sim \sum_{k=1}^{\infty} a(f, e_k) e^{i e_k t} + a(f, -e_k) e^{-i e_k t} \quad g(t) \sim \sum_{k=1}^{\infty} a(g, e_k) e^{i e_k t} + a(g, -e_k) e^{-i e_k t}.$$

By [32, Theorem 1.19], the functions f and g are identically zero if and only if all the coefficients in their Fourier series are all null, that is $x^0_{e_k} = y^0_{e_k} = 0$ for every k. Then the largest invariant subset of $\mathcal{M}$ is $\mathcal{I} = \{\boldsymbol{X} : x_{e_k} = y_{e_k} = 0, |z_{e_k}| = 1\ \forall\ k\}$.

Applying Proposition 3 to these facts, we get the following result.

Corollary 1. *Let $\boldsymbol{X}^0 \in \boldsymbol{S}$, and let $\boldsymbol{X}(t) = \phi^t(\boldsymbol{X}^0)$, with the usual notation $\boldsymbol{X}(t) = \{X_{e_k}(t)\}_k$ and $X_{e_k}(t) = (x_{e_k}(t), y_{e_k}(t), z_{e_k}(t))$. Then*

$$\lim_{t \to +\infty} x_{e_k}(t) = \lim_{t \to +\infty} y_{e_k}(t) = 0$$
$$|\lim_{t \to +\infty} z_{e_k}(t)| = 1$$

for every $k \geq 1$.

Remark 5. *It is easy to see that every point $\boldsymbol{X} \in \mathcal{I}$ is an accumulation point for the set $\mathcal{I}$, but $\mathcal{I}$ is not dense in $\boldsymbol{S}$. In particular, $\boldsymbol{X}^-$ is an accumulation point for $\mathcal{I}$ too.*

In the following, for every $N \geq 1$ we consider the *truncated feedback control* $\mathbf{u}^N = (u_1^N, u_2^N)$ with $u_1^N = \sum_{k=1}^N 2^{-k} y_{e_k}$ and $u_2^N = \sum_{k=1}^N 2^{-k} x_{e_k}$, and we call $\Xi^N(\cdot)$ the solution of the differential equation $\dot{\Xi}^N = \boldsymbol{F}(\Xi^N, \mathbf{u}^N)$, $\Xi^N \in \mathfrak{X}$. We remark that, for initial conditions in $\boldsymbol{S}$, the solution of the Cauchy problem exists and remains in $\boldsymbol{S}$ for all t. As above, we use the notations $\Xi^N(\cdot) = \{\Xi_{e_k}^N(\cdot)\}_k$ with $\Xi_{e_k}^N = \{(x_{e_k}^N(\cdot), y_{e_k}^N(\cdot), z_{e_k}^N(\cdot))\}_k$.

Applying the same arguments as in Section 3, we can prove the following result.

Proposition 4. *For every $N > 0$, there exists an open dense set $A \subset \boldsymbol{S}$ such that for every $\boldsymbol{X} \in A$ the solution $\Xi^N(\cdot)$ of the equation $\dot{\Xi}^N = \boldsymbol{F}(\Xi^N, \mathbf{u}^N)$ with initial condition equal to $\boldsymbol{X}$ has the following asymptotic behavior:*

$$\lim_{t\to+\infty} x_{e_k}^N(t) = 0 \qquad \lim_{t\to+\infty} y_{e_k}^N(t) = 0 \tag{11}$$

$$\lim_{t\to+\infty} z_{e_k}^N(t) = -1 \tag{12}$$

for $1 \leq k \leq N$.

Proof. The proof relies on the fact that the restriction of Ξ^N to the first N components obeys to the dynamical system (2)-(4). Then we can apply Theorem 1 and conclude that there exists an open dense subset A' of $\prod_{k=1}^N S^2$ such that for every $\boldsymbol{X} \in \boldsymbol{S}$ with $\{X_{e_1}, \ldots, X_{e_N}\} \in A'$ the solution $\Xi^N(\cdot)$ of the equation $\dot{\Xi}^N = \boldsymbol{F}(\Xi^N, \mathbf{u}^N)$ with initial condition equal to $\boldsymbol{X}$ satisfies the behavior described in (11), independently on the value of $\{X_k\}_{k\geq N+1}$. □

Proposition 4 leads to the asymptotic pointwise convergence of the trajectories of $(\boldsymbol{X}, \mathbf{u})$ to $\boldsymbol{X}^-$, according to the following definition:

Definition 2. *The sequence of functions $\{\Xi_k\}_k$, with $\Xi_k(\cdot) : \mathbb{R} \to \boldsymbol{S}$ for every k, converges asymptotically pointwise to the point $\boldsymbol{X} \in \boldsymbol{S}$ if for every $\epsilon > 0$ there exist an integer $\bar{N} > 0$ such that for every $N \geq \bar{N}$ there exists a time $t = t(N, \epsilon)$ such that if $t \geq t(N, \epsilon)$ then $\boldsymbol{d}(\Xi_N(t), \boldsymbol{X}) \leq \epsilon$.*

We can then state the following result.

Theorem 3. *There exists a residual set $\mathcal{G} \subset \boldsymbol{S}$ such that for every $\boldsymbol{X} \in \mathcal{G}$ there exists a sequence $\{\mathbf{u}^k\}_k$ of controls such that the sequence $\{\Xi_k\}_k$ of solutions of the equation $\dot{\Xi}_k = \boldsymbol{F}(\Xi_k, \mathbf{u}_k)$ with initial condition equal to $\boldsymbol{X}$ converges asymptotically pointwise to $\boldsymbol{X}^-$.*

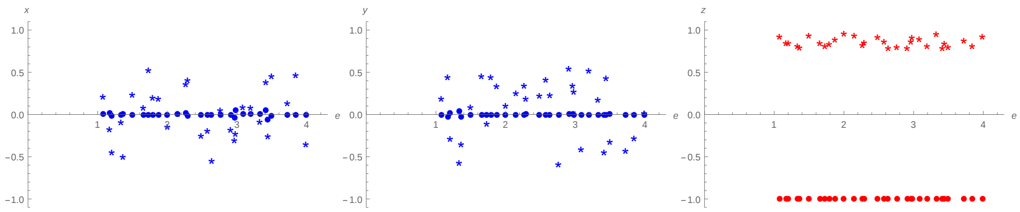
Proof. Let $\boldsymbol{G}^N \subset \prod_{k=1}^N S_k^2$ be the set of "good initial conditions" for the N-dimensional system, as defined in Theorem 1, and let us define $\widehat{\boldsymbol{G}}^N = \boldsymbol{G}^N \times \prod_{k\geq N+1} S^2 \subset \boldsymbol{S}$. Proposition 4 states that the solution of the truncated system $\dot{\Xi}^N = \boldsymbol{F}(\Xi^N, \mathbf{u}^N)$ with initial condition in $\widehat{\boldsymbol{G}}^N$ has the limit (11). Since $\widehat{\boldsymbol{G}}^N$ is an open dense subset of $\boldsymbol{S}$ for every N, and $\boldsymbol{S}$ has the Baire property ([30]), then $\mathcal{G} = \cap_N \widehat{\boldsymbol{G}}^N$ is a dense subset of $\boldsymbol{S}$.

Let $\boldsymbol{X} \in \mathcal{G}$ and fix $\epsilon > 0$. For some integer N such that $2^{-N+1} < \epsilon$, consider the truncated feedback $\mathbf{u}^N$, defined as above, and the corresponding trajectory Ξ^N with $\Xi^N(0) = \boldsymbol{X}$. Since $\boldsymbol{X} \in \widehat{\boldsymbol{G}}^N$, by Proposition 4 there exists a time $t = t(N, \epsilon)$ such that for $t \geq t(N, \epsilon)$ it holds $\sum_{k=1}^N 2^{-k} |X_{e_k}^N(t) - (0, 0, -1)^T| \leq \epsilon/2$, then, since $\sum_{k\geq N+1} 2^{-k} |X_{e_k}^N(t) - (0, 0, -1)^T| \leq 2^{-N} \leq \epsilon/2$, we get that $\boldsymbol{d}(\boldsymbol{X}^N(t), \boldsymbol{X}^-) \leq \epsilon$ for $t \geq t(N, \epsilon)$. □

5. Closed-loop simulations

Let $\mathcal{E}$ be a collection of $N = 30$ randomly chosen points contained in the interval $[1, 4]$, and we consider N randomly chosen initial conditions $X_e(0)$ with $z_e(0) \in [0.8, 1]$ and $|X_e(0)| = 1$. We perform closed-loop simulation of the dynamical system (2) with feedback control $u_1(t) = \sum_{k=1}^{N} y_{e_k}(t)$ and $u_2(t) = \sum_{k=1}^{N} x_{e_k}(t)$, up to a final time $T = 20000$.

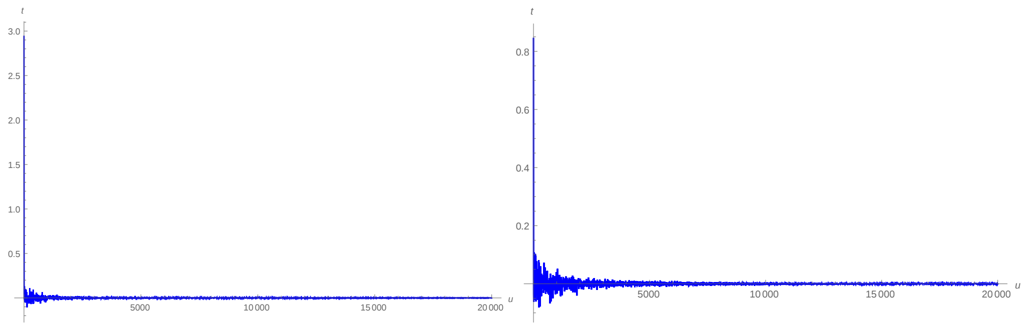
In Figure 1 we show the convergence to the target point of the collections X_e, $e \in \mathcal{E}$.



(a) x_e (b) y_e (c) z_e

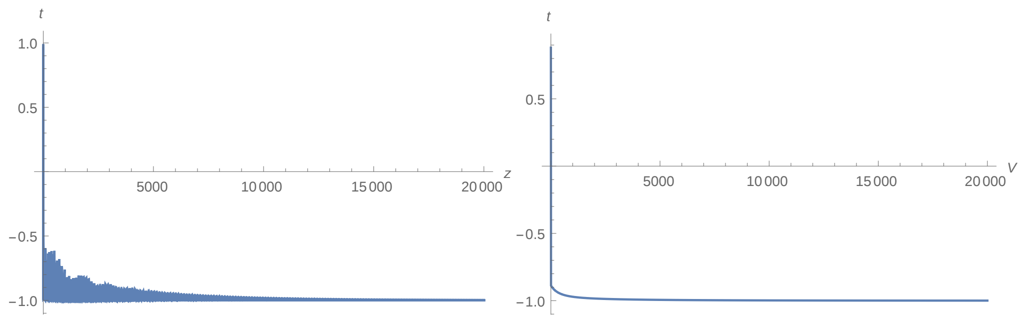
Figure 1: Initial and final states with respect to different values of frequencies (stars denote the initial point, bullets the final point).

Figure 2 plots the time evolution of the feedback control function, while in Figures 3a and 3b we plot respectively the values of the last coordinate $z_e(t)$, for all $e \in \mathcal{E}$, and of the Lyapunov function $V(t)$, normalized by N.



(a) u_1 (b) u_2

Figure 2: Time evolution of the control function

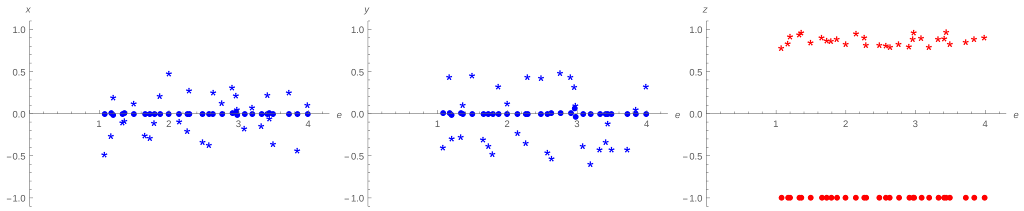


(a) $z_e(t)$ for different values of e (b) Lyapunov function (normalized)

Figure 3: Time evolution of z_e and the Lyapunov function

We then take the same collection $\mathcal{E}$ as before, and we consider N randomly chosen initial conditions $X_e(0)$ with $z_e(0) \in [0.8, 1]$ and $|X_e(0)| = 1$. We now perform closed-loop simulation of the dynamical system (2) with feedback control $u_1(t) = \sum_{k=1}^{N} w_k y_{e_k}(t)$ and $u_2(t) = \sum_{k=1}^{N} w_k x_{e_k}(t)$, with $w_k = (1.1)^{-k}$, up to a final time $T = 20000$. The purpose of this new run is to visualize the influence of the weights w_k on the convergence of the systems. As we can see from Figure 6a, the weights slow down the convergence of the systems (this cannot be seen from Figure 6b, since the slower components in the Lyapunov function are multiplied by a small weight).

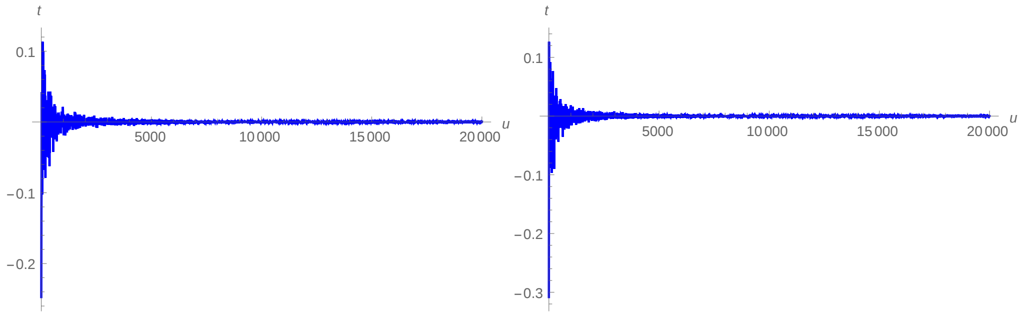
In Figure 4 we show the convergence to the target point of the collections X_e, $e \in \mathcal{E}$.



(a) x_e (b) y_e (c) z_e

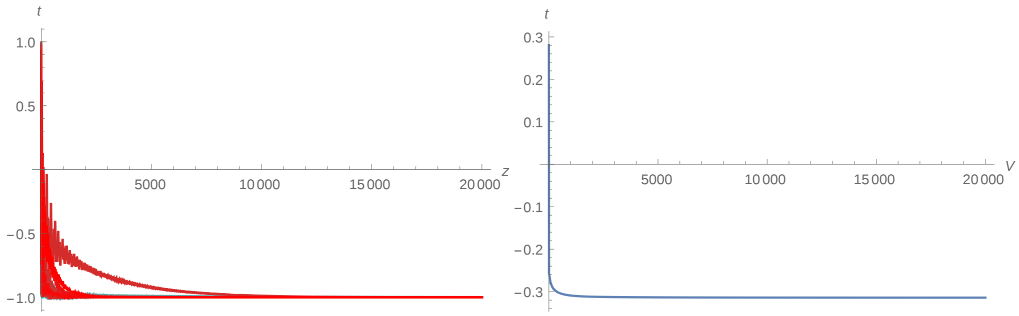
Figure 4: Initial and final states with respect to different values of frequencies (stars denote the initial point, bullets the final point).

As above, in Figure 5 we plot the time evolution of the feedback control function, in Figures 6a $z_e(t)$, and in 6b the Lyapunov function $V(t)$, normalized by N.



(a) u_1 (b) u_2

Figure 5: Time evolution of the control function



(a) $z_e(t)$ for different values of e (b) Lyapunov function (normalized)

Figure 6: Time evolution of z_e and the Lyapunov function

6. Conclusions

In this paper, we have investigated the stabilization of an ensemble of non-interacting half-spins to the uniform state $-1/2$ (represented by the state $(0, 0, -1)$ in the Bloch sphere); in particular, we provided a feedback control that stabilizes a generic initial condition to the target state, asymptotically in time.

In the finite-dimensional case, we remark a close link between the proposed control (4) and the *radiation damping effect (RDE)* (see for instance [24, 25] for a detailed description of the phenomenon). In an NMR setup, the radiation damping is a reciprocal interaction between the spins and the radio-frequency source (a coil): this coupling can be taken into account by adding a non-linear term to the uncontrolled Bloch equation (see for instance [33, 34] and references therein). In particular, in our notations the uncontrolled Bloch equation with RDE reads

$$\begin{cases} \dot{x}_e = -e_k y_{e_k} - \ell z_{e_k} \overline{X} \\ \dot{y}_e = e x_{e_k} - \ell z_{e_k} \overline{Y} \\ \dot{z}_e = \ell\big(\overline{X}^2 + \overline{Y}^2\big) \end{cases}, \tag{13}$$

where ℓ is the radiation damping rate (depending on the apparatus) and $\overline{X} = \frac{1}{p}\sum_{k=1}^{p} x_{e_k}$, $\overline{Y} = \frac{1}{p}\sum_{k=1}^{p} y_{e_k}$ are the average values of the magnetization. The analysis carried out in Section 3 applies also in this case, with the only difference that $\boldsymbol{X}^-$ is a repeller and $\boldsymbol{X}^+$ is an attractor of equation (13). This gives a rigorous justification of the stabilizing properties of RDE.

If we want to exploit RDE for stabilizing the system towards $\boldsymbol{X}^-$, it is sufficient to invert the z-component of the magnetic field: this yields a change of the sign of the right-hand side of equation (13), thus, up to a change in the sign of the frequencies (which does not affect the dynamics, being the set $\mathcal{E}$ arbitrary) and to a multiplicative factor ℓ/p on the control, we obtain the dynamical system (2)-(4). The multiplicative factor ℓ/p affects only the magnitude of the real part of the eigenvalues (see equation (6)), that is, the rate of convergence towards the equilibria.

If it is not possible to invert the z-component of the magnetic field, so that the RDE tends to stabilize the system to $\boldsymbol{X}^+$, the stabilization to $\boldsymbol{X}^-$ can be still achieved by choosing a sufficiently strong control (see for instance [33] for a similar result in the single spin case).

In the countable case, we use the same approach to provide a (continuous bounded) feedback control which asymptotically stabilizes, in the pointwise convergence norm, a generic set of initial conditions.

Concerning the case where $\mathcal{E}$ is an interval, and $\boldsymbol{X} \in L^2(\mathcal{E}, S^2)$, the question addressed in [4] about controllability of the system by means of bounded controls is still left open. This topic makes the subject of further investigations of the authors.

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