OUTPUT FEEDBACK STABILIZATION OF NON-UNIFORMLY OBSERVABLE SYSTEMS

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Abstract. Stabilizing the state of a system relying only on the knowledge of a measured output is a classical control theory problem. Designing a stable closed-loop based on an observer design requires that some necessary information on the state can be accessed through the output trajectory. For non-linear systems, this may not be true for all controls. The existence of singular controls (from the point of view of observability) is even generic in many cases. Then, the design of asymptotically stable closed-loops becomes a challenge that remains to be fully answered. Using various examples, we propose to review some strategies that showed to be efficient in tackling the difficulties posed by non-uniform observability (i.e., existence of singular controls) in the context of dynamic output feedback stabilization.

1. Introduction

In many physical systems, only an output of the system is known, and it is usual to rely on estimations of the state to achieve control objectives on the system. For instance, stabilizing the state of a dynamical system to a target point from the knowledge of its output is a classical goal in control theory [3]. However, stabilizing a system and estimating the state are two competing processes that need to happen simultaneously in order to stabilize a partially measured system in closed loop. For non-linear systems, it turns out that the choice of the control can influence the observability of the system, that is our ability to adequately reconstruct the state from the output. This can pose a major problem for closed loop systems. It has been known since the nineties that if a non-linear system is both stabilizable by means of a state feedback and observable for any input, i.e., uniformly observable, then it should also be stabilizable by a dynamic feedback depending only on knowledge of the input and output [20, 33]. It is, however, much more difficult to give a definitive answer when systems admit singular inputs for which they are unobservable.

In the present work, we propose to review and discuss some recent results by the authors and their collaborators on this issue. Despite the generality of non-uniform observability among non-linear systems (it is generic when the dimension of the output is less or equal to the dimension of the input [16]), little is known. Still, early results on the problem were obtained by Coron [12] and Shim and Teel [29], respectively on local and practical stabilization by means of periodic time-varying output feedbacks. We focus on the design of autonomous closed-loop, preventing the use of such switching methods. With the present text, our goal is to present the strategies we developed based on feedback perturbations (to improve observability of the closed-loop system), dissipative systems (that present some robustness with respect to observability singularities) and embedding design (into dissipative systems).

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2. Problem discussion

2.1. **Output feedback.** Let n, m and p be positive integers, $f \in C^0(\mathbb{R}^n \times \mathbb{R}^p, \mathbb{R}^n)$ be uniformly locally Lipschitz with respect to its first variable and $h : \mathbb{R}^n \to \mathbb{R}^m$. For all $u \in C^0(\mathbb{R}_+, \mathbb{R}^p)$, consider the general observation-control system

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{1}$$

where x is the state of the system, u is the control (or input) and y is the observation (or measured output). We are interested in the problem of stabilizing (1) to the origin by using only knowledge of f, h, and online measurement of y.

Definition 2.1 (Dynamic output feedback stabilizability)**.** System (1) is said to be *locally* (resp. *globally*) *stabilizable by means of a dynamic output feedback* if and only if the following holds.

There exist two continuous maps $\nu : \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^q$ and $\lambda : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^p$ for some non-negative integer q such that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^q$ is a locally (resp. globally) asymptotically stable equilibrium point of the following system:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}, \qquad \begin{cases} \dot{w} = \nu(w, u, y) \\ u = \lambda(w, y). \end{cases} \tag{2}$$

Additionally, if for any compact set $\mathcal{K}_x \subset \mathbb{R}^n$, there exist two continuous maps $\nu : \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^q$ and $\lambda : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^p$ for some non-negative integer q, and a compact set $\mathcal{K}_w \subset \mathbb{R}^q$ such that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^q$ is an asymptotically stable equilibrium point of (2) with basin of attraction containing $\mathcal{K}_x \times \mathcal{K}_w$, then (1) is said to be *semi-globally stabilizable by means of a dynamic output feedback.*

We focus on autonomous stabilization strategies, while a time-varying generalization (where ν and λ in (2) depend on time) can be considered even for autonomous f and h as shown in [12]. We particularly investigate the case of semi-global stabilization, that is of deep interest in engineering applications. Relying on a dynamical observer to estimate the state of the system and produce a suitable stabilizing control is a classical strategy. Naturally, this requires the estimation problem to be well-posed, that is, that the system is observable.

Definition 2.2 (Observability)**.** System (1) is said to be *observable* in time T for an input $u \in C^0(\mathbb{R}_+, \mathbb{R}^p)$ if and only if, for all initial conditions $x_0 \neq \tilde{x}_0 \in \mathbb{R}^n$, the set

$$\left\{ t \in [0, \bar{T}) \ : \ h(x(t)) \neq h(\tilde{x}(t)) \right\}$$

has positive measure, where x and $\tilde{x}$ denote respectively the solutions of (2) starting from x_0 and $\tilde{x}_0$ and $\bar{T}$ denotes the minimum between T and the existence time of x and $\tilde{x}$. If system (1) is observable in any time $T > 0$ for all inputs u, then it is said to be *uniformly observable* in small time.

A notion stronger than uniform observability in small time can be defined for regular systems.

Definition 2.3 (Complete uniform observability)**.** Assume that f and h are sufficiently regular. System (1) is said to be *completely uniformly observable* if and only if there exist two non-negative integers n_y and n_u and a smooth function $\eta : \mathbb{R}^{m(n_y+1)} \times \mathbb{R}^{p(n_u+1)} \to \mathbb{R}^n$ such that, for all smooth inputs $u : \mathbb{R}_+ \to \mathbb{R}^p$ and all solutions (x, y) of (1), we have for all $t \geqslant 0$ such that $x(t)$ is defined,

$$x(t) = \eta((y^{(i)}(t))_{0 \leqslant i \leqslant n_y}, (u^{(i)}(t))_{0 \leqslant i \leqslant n_u})$$

where $(y^{(i)}(t))_{0 \leqslant i \leqslant n_y}$ and $(u^{(i)}(t))_{0 \leqslant i \leqslant n_u}$ denote the n_y and n_u first derivatives at time t of y and u, respectively.

This notion proved to be particularly useful in the context of output feedback stabilization, as shown by Teel and Praly in [33, 34].

Theorem 2.4 ([33])**.** *If system* (1) *is*

- *semi-globally stabilizable by means of a smooth state feedback,*
- *completely uniformly observable,*

then it is semi-globally stabilizable by means of a dynamic output feedback.

A related result can be found in [19, 20] under a strong differential observability assumption. The main drawback of this strategy is that uniform observability is a restrictive assumption that is generically not satisfied when the dimension of the output is less than or equal to the dimension of the input (i.e., $m \leqslant p$), as discussed in [16, Chapter 3]. Hence, in most cases, one can not rely on Theorem 2.4 (based on a high gain strategy) to stabilize a control systems by means of a dynamic output feedback.

2.2. **Non-uniformly observable systems.** We are interested in possible extensions of these well known output stabilization strategies to the case of systems that admits inputs for which they are unobservable. Such systems are called non-uniformly observable. These singular inputs pose a major issue in the simultaneous estimation and control of the systems. It falls back on the design of the feedback law and the observer to maintain a sufficient level of observability along trajectories of the closed-loop. Two sub-cases of non-uniformly observable systems can be distinguished, depending on whether or not their target corresponds to an observable input. That is, when the target is reached, whether or not the constant input generated in closed-loop makes the system observable. Each case leads to different issues in the design of the closed-loop. If the target is observable, then stability of the closed-loop implies that the input eventually becomes non-singular. Hence, observability issues occur only during the transient response. If the target is unobservable, then the observability singularity is somewhat unavoidable: if stabilization is achieved, then the input tends towards a singular one.

As recalled in Section 2.1, efficient tried-and-tested methods for output feedback stabilization of uniformly observable systems exist. Much less has been obtained for non-uniformly observable ones. Nevertheless, the issue of observability singularities appears in numerous modern applications (see, e.g., [1,2,11,13,18,27,31,32]), leading to a renewal of interest in the issue in recent years. A recurring strategy is the use of modifications of the input in closed-loop. In Theorem 2.4, the input of the closed-loop system is chosen as $u = \lambda(\hat{x})$ where $\hat{x}$ is designed to be an observer of the state, so that $\hat{x} - x \to 0$, and λ is a stabilizing state feedback law. Notable efforts in that direction include [11,12,29]. In [12], Coron proposed a switching strategy in the design of the feedback law. The input switches between non-singular inputs making the system observable, and potentially singular ones ensuring the stabilization of the system. This allowed to prove local stabilizability by means of time-varying periodic dynamic output feedback. In a similar manner, Shim and Teel proposed in [29] a Lyapunov based strategy to achieve practical semi-global stabilization (that is, in any neighborhood of the target). This switching method has also been investigated in [30] in the context of output regulation and [26] for systems with positive outputs. More recently, it has been proposed in the sequence of papers [11,31,32] to modify the input by adding small excitatory signals improving the observability of the system: the input is chosen as $u = \lambda(\hat{x}) + d(t)$, where $d(t)$ is a signal of small amplitude and high frequency.

One of the drawbacks of these strategies is that they result in time-varying closed-loop systems. In a different line, we seek autonomous strategies. We essentially rely on autonomous perturbations of the feedback law of the form $u = \lambda(\hat{x}) + \delta(\hat{x})$ where δ vanishes near the target point, as in the next section.

2.3. **The transversality point of view.** In this section, we restrict ourselves to the analysis of single-input single-output bilinear systems, i.e. of the form

$$\begin{cases} \dot{x} = (A + uB)x + bu \\ y = Cx. \end{cases} \tag{3}$$

where $A, B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$, $b \in \mathbb{R}^n$ and $u \in C^\infty(\mathbb{R}_+, \mathbb{R})$. The following theorem by Fliess and Kupka motivates this primary focus on bilinear systems. Recall that the observation space of a control-affine system $\dot{x} = f(x) + \sum_{i=1}^{p} u_i g_i(x)$ with measured output $y = h(x)$ is the smallest vector subspace of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ containing h and closed under the Lie derivation along elements of $\{f + \sum_{i=1}^p u_i g_i, u \in \mathbb{R}^p\}$.

Theorem 2.5 ([14])**.** *A C^∞ control-affine system can be immersed into a bilinear one if and only if its observation space is finite-dimensional.*

Bilinear systems are also used to model various physical phenomena (see [25] for a review on the subject). On such systems, observability singularities are very common. In fact, direct analysis yields the following result on the prevalence of systems with singularities among all possible systems.

Theorem 2.6. *The set Σ of matrices $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{1\times n}$ such that (3) is not uniformly observable contains an open dense set.*

Proof. Let $U = \{(A, B, C) \in \Sigma \ : \ (C, A) \text{ is observable}\}$. First, we show that U is open by using the usual normal observability form of bilinear systems. For $(A, B, C) \in U$, since (C, A) is observable, there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$, continuously depending on (C, A), such that TAT^{-1} is a companion matrix and $CT^{-1} = (1, 0, \ldots, 0)$ (see, e.g., [21, Chapter 4, Theorem 1]). According to [15, Theorem 2], $(A, B, C) \in \Sigma$ is equivalent to TBT^{-1} not being lower triangular, which then implies that U is open in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{1\times n}$.

The same line of reasoning allows to show that Σ is dense. Indeed if $(A, B, C) \notin \Sigma$, then (C, A) is observable and TBT^{-1} is now lower triangular. Consider a sequence $(B_k)_{k\in\mathbb{N}}$ converging to B, such that TB_kT^{-1} is not lower triangular for any k. Then $(A, B_k, C) \in \Sigma$ and tends towards (A, B, C). Since the set of observable pairs (C, A) is open and dense in $\mathbb{R}^{n\times n} \times \mathbb{R}^n$ (as the preimage of zero by the determinant of the Kalman observability matrix), it follows that U is also dense as the intersection of an open dense set with the dense set Σ. ■

When a bilinear system has observability singularities, different strategies may be used to approach this difficulty. Closed-loop feedback relying on an observer may be sufficient to avoid, in a sense, the singularities if they are not at the target. To illustrate this idea, let us introduce the usual Kalman observer coupling for (3) on $\mathbb{R}^n \times \mathbb{R}^n \times S^n_{++}$:

$$
\begin{cases}
\dot{x} = (A + uB)x + bu \\
\dot{\hat{x}} = (A + uB)\hat{x} + bu - PC'(C\hat{x} - y) \\
\dot{P} = P(A + uB)' + (A + uB)P + Q - PC'CP
\end{cases}
\tag{4}
$$

with $Q \in S^n_{++}$ being the observer gain, and S^n_{++} denoting the set of positive definite symmetric matrices. State feedback stabilization is an important issue for SISO bilinear systems and various strategies have been developed (see, e.g., [4]). In the context of dynamic output feedback stabilization, we assume the existence of a smooth locally stabilizing state feedback $\lambda \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that 0 is an asymptotically stable equilibrium point of (3) when $u = \lambda \circ x$ for some open basin of attraction $\mathcal{D}(\lambda)$. We always assume that $\lambda(0) = 0$, which is true up to a substitution of A with $A + \lambda(0)B$.

Then, one may wonder if the controls actually generated through the corresponding output feedback loop, of the form $u = \lambda \circ \hat{x}$, actually belong to the set of singular controls. This possibility is essential, as it partly justifies the idea of pursuing observability in order to close the feedback loop when singularities threaten the whole strategy. A transversality approach can shed a light on this matter in the case of observable targets.

Theorem 2.7 ([7])**.** *Assume that the pairs (C, A) and (C, B) are observable. Let $\mathcal{K} = \mathcal{K}_x \times \mathcal{K}_{\hat{x}} \times \mathcal{K}_P \subset \mathbb{R}^n \times \mathbb{R}^n \times S^n_{++}$ be a compact set. Assume that 0 is in the interior of $\mathcal{K}_{\hat{x}}$. Denote by Λ the set of feedbacks $\lambda \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that 0 is a locally asymptotically stable equilibrium point of (3) when $u = \lambda \circ x$. Let $T > 0$ and $\Lambda_T \subset \Lambda$ be the set of feedbacks $\lambda \in \Lambda$ such that (3) is observable in time T for the control $u = \lambda \circ \hat{x}$, where $\hat{x}$ follows (4) with initial conditions $(x_0, \hat{x}_0, P_0)$ in $\mathcal{K}$. Then Λ_T is a dense open subset of Λ in the Whitney C^∞ topology.*

The proof of Theorem 2.7 relies on deep transversality arguments on the function mapping any initial condition to the jets of the corresponding output. More precisely, the proof requires chained applications of Goresky-MacPherson transversality theorems [17] on a finite increasing sequence of subspaces of the considered initial conditions that reach the full compact set in a finite number of iterations, and computation of arbitrarily high order derivatives of the output. This result is a first step toward the achievement of a generic separation principle for SISO bilinear systems, since the observability of (C, A) and (C, B) is generic. It states that if a system is state feedback stabilizable, then generically on the feedback and on the system, the inputs produced by the closed-loop system make it observable. Actually, one can prove the convergence to the target of bounded trajectories when this observability is guaranteed. In order to follow the strategy of [15] in the uniformly observable case, we change the Kalman observer for a Kalman-like one: $\dot{P} = P(A + uB)' + (A + uB)P + \theta P - PC'CP$, where $\theta > 0$. Theorem 2.7 easily extends to this case, and we have the following result.

Theorem 2.8. *Let $\lambda \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that* (3) *coupled with $u = \lambda \circ x$ is locally asymptotically stable at* 0 *with basin of attraction $\mathcal{D}(\lambda)$. Let $\mathcal{K} = \mathcal{K}_x \times \mathcal{K}_{\hat{x}} \times \mathcal{K}_P \subset \mathbb{R}^n \times \mathbb{R}^n \times S^n_{++}$ be a compact set such that $K_{\hat{x}} \subset \mathcal{D}(\lambda)$. Assume that λ is bounded over $\mathcal{D}(\lambda)$, $\mathcal{K}$ is positively invariant under* (4)*, and* (3) *is observable in any positive time for the control $u = \lambda \circ \hat{x}$. Then $(0, 0, P_\infty)$ is a locally asymptotically stable equilibrium point of* (4) *with basin of attraction containing $\mathcal{K}$, where P_∞ is the unique solution of $P_\infty(A + uB)' + (A + uB)P_\infty + \theta P_\infty = P_\infty C'CP_\infty$.*

Proof. The proof is similar to the strategy used in [15] in the uniformly observable case. Let $(x, \hat{x}, P)$ be a trajectory and $u = \lambda \circ \hat{x}$. Set $S = P^{-1}$ and $\varepsilon = \hat{x} - x$. Then

$$\dot{S} = -(A + uB)'S - S(A + uB) - \theta S + C'C.$$

Classically, this implies that $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon' S\varepsilon \leqslant -\theta\varepsilon' S\varepsilon$, hence $\varepsilon' S\varepsilon(t) \leqslant e^{-\theta t}(\varepsilon' S\varepsilon)(0)$ for all $t \geqslant 0$. Moreover, by the variation of constant formula,

$$S(t) = e^{-\theta t}(\Phi'_u(t))^{-1} S_0 (\Phi_u(t))^{-1} + W_u(t)$$

where $\Phi_u(t)$ is the resolvent matrix of $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_u(t) = (A + uB)\Phi_u(t)$ with $\Phi_u(0) = \mathrm{Id}$ and $W_u(t)$ is the Gramian-like observability matrix defined by

$$\begin{aligned} W_u(t) &:= \int_0^t e^{-\theta(t-s)}(\Phi'_{u(\cdot+s)}(t-s))^{-1} C'C (\Phi_{u(\cdot+s)}(t-s))^{-1}\mathrm{d}s \\ &\geqslant e^{-\theta\tau}\int_{t-\tau}^t (\Phi'_{u(\cdot+s)}(t-s))^{-1} C'C (\Phi_{u(\cdot+s)}(t-s))^{-1}\mathrm{d}s \end{aligned}$$

for any $\tau \in (0, t)$. For all $(x_0, \hat{x}_0, P_0) \in \mathcal{K}$, the corresponding input $u = \lambda \circ \hat{x}$ is such that

$$\int_0^\tau |C(\Phi_u(\tau - s))^{-1} x|^2 \mathrm{d}s \geqslant \alpha_u |x|^2$$

for all $x \in \mathbb{R}^n$ for some positive constant α_u since u makes (3) observable in any positive time. The function $(x_0, \hat{x}_0, P_0) \mapsto \alpha_{\lambda\circ\hat{x}}$ has a positive minimum α over $\mathcal{K}$ since it is continuous (see [15]). Note that if $u = \lambda \circ \hat{x}$, then $u(\cdot + t)$ can also be written as $\lambda \circ \hat{x}$ with initial conditions $(x, \hat{x}, P)(t) \in \mathcal{K}$. Hence, $S(t) \geqslant W_u(t) \geqslant e^{-\theta\tau}\alpha\mathrm{Id}$, which yields

$$|\varepsilon(t)|^2 \leqslant \frac{1}{\alpha} e^{-\theta(t-\tau)}(\varepsilon' S\varepsilon)(0).$$

Thus ε is exponentially converging towards zero. The rest of the proof is identical to [15, Theorem 3] and we dot not recall it here. The strategy is the following: in the ω-limit set of any trajectory, $\varepsilon \equiv 0$, hence the stabilizing property of λ makes $\hat{x}$ tends towards zero, and P to P_∞. The local asymptotic stability is obtained by the center manifold theorem. ∎

Hence, to achieve semi-global output feedback stabilization, the remaining difficulty lies in proving that the trajectories of (3) coupled with $u = \lambda \circ x$ are bounded. In the uniformly observable case, it is sufficient to choose θ sufficiently large. Then the exponential decrease of $\varepsilon' S\varepsilon$ and the uniform lower bound on the observability Gramian yields boundedness of trajectories. However, in the non-uniformly observable case, one need to invoke Theorem 2.7 in order to find a feedback λ making the system observable. Then λ depends on θ, and the lower bound of the observability Gramian depends on λ. Therefore, when increasing θ, the lower bound of S could tend towards zero, hence nothing shows that increasing θ actually increases the rate of convergence of ε towards 0.

Open question. In the generic case of non-uniformly observable SISO bilinear systems, there does not exist any proof that increasing the gain of a Kalman filter can lead to an arbitrary increase in the speed of convergence of the observer. It is a major obstacle to obtain a generic separation principle for these systems.

This difficulty leads us to consider a more restrictive class of systems, for which at least the observer error ε remains bounded, independently of the observability assumptions. The question is then: for dissipative systems, are we able to use the perturbation strategy developed in this section to set up a separation principle? We will see that for such systems, no perturbation is needed to achieve this goal.

3. Error-dissipative approach

3.1. Eventually detectable systems.

Let us temporarily remove the SISO assumption that has prevailed up to now: consider control systems with p inputs and m outputs. Let $A : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ and $b : \mathbb{R}^p \to \mathbb{R}^n$ be two locally Lipschitz maps, $C \in \mathbb{R}^{m\times n}$.

The state-affine system

$$\begin{cases} \dot{x} = A(u)x + b(u) \\ y = Cx \end{cases} \tag{5}$$

is said to be *dissipative* over an admissible set $\mathcal{U} \subset \mathbb{R}^p$ if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that for all $u \in \mathcal{U}$,

In order to address the issue of dynamic output feedback stabilization of (5), let us consider a locally Lipschitz feedback law $\lambda : \mathbb{R}^n \to \mathbb{R}^p$ such that 0 is a locally asymptotically stable equilibrium point of (5) when $u = \lambda \circ x$. Let $\mathcal{D}(\lambda)$ be the corresponding basin of attraction, and assume that (5) is dissipative over $\mathcal{U} = \lambda(\mathcal{D}(\lambda))$. Then, the dynamic output feedback stabilizability of (5) is fully characterized by the detectability of the pair $(C, A(0))$. That is, when any trajectory of $\dot{x} = A(0)x$ such that Cx is constantly null is such that $x \to 0$.

Theorem 3.1 ([28])**.** *System* (5) *is globally asymptotically stabilizable by means of a dynamic output feedback if and only if the pair* $(C, A(0))$ *detectable. In that case, the globally asymptotically stabilizing dynamic output feedback may be designed with the following Luenberger observer with dynamic gain:*

$$\begin{cases} \dot{x} = A(\lambda(\hat{x}))x + B(\lambda(\hat{x})) \\ \dot{\hat{x}} = A(\lambda(\hat{x}))\hat{x} + B(\lambda(\hat{x})) - \alpha(\hat{x}, C(\hat{x} - x))PC'C(\hat{x} - x) \end{cases} \tag{6}$$

where $\hat{x}(0)$ *lies in* $\mathcal{D}(\lambda)$ *and* α *is a positive locally Lipschitz function given by* (9)*.*

Theorem 3.1 exhibits a key feature of dissipative systems: the dynamic output feedback stabilizability of the system is determined by a detectability property at the target point (namely, of the pair $(C, A(0))$). Actually, the generalized condition called 0-detectability is necessary for control systems of the form (1).

Theorem 3.2 ([8])**.** *Let* $\mathcal{X}_0 \subset \mathbb{R}^n$ *be the set of initial conditions of* (1) *such that the corresponding solutions have a constantly null output. If* (1) *is locally (resp. semi-globally, globally) stabilizable by means of a dynamic output feedback, then* 0 *is a locally (resp. globally, globally) asymptotically stable equilibrium point of the vector field* $\mathcal{X}_0 \ni x \mapsto f(x, 0)$*.*

Example 3.3. The following (non-dissipative) linear dynamics with quadratic output

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y = x' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x \end{cases} \tag{7}$$

is not 0-detectable since any solution of (7) with $u \equiv 0$ starting from (x_1, x_1) for some $x_1 \in \mathbb{R}\setminus\{0\}$ is such that $y \equiv 0$ but x does not tend towards 0. Therefore, it is not stabilizable by means of a dynamic output feedback.

Since system (5) is dissipative, this necessary condition also becomes sufficient, as stated in Theorem 3.1. In particular, no observability or detectability properties of the system are required along trajectories of the closed-loop system, namely, on $(C, A(\lambda \circ \hat{x}))$. Therefore, a perturbation strategy of the feedback law as the one developed in Section 2.3 does not have to be employed. Instead, a direct proof of global asymptotic stability can be performed, following three main steps that exploit dissipativity, Lyapunov analysis and ω-limit arguments.

Let us specify the map α given in Theorem 3.1. According to a converse Lyapunov theorem (see, e.g., [22, 24, 35]), there exists a proper function $W \in C^\infty(\mathcal{D}, \mathbb{R}_+)$ such that $W(0) = 0$ and

$$\frac{\partial W}{\partial x}(x)\left(A(\lambda(x))x + B(\lambda(x))\right) \leqslant -W(x), \quad \forall x \in \mathcal{D}. \tag{8}$$

We choose the positive locally Lipschitz map α used in the closed-loop (6) as

$$\alpha(\hat{x}, y) = \frac{\max\{W(\hat{x}), 1\}}{2\left(1 + \left|\frac{\partial W}{\partial x}(\hat{x})\right|\right)\left(1 + |PC'y|\right)}. \tag{9}$$

Alternatively, α can be chosen as a positive constant sufficiently small with respect to a given set of initial conditions in order to obtain semi-global asymptotic stability of (6) (see [28]).

Let us give a sketch of the proof of Theorem 3.1 given in [28]. It mainly relies on the fact that the estimation error $\varepsilon = \hat{x} - x$ has the dissipative dynamics $\dot{\varepsilon} = (A(u) - \alpha PC'C)\varepsilon$, which ensures that $\varepsilon' P^{-1}\varepsilon$ is non-increasing since

$$\frac{d}{dt}\varepsilon' P^{-1}\varepsilon \leqslant -2\alpha(\hat{x}, C\varepsilon)|C\varepsilon|^2 \leqslant 0. \tag{10}$$

- In $(\varepsilon, \hat{x})$-coordinates, linearization of (6) at the origin yields a lower triangular structure. LaSalle's invariance principle on the Lyapunov function $\varepsilon' P^{-1}\varepsilon$ and detectability of $(C, A(0))$ imply local asymptotic stability of the ε-subsystem. The locally stabilizing feature of the feedback law λ, implies the existence of a center manifold at the target, which can be extended to prove local asymptotic stability of the $(\varepsilon, \hat{x})$ system at the origin.
- Then, boundedness of trajectories is proved. The boundedness of ε follows from (10). Using the definition of α and standard Lyapunov machinery, one can show that W given in (8) is bounded along trajectories of $\hat{x}$, bounding $\hat{x}$.
- Finally, global attractivity of the origin is obtained by means of ω-limit arguments. By construction of the observer, $C\varepsilon \to 0$. As a consequence of the stabilizing feature of λ and the 0-detectability condition, we obtain that the ω-limit set of any trajectory contains 0. Hence, by local asymptotic stability, the origin is globally attractive.

This last point illustrates how detectability "in the limit" is the only necessary observability condition. Instead of relying on some uniform observability assumption, the observability analysis is carried out in the ω-limit sets of trajectories. Here, dissipativity is used at each step of this proof, either to guarantee that ε is non-increasing or that $C\varepsilon$ tends toward zero. The use of dissipativity in the context of dynamic output feedback stabilization is a powerful property beyond Theorem 3.1. This is illustrated in the following section, by applying dissipativity techniques on an example that is not immediately state-affine dissipative.

3.2. **A kinematic drone model example.** In this section, we consider an example of a kinematic drone model admitting an immersion into a bilinear system (see Theorem 2.5). It is a Dubins model variation for a fixed wings drone (or UAV), flying at constant altitude, with constant linear velocity:

$$\begin{cases} \dot{x}_1 = \cos\theta, \\ \dot{x}_2 = \sin\theta, \\ \dot{\theta} = u, \qquad -u_{\max} \leqslant u \leqslant u_{\max}. \end{cases}$$

We are interested in the following output feedback stabilisation problem: endowed with the only information given by $y = x_1^2 + x_2^2$, the square of the distance to the origin, we ask "is it possible to stabilize this system on a circular trajectory of minimal radius $1/u_{\max}$ around the origin?"

This system admits rotational symmetry and therefore cannot be observable. This symmetry can be reduced by introducing new coordinates in a moving frame. We set

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and the system reduces to

$$\begin{cases} \dot{\tilde{x}}_1 = u\,\tilde{x}_2 + 1, \\ \dot{\tilde{x}}_2 = -u\,\tilde{x}_1. \\ \ y = \tilde{x}_1^2 + \tilde{x}_2^2 \end{cases} \tag{11}$$

As a consequence, the targeted set of minimal radius (traveled counter-clockwise) is reduced to the point $(0, 1/u_{\max})$ in these new coordinates.

In that form, it becomes clear that with constant control $u = 0$, the observation is not sufficient to disambiguate the two trajectories with initial conditions $(\tilde{x}_1^0, \tilde{x}_2^0)$ and $(\tilde{x}_1^0, -\tilde{x}_2^0)$. This means that the reduced system is not uniformly observable, and $u = 0$ is the only observability singularity.

To stabilize $\tilde{x}$ at $(0, 1/u_{\max})$, we propose the classical bilinear embedding $\tau : \mathbb{R}^2 \to \mathbb{R}^3$ setting $z = \tau(\tilde{x}) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_1^2 + \tilde{x}_2^2)$. Then we get the bilinear system with linear observation

$$\begin{cases} \dot{z} = Az + uBz + b, \\ y = Cz, \quad u \in [-u_{\max}, u_{\max}] \end{cases} \tag{12}$$

with $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. Having in mind dissipativity properties like in the previous section, we pick a Luenberger observer

$$\dot{\hat{z}} = A\hat{z} + uB\hat{z} + b - K(C\hat{z} - y), \qquad \text{with } K' = \begin{pmatrix} 2 & 0 & \alpha \end{pmatrix},\ \alpha > 0. \tag{13}$$

This choice of K is fundamental, as it implies dissipative dynamics on the observer error $\hat{z} - z$, while the parameter α offers a degree of freedom allowing to partially tune the convergence of the observer. In particular $\frac{\mathrm{d}}{\mathrm{d}t}|\hat{z} - z|^2 = -2\alpha|\hat{z}_3 - z_3|^2$, hence the observer error is bounded. Relying on this observer, we are able to prove the following closed-loop result.

Theorem 3.4 ([2])**.** *System* (11) *is semi-globally stabilizable by means of dynamic output feedback. More precisely, for any smooth feedback globally stabilizing* (11) *at the target, there exists a choice of $\alpha > 0$ for which the coupled state-observer system* (11)-(13) *in closed loop is asymptotically stable at the target, with an arbitrarily large basin of attraction.*

The proof follows similar structure to that of Theorem 3.1. Let us emphasize the main differences. System (12) is not a dissipative system, which prevents to apply directly Theorem 3.1. However, thanks to the design of the observer gain K in (13), the observer error $\hat{z} - z$ has a dissipative dynamics. This is the key tool that allows to adapt the techniques of Section 3.1 even if the original system is not dissipative. The system (11) to be stabilized is not a state-affine dissipative system. However, using Theorem 2.5, it can be immersed into the bilinear system (12). Hence, initial conditions of (12) must be taken inside the submanifold $\tau(\mathbb{R}^2)$. In Theorem 3.1, the observer gain α was chosen small enough in order to increase the basin of attraction by guaranteeing that the correction term $\alpha PC'C(\hat{x} - x)$ is small since $\hat{x} - x$ is bounded. On the contrary, in theorem 3.4, α is chosen large enough to guarantee that $C\hat{z} - y$ can be made small fast enough, which is useful for Lyapunov based arguments. The constraints on the form of K leading to dissipativity result in this alternative strategy in the choice of α.

3.3. **A quantum control unobservable target.** Consider the Bloch equations

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx \end{cases}, \quad A(u) = \begin{pmatrix} 0 & 1 & u_1 \\ -1 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \tag{14}$$

where x in the unit sphere $\mathbb{S}^2$ is the state, u in $\mathbb{R}^2$ is the control and y in $\mathbb{R}$ is the output. Since $A(u)$ is skew-symmetric for all $u \in \mathbb{R}^2$, trajectories of (14) are bound to the unit sphere $\mathbb{S}^2$. The goal is to stabilize (14) at the target point $x^\star = (0, 0, -1)$. Clearly, the system is unobservable for the control $u = 0$, which is the value of the stabilizing control at the target point. Although (14) is dissipative, one can not readily apply Theorem 3.1 since the target is not detectable. A new stabilization strategy must be devised.

It can be shown by applying LaSalle's invariance principle on the candidate Lyapunov function $V(x) = x_3$ that the state feedback law $\lambda(x) = (x_1, x_2)$ is asymptotically stabilizing system (14) with basin of attraction $\mathbb{S}^2 \setminus \{-x^\star\}$. Then, by choosing a feedback perturbation of the form $\delta(\hat{x}_3^2 - 1)$ (this choice is discussed in Section 4.1), the following result can be obtained.

Theorem 3.5 ([23])**.** *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, the system*

$$\begin{cases} \dot{x} = A(u)x \\ y = Cx \end{cases}, \qquad \begin{cases} \dot{\hat{x}} = A(u)\hat{x} - C'(C\hat{x} - y) \\ u = \lambda(\hat{x}) + \delta(\hat{x}_3^2 - 1) \end{cases} \tag{15}$$

is locally asymptotically stable at $(x^\star, x^\star)$ with a basin of attraction that is open, dense, and of full measure in $\mathbb{S}^2 \times \mathbb{R}^3$.

The dissipativity of (14) remains crucial to design a Luenberger observer with non-increasing error, but the observability analysis is performed along the trajectories of the system. More precisely, it is shown in [23] that for any input u generated by the closed-loop (15), system (14) is observable except if (15) is initialized at the target $(x^\star, x^\star)$. As in Section 2.3, a perturbation of the feedback law is used. However, this perturbation is explicitly designed, and does not help to avoid observability singularities along trajectories but rather to get new observability properties near the target point in closed-loop.

4. An embedding strategy

We have seen in the previous section how dissipativity may be used in the context of output feedback stabilization. Inspired by examples such as the kinematic drone model (Section 3.2), we propose embedding strategies into dissipative systems. We investigate the case of a harmonic oscillator where the measured output is the norm of the state. This example can be seen as an extension of the one-dimensional example $\dot{x} = u,\ y = x^2$ investigated in [12]. First, we present a method similar to Section 3.2, where after linearization of the output, the observer can be designed to be error-dissipative. In this case however, the singularity at the target imposes a perturbation of the control, chosen to depend on the distance to the target similarly to Section 3.3. In a second part, we present a different embedding of the harmonic oscillator based on unitary representation theory inspired by [10]. This infinite dimensional embedding is not tied to the measured output, and allows to consider complex non-linear outputs appearing as bounded linear forms in the embedded system.

4.1. **The harmonic oscillator problem.** Consider the problem of dynamic output feedback stabilization of the following system

$$\begin{cases} \dot{x} = Ax + bu \\ y = \dfrac{1}{2}|x|^2 \end{cases}, \qquad A' = -A. \tag{16}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $u \in \mathbb{R}$. The difficulty comes from the unobservability of the target point 0. Indeed, the constant input $u = 0$ makes the system unobservable in any positive time, due to the skew-symmetry of A combined with the symmetry of the output. Actually, this system is not stabilizable by means of a dynamic output feedback when A is not invertible, as shown in [8]. In

[12], time-varying stabilization strategies are considered to tackle this issue. In a different line, we wish to investigate the case where A is invertible and still restrict ourselves to time-independent strategies. Note that the invertibility of A does not influence the unobservability of the system.

As in Section 3.2, we can embed the system into a bilinear one with linear output by considering the change of coordinates $z = (x, y) \in \mathbb{R}^{n+1}$. The resulting dynamics takes the form

$$\begin{cases} \dot{z} = \mathcal{A}(u)z + \mathcal{B}u \\ y = \mathcal{C}z. \end{cases} \tag{17}$$

where $\mathcal{A}(u) = \begin{pmatrix} A & 0 \\ ub' & 0 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$ and with initial conditions in $\{(x, \frac{1}{2}|x|^2), x \in \mathbb{R}^n\}$. System (17) is not dissipative. However, it is possible to define a Luenberger observer with dissipative error dynamics by considering the time-varying observer gain $\mathcal{L}(u) = \begin{pmatrix} bu \\ \alpha \end{pmatrix}$. The resulting observer $\hat{z}$ satisfies

$$\dot{\hat{z}} = \mathcal{A}(u)\hat{z} + \mathcal{B}u - \mathcal{L}(u)(\mathcal{C}\hat{z} - y),$$

hence the error $\varepsilon = \hat{z} - z$ is dissipative since $\frac{1}{2}\frac{\mathrm{d}\|\varepsilon\|^2}{\mathrm{d}t} = -\alpha \|\mathcal{C}\varepsilon\|^2 \leqslant 0$.

Then, a natural strategy to close the loop would be to apply the feedback law $u = K\hat{x}$, where K is such that $A + bK$ is Hurwitz and $\hat{x}$ corresponds to the n firsts coordinates of $\hat{z}$. However, this strategy fails to be applied due to the unobservability of the target point. Indeed, any trajectory of the closed-loop system starting from $(x(0), \hat{z}) = (x_0, 0, \frac{1}{2}|x_0|^2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ for some $x_0 \in \mathbb{R}^n \setminus \{0\}$ is such that u remains constantly equal to 0 and $\mathcal{C}\hat{z}$ remains constantly equal to $\frac{1}{2}|x_0|^2$. Therefore, x does not tend towards 0.

This issue can be dealt with by adding a perturbation of the feedback law of the form $u = K\hat{x} + \delta\hat{y}$, where $\hat{y}$ stands for the last coordinate of $\hat{z}$. By choosing δ small enough, the stabilizing nature of the feedback law is not affected. Yet, the perturbation term improve the observability properties of the closed-loop, in the sense that the only trajectory that generates the constant input $u = 0$ is $(x, \hat{z}) = (0, 0)$. A similar result holds for the perturbation of the quantum control system in Section 3.2. Thanks to this observation, the following semi-global dynamic output feedback stabilization result can be obtained.

Theorem 4.1 ([8]). *If A is skew-symmetric and invertible and (A, b) is stabilizable, then*

$$\begin{cases} \dot{x} = Ax + bu \\ y = \dfrac{1}{2}|x|^2 \end{cases}, \quad \begin{cases} \dot{\hat{z}} = \mathcal{A}(u)\hat{z} + \mathcal{B}u - \mathcal{L}(u)(\mathcal{C}\hat{z} - y), \\ u = \begin{pmatrix} K & \delta \end{pmatrix} \hat{z} \end{cases}$$

is locally asymptotically stable with basin of attraction containing an arbitrarily large basin of attraction by choosing δ small enough and α large enough.

With this method, we were able to deal with the singularity of this non-linear output through embedding into a bilinear system and perturbation of the output. However this result relies on the precise shape of the output to find an embedding that is error-dissipative. We propose an alternative embedding that allows to linearize many more potential outputs. The price to pay is to consider infinite-dimensional embeddings. For this presentation, we still focus on the particular output $y = |x|^2/2$, with some discussion of generalization at the end.

4.2. **A unitary group representation point of view for error dissipative embedding.**

4.2.1. *Unitary embedding.* In [10], the authors investigated the problem of observer design for (16) by means of infinite-dimensional embeddings. We briefly recall their strategy, that relies on representation theory (see, e.g., [5, 36]). The Lie group G of flows generated by the dynamical system (16) is isomorphic to $\mathbb{R}^2 \rtimes_{\mathcal{R}} \mathbb{S}^1$, where $\mathbb{S}^1 \simeq \{e^{tA}, t \in \mathbb{R}_+\}$ is the group of rotations, $\mathcal{R} : \mathbb{S}^1 \ni \theta \mapsto e^{\theta A}$ is an automorphism of $\mathbb{R}^2$ and $\rtimes_{\mathcal{R}}$ denotes the outer semi-direct product with respect to $\mathcal{R}$. Hence G is the group of motions of the plane. According to [36, Section IV.2], its unitary irreducible representations are given by a family $(\rho_\mu)_{\mu>0}$, where for each $\mu > 0$,

$$\begin{array}{rccl}\rho_\mu : & G & \longrightarrow & \mathscr{L}(L^2(\mathbb{S}^1,\mathbb{C})) \\ & (x,\vartheta) & \longmapsto & \left(\xi \in L^2(\mathbb{S}^1,\mathbb{C}) \mapsto \left(\mathbb{S}^1 \ni s \mapsto e^{i\mu(1,0)e^{sA'}x}\xi(s-\vartheta)\right)\right).\end{array}$$

Let $X = L^2(\mathbb{S}^1,\mathbb{C})$ be the set of real-valued square-integrable functions over $\mathbb{S}^1$. Then X is a Hilbert space endowed with the scalar product defined by $\langle \xi,\zeta\rangle_X = \frac{1}{2\pi}\int_0^{2\pi}\xi(s)\bar{\zeta}(s)\mathrm{d}s$ and the induced norm $\|\cdot\|_X$. Since $\mathbb{S}^1$ is compact, the constant function $\mathbb{1} : s \mapsto 1$ lies in X. Let $\mu > 0$ to be fixed later. Set $\tau : \mathbb{R}^2 \to X$ such that $\tau(x) = \rho_\mu(x,0)\mathbb{1}$ for all $x \in \mathbb{R}^2$ (naturally, τ depends on μ but we omit it for readability). Since ρ_μ is a unitary representation, $\|\tau(x)\|_X = 1$ for all $x \in \mathbb{R}^2$ and $\tau(0) = \mathbb{1}$. For all $x = (x_1,x_2) = (r\cos\theta, r\sin\theta)$ in $\mathbb{R}^2$, we have

$$\tau(x) : \mathbb{S}^1 \ni s \mapsto e^{i\mu(x_1\cos(s)+x_2\sin(s))} = e^{i\mu r\cos(s-\theta)}.$$

Clearly, τ is injective. Let x be a solution of (16) and set $z = \tau(x)$. Then

$$\dot{z} = i\mu\left(-x_2\cos(s) + x_1\sin(s) + u\sin(s)\right)z = -\frac{\partial z}{\partial s} + iu\mu\sin(s)z =: \mathcal{A}(u)z$$

where $\mathcal{A}(u)$ is defined on the dense domain $\mathcal{D} = H^1(\mathbb{S}^1,\mathbb{C}) = \{f \in X \ :\ f' \in X\}$ and is the skew-adjoint generator of a strongly continuous unitary group on X for any $u \in \mathbb{R}$.

Denote $(e_k)_{k\in\mathbb{Z}}$ the canonical Hilbert basis of X. Then

$$\langle \tau(x), e_k\rangle_X = \frac{1}{2\pi}\int_0^{2\pi} e^{i\mu r\cos(s-\theta)-iks}\mathrm{d}s = i^k J_k(\mu r)e^{-ik\theta}$$

where J_k denotes the k-th Bessel function of the first type. In particular, it appears that $\langle\tau(x),e_0\rangle_X = J_0(\mu|x|)$. Denoting j_0' the first zero of J_0', the knowledge of $|x|$ is equivalent to the knowledge of $J_0(\mu r)$ in the disk $\{x \in \mathbb{R}^2 \ :\ |x| < j_0'/\mu\}$. This allows to embed (16) into the unitary bilinear infinite-dimensional system

$$\begin{cases}\dot{z} = \mathcal{A}(u(t))z \\ \mathfrak{y} = \mathcal{C}z,\end{cases} \tag{18}$$

locally around 0 (when x lies in this disk), where $\mathcal{C} = \langle\cdot,e_0\rangle$ is a bounded linear form. By selecting μ small enough, this domain can be made arbitrarily large, which help us to derive a semi-global stabilization result.

4.2.2. *Observer design.* An infinite-dimensional Luenberger observer of (18) takes the form

$$\dot{\hat{z}} = \mathcal{A}(u(t))\hat{z} - r\mathcal{C}^*(\mathcal{C}\hat{z} - \mathfrak{y})$$

where r is some positive observer gain and $\mathcal{C}^*$ denotes the adjoint of $\mathcal{C}$. Since $\mathcal{A}(u)$ is a unitary operator for any u, the error system $\varepsilon = \hat{z} - z$ that satisfies $\dot{\varepsilon} = (\mathcal{A}(u(t)) - r\mathcal{C}^*\mathcal{C})\varepsilon$ is such that $\frac{1}{2}\frac{\mathrm{d}\|\varepsilon\|^2}{\mathrm{d}t}(t) = -\alpha\left\|\mathcal{C}\varepsilon(t)\right\|^2 \leqslant 0$. It is well-known that, under asymptotic almost periodicity assumption and approximate or exact observability hypotheses, one can deduce respectively the strong or weak convergence of ε towards 0 (see, e.g., [6]). However, our goal here is to use this infinite-dimensional observer in closed-loop to stabilize (16). To do so, we need to obtain, from the estimation $\hat{z}$ of z, an estimation $\hat{x}$ of x, the state of the original system. This is obtained thanks to a left-inverse of the embedding τ, whose existence follows from the next theorem proved in [8]. Indeed, noetherianity of analytic maps allows to show that finitely many bounded linear forms are sufficient to discriminate any two points of an embedded compact.

Theorem 4.2 ([8])**.** *Let X be a separable Hilbert space, $\tau : \mathbb{R}^n \to X$ be an analytic map and $\mathcal{K}_x \subset \mathbb{R}^n$ be a compact set . If $\tau|_{\mathcal{K}_x}$ is injective, there exists a continuous map $\pi : X \to \mathcal{K}_x$, a class K_∞ function[1] ρ^* and $Q \in \mathscr{L}(X,\mathbb{C}^q)$ for some a positive integer q such that, for all $(x,\xi) \in \mathcal{K}_x \times X$,*

$$|\pi(\xi) - x| \leqslant \rho^*(|Q(\xi - \tau(x))|).$$

[1]A class K_∞ function is a continuous function $\rho^* : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho^*(0) = 0$, ρ^* is strictly increasing and tends to infinity at infinity.

As a consequence, if $(x_n)_{n\in\mathbb{N}}$ and $(\xi_n)_{n\in\mathbb{N}}$ are two sequences in $\mathcal{K}_x$ and X, respectively, such that $\xi_n - \tau(x_n) \overset{w}{\rightharpoonup} 0$, then $|\pi(\xi_n) - x_n| \to 0$. As such, π is known as a strong left-inverse of τ on $\mathcal{K}_x$.

In the specific case of system (16), a left-inverse of τ can be made explicit. Indeed, $x \mapsto \langle \tau(x), e_1\rangle_X = iJ_1(\mu|x|)e^{-i\theta}$ is invertible on $B_{\mathbb{R}^2}(0, \frac{j_1'}{\mu})$ where j_1' denotes the first zero of J_1'. This allows to define $\pi(\xi)$ as a continuously differentiable and globally Lipschitz map applied to $\langle \xi, e_1\rangle_X$:

$$\begin{array}{rccc}\pi: & X & \longrightarrow & \mathbb{R}^2 \\ & \xi & \longmapsto & \mathfrak{f}(\langle \xi, e_1\rangle_X)\end{array}$$

Then π defines a strong left-inverse of τ over $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu})$, where $j < j_1'$.

4.2.3. *Feedback perturbation and sampling.* To close the loop of the system, it remains to apply a stabilizing state feedback law $u = Kx$ (i.e., such that $A + bK$ is Hurwitz) to the estimation of the state obtained by $\hat{x} = \pi(\hat{z})$. However, due the unobservability of the system when $u = 0$, this strategy fails to be directly applied. An additional step is required to improve the observability properties of the resulting closed-loop system. We sample the input in order to reduce the observability issues to constant controls. That is, for a given constant time step subdivision (t_k) of $[0, +\infty)$, we assume that the control is constant and periodically updated every t_k according to some feedback law. Then, the input $u = 0$ remains an isolated observability singularity, in the sense that any small non-zero constant input makes the system observable. In order to deal with this last singularity at the target, we proceed as in Sections 3.3 and 4.1 by means of a feedback perturbation. We add to the feedback law a term that vanishes only when $\hat{z}(t)$ tends towards the embedded target $\tau(0) = \mathbb{1}$. Since the output operator $\mathcal{C}$ is bounded, only weak convergence of the observer can be guaranteed (see [6]). Therefore, we choose a perturbation of the form $\delta\mathcal{N}^2(\hat{z} - \mathbb{1})$, where δ is a sufficiently small constant parameter and $\mathcal{N}^2(\xi) = \sum_{k\in\mathbb{Z}} \frac{|\langle \xi, e_k\rangle_X|^2}{k^2+1}$, so that $\mathcal{N}$ is a norm associated to the weak topology on bounded sets. Doing so, we obtain the following semi-global dynamic output feedback stabilization result.

Theorem 4.3 ([8])**.** *The closed-loop system*

$$\begin{cases} \dot{x} = Ax + bu, \\ y = \dfrac{1}{2}|x|^2. \end{cases}, \qquad \begin{cases} \dot{\hat{z}} = \mathcal{A}(u(t))\hat{z} - \alpha\mathcal{C}^*(\mathcal{C}\hat{z} - J_0(\mu\sqrt{2y})), \\ u(t_k) = K\pi(\hat{z}(t_k^-)) + \delta\mathcal{N}^2(\hat{z}(t_k^-) - \mathbb{1}) \\ u(t) = u(t_k), \qquad t \in [t_k, t_{k+1}) \end{cases}$$

is locally stable at the equilibrium point $(0, \mathbb{1})$ in $\mathbb{R}^2 \times (X, \|\cdot\|_X)$ and attractive in $\mathbb{R}^2 \times (X, \mathcal{N})$ with basin of attraction containing $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu}) \times \tau\left(\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu})\right)$.

Although the problem of output feedback stabilization of (16) was tackled by Theorem 4.1, the infinite-dimensional strategy of Theorem 4.3 offers new perspectives. Indeed, it actually allows to take into account more general output maps than $y = \frac{1}{2}|x|^2$. While the embedding obtained in Section 4.1 was highly dependent of this choice, the framework of Section 4.2 is sufficiently general to deal more output maps. If there exists a map $\mathfrak{h}$ such that $\mathfrak{h}(y) = \mathcal{C}\tau(x)$, then the output feedback stabilization strategy of Theorem 4.3 still holds (see [8, Theorem 4.4]) when $\mathcal{C}^*$ lies in the span of a finite number of elements of the canonical Hilbert basis of X. Due to Gelfand–Raikov theorem, output maps satisfying this condition are dense for the uniform convergence on compact sets of $\mathbb{R}^2$. In the case where $y = \frac{1}{2}|x|^2$, one can choose $\mathfrak{h}(y) = J_0(\mu\sqrt{2y})$. This shows that the output maps $y = J_0(\mu|x|)$ or $y = J_2(\mu|x|)\cos(2\theta)$, with θ the argument of x, both present an observability singularity when $u = 0$ and are covered by Theorem 4.3, while the strategy of Theorem 4.1 seems unapproachable.

5. Perspectives

In this work, we have reviewed various techniques for output feedback stabilization of non-uniformly observable systems, from feedback perturbation strategies to embedding into systems admitting observers with dissipative errors. Depending on the class of systems under consideration and the nature of observability singularities, these strategies must be combined in different ways to come up with efficient solutions to the output feedback stabilization issue. Uniting these techniques to obtain more universal results on dynamic output feedback stabilization is still an ongoing effort. We have essentially restricted ourselves to autonomous approaches, except in the last section where the input is sampled. Time-varying approaches such as sampling, switching or quantification of the input are promising ways to deal with observability singularities. In [9], we have illustrated this methodology and combined it with embeddings into dissipative systems. In future works, we wish to develop this point of view, that takes its roots in the seminal works [12, 29].

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